ANOTHER PROOF THAT BOUNDED MODULAR FUNCTIONS ARE CONSTANTS

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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This note deals with the same subject as a recent paper by Rademacher.[1]

Let H denote the upper half-plane, Γ the modular group. The theorem in question may be stated as follows:

THEOREM 1. *Let* Γ' *be a subgroup of finite index of* Γ. *Let* f *be regular and bounded in* H, *and let*

$$f(T'\tau) = f(\tau) \tag{1}$$

for every transformation T' *of* Γ' *and every point* τ *of* H. *Then* f *is constant in* H.

This will be proved in two stages: First the particular case $\Gamma' = \Gamma$ will be deduced from well-known properties of regular functions, then the general result from the particular case.[2]

THEOREM 2. *Let* S *be a closed bounded set of points, contained in the domain* D. *Let* g *be regular in* D, *and suppose that every value assumed by* $|g(z)|$ *in* D *is already assumed in* S. *Then* g *is constant in* D.

Proof. The set of the values assumed by $|g(z)|$ in S has a greatest member, but the set of the values assumed by $|g(z)|$ in D has no greatest member unless g is constant in D. By hypothesis, these two sets are the same. Hence the result.

THEOREM 3. *Let* ϕ *be regular and bounded in* H, *and let*

$$\phi(T\tau) = \phi(\tau) \tag{2}$$

for every transformation T *of* Γ *and every point* τ *of* H. *Then* ϕ *is constant in* H.

Proof. Let

$$g(z) = \phi\{(\log z)/(2\pi i)\} \qquad (0 < |z| < 1). \tag{3}$$

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[1] *A proof of a theorem on modular functions*, Amer. J. Math., vol. 82 (1960), pp. 338–340.

[2] *Added May 23, 1961.* Professor P. T. Bateman has pointed out to me that my proof for the particular case has some similarity to that given on p. 160 of G. H. Hardy's book, *Ramanujan* (Cambridge, 1940) and attributed by Hardy to Heilbronn. With regard to my deduction of the general result from the particular case, it is also probable that others have had similar ideas, but they do not seem to appear in the literature.

Then, since we may take $T\tau = \tau + 1$, it follows from the hypotheses that $g(z)$ is regular and bounded for $0 < |z| < 1$. Hence $\lim_{z\to 0} g(z)$ exists, and, defining $g(0)$ as this limit, we have g regular also at 0. Thus, defining D as the unit disc, we have g regular in D. Now let S consist of 0 and those points z for which $0 < |z| < 1$ and $|\log z| \geqq 2\pi$, the logarithm having its principal value. Then S is closed and bounded and contained in D. Thus all the hypotheses of Theorem 2, other than the last, are satisfied. So is the last one if, for every point z' of D, there is a point z'' of S such that $g(z'') = g(z')$. If $z' = 0$, take $z'' = 0$. For any other point z' of D, we have $(\log z')/(2\pi i) \in H$. Hence there is a modular transformation T such that, putting

$$T\{(\log z')/(2\pi i)\} = \tau,$$

we have

$$\tau \in H, \tag{4}$$

$$-\tfrac{1}{2} < \operatorname{re} \tau \leqq \tfrac{1}{2}, \tag{5}$$

$$|\tau| \geqq 1, \tag{6}$$

and, by (2) and (3),

$$\phi(\tau) = \phi\{(\log z')/(2\pi i)\} = g(z'). \tag{7}$$

Now let $z'' = e^{2\pi i \tau}$. Then, by (4),

$$0 < |z''| < 1. \tag{8}$$

By (5), $-\pi < \operatorname{im}(2\pi i\tau) \leqq \pi$, so that

$$\log z'' = 2\pi i \tau, \tag{9}$$

and hence, by (6), $|\log z''| \geqq 2\pi$. This shows that $z'' \in S$. Also, by (3), (8), (9), and (7), $g(z'') = g(z')$. Thus the last hypothesis of Theorem 2 is also satisfied, and, by that theorem, g is constant in D, from which it easily follows that ϕ is constant in H.

Proof of Theorem 1. Since f is continuous in H, it is sufficient to prove that f cannot assume infinitely many values in H.

Let the index of Γ' be k, and let the right cosets of Γ' be $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$, where $\Gamma_1 = \Gamma'$. For $m = 1, 2, \cdots, k$, choose T_m from Γ_m, and let

$$f_m(\tau) = f(T_m \tau). \tag{10}$$

In view of (1), $f_m(\tau)$ is independent of the choice of T_m. Then

$$f_1(\tau) = f(\tau) \qquad (\tau \in H), \tag{11}$$

and, for any member T of Γ, there is a permutation $n_1, n_2, \cdots, n_k$ of the numbers $1, 2, \cdots, k$, such that $T_m T \in \Gamma_{n_m}$ $(m = 1, 2, \cdots, k)$. The formula $T_m T \in \Gamma_{n_m}$ means that there is a member T' of Γ' such that $T_m T = T' T_{n_m}$.

From this and (10) and (1) it follows that

$$f_m(T\tau) = f(T_m T\tau) = f(T'T_{n_m}\tau) = f(T_{n_m}\tau) = f_{n_m}(\tau).$$

Thus

$$f_m(T\tau) = f_{n_m}(\tau) \quad (m = 1, 2, \cdots, k;\ \tau \in H). \tag{12}$$

Now let w be any point of H, and

$$\phi(\tau) = \prod_{m=1}^{k} \{f_m(\tau) - f(w)\}. \tag{13}$$

Then, by (12),

$$\phi(T\tau) = \prod_{m=1}^{k} \{f_{n_m}(\tau) - f(w)\} = \prod_{m=1}^{k} \{f_m(\tau) - f(w)\} = \phi(\tau) \quad (\tau \in H).$$

Thus (2) holds for every transformation T of Γ and every point τ of H, and it is trivial that ϕ is regular and bounded in H. Hence, by Theorem 3, ϕ is constant in H. In particular, $\phi(w) = \phi(i)$. Now, by (13) and (11), $\phi(w) = 0$. Hence $\phi(i) = 0$. From this and (13) it follows that

$$\prod_{m=1}^{k} \{f_m(i) - f(w)\} = 0,$$

which means that $f(w)$ cannot have any value other than $f_1(i), f_2(i), \cdots, f_k(i)$. Since this holds for every point w of H, the theorem is proved.

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