EQUIVALENCE OF REPRESENTATIONS UNDER EXTENSIONS OF LOCAL GROUND RINGS[1]

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We shall use the following notations: K = algebraic number field, R = valuation ring in K with maximal ideal P, K' = finite extension field over K, R' = valuation ring of K' containing R; A = finite-dimensional algebra over K, G = R-order in A (that is, G is a subring of A containing the unity element of A as well as a K-basis of A, and such that G has a finite R-basis). We define

$$A' = K' \otimes_K A, \qquad G' = R' \otimes_R G,$$

so that G' is an R'-order in the K'-algebra A'. By a *G-module* we shall mean a left unital G-module having a finite R-basis. To each G-module M there corresponds a G'-module M' defined by

$$M' = R' \otimes_R M.$$

Finally we assume that all G-modules have finite height at P (see Higman [2]). Thus for each pair M, N of G-modules there exists an integer $s \geqq 0$ such that

$$P^s \operatorname{Ext}^1(M, N) = 0. \tag{1}$$

The most interesting case is that in which $G = RH$ is the group ring of a finite group H; in this case we may choose for s any integer such that the group order $[H:1]$ lies in P^s. (In this connection see also Maranda [4].)

Our aim is to establish the following:

THEOREM. *Let M and N be G-modules. Then $M' \cong N'$ as G'-modules if and only if $M \cong N$ as G-modules.*

On the one hand we may regard this result as a generalization of the Noether-Deuring Theorem [1] which applies when $R = K$, and indeed the central idea of their proof is also used here. On the other hand the present theorem generalizes a result of the first author [5] in which the theorem was established under various restrictive hypotheses.

In order to prove this theorem it is sufficient to show that $M' \cong N'$ implies $M \cong N$, the reverse implication being obvious. Let s satisfy (1), set $t = s + 1$, and define

$$\bar{R} = R/P^t, \qquad \bar{R}' = R'/P^tR'.$$

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We may then view $\bar{R}$ as a subring of $\bar{R}'$. Furthermore R' is a free R-module with a finite basis, and so $\bar{R}'$ is a free $\bar{R}$-module with a finite basis. If we set

$$\bar{G} = G/P^tG, \qquad \bar{G}' = G'/P^tG',$$

we find readily that

$$\bar{G}' = \bar{R}' \otimes_{\bar{R}} \bar{G}.$$

Likewise for the G-module M we let

$$\bar{M} = M/P^tM, \qquad \bar{M}' = M'/P^tM',$$

and we have

$$\bar{M}' = \bar{R}' \otimes_{\bar{R}} \bar{M}. \tag{2}$$

Thus $\bar{M}$ is a $\bar{G}$-module, and by extension of the ground ring from $\bar{R}$ to $\bar{R}'$ we obtain the $\bar{G}'$-module $\bar{M}'$.

Suppose now that $M' \cong N'$; then $\bar{M}' \cong \bar{N}'$ as $\bar{G}'$-modules. If k is the number of elements in an $\bar{R}$-basis of $\bar{R}'$, it follows from (2) that as $\bar{G}$-module $\bar{M}'$ is isomorphic to a direct sum of k copies of $\bar{M}$, and likewise $\bar{N}'$ is isomorphic to a direct sum of k copies of $\bar{N}$. Thus

$$\bar{M} \oplus \cdots \oplus \bar{M} = \bar{N} \oplus \cdots \oplus \bar{N} \qquad \text{as } \bar{G}\text{-modules,}$$

where k summands occur on each side. But now let

$$\bar{M} = M_1 \oplus \cdots \oplus M_a, \qquad \bar{N} = N_1 \oplus \cdots \oplus N_b$$

be the decompositions of $\bar{M}$ and $\bar{N}$ into indecomposable $\bar{G}$-submodules. Then we have

$$k(M_1 \oplus \cdots \oplus M_a) \cong k(N_1 \oplus \cdots \oplus N_b). \tag{3}$$

However $\bar{G}$ is a ring with minimum condition, and therefore (see Jacobson [3]) the Krull-Schmidt Theorem is valid for $\bar{G}$-modules. From (3) we conclude that the $\{M_i\}$ are up to isomorphism just a rearrangement of the $\{N_j\}$, and thus $\bar{M} \cong \bar{N}$.

To complete the proof we need only observe that $\bar{M} \cong \bar{N}$ implies $M \cong N$. a result due originally to Maranda [4] and generalized to the present context by Higman [2].

It is easy to see that the theorem is still valid under slightly more general hypotheses. For example K need not be an algebraic number field, so long as we know that R' has a finite R-basis and that $R' \cap K = R$. If furthermore K' is algebraic over K, the restriction that $(K':K)$ be finite can be dropped.

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