MARKOV PROCESSES WITH IDENTICAL HITTING DISTRIBUTIONS

BY

R. M. BLUMENTHAL, R. K. GETOOR, AND H. P. McKEAN, JR.[1]

1. Introduction

The results in this paper were announced in [1]. Let X and X^* be time homogeneous Markov processes taking values in a locally compact space E with a countable base. Suppose both processes satisfy Hunt's condition (A) [6, pp. 48–50]. We are interested in knowing when there exists a continuous random time change $\tau(t)$ in the sense of [10, p. 104] such that $X(\tau(t))$ and $X^*(t)$ have the same transition function. An obvious necessary condition, at least if $\tau(t) \rightarrow \infty$ as $t \rightarrow \infty$, is that the two processes have the same hitting distributions. Our main theorem is that this condition is also sufficient.

In order to make this paper more nearly self-contained we will state some facts and reproduce some proofs which appear elsewhere in the literature. Most of the preliminary material is in Section 2, while Section 8 contains a remark on the hypotheses. The rest of the paper is devoted primarily to the construction of the time change. This paper is closely related to [9], which contains a more explicit form of our theorem in case X is Brownian motion and X^* is a diffusion process in Euclidean space. The construction in [9] makes use of potential-theoretic facts which are available for transition functions having a sort of symmetry, but not for those as general as the ones we consider here.

In case the state space is the real line and X and X^* are regular diffusions with the same canonical scale, and hence the same hitting distributions, the additive functional whose inverse is the desired time change is given explicitly in [7] as an integral of the local time for X with respect to the speed measure of X^* relative to X. The existence of such a time change in this case, but not of the integral representation, was also proved by Volkonskii [13].

2. Preliminaries

Let Δ be a point adjoined to E as the point at infinity if E is not compact, and as an isolated point if E is compact. Let $\bar{E} = E \cup \Delta$. Let $\mathcal{B}$ denote the topological Borel field of E and $\bar{\mathcal{B}}$ the Borel sets of $\bar{E}$. A real valued function f defined on E will always be regarded as extended to $\bar{E}$ via the convention $f(\Delta) = 0$.

We will take as sample space for our processes a set W of functions w from $[0, \infty)$ to $\bar{E}$. For any such function w and any positive r define the *shift transformation* θ_r by $\theta_r w.(t) = w(t + r)$. We assume that W is closed under

---

Received March 26, 1962.

[1] The research of all three authors was supported, in part, by the National Science Foundation.

the shift and that every function in W is everywhere right continuous, has left-hand limits, and satisfies $w(t) = \Delta$ whenever $w(r) = \Delta$ and t is greater than r. Set $X(t) = X(t, w) = w(t)$. Let $\mathfrak{F}(t)$ denote the σ-field of subsets of W generated by the sets $\{X(s) \in B\}$ with $s \leqq t$ and B in $\bar{\mathfrak{B}}$, and let $\mathfrak{F}$ be the σ-field generated by the union of the $\mathfrak{F}(t)$.

We suppose given for each x in $\bar{E}$ a probability measure P_x on $\mathfrak{F}$ satisfying: (p_1) for each Λ in $\mathfrak{F}$, $P_x(\Lambda)$ is $\bar{\mathfrak{B}}$-measurable, and (p_2) for each x in $\bar{E}$, $P_x(X(0) = x) = 1$. For A a subset of $\bar{E}$ define the *hitting time* $T_A(w)$ as the infimum of the strictly positive t for which $X(t, w)$ is in A, so that $T_A(w) = \infty$ if there are no such t. We set $\sigma = T_\Delta$ and call $\sigma(w)$ the *terminal time* of the path w. Clearly σ is $\mathfrak{F}$-measurable. The collection (W, P_x, σ), usually denoted just by X, is called a *simple Markov process* if for each x in E, $t \geqq 0$, Λ in $\mathfrak{F}(t)$, and bounded $\mathfrak{F}$-measurable f we have

$$(\mathrm{p}_3) \qquad E_x(f(\theta_t w); \Lambda) = E_x(E_{X(t)} f; \Lambda),$$

that is, $E_{X(t)} f$ is one version of $E_x(f(\theta_t w) \mid \mathfrak{F}(t))$. The expression $E_x(g; \Gamma)$ denotes the integral with respect to P_x of the $\mathfrak{F}$-measurable function g over the $\mathfrak{F}$-set Γ. Clearly (p_1) implies the $\mathfrak{B}$-measurability of this as a function of x.

The statements in the following five paragraphs are familiar and easy to prove. Elaboration can be found in [4] and Sections 1 through 3 of [6].

The mapping $(t, w) \to X(t, w)$ is measurable with respect to $\mathfrak{J} \times \mathfrak{F}$ and $\bar{\mathfrak{B}}$, where $\mathfrak{J}$ denotes the Borel subsets of $[0, \infty)$. Also if f is a bounded $\mathfrak{B}$-measurable function, then $E_x f(X(t))$ is $\mathfrak{J}\times\bar{\mathfrak{B}}$-measurable in (t, x). If for B in $\mathfrak{B}$ we set $P(t, x, B) = P_x(X(t) \in B)$, then $P(t, x, B)$ is a sub-Markovian transition function called the *transition function* of X. For each B it is measurable in (t, x), and of course $P(t, x, E) = P_x(\sigma > t)$.

Let μ be a finite measure on $\mathfrak{B}$, and let $\mathfrak{B}(\mu)$ be the completion of $\mathfrak{B}$ with respect to μ. The intersection of the $\mathfrak{B}(\mu)$ as μ ranges over all such measures is a σ-field $\mathfrak{A}$ containing at least the analytic sets of E. Let $\bar{\mathfrak{A}}$ consist of those sets in $\bar{E}$ whose trace on E is in $\mathfrak{A}$. For μ as above and Λ in $\mathfrak{F}$ we set

$$P_\mu(\Lambda) = \int P_x(\Lambda)\mu(dx).$$

Let $\mathfrak{G}$, $(\mathfrak{G}(t))$, be the σ-field of subsets of W which are in the completion of $\mathfrak{F}$, $(\mathfrak{F}(t))$, with respect to each P_μ. The measures P_μ naturally extend to $\mathfrak{G}$, and with this extension $P_x(\Lambda)$ is $\mathfrak{A}$-measurable in x for each Λ in $\mathfrak{G}$. The mapping $(t, w) \to X(t, w)$ is measurable with respect to $\mathfrak{J} \times \mathfrak{G}$ and $\bar{\mathfrak{A}}$, and the simple Markov property (p_3) remains valid if we assume merely that f is $\mathfrak{G}$-measurable and Λ is in $\mathfrak{G}(t)$.

A nonnegative function T on W, possibly taking on the value ∞, is called a *stopping time* if $\{T < t\}$ is in $\mathfrak{G}(t)$ for each positive t. For example, σ is a stopping time, even satisfying the stronger condition $\{\sigma < t\}$ in $\mathfrak{F}(t)$. The maximum, minimum, and sum of two stopping times is again one, and so is the pointwise limit of a sequence of stopping times. If R is a positive function on W, we define the shift θ_R on $\{R < \infty\}$ by $\theta_R w.(t) = w(t + R)$. If R and S are both stopping times and we set $T(w) = R(w) + S(\theta_R w)$, taking T to be infinite if R is, then T is a stopping time. Given a stopping time T we denote by $\mathcal{G}(T+)$ the σ-field consisting of all Λ in $\mathcal{G}$ such that $\Lambda \cap \{T < t\}$ is in $\mathcal{G}(t)$ for each positive t. Note that if $T \equiv t$, then $\mathcal{G}(t+)$ is the intersection of the fields $\mathcal{G}(s)$ with $s > t$. If T and S are stopping times, then $\{T < S\}$ is in $\mathcal{G}(T+)$ and also in $\mathcal{G}(S+)$. The stopping time T is $\mathcal{G}(T+)$-measurable, and the mapping $w \to X(T(w), w)$ defined on $\{T < \infty\}$ is measurable relative to $\mathcal{G}(T+)$ and $\bar{\mathfrak{A}}$.

A simple Markov process $X = (W, P_x, \sigma)$ is called a *strong Markov process* if for each x in E, stopping time T, Λ in $\mathcal{G}(T+)$, and bounded $\mathcal{G}$-measurable f we have

$$(\mathrm{p}_4) \qquad E_x(f(\theta_T w); \Lambda, T < \infty) = E_x(E_{X(T)} f; \Lambda, T < \infty).$$

Such processes have a zero-one law; namely that if Λ is in $\mathcal{G}(0+)$, then for each x, $P_x(\Lambda)$ is 0 or 1. If (p_4) is satisfied, and we are going to assume it is, then one sees rather easily that $\mathcal{G}(s+) = \mathcal{G}(s)$ for each positive s. Thus we may drop the plus sign in the notation of this and the preceding paragraph.

A process X is called a *Hunt process* if it is a strong Markov process and if whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T, then for each x, $X(T_n) \to X(T)$ almost everywhere P_x on $\{T < \infty\}$. (Henceforth we will use the phrase almost everywhere as an abbreviation for the phrase for each x almost everywhere P_x.) These conditions together with the right continuity of the paths are referred to in [6] as hypothesis (A). A consequence of the right continuity of the paths is that $P_x(\sigma > 0) = 1$ for all x in E.

From now on X is assumed to be a Hunt process. The material in the rest of this section is taken from [6]. The hitting time T_A of an analytic set A in $\bar{E}$ is a stopping time. Moreover for each finite measure μ on $\mathfrak{B}$ there is an increasing sequence $\{K_n\}$ of compact subsets of A such that T_{K_n} decreases to T_A almost everywhere P_μ. In addition if $\mu(A) = 0$, there is a decreasing sequence $\{G_n\}$ of open sets containing A such that T_{G_n} increases to T_A almost everywhere P_μ. Note that if A is contained in E, then $\{T_A < \infty\} = \{T_A < \sigma\}$. By the zero-one law $P_x(T_A = 0) = 0$ or 1, and in the latter case we say that x is *regular for* A. The set of points regular for A is in $\bar{\mathfrak{A}}$.

If A is analytic, we define the *hitting distribution of* A *starting from* x by

$$H_A(x, B) = P_x(X(T_A) \in B, T_A < \infty)$$

for B in $\mathfrak{A}$. If A is contained in E, then clearly the support of this measure is contained in the closure in E of A. In any event this measure attributes no mass to the points of A complement which are not regular for A.

Given such an A contained in E, let S be the minimum of T_A and σ, so that S is the hitting time of $A \cup \{\Delta\}$. Given a number $\lambda \geqq 0$, a nonnegative $\mathfrak{A}$-measurable function f on E is called λ-*excessive* (or just *excessive* when

$\lambda = 0$) *relative to* (X, S) if

$$e^{-\lambda t}E_x(f(X(t)); t < S) \leqq f(x)$$

for all $t \geqq 0$ and x in E, and if also the left side of the inequality increases to the right side as t decreases to 0. The transition function and potential kernel for (X, S) are

$$Q(t, x, B) = P_x(X(t) \in B, t < S),$$

$$V_\lambda(x, B) = \int_0^\infty e^{-\lambda t}Q(t, x, B)\, dt.$$

For a bounded function f we write $Q(t, x, f)$ for $\int Q(t, x, dy)f(y)$, and $V_\lambda f.(x)$ for $\int V_\lambda(x, dy)f(y)$. For example if f is nonnegative and $\mathcal{A}$-measurable, its potential $V_\lambda f$ is λ-excessive relative to (X, S). An important fact [6, Theorem 11.3] is that if f is λ-excessive relative to (X, S), then for each x with P_x probability one the function $t \to f(X(t))$ is right continuous throughout $[0, S)$. We will also use the fact that if f is excessive, and if T and R are stopping times with $T \leqq R$, then $E_x(f(X(T)); T < S)$ is no smaller than $E_x(f(X(R)); R < S)$.

We wish to be rather specific about the realization of a process X terminated in the above manner. Let T be any stopping time such that $T \leqq \sigma$ and $T(\theta_t w) = T(w) - t$ whenever $T(w) \geqq t$. Set

$$\begin{aligned} Y(t, w) &= X(t, w), & t &< T(w), \\ &= \Delta, & t &\geqq T(w)\cdot \end{aligned}$$

The mapping $X \to Y$ gives rise to a new space of paths W' having the same characteristics as W, and to induced measures P'_x on it. We will not phrase the properties of Y in terms of these, but will merely observe that if $\mathcal{H}(t)$ is the σ-field in W generated by the sets $\{Y(s) \in B\}$ with $s \leqq t$ and B in $\bar{\mathcal{B}}$, and if $\mathcal{H}$ is generated by the union of the $\mathcal{H}(t)$, then for each x in $\bar{E}$, $t \geqq 0$, Λ in $\mathcal{H}(t)$, and bounded $\mathcal{H}$-measurable f, we have

$$E_x(f(\theta_t w); \Lambda) = E_x(E_{Y(t)} f; \Lambda),$$

so that Y is a simple Markov process, terminated at time T, which we denote by (X, T). Indeed, (X, T) is a strong Markov process. The validity of this assertion in a more general situation is proved in [10, Part 1, Section 4]. Finally, if $\{T_n\}$ is an increasing sequence of stopping times (for (X, T) and hence for X) with limit R, then $Y(T_n) \to Y(R)$ almost everywhere on $\{R < T\}$ since X and Y agree there. Strictly speaking Y need not be a Hunt process, for this convergence may fail on $\{R = T < \infty\}$. However we can say that on the set $\{T_n < T < \infty, T_n \to T\}$ we have $Y(T_n) \to X(T)$ almost everywhere because of the corresponding properties of X.

We will use the material in this section without special mention.

3. Processes with identical hitting distributions

Let $X = (W, P_x, \sigma)$ and $X^* = (W, P_x^*, \sigma)$ be two Hunt processes with the same enlarged state space $\bar{E}$. Observe that our notation refers to two different families of measures on the same sample space. Quantities analogous to the E_x, P_x, etc. of Section 2, but defined relative to X^*, will be denoted by E_x^*, P_x^*, etc. For example $H_K^*(x, B) = P_x^*(X(T_K) \in B, T_K < \infty)$. We say that X *and* X^* *have identical hitting distributions* if $H_K(x, B) = H_K^*(x, B)$ for all x in E, compact K, and B in $\mathfrak{B}$. From now on we will assume that X and X^* have identical hitting distributions. From the fact that the time of hitting an analytic set can be approximated by the time of hitting compact sets contained in it, and the fact that the paths are right continuous, it follows easily that $H_A(x, B) = H_A^*(x, B)$ for all analytic A, x in $\bar{E}$, and B in $\bar{\mathfrak{A}}$.

PROPOSITION 3.1. *If A is analytic and x is not in A, then x is regular for A relative to X if and only if x is regular for A relative to X^*.*

Proof. It is easy to see that such an x is regular for A relative to X if and only if $H_A(x, \{x\}) = 1$. The same thing is true relative to X^*, so the identity of the hitting distributions gives the result.

A Borel subset B of $\bar{E}$ is said to be *nearly open* relative to X if

$$P_x(T_{B^c} > 0) = 1$$

for all x in B. The superscript c always denotes complement in $\bar{E}$. Proposition 3.1 implies that a set is nearly open relative to X if and only if it is so relative to X^*, so the phrase nearly open needs no further qualification.

Let K be a Borel subset of E whose complement is nearly open, and let $T = T_{K^c}$. For each x and t we have

$$P_x^*(T < t) = P_x^*(\cup \{X(r) \notin K\}),$$

the union being over the rational $r < t$. Thus $P_x^*(T < t)$ is Borel-measurable in x. Suppose now that K is compact and contained in E, and let B be the set of points not regular for K^c. Clearly B is a Borel set and is contained in K. Also B is nearly open, and if $S = T_{B^c}$, then $P_x(T = S) = P_x^*(T = S) = 1$ for all x. The set B is the proper state space for either process terminated when it leaves K. The next proposition states that the property of having identical hitting distributions is preserved when the processes are suitably terminated.

PROPOSITION 3.2. *If K, T, and B are as above, and if D in $\mathfrak{B}$ is a subset of B, then for each bounded $\mathfrak{A}$-measurable f and x in E we have*

$$E_x(f(X(T_D)); T_D < T) = E_x^*(f(X(T_D)); T_D < T)$$

$$= H_D f.(x) - \int_{B^c} H_{K^c \cup D}(x, dy) H_D f.(y).$$

Proof. We may assume that f is $\mathcal{B}$-measurable and, by the usual approximation of hitting times, that D is compact. Then $P_x(T = T_D < \infty) = 0$ since $X(T_D)$ is in D and $X(T)$ is in B^c almost everywhere P_x on the sets $\{T_D < \infty\}$ and $\{T < \infty\}$ respectively. Thus we have

$$E_x(f(X(T_D)); T_D < \infty) = E_x(f(X(T_D)); T < T_D < \infty) + E_x(f(X(T_D)); T_D < T).$$

The first term on the right may be calculated by using the facts that $T_D(w) = T(w) + T_D(\theta_T w)$ on $\{T < T_D\}$ and that $P_{X(T)}(T_D = 0) = 0$ together with the strong Markov property. One obtains

$$E_x(E_{X(T)}(f(X(T_D)); 0 < T_D < \infty); T < T_D < \infty) = \int_{B^c} H_{K^c \cup D}(x, dy) H_D f.(y).$$

The same calculation using X^* and the identity of the hitting distributions gives the desired result.

One should note that

$$E_x(f(X(T_D)); T_D < T) = E_x(f(X(T_D)); T_D < \min(T, \sigma)),$$

and that the same statement holds relative to X^*.

4. A particular excessive function and a theorem of Dynkin

Let C be an open set with compact closure K contained in E, and suppose that for each x in K, $P_x^*(T_{K^c} < \infty) = 1$. Let T be the hitting time of $\bar{E} - K$, and as before, $B = \{x: P_x(T = 0) = 0\}$, so that B is the state space for (X^*, T). For λ strictly positive, set

$$f_\lambda(x) = E_x^*(1 - e^{-\lambda T}) \quad \text{and} \quad g_\lambda(x) = 1 - f_\lambda(x).$$

Clearly f_λ vanishes on B^c, while the computation

$$E_x^*(f_\lambda(X(t)); t < T) = E_x^*(1 - e^{-\lambda(T-t)}; t < T)$$

for x in B shows that f_λ is excessive relative to (X^*, T).

We wish to show that f_λ is also excessive relative to (X, T), and this follows from a theorem of Dynkin announced in [5]. Professor Dynkin was kind enough to send us a proof of his announcement, and we shall give it now.

THEOREM 4.1. *Let f be a bounded $\mathcal{B}$-measurable function such that for each compact subset D of B and each x we have*

$$E_x(f(X(T_D)); T_D < T) \leqq f(x),$$

and $E_x(f(X(t)); t < T) \to f(x)$ as $t \to 0$. Then f is excessive relative to (X, T).

Proof. As in Section 2 we denote the transition function of (X, T) by Q, and the potential operators by V_λ. Let λ and η be strictly positive, and let $h = V_{\lambda+\eta} f$, $F = \lambda h - f$. An application of the resolvent equation shows that for $\mu > 0$ we have

$$h = V_\mu[f + (\mu - \lambda - \eta)h] = V_\mu[(\mu - \eta)h - F].$$

Let $A = \{y: F(y) \leqq 0\}$, and let $\{D_n\}$ be a sequence of compact subsets of A such that T_{D_n} decreases to T_A, P_x almost everywhere, x now being a fixed point of B. Now letting $S_n = \min(T_{D_n}, T)$ and $S = \min(T_A, T)$ and applying Dynkin's lemma (see 41.1 B of [8]) we have for each strictly positive μ

$$E_x(e^{-\mu S_n} h(X(S_n)); S_n < T) - h(x) = E_x \int_0^{S_n} e^{-\mu t}[F(X(t)) + (\eta - \mu)h(X(t))]\, dt.$$

Consequently

$$E_x(e^{-\mu S_n} F(X(S_n)); S_n < T) - F(x) = \lambda E_x \int_0^{S_n} e^{-\mu t}[F(X(t)) + (\eta - \mu)h(X(t))]\, dt - E_x(e^{-\mu S_n} f(X(S_n)); S_n < T) + f(x).$$

In this last displayed equality we take $\mu \leqq \eta$ and apply the inequality of the hypotheses to the right side. As a result the left side is no smaller than

$$\lambda E_x \int_0^{S_n} e^{-\mu t} F(X(t))\, dt,$$

and since $F(X(t)) \geqq 0$ for $t < S$, we have

$$\liminf_{n\to\infty} E_x(e^{-\mu S_n} F(X(S_n)); S_n < T) - F(x) \geqq 0.$$

Now if $S_n < T$, then $F(X(S_n)) \leqq 0$, and so $F(x) \leqq 0$, x being an arbitrary point of B. We have shown that $\lambda V_{\lambda+\eta} f \leqq f$ for all $\eta > 0$ and hence also for $\eta = 0$. If we now set $k = V_\lambda f$ and $g = \lambda k - f$ and apply Dynkin's lemma again, taking the stopping time as the constant t, we obtain

$$Q(t, x, k) - k(x) = \int_0^t Q(u, x, g)\, du \leqq 0,$$

and so $k(x) \geqq Q(t, x, k)$. By the second hypothesis $\lambda V_\lambda f$ approaches f boundedly as $\lambda \to \infty$, and so the assertion of the theorem follows immediately. Perhaps we should note explicitly that a function excessive for (X, T) vanishes off B.

To apply this theorem to f_λ we observe that f_λ is (X^*, T)-excessive, and so for each compact D in B we have $E_x^*(f_\lambda(X(T_D)); T_D < T) \leqq f_\lambda(x)$. Thus by Proposition 3.2 it satisfies the first hypothesis of Theorem 4.1. Also for any x and $\varepsilon > 0$ the point x is not regular relative to (X^*, T) for the set $\{y: | f_\lambda(y) - f_\lambda(x) | > \varepsilon\}$. By Proposition 3.1 the same assertion holds relative to (X, T), and the second hypothesis follows easily. Consequently, f_λ is excessive relative to (X, T).

5. Construction of an additive functional

The sets K and B and the function f_λ are those of the previous section. In this section we will show that f_λ is the potential of an additive functional ϕ_λ and will investigate properties of the functionals corresponding to different values of λ. We rely heavily on the definitions and results of [10]. The construction itself makes use of the fundamental lemma in [11]. We have included some technical improvements on the rest of Šur's construction and have treated a measurability difficulty which he left aside. For a while the λ will be fixed, and we will suppress it in the notation.

Let

$$f_n(x) = n \int_0^{1/n} Q(t, x, f)\, dt,$$

and note that f_n is excessive and that f_n increases to f as $n \to \infty$.

PROPOSITION 5.1. *For a fixed strictly positive ε let*

$$B_n = \{y: f(y) - f_n(y) \geqq \varepsilon\}$$

and $T_n = T_{B_n}$. If $R = \lim_{n\to\infty} T_n$, then

$$E_x(f(X(T_n)); T_n < T) \to E_x(f(X(R)); R < T)$$

for each x, as $n \to \infty$.

Proof. Let

$$\mu_n(A) = P_x(X(T_n) \in A, T_n < T) \quad \text{and} \quad \mu(A) = P_x(X(R) \in A, R < T)$$

for a fixed x in B, and let μ_n^* and μ^* be the same measures defined relative to X^*. The asserted convergence with E_x replaced by E_x^* is an immediate consequence of the strong Markov property, that is,

$$\int f(y)\mu_n^*\,(dy) \to \int f(y)\mu^*\,(dy) \qquad \text{as } n \to \infty$$

Of course μ_n^* is equal to μ_n by Proposition 3.2. Next let $\{G_j\}$ be a decreasing sequence of open sets containing $\bar{E} - B$ and such that if $S_j = T_{G_j}$, then S_j increases to T almost everywhere relative to P_x and P_x^*. Such a sequence exists according to Proposition 2.2 of [6]. Let D_j consist of G_j and the points regular for it, so that $T_{D_j} = S_j$ almost everywhere. Now for a fixed j and n

we have

$$\begin{aligned} E_x^*(1 - e^{-\lambda(T-S_j)}; X(T_n) \in B \cap D_j) &\geqq E_x^*(1 - e^{-\lambda(T-T_n)}; X(T_n) \in B \cap D_j) \\ &= E_x^*(f(X(T_n)); X(T_n) \in B \cap D_j) \\ &\geqq \varepsilon P_x^*(X(T_n) \in B \cap D_j). \end{aligned}$$

Thus $\limsup_{n\to\infty} P_x^*(X(T_n) \in B \cap D_j) \to 0$ as $j \to \infty$, and by the equality of the hitting distributions the same thing holds with P_x^* replaced by P_x. We know that on the set $\{T_n < T \text{ for all } n, T_n \to T\}$ we have $\lim_{n\to\infty} X(T_n) \notin B$ almost everywhere, and consequently we have just shown that the event $\{T_n < T \text{ for all } n, T_n \to T\}$ has P_x and P_x^* probability 0. Now almost everywhere we have $X(T_n) \to X(R)$ on $\{R < T\}$, and by the previous sentence almost everywhere $\{T_n < T\}$ decreases to $\{R < T\}$. Thus μ and μ^* are equal, so the proof is complete. We point out, for use in Proposition 5.6, that the only important property of the B_n used here, aside from the fact that they are decreasing, is the fact that $f(X(T_n))$ is bounded away from 0 on $\{T_n < T\}$. In the remainder of this section and throughout the next one, X and X^* will denote the original processes terminated at time T. The next proposition is the basic lemma from [11].

PROPOSITION 5.2. *If B_n and T_n are as in the previous proposition, then for each x, $P_x(T_n = \infty) \to 1$ as $n \to \infty$.*

Proof. Clearly it suffices to replace T_n by its minimum with T, making a corresponding change in the proposition to be proved. We then have

$$\begin{aligned} &\varepsilon \cdot P_x(T_n < T \text{ for all } n) \\ &\qquad \leqq E_x(\liminf_{k\to\infty} f(X(T_k)) - f_k(X(T_k)); T_n < T \text{ for all } n) \\ &\qquad \leqq \liminf_{k\to\infty} (E_x f(X(T_k)) - E_x f_j(X(T_k))), \end{aligned}$$

where j is any positive integer. Because f_j is excessive, we may, in the expression $E_x f_j(X(T_k))$, replace T_k by $R = \lim T_k$ without destroying the inequality. Applying Proposition 5.1 we then find that the first line in the display is no greater than $E_x f(X(R)) - E_x f_j(X(R))$. This last expression approaches 0 as $j \to \infty$, so the proof is complete.

We begin the construction of the desired additive functional as follows: let

$$g_n(x) = n(f(x) - Q(1/n, x, f)) \quad \text{and} \quad \phi_n(t, w) = \int_0^t g_n(X(r, w))\, dr.$$

Clearly for each n, ϕ_n is a continuous additive functional relative to (X, T) with $\phi_n(t) = \phi_n(T)$ if $t \geqq T$. An easy calculation shows that $E_x \phi_n(\infty) = f_n(x)$. Furthermore $E_x(\phi_n(\infty) \mid \mathcal{G}(t))$ is equal to $\phi_n(t) + f_n(X(t))$, so relative to the measure P_x this latter family of random variables is a martingale. It is separable since f_n composed with the process is right continuous, and so, setting $e_n(t) = \phi_n(t) + f_n(X(t))$ and applying the martingale inequality of Doob [2, p. 353], we have for $\delta > 0$

$$P_x(\sup_t | e_n(t) - e_m(t) | \geqq \delta) \leqq \delta^{-2} E_x(e_n(\infty) - e_m(\infty))^2.$$

We may calculate this last expectation as is done in [11] or [12]. Using the facts that $E_x \phi_n(\infty) = f_n(x)$ and that f_n increases to f we have for $m \leqq n$

$$\begin{aligned}
&E_x(e_n(\infty) - e_m(\infty))^2 \\
&\quad = 2E_x \int_0^\infty [g_n(X(r)) - g_m(X(r))]\, dr \int_r^\infty [g_n(X(s)) - g_m(X(s))]\, ds \\
&\quad = 2E_x \int_0^\infty [g_n(X(r)) - g_m(X(r))]\, dr\, E_{X(r)} \int_0^\infty [g_n(X(s)) - g_m(X(s))]\, ds \\
&\quad \leqq 2E_x \int_0^\infty g_n(X(r))[f(X(r)) - f_m(X(r))]\, dr \\
&\quad \leqq 2E_x \left(\sup_t [f(X(t)) - f_m(X(t))] \int_0^\infty g_n(X(r))\, dr \right) \\
&\quad \leqq 2 \left\{ E_x \left(\int_0^\infty g_n(X(r))\, dr \right)^2 E_x(\sup_t [f(X(t)) - f_m(X(t))])^2 \right\}^{1/2} \\
&\quad \leqq 2^{3/2} k \{ E_x(\sup_t [f(X(t)) - f_m(X(t))]^2) \}^{1/2},
\end{aligned}$$

where k is an upper bound for f. By Proposition 5.2 the last term approaches 0 as $m \to \infty$. This estimate and another application of Proposition 5.2 shows that for each x and strictly positive δ,

$$P_x(\sup_t | \phi_n(t) - \phi_m(t) | \geqq \delta) \to 0, \qquad m, n \to \infty.$$

THEOREM 5.3. *There is a continuous additive functional ϕ of (X, T) with $\phi(t) = \phi(T)$ for all $t \geqq T$ such that $E_x(\phi(\infty)) = E_x \phi(T) = f(x)$.*

Proof. We make use of the theory in the first part of [10]. Form the multiplicative functionals $M_n(t) = \exp(-\phi_n(t))$ and the corresponding semigroups $Q_n(t, x, f) = E_x(f(X(t)) \cdot M_n(t); t < T)$ all of which are subordinate to Q. If $|f| \leqq 1$, then

$$| Q_n(t, x, f) - Q_m(t, x, f) | \leqq E_x(| M_n(t) - M_m(t) |; t < T),$$

so by the calculations of the previous paragraph, for each x, $Q_n(t, x, f)$ is a Cauchy sequence, uniformly in t positive, and $|f| \leqq 1$. If we set $Q_0(t, x, f) = \lim_{n\to\infty} Q_n(t, x, f)$, then the uniformity of the convergence implies that Q_0 is a semigroup, and that $Q_0(t, x, 1)$ is continuous in t at the point $t = 0$. It is subordinate to Q, obviously, and so by Theorem 2.2 of [10, first part] there is a multiplicative functional $M(t)$ such that

$$Q_0(t, x, f) = E_x(f(X(t)) \cdot M(t); t < T)$$

for each t, x and bounded f. For each x and t the sequence $\{M_n(t)\}$ is a Cauchy sequence in P_x measure. Hence it has a limit Y in measure. But by a standard iteration of conditional expectations and the uniform convergence of the semigroups one easily establishes $E_x(M(t); t < T, A) = E_x(Y; t < T, A)$ for all A in $\mathfrak{F}(t)$, and so the limit in P_x measure of $M_n(t)$ is $M(t)$. Let $\phi(t) = -\log M(t)$. Then ϕ is an additive functional, and for each t and x, $\phi_n(t)$ converges to $\phi(t)$ in P_x measure as $n \to \infty$. The bound $E_x[\phi_n(\infty)]^2 \leqq 2k^2$ shows that the $\phi_n(\infty)$ are uniformly integrable, and so $E_x\phi(\infty) = \lim_{n\to\infty} E_x\phi_n(\infty) = f(x)$. To prove that ϕ is continuous, fix an x and choose a sequence $\{n_k\}$ such that $P_x(\sup_t |\phi_n(t) - \phi_m(t)| > 2^{-k}) < 2^{-k}$ if n and m exceed n_k. Then $\phi_{n_k}(t)$ converges uniformly as $k \to \infty$ to a continuous function agreeing with $\phi(t)$ for all rational values of t, these statements holding with P_x probability one. But ϕ is by definition right continuous, so it agrees with the continuous limit for all values of t.

Now we reintroduce the subscript λ and refer to the functional we have just constructed, using f_λ, as ϕ_λ.

PROPOSITION 5.4. *Let A in $\mathfrak{B}$ be a subset of B, and let S be the minimum of T_A and T. Then $E_x\phi_\lambda(S) = E_x^*(e^{-\lambda T}[e^{\lambda S} - 1])$.*

Proof. By the strong Markov property for additive functionals [10, first part; Theorem 4.2] we have $\phi_\lambda(T, w) = \phi_\lambda(S, w) + \phi_\lambda(T - S, \theta_S w)$. But $T(w) - S(w) = T(\theta_S w)$ almost everywhere P_x, so the proposition follows by a routine calculation using the equality of the hitting distributions and the definition of f_λ.

PROPOSITION 5.5. *For each x with P_x probability one, $\phi_\lambda(t)$ is strictly increasing in $[0, T]$.*

Proof. Let $R = \inf\{t \geqq 0 : \phi_\lambda(t) > 0\}$. For each x, $P_x(R = 0)$ is 0 or 1, and we let A be the set on which it is 1. It is easy to see that A is in $\mathfrak{B}$ and that $B - A$ is nearly open, or more precisely, that $P_x(R = T_A) = 1$. If x is in $B - A$ and S is the minimum of T_A and R, then by Proposition 5.4

$$E_x\phi_\lambda(S) = E_x^*(e^{-\lambda T}[e^{\lambda S} - 1]).$$

But the left side of this expression is 0 because ϕ_λ is continuous, while the right side is strictly positive because S is. Thus $B - A$ is empty, and, because ϕ_λ is additive, this implies the proposition.

PROPOSITION 5.6. *Let $B_n = \{x : f_\lambda(x) \geqq 1 - 1/n\}$, and let $T_n = T_{B_n}$. Then $P_x(T_n < T$ for all $n) = 0$ for all x.*

Proof. Let δ denote the probability in question, and let R denote the limit of the T_n. By Proposition 5.1 and the remark following its proof we have

$$E_x(f_\lambda(X(T_n)); T_n < T) \to E_x(f_\lambda(X(R)); R < T), \quad n \to \infty.$$

The n^{th} term of the sequence is no smaller than $\delta(1 - 1/n)$ because $f_\lambda(X(t))$

is right continuous, while the limit is no larger than $P_x(R < T)$. Thus $P_x(R < T) = \delta$, and it then follows that $\delta = 0$ since f_λ is everywhere strictly less than 1.

An obvious consequence of this proposition is that with P_x probability 1 the function $t \longrightarrow g_\lambda(X(t))$ is bounded away from 0 on $[0, T]$.

The next theorem contains the fundamental calculation of this section.

THEOREM 5.7. *If f is a bounded $\mathcal{B}$-measurable function, then*

$$E_x \int_0^T f(X(t))\, d\phi_\lambda(t) = \lambda E_x^* \int_0^T f(X(t)) g_\lambda(X(t))\, dt. \tag{5.1}$$

Proof. We note that replacing the upper limit T by ∞ has no effect on the integrals since the processes move to Δ at time T. Also if f is identically 1, a simple calculation shows the two sides of (5.1) have the common value $f_\lambda(x)$, so the expressions in question are finite. Obviously it is enough to establish the equality whenever f is continuous and bounded in absolute value by 1. Assuming it is, and fixing a strictly positive ε we define as follows

$$\begin{aligned} S_1' &= \inf\{t > 0 : |f(X(t)) - f(X(0))| \geqq \varepsilon\}, \\ S_1 &= \min(S_1', T), \\ S_{n+1} &= S_n + S_1(\theta_{S_n} w), \qquad n \geqq 1. \end{aligned}$$

The $\{S_n\}$ form an increasing sequence of stopping times. If $S_{n+1} < T$, then $|f(X(S_n)) - f(X(S_{n+1}))| \geqq \varepsilon$, and since f is continuous and, almost everywhere P_x, $X(S_n)$ approaches a limit on $\{S_n < T \text{ for all } n\}$, it follows that almost everywhere P_x, $S_n = T$ for large enough n. Now fix x. Since in the interval $[S_n, S_{n+1})$, $f(X(t))$ differs from $f(X(S_n))$ by less than ε, the left side of (5.1) differs by less than $\varepsilon \cdot f_\lambda(x)$ from

$$\textstyle\sum_{n=0}^{\infty} E_x f(X(S_n))[\phi_\lambda(S_{n+1}) - \phi_\lambda(S_n)], \tag{5.2}$$

while the right side differs by less than $\varepsilon \cdot f_\lambda(x)$ from

$$\sum_{n=0}^{\infty} \lambda E_x \int_{S_n}^{S_{n+1}} g_\lambda(X(t))\, dt \cdot f(X(S_n)). \tag{5.3}$$

Here we are taking $S_0 = 0$. But (5.2) and (5.3) are equal, as the following argument shows. Set $G(y) = E_y^*(e^{-\lambda T}[e^{\lambda S_1} - 1])$. We have seen in Proposition 5.4 that $G(y)$ also equals $E_y \phi_\lambda(S_1)$, while an easy calculation shows it to be also equal to

$$\lambda E_y^* \int_0^{S_1} g_\lambda(X(t))\, dt.$$

Now $\phi_\lambda(S_{n+1}(w), w) - \phi_\lambda(S_n(w), w)$ is equal, by the strong Markov property for additive functionals and the definition of the S_n, to $\phi_\lambda(S_1(\theta_{S_n} w), \theta_{S_n} w)$, and so the general term of (5.2) is equal to $\int f(y) G(y) \mu_n(dy)$, μ_n being the distribution under P_x of $X(S_n)$. Similarly

$$\lambda \int_{S_n}^{S_{n+1}} g(X(t))\, dt = \lambda \int_0^{S_1(\theta_{S_n} w)} g_\lambda(X(t, \theta_{S_n} w)\, dt,$$

and so the general term of (5.3) is $\int f(y) G(y) \nu_n(dy)$, where ν_n is the distribution under P_x^* of $X(S_n)$. One sees by induction, using the identity of the hitting distributions, that $\mu_n = \nu_n$ for all n, and so (5.2) and (5.3) are indeed the same. Thus for each $\varepsilon > 0$ the two sides of (5.1) differ by less than $2\,\varepsilon f_\lambda(x)$, and the proof is complete.

THEOREM 5.8. *The expression*

$$\int_0^t [\lambda g_\lambda(X(u))]^{-1}\, d\phi_\lambda(u)$$

is independent of λ *and defines a continuous additive functional* ϕ *which is strictly increasing in* $[0, T]$ *and for which* $\phi(T)$ *is finite, the assertions holding almost everywhere.*

Proof. Clearly for each λ the integral defines an additive functional, which is strictly increasing in $[0, T]$ since ϕ_λ is, and g_λ is bounded. By Proposition 5.6, $g_\lambda(X(t))$ is bounded away from 0 in $[0, T]$ with P_x probability 1. This together with the facts that ϕ_λ is continuous and that $\phi_\lambda(T)$ is finite establish the continuity and finiteness assertions. By Theorem 5.7 for each $\lambda > 0$, $\mu > 0$, x, and bounded f, the expression

$$E_x \int_0^T \mu f(X(t)) g_\mu(X(t))\, d\phi_\lambda(t)$$

is symmetric in λ and μ. Thus by the uniqueness theorem of [10] (second part, Theorem 4.4) the additive functional

$$\mu \int_0^t g_\mu(X(u))\, d\phi_\lambda(u)$$

is invariant under permutation of μ and λ. This proves the assertion.

Note that g_λ increases to 1 as λ decreases to 0. Consequently for each x with P_x probability 1, $\lambda^{-1}\phi_\lambda(t)$ increases to $\phi(t)$ for all t. Also with P_x^* probability 1 the continuous additive functional

$$\int_0^t g_\lambda(X(u))\, du$$

increases to t for all t as λ decreases to 0. We will need these remarks in the next section.

6. The time change in a neighborhood

We will continue to work within the framework of Sections 4 and 5. Let $\psi(t)$ be any continuous strictly increasing additive functional relative to

(X, T) with $\psi(T)$ finite, and let τ denote the functional inverse to ψ, that is,

$$\begin{aligned}\tau(t) &= s \quad \text{if} \quad \psi(s) = t, && t \leqq \psi(T),\\ &= T, && t > \psi(T).\end{aligned}$$

The functional τ is a time change in the sense of [10, second part, Section 7], and the process $X(\tau(t))$ is a simple Markov process terminated at time $\psi(T)$ relative to the measures P_x and to the σ-fields $\mathcal{G}(\tau(t))$ induced by the stopping times $\tau(t)$. Indeed this process is actually a strong Markov process, and it is of course obvious that for each x with P_x probability 1, $\tau(t)$ is strictly increasing in $[0, \psi(T)]$ and is continuous. These matters are discussed fully in the part of [10] just referred to, and proofs are given in [13] with some amplification in [12].

Referring back to the functionals of Section 5, let $\tau(t)$ be the functional inverse to $\phi(t)$, and let $r_\lambda(t)$ be the one inverse to $\lambda^{-1}\phi_\lambda(t)$. Let

$$\psi_\lambda(t) = \int_0^t g_\lambda(X(u))\, du$$

be regarded as an additive functional of (X^*, T), and let $s_\lambda(t)$ be the functional inverse to it. From the last paragraph of Section 5 it follows that for each x with P_x probability 1, $r_\lambda(t)$ decreases to $\tau(t)$, and with P_x^* probability 1, $s_\lambda(t)$ decreases to t as λ decreases to 0.

THEOREM 6.1. *The process (X^*, T) has the same transition function as the process $(Y, \phi(T))$, where $Y(t) = X(\tau(t))$.*

Proof. We need to show that for each x, t and bounded continuous f

$$E_x(f(X(\tau(t))); t < \phi(T)) = E_x^*(f(X(t)); t < T). \tag{6.1}$$

But by the approximation just referred to and the right continuity of the paths it will suffice to show that (6.1) is valid when $\tau(t)$ is replaced by $r_\lambda(t)$ and $\phi(T)$ by $\lambda^{-1}\phi_\lambda(T)$ on the left, and when t is replaced by $s_\lambda(t)$ and T by $\psi_\lambda(T)$ on the right. To do this define the potential operators V and W on bounded continuous functions by

$$Vf.(x) = E_x \int_0^{\lambda^{-1}\phi_\lambda(T)} f(X(r_\lambda(t)))\, dt, \tag{6.2}$$

and

$$Wf.(x) = E_x^* \int_0^{\psi_\lambda(T)} f(X(s_\lambda(t)))\, dt. \tag{6.3}$$

These are the potential operators for $X(r_\lambda(t))$ under P_x and $X(s_\lambda(t))$ under

P_x^* respectively. Now by changes of variables

$$Vf.(x) = E_x \int_0^T f(X(u))\lambda^{-1}\, d\phi_\lambda(u), \quad \text{and}$$

$$Wf.(x) = E_x^* \int_0^T f(X(u))g_\lambda(X(u))\, du.$$

By Theorem 5.7 these two transformations are then the same. They are bounded transformations, having a norm not exceeding $1/\lambda$. Thus one may conclude the proof using the uniqueness argument carried out by Hunt on pages 352–353 of [6, II]. Specifically, his argument shows that if

$$K(t, x, f) = E_x(f(X(r_\lambda(t)))\,;\, t < \lambda^{-1}\phi_\lambda(T)) \quad \text{and}$$

$$L(t, x, f) = E_x^*(f(X(s_\lambda(t)))\,;\, t < \psi_\lambda(T)),$$

then for each positive η,

$$\int_0^\infty e^{-\eta t} K(t, x, f)\, dt = \int_0^\infty e^{-\eta t} L(t, x, f)\, dt.$$

The uniqueness theorem for Laplace transforms, together with the fact that $K(t, x, f)$ and $L(t, x, f)$ are right continuous in t, when f is continuous, shows that K and L agree. This completes the proof.

What we have proved so far is that locally the processes X and X^* are equivalent after a suitable continuous time change in one of them.

7. The complete time change

In this section we will connect the local time changes relating X to X^* to obtain a global time change. We need some preliminary material.

Let C and C' be open subsets of E with compact closures K and K' contained in E, and let T and T' be the hitting times of $\bar{E} - K$ and $\bar{E} - K'$ respectively. Suppose that for each x the hitting times T and T' are finite almost everywhere P_x^*. Let $g_\lambda(x) = E_x^*(e^{-\lambda T})$ and $g_\lambda'(x) = E_x^*(e^{-\lambda T'})$, and let ϕ and ϕ' be the continuous additive functionals constructed in Section 5 relative to T and T' respectively. We wish to show that these two functionals are compatible in the sense that if S denotes the minimum of T and T', then for each x with P_x probability 1, $\phi(t) = \phi'(t)$ for all $t \leqq S$. Note that S is the hitting time of $\bar{E} - (K \cap K')$. Since the functional which equals ϕ on the interval $[0, S]$ and is equal to $\phi(S)$ thereafter is additive relative to (X, S), with a similar statement holding for ϕ', the desired compatibility will follow from Meyer's uniqueness theorem and Theorem 5.8 once we prove the following proposition.

PROPOSITION 7.1. *For each x and bounded $\mathcal{B}$-measurable f we have*

$$E_x \int_0^S f(X(t))g_\lambda'(X(t))\, d\phi_\lambda(t) = E_x \int_0^S f(X(t))g_\lambda(X(t))\, d\phi_\lambda'(t).$$

Proof. If we write the left side of the desired equality as

$$E_x \int_0^T - E_x \int_S^T ,$$

and the right side in the same way with T replaced by T', then use the strong Markov property, the identity of the hitting distributions, and Theorem 5.7, we find that each side is equal to

$$\lambda E_x^* \int_0^S f(X(t)) g_\lambda'(X(t)) g_\lambda(X(t))\, dt.$$

A point x is called a *trap* if $P_x^*(T_{\{x\}^c} < \infty) = 0$. Note that Δ is a trap. Let $\{N_i\}$ be a family of open sets forming a base for the topology of E, and let

$$u_i(x) = \int_0^\infty e^{-t} P^*(t, x, \bar{N}_i)\, dt.$$

Observe that u_i is $\mathcal{B}$-measurable, and that since it is 1-excessive the sets $\{u_i > a\}$ and $\{u_i < a\}$ are nearly open. The point x is not a trap if and only if it lies in some N_i such that $u_i(x) < 1$. Thus the traps form a Borel set C, and we let $\sigma' = T_C$. Let $W_{ij} = \bar{N}_i \cap \{u_i \leqq 1 - 1/j\}$, and with i and j fixed for a moment let T be the hitting time of $\bar{E} - W_{ij}$. This set is nearly open, so by some remarks from Section 2, $E_x^* T$ is $\mathcal{B}$-measurable. It is also bounded according to the argument on p. 640 of [8]. As usual we delete from each W_{ij} the points regular for its complement and enumerate the resulting sets in some order as $\{V_n\}$. Define a function N from $E - C$ to the positive integers by $N(x) = \min\{i : x \in V_i\}$. Each V_i is a nearly open Borel set; and we have associated with each x not in C a nearly open set $V_{N(x)}$ containing it in such a way that $\{x : V_{N(x)} = V_i\}$ is in $\mathcal{B}$. Let $T_i = T_{V_i^c}$, and let ϕ_i be the continuous additive functional constructed as in Section 5, but relative to the terminal time T_i.

In what follows, continuity of an additive functional is defined relative to the topology of the extended real line. Also we will set $X(\infty) = \Delta$.

THEOREM 7.2. *Let X and X^* be Hunt processes with identical hitting distributions. Then there is a continuous additive functional ϕ of X with $\phi(t) = \phi(\sigma')$ for $t \geqq \sigma'$ such that* (i) *for each x with P_x probability* 1, *ϕ is strictly increasing in* $[0, \sigma']$, *and* (ii) *if τ is the inverse to ϕ, that is,*

$$\begin{aligned} \tau(t) &= s \quad \text{if} \quad \phi(s) = t, && t \leqq \phi(\sigma'), \\ &= \sigma', && t > \phi(\sigma'), \end{aligned}$$

then for each positive t, x in E and B in $\mathcal{B}$ we have

$$P_x(X(\tau(t)) \in B) = P_x^*(X(t) \in B).$$

Proof. We need only show that

$$P_x(X(\tau(t)) \in B, t < \phi(\sigma')) = P_x^*(X(t) \in B, t < \sigma')$$

because X and X^* have identical hitting distributions and $\phi(\sigma')$ is the time at which $X(\tau(t))$ reaches a trap. Also for the sake of clarity we intend to carry out the proof of Theorem 7.2 under a mild additional assumption. Then we will give the argument for the general case, omitting some of the details.

So assume now that there is a sequence $\{G_n\}$ of open subsets of E with compact closures $\bar{G}_n$ such that (1) $E = \bigcup_{n=1}^{\infty} G_n$, (2) $\bar{G}_n$ is contained in G_{n+1}, (3) for each n and x we have $P_x(T^n < \infty) = 1$, where T^n is the hitting time of $\bar{E} - \bar{G}_n$. Let ϕ^n be the additive functional constructed in the previous sections, where the K of those sections is taken to be $\bar{G}_n$. The compatibility argument at the beginning of this section shows that if m is greater than n, then for each x with P_x probability 1, $\phi^m(t) = \phi^n(t)$ for all $t \leqq T^n$. Now we may define $\phi(t, w)$ as $\lim_{n\to\infty} \phi^n(t, w)$. The limit exists because of the compatibility just established. Clearly $P_x(T^n \to \sigma) = 1$ for all x since the original process is a Hunt process, and so the ϕ we have just defined is an additive functional continuous and strictly increasing in $[0, \sigma]$, the last assertion holding almost everywhere. The functional τ inverse to ϕ is continuous and strictly increasing in $[0, \phi(\sigma)]$. By the results of Section 6 we have for each n, x, t and B in $\mathcal{B}$

$$P_x(X(\tau(t)) \in B, t < \phi(T^n)) = P_x^*(X(t) \in B, t < T^n).$$

The proof is then completed by letting $n \to \infty$, noting that $\sigma' = \sigma$ in the present situation and using the fact that $P_x^*(T^n \to \sigma) = P_x(\phi(T^n) \to \phi(\sigma)) = 1$.

We now proceed to the general case. Recalling the material in the second paragraph before the statement of the theorem, we wish to connect the functionals ϕ_i in such a way that we get the desired time change. To do this let

$$\begin{aligned} R_1(w) &= T_{N(X(0,w))}, \qquad && X(0, w) \notin C, \\ &= 0, && X(0, w) \in C, \end{aligned}$$

and proceed inductively as follows: if α is an ordinal of the first or second class, and if R_β has been defined for all $\beta < \alpha$, then set

$$R_\alpha(w) = R_{\alpha-1}(w) + R_1(\theta_{R_{\alpha-1}}(w))$$

provided α has an immediate predecessor $\alpha - 1$. Otherwise set $R_\alpha = \sup R_\beta$, the supremum being over all $\beta < \alpha$. Clearly each R_α is a stopping time, and, since the V_i are nearly open, for each $\alpha < \beta$ and x we have $P_x(R_\alpha < R_\beta) > 0$ unless $P_x(R_\alpha = \sigma') = 1$. The argument in Theorem 7.2 of [3] shows that for each x, $P_x(R_\alpha = \sigma') = 1$ for some ordinal α of the first or second class. We define ϕ on $[0, \sigma']$ by induction:

$$\phi(t, w) = \phi_{N(X(0,w))}(t, w), \qquad 0 \leqq t \leqq R_1(w).$$

Suppose ϕ has been defined in $[0, R_\beta]$ for each $\beta < \alpha$ in such a way that (1) the functional agreeing with ϕ in $[0, R_\beta]$ and equal to $\phi(R_\beta)$ thereafter is a continuous additive functional of (X, R_β), (2) the function $t \to \phi(t)$ is with P_x probability 1 strictly increasing in $[0, R_\beta]$, (3) $X(\phi^{-1}(t))$ terminated at $\phi(R_\beta)$ is equivalent to $X^*(t)$ terminated at R_β. We then extend the definition of ϕ to $[0, R_\alpha]$ by setting $\phi(R_\alpha, w) = \lim \phi(t, w)$ as t increases to $R_\alpha(w)$, provided α is a limit ordinal, and otherwise we define

$$\phi(t, w) = \phi(R_{\alpha-1}, w) + \phi_{N(X(0,\theta_{R_{\alpha-1}}w))}(t - R_{\alpha-1}(w), \theta_{R_{\alpha-1}}w)$$

in the interval $R_{\alpha-1} < t \leqq R_\alpha$. Obviously the desired properties (1) through (3) carry over if α is a limit ordinal, but otherwise their validity is rather tedious to verify. One finds that the verification amounts to showing that for each i, j and x not in C we have $\phi_i(t) = \phi_j(t)$ for every $t \leqq \min(T_i, T_j)$, almost everywhere P_x. But this is merely the compatibility established at the beginning of this section. Thus we have defined ϕ on $[0, \sigma']$, and we may take $\phi(t)$ to equal $\phi(\sigma')$ for $t > \sigma'$. The functional inverse to ϕ is the desired time change, so the proof is complete.

8. A remark on the hypothesis

If X and X^* have identical hitting distributions, then they have the same class of excessive functions. This is an immediate consequence of Dynkin's theorem in Section 4. It is worthwhile to remark that in some cases the converse is also true.

THEOREM 8.1. *Suppose X and X^* are Hunt processes which also satisfy hypotheses (C) and (E) of Hunt* [6, p. 89 *and* p. 330]. *If X and X^* have the same class of excessive functions, then they have identical hitting distributions.*

Proof. For simplicity we assume that the terminal time σ is infinite. Let B be an open subset of E with compact closure. By Proposition 6.1 of [6] we have $H_B f = H_B^* f$ if f is excessive. Thus if U denotes the potential kernel for X, we have

$$\int H_B(x, dy)U(y, A) = \int H_B^*(x, dy)U(y, A)$$

for all x in E. By Proposition 14.1 of [6] we then have $H_B^*(x, \cdot) = H_B(x, \cdot)$ Now let K be compact, and let $\{B_n\}$ be open sets which contain K, which have compact closures and which shrink down to K. If $T_n = T_{B_n}$ and $T = T_K$, then $P_x(T_n \to T) = P_x^*(T_n \to T) = 1$, and by Proposition 12.5 of [6], $P_x(T_n < \infty \text{ for all } n, T = \infty) = 0$, and similarly for P_x^*. Since $X(T_n) \to X(T)$ almost everywhere P_x and P_x^* on $\{T < \infty\}$, we have for all bounded continuous f

$$\begin{aligned} E_x(f(X(T)); T < \infty) &= \lim_{n\to\infty} E_x(f(X(T_n)); T_n < \infty) \\ &= E_x^*(f(X(T)); T < \infty), \end{aligned}$$

which proves the equality of the hitting distributions.

The theorem must be reformulated for recurrent processes, as one sees by considering the stable processes of index greater than 1 on the real line.

BIBLIOGRAPHY

1. R. M. BLUMENTHAL, R. K. GETOOR, AND H. P. MCKEAN, JR., *Markov processes with identical hitting distributions*, Bull. Amer. Math. Soc., vol. 68 (1962), pp. 372–373.
2. J. L. DOOB, *Stochastic processes*, New York, Wiley, 1953.
3. ———, *A probability approach to the heat equation*, Trans. Amer. Math. Soc., vol. 80 (1955), pp. 216–280.
4. E. B. DYNKIN, *Theory of Markov processes* (English translation), Engelwood Cliffs, N. J., Prentice-Hall, 1961.
5. ———, *Intrinsic topology and excessive functions associated with a Markov process*, Dokl. Akad. Nauk SSSR, vol. 127 (1959), pp. 17–19 (in Russian).
6. G. A. HUNT, *Markov processes and potentials I, II*, Illinois J. Math., vol. 1 (1957), pp. 44–93 and 316–369.
7. K. ITÔ AND H. P. MCKEAN, JR., *Diffusion theory*, to appear in book form.
8. M. LOÈVE, *Probability theory*, 2nd ed., Princeton, Van Nostrand, 1960.
9. H. P. MCKEAN, JR. AND H. TANAKA, *Additive functionals of the Brownian path*, Mem. Coll. Sci. Univ. Kyoto, Ser. A, Math., vol. 33 (1961), pp. 479–506.
10. P. A. MEYER, *Fonctionelles multiplicatives et additives de Markov*, Ann. Inst. Fourier, Grenoble, vol. 12 (1962), pp. 125–230.
11. M. G. ŠUR, *Continuous additive functionals of Markov processes and excessive functions*, Dokl. Akad. Nauk SSSR, vol. 137 (1961), pp. 800–803 (in Russian). Translated in Soviet Mathematics, vol. 2 (1961), pp. 365–368.
12. V. A. VOLKONSKII, *Additive functionals of Markov processes*, Trudy Moskov. Mat. Obšč., vol. 9 (1960), pp. 143–189 (in Russian).
13. ———, *Random substitution of time in strong Markov processes*, Teor. Veoroyatnost. i Primenen., vol. 3 (1958), pp. 332–350 (in Russian). Translated in Theory of Probability and Its Applications, vol. 3, pp. 310–326.

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS