TWO THEOREMS ON AUTOMORPHIC FUNCTIONS

Dedicated to Hans Rademacher on his seventieth birthday

BY
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1. Recently Rademacher gave a new proof of the fundamental theorem that a modular function belonging to a modular congruence subgroup and which is regular and bounded in the upper half-plane is a constant [6]. His argument relies on the divergence of the Poincaré series

$$\sum_{V} | cz + d |^{-2}, \qquad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where V runs over the principal congruence subgroup of level N with the restriction $-\frac{1}{2}N \leqq \Re(V(i)) < \frac{1}{2}N$.

The natural generalization of the modular group is the class of horocyclic groups (Grenzkreisgruppen, Fuchsian groups of the first kind) that have fundamental regions with a finite number of sides. *We call this class* $\mathfrak{F}$. In attempting to apply Rademacher's reasoning one must first prove the divergence of the analogous Poincaré series, a fact we state as Theorem 1. For convenience we shall assume our groups are defined on the unit disk $\mathfrak{U}$. As is well known, every linear transformation mapping $\mathfrak{U}$ on itself can be written as

$$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}, \qquad a\bar{a} - c\bar{c} = 1.$$

THEOREM 1. *If* $\Gamma \in \mathfrak{F}$, *then*

$$\sum_{n=0}^{\infty} | c_n z + \bar{a}_n |^{-2} = \infty \tag{1}$$

for each $z \in \mathfrak{U}$. *Here*

$$\left\{ V_n = \begin{pmatrix} a_n & \bar{c}_n \\ c_n & \bar{a}_n \end{pmatrix}, n \geqq 0; V_0 = I \right\}$$

is an enumeration of the elements of Γ.

Theorem 1 is classical ([3], pp. 255–258). The first object of this paper is to present a new proof. After this it will be easy to extend Rademacher's argument and so obtain

THEOREM 2. *A function regular and bounded in* $\mathfrak{U}$ *and automorphic on a group* $\Gamma \in \mathfrak{F}$ *is a constant.*

I am indebted to W. K. Hayman for a helpful conversation on some points in Section 2.

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2. Let $\Gamma \in \mathfrak{F}$. Let the elements of Γ be denoted as in Theorem 1. It is known that there is only a finite number of n for which $c_n = 0$; for these n, and only these n, $V_n(0) = 0$. The set $\{\bar{a}_n/c_n, c_n \neq 0\}$ is bounded above, and it is trivial that $|\bar{a}_n/c_n| > 1$.

We shall prove Theorem 1 by contradiction: Assume

$$\sum |c_n z + \bar{a}_n|^{-2} < \infty \tag{2}$$

for some $z = z_0 \in \mathfrak{U}$. Since for each $z \in \mathfrak{U}$ the ratio $|c_n z_0 + \bar{a}_n|/|c_n z + \bar{a}_n|$ is bounded above, this implies that (2) converges throughout $\mathfrak{U}$.

Now

$$\frac{1 - |V_n z|^2}{1 - |z|^2} = \frac{1}{|c_n z + a_n|^2};$$

hence (2) implies the convergence of $\sum (1 - |V_n z|^2)$, from which it follows that

$$\sum (1 - |V_n z|) < \infty. \tag{3}$$

Consider the Blaschke product

$$\pi(z) = \prod_{n=0}^{\infty} \frac{z - z_n}{\bar{z}_n z - 1} \exp(-i\beta_n), \tag{4}$$

where

$$z_n = V_n(0), \qquad \beta_n = \arg z_n .$$

It can be shown [2] that the product converges uniformly in $|z| \leqq \rho < 1$ whenever (3) holds. Hence, $\pi(z)$ is regular in $\mathfrak{U}$. Obviously $|\pi(z)| \leqq 1$.

It is also true that

$$|\pi(z)| = \prod_n |(z - z_n)/(\bar{z}_n z - 1)|$$

converges absolutely, and so the factors may be rearranged. Now

$$|(z - z_n)/(\bar{z}_n z - 1)| = |V_n^{-1} z| .$$

It follows that $|\pi|$ is invariant under Γ:

$$|\pi(Lz)| = |\pi(z)| \tag{5}$$

for each $L \in \Gamma$ and each $z \in \mathfrak{U}$.

It is known that $\pi(z)$ possesses radial limits of absolute value 1 almost everywhere ([5], p. 196). π vanishes exactly on the set $\{z_n\}$. Let

$$\mathfrak{D} = \mathfrak{U} - \{z_n\}.$$

Then the function

$$\phi(z) = \log |\pi(z)| \tag{6}$$

is harmonic in $\mathfrak{D}$ and invariant under Γ. The limit function

$$\omega(\theta) = \lim_{r \to 1} \phi(re^{i\theta}) \tag{7}$$

vanishes for almost all θ.

3. We now proceed as follows. An upper bound for ϕ in $\mathfrak{U}$ is clearly 0. If Γ has a compact fundamental region R (i.e., R is contained in a disk $|z| \leqq \rho < 1$), it is clear that the maximum of ϕ will be attained at a point of R. For if $\omega(\theta_0)$ exists and equals 0, let $\zeta_n = r_n \exp(i\theta_0)$ be a sequence tending to $\exp(i\theta_0)$. Each ζ_n has an image—call it ζ_n'—that lies in $\bar{R}$. Let ζ_p' be a subsequence converging to ζ^*, a point of $\bar{R}$. Because of the invariance of ϕ, we deduce $\phi(\zeta_p') \to 0$ from $\phi(\zeta_p) \to 0$. But $\zeta^* \notin \{z_n\}$, for $\phi(z) \to -\infty$ as $z \to z_n$. Hence ϕ is continuous at ζ^*, and $\phi(\zeta^*) = 0$. ϕ assumes its maximum at ζ^*, an interior point of $\mathfrak{D}$. This shows that $\phi(z)$ is constant, which is a contradiction and proves Theorem 1 for this case.

When R has vertices on the unit circle, the method fails, for the images ζ_n' may tend to the vertices. To get around the difficulty we need a geometrical lemma.

4. Let $\mathfrak{Q}$ be the unit circle.

LEMMA. *Let $\mathcal{P}$ be the set of parabolic vertices of Γ. Let $\alpha \in \mathfrak{Q} - \mathcal{P}$, and let λ_α be a radius terminating in α. Then there is a constant ρ $(0 < \rho < 1)$, depending only on Γ, such that on λ_α there is a sequence $\zeta_n \to \alpha$ having Γ-images ζ_n' lying in the disk $|z| \leqq \rho$.*

This result is due to Hedlund ([4], p. 538). For the sake of completeness we reproduce the proof.

A horocycle $C(p, r)$ is a euclidean circle of radius $r < 1$ tangent to $\mathfrak{Q}$ at p. Let $p_1, p_2, \cdots, p_s$ be the parabolic vertices of R. Since R has a finite number of sides, s is finite. Draw horocycles $C_i = C(p_i, r_i)$, $i = 1, 2, \cdots, s$, so that the union of their interiors covers the interior of R. This can be done by taking r_i near enough to 1.

Let P_i be a parabolic element of Γ generating the subgroup of Γ that fixes p_i. Then C_i is invariant under P_i. If z_0 is a point on C_i different from p_i, there are two images of z_0 on C_i, say z_1, z_2, whose euclidean distances from the origin are minimal. The horocycle C_i is partitioned into a countable number of arcs, each of which is the image of $z_1 z_2$ by some power of P_i. Every point $z \in C_i - p_i$ lies on one of these arcs, and so z is equivalent under Γ to a point on the arc $z_1 z_2$. Let d_i be the maximum of $|z_1|, |z_2|$. Then every point z on C_i other than p_i has a Γ-image whose distance from the origin is not more than d_i. Here $d_i < 1$ and depends on i but not on the point $z \in C_i$. Set $\rho = \max(d_1, \cdots, d_s)$. We have $\rho < 1$, and every point on any of the horocycles C_i except a point of tangency has a Γ-image lying in the disk $|z| \leqq \rho$.

Let C be the set of horocycles $\{C_i, i = 1, \cdots, s\}$ together with all their images under Γ. Every point of $\mathfrak{U}$ is interior to some element of the set C. If α' is any point other than α on the ray $O\alpha$, it lies interior to one of the horocycles of the set C. But the ray $\alpha'\alpha$ cannot lie entirely in any one member of C, for this would imply that α is a parabolic vertex. Hence $\alpha'\alpha$ must intersect one of the elements of C. It is now clear there is a sequence of points ζ_n on

λ_α such that $\zeta_n \to \alpha$ and each ζ_n is the intersection of an element of C with λ_α. By what has been proved, each ζ_n has a Γ-image in the disk $|z| \leqq \rho$.

5. We can now complete the proof of Theorem 1. Since ϕ has the radial limit 0 almost everywhere, there is a θ_0 such that $\alpha = \exp(i\theta_0)$ is not in the countable set $\mathcal{P}$ and such that $\phi(r\alpha) \to 0$ with $r \to 1$. If ζ_n, ζ_n' are the sequences of the lemma, we have $\phi(\zeta_p') \to 0$ and $\zeta_p' \to \zeta^*$ with $|\zeta^*| \leqq \rho < 1$. The remainder of the proof is the same as in Section 3.

6. We can now prove Theorem 2 by Rademacher's method [6]. Since $\Gamma \in \mathfrak{F}$, the series (1) diverges (Theorem 1). Let $z_n = V_n(0)$ be different from 0 for $n \geqq N$ (cf. beginning of Section 2). The divergence of (1) at $z = 0$ implies the divergence of $\sum (1 - |z_n|)$, or what is the same thing,

$$\prod_{n=N}^{\infty} |z_n| = 0. \tag{8}$$

Let $F(z)$ be automorphic on Γ and regular and bounded on $\mathfrak{U}$. If F is not constant, we can find an integer k such that

$$G(z) = (F(z) - F(0))/z^k$$

has the property $G(0) \neq 0$, G is regular and bounded in $\mathfrak{U}$. For $z_n \neq 0$ we have

$$G(z_n) = 0$$

by virtue of $F(z_n) = F(0)$. Jensen's inequality ([1], p. 109) yields

$$|z_N z_{N+1} \cdots z_m| \geqq |G(0)|/M > 0, \qquad m > N,$$

where $|G(z)| < M$ for $z \in \mathfrak{U}$. For $m \to \infty$ this contradicts (8) unless $G(0) = 0$. Hence there is no integer k, and F is constant.

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