EXTENSIONS OF SHEAVES OF ALGEBRAS

BY
JOHN W. GRAY[1]

Introduction

It is well known that, since sheaves of modules over a fixed sheaf of rings R on a topological space X form an abelian category, the set of equivalence classes of extensions of a sheaf A'' by a sheaf A', i.e., of exact sequences

$$0 \to A' \to A \to A'' \to 0,$$

is in 1–1 correspondence with the first derived functor $\operatorname{Ext}_R^1(A'', A')$ of $\operatorname{Hom}_R(A'', A')$. This statement about sheaves corresponds exactly to the analogous statement about modules. Similarly, there is a classification theorem for extensions of an associative algebra Λ by a kernel A with trivial multiplication which asserts that the set $F(\Lambda, A)$ of equivalence classes of such extensions is in 1–1 correspondence with $\operatorname{Ext}_{\Lambda^e}^2(\Lambda, A)$, where Λ^e denotes the enveloping algebra of Λ. It is natural to ask whether or not the same result holds for sheaves of associative algebras. It will be shown that, in general, this is not the case, but that under appropriate hypotheses there is an exact sequence

$$\to \check{H}^1(X, A) \to F(\Lambda, A) \to \operatorname{Ext}_{\Lambda^e}^2(\Lambda, A) \to \check{H}^2(X, A) \to \cdots$$

where the symbols refer to sheaves of algebras and where $\check{H}^*(X, A)$ is the Čech cohomology of X with coefficients in A.

The paper is divided into three parts. In the first section it is shown that the groups $\operatorname{Ext}_R^n(B, A)$ can be calculated from a "weakly projective and coherent" resolution of B instead of an injective resolution of A. The main result of the second section is that if Λ is itself a weakly projective and coherent sheaf of associative algebras, then the usual standard complex is a suitable resolution of Λ. Similar results are given for sheaves of supplemented algebras and for sheaves of Lie algebras. In the third section it is shown that the corresponding extensions can be classified by the cohomology groups of a subcomplex of the bicomplex of Čech cochains of X with coefficients in the appropriate standard complexes. This leads to the indicated exact sequence.

It is assumed throughout the paper that X is paracompact Hausdorff, and essential use is made of the uniqueness theorem for cohomology with coefficients in a differential sheaf. As this theorem does not seem to be readily available, a short proof of it which was suggested by J. C. Moore is given in the first section. Also, Proposition 1.1 was included in a series of his lectures.

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In a subsequent paper, I hope to discuss the classification of extensions by kernels with nontrivial multiplication. I hope, furthermore, that these results will eventually apply to the problem of classifying deformations of pseudogroup structures.

1. Calculation of $\mathrm{Ext}^n_R(B, A)$

1.1 *Preliminaries.* In our treatment of sheaves, we will follow the usage of Godement [3], particularly Ch. 7, and Grothendieck [4], Ch. 4. Thus, if A is a sheaf of (left) R-modules, where R is a sheaf of rings (commutative with units) on a topological space X, then A is a subsheaf of an R-injective sheaf of modules I, where I is the sheaf corresponding to the presheaf $U \to \prod_{x \in U} I(x)$, $I(x)$ being an R_x-injective module containing $A_x = \pi^{-1}(x)$. Note that if B is any sheaf of R-modules, then

$$\mathrm{Hom}_R(B, I) = \prod_{x \in X} \mathrm{Hom}_{R_x}(B_x, I(x)).$$

This construction yields an injective resolution I^* of A. The cohomology groups of the cochain complex $\mathrm{Hom}_R(B, I^*)$ are denoted by $\mathrm{Ext}^n_R(B, A)$. In particular, the cohomology groups $H^n(X, A)$ of the space X with coefficients in A are, by definition, the groups $\mathrm{Ext}^n_R(R, A)$. Similarly, if $\mathbf{Hom}_R(B, A)$ denotes the sheaf of germs of R-homomorphisms of B into A, then $\mathbf{Ext}^n_R(B, A)$ will denote the cohomology sheaves of the complex $\mathbf{Hom}_R(B, I^*)$.

A sheaf B of R-modules is called R-coherent if, for each $x \in X$, there exist a neighborhood U of x and integers p and q such that

$$R^p \mid U \to R^q \mid U \to B \mid U \to 0$$

is exact. If B is R-coherent, then

$$\mathbf{Hom}_R(B, A)_x = \mathrm{Hom}_{R_x}(B_x, A_x).$$

A sheaf B of R-modules is called weakly R-projective if B_x is R_x-projective for all $x \in X$, and it is called wilted (flasque) if for every open set $U \subset X$, the map $\Gamma(X, B) \to \Gamma(U, B)$ is surjective. Clearly, if I is injective, then $\mathbf{Hom}_R(B, I)$ is wilted. If B is wilted, then $H^q(X, B) = 0$ for $q > 0$.

PROPOSITION. *If B is weakly R-projective and R-coherent, then*

$$\mathbf{Ext}^n_R(B, A) = 0$$

for $n > 0$.

Proof. As in Grothendieck [4], Ch. 4, there is a diagram

$$\begin{array}{ccccccccc}
0 \to & \mathbf{Hom}_R(B, A)_x & \to & \mathbf{Hom}_R(R^q, A)_x & \to & \mathbf{Hom}_R(D, A)_x & \to & \mathbf{Ext}^1_R(B, A)_x & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Hom}_{R_x}(B_x, A_x) & \to & \mathrm{Hom}_{R_x}(R^q_x, A_x) & \to & \mathrm{Hom}_{R_x}(D_x, A_x) & \to & \mathrm{Ext}^1_{R_x}(B_x, A_x) & \to 0
\end{array}$$

where D is the kernel,

$$0 \to D \mid U \to R^q \mid U \to B \mid U \to 0.$$

The first two vertical arrows are isomorphisms, and the third is a monomorphism. Hence the map $\mathbf{Ext}_R^1(B, A)_x \to \mathrm{Ext}_{R_x}^1(B_x, A_x)$ is a monomorphism. Since B_x is R_x-projective, $\mathrm{Ext}_{R_x}^1(B_x, A_x) = 0$, from which the desired result follows.

1.2 *Differential Sheaves.* A differential graded sheaf is a positive graded sheaf $A^* = \sum_{n=0}^{\infty} A^n$ provided with a differential operator d of degree $+1$. We define the cohomology sheaves of A^* by

$$\mathcal{H}^n(A^*) = \ker(d{:}A^n \to A^{n+1})/\mathrm{im}(d{:}A^{n-1} \to A^n).$$

As in Cartan-Eilenberg [1], Ch. XVII, every differential graded sheaf of R-modules admits an R-injective resolution $I = \{I^p(A^q)\}$ such that, in particular, $I^*(A^q)$ is an injective resolution of A^q, and the sheaves

$$\mathcal{H}^q(I^p(A^*)) = \ker(d_2{:}I^p(A^q) \to I^p(A^{q+1}))/\mathrm{im}(d_2{:}I^p(A^{q-1}) \to I^p(A^q)),$$

where d_2 is the differential operator induced by $d{:}A^q \to A^{q+1}$, for fixed q form an injective resolution of $\mathcal{H}^q(A^*)$.

Let $C^p(X, A^q) = \Gamma(X, I^p(A^q))$, and let

$$C^p(X, \mathcal{H}^q(A)) = \Gamma(X, \mathcal{H}^q(I^p(A^*))).$$

Denote again by d_2 the differential operator $d_2{:}C^p(X, A^q) \to C^p(X, A^{q+1})$ induced by the operator $d{:}A^q \to A^{q+1}$, and denote by

$$d_1{:}C^p(X, A^q) \to C^{p+1}(X, A^q)$$

the operator induced from the mapping $I^p(A^q) \to I^{p+1}(A^q)$, multiplied by $(-1)^q$. Clearly, if H_{II} denotes the cohomology bicomplex with respect to d_2, then $H_{II}^q[C^p(X, A^*)] = C^p(X, \mathcal{H}^q(A^*))$. We define $H^*(X, A^*)$ to be the cohomology groups of $I^p(A^q)$ with respect to the total differential operator $\delta = d_1 + d_2$.

Cohomology groups $\check{H}^*(X, A^*)$ can also be defined by the Čech process (see [3], Ch. 5) as follows: If A is a sheaf of R-modules, let $\check{C}^*(\mathfrak{U}, A)$ denote the complex of Čech cochains with respect to an open covering $\mathfrak{U}$ of X. The direct limit of the $\check{C}^*(\mathfrak{U}, A)$ over coverings indexed by the points $x \in X$ and satisfying $x \in U_x$ will be denoted by $\check{C}^*(X, A)$. We note that if X is paracompact Hausdorff, then $\check{C}^*(X, —)$ is an exact functor. If A^* is a differential graded sheaf, then the cohomology groups of the bicomplex $\check{C}^i(X, A^j)$ will be denoted by $\check{H}^*(X, A^*)$.

UNIQUENESS THEOREM. *If X is paracompact Hausdorff, then*

$$H^n(X, A^*) \approx \check{H}^n(X, A^*).$$

Proof. If A is a sheaf of R-modules, then, as in [3], Ch. 5, let $\check{\mathcal{C}}^*(X, A)$ denote the direct limit over coverings $\mathfrak{U}$ indexed by X of the sheaves $\check{\mathcal{C}}^*(\mathfrak{U}, A)$ corresponding to the presheaves $V \to \check{C}^*(\mathfrak{U} \cap V, A)$, where $\mathfrak{U} \cap V$ denotes the covering of V by sets of the form $U \cap V$, $U \in \mathfrak{U}$. Note that if X is paracompact Hausdorff then $\check{\mathcal{C}}^*(X, \text{—})$ is an exact functor, $\check{C}^*(X, A) = \Gamma\check{\mathcal{C}}^*(X, A)$, and $\check{\mathcal{C}}^*(X, A)$ is a resolution of A. Thus, if A^* is a graded differential sheaf, then $\check{\mathcal{C}}^*(X, A^*)$ is a resolution of A^*, and hence, by [1], Ch. XVII, since $I^*(A^*)$ is an injective resolution of A^*, there exists a map

$$\varphi : \check{\mathcal{C}}^i(X, A^j) \to I^i(A^j)$$

covering the identity map $A^* \to A^*$. We wish to compare the cohomology groups of the complexes $\check{C}^i(X, A^j)$ and $C^i(X, A^j)$. If each complex is filtered by the first index, then in the resulting spectral sequences the first terms are respectively

$$E_1^{p,q}(\check{C}^*(X, A^*)) = H^q\check{C}^p(X, A^*) = \check{C}^p(X, \mathcal{H}^q(A^*))$$

and

$$E_1^{p,q}(C^*(X, A^*)) = C^p(X, \mathcal{H}^q(A^*)).$$

Thus

$$E_2^{p,q}(\check{C}^*(X, A^*)) = \check{H}^p(X, \mathcal{H}^q(A^*))$$

and

$$E_2^{p,q}(C^*(X, A^*)) = H^p(X, \mathcal{H}^q(A^*)).$$

By the usual uniqueness theorem, if X is paracompact Hausdorff, then φ induces an isomorphism

$$E_2^{p,q}(\check{C}^*(X, A^*)) \approx E_2^{p,q}(C^*(X, A^*)).$$

Therefore, by Theorem 3.2 of [1], Ch. XV, φ induces an isomorphism

$$\check{H}^n(X, A^*) \approx H^n(X, A^*).$$

1.3 *The Representation Theorem.* Let A and B be sheaves of R-modules on X. A weakly R-projective and R-coherent resolution P_* of B is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to B \to 0$$

such that each P_i is weakly R-projective and R-coherent. Of course, in general, there is no way to know whether or not such a resolution exists. We should like to show, however, that if B admits such a resolution then $\mathrm{Ext}_R^n(B, A) = H^n(X, \mathbf{Hom}_R(P_*, A))$. We begin with the following preliminary result.

LEMMA. *If P_* is a weakly R-projective and R-coherent resolution of B, and if I is an injective sheaf containing A, then $H^n(X, \mathbf{Hom}_R(P_*, I)) = 0$ for $n > 0$, and $H^0(X, \mathbf{Hom}_R(P_*, I)) = \mathrm{Hom}_R(B, I)$.*

Proof. If the complex $C^i(X, \mathbf{Hom}_R(P_j, I))$ is filtered by j, then a spectral sequence is obtained in which

$$E_1^{p,q} = H^q(X, \mathbf{Hom}_R(P_p, I)).$$

Since I is injective, the sheaves $\mathbf{Hom}_R(P_p, I)$ are wilted, and hence $E_1^{p,q} = 0$

if $q > 0$, and $E_1^{p,0} = \mathrm{Hom}_R(P_p, I)$. It follows that $E_2^{p,q} = 0$ if $q > 0$, and $E_2^{p,0} = H^p[\mathrm{Hom}_R(P_*, I)]$. But, as was remarked in paragraph 1.1,

$$\mathrm{Hom}_R(P_p, I) = \prod_{x \epsilon X} \mathrm{Hom}_{R_x}((P_p)_x, I(x)).$$

Now, for each $x \epsilon X$, the complex $\mathrm{Hom}_{R_x}((P_*)_x, I(x))$ is acyclic in positive dimensions since $(P_*)_x$ is an R_x-projective resolution of B_x and $I(x)$ is R_x-injective. Thus, since direct product commutes with cohomology, it follows that

$$H^p[\mathrm{Hom}_R(P_*, I)] = H^p\left[\prod_{x \epsilon X} \mathrm{Hom}_{R_x}((P_*)_x, I(x))\right] = 0$$

if $p > 0$, and

$$H^0[\mathrm{Hom}_R(P_*, I)] = \prod_{x \epsilon X} \mathrm{Hom}_{R_x}(B_x, I(x)) = \mathrm{Hom}_R(B, I).$$

Therefore $H^n(X, \mathbf{Hom}_R(P_*, I)) = E_2^{n,0} = 0$ if $n > 0$, and

$$H^0(X, \mathbf{Hom}_R(P_*, I)) = \mathrm{Hom}_R(B, I).$$

THEOREM. *If P_* is a weakly R-projective and R-coherent resolution of B, then* $\mathrm{Ext}_R^n(B, A) = H^n(X, \mathbf{Hom}_R(P_*, A))$. *If, in addition, X is paracompact Hausdorff, then* $\mathrm{Ext}_R^n(B, A) = \check{H}^n(X, \mathbf{Hom}_R(P_*, A))$.

Proof. Let Y^* be an injective resolution of A. Then we shall show that both of these sets of cohomology groups are equal to the cohomology groups of the tricomplex K given by

$$K^{i,j,k} = C^i(X, \mathbf{Hom}_R(P_j, Y^k))$$

with the obvious induced differential operators taken with appropriate signs. For, if K is filtered by k, then a spectral sequence is obtained in which

$$''E_1^{p,q} = H^q[C^*(X, \mathbf{Hom}_R(P_*, Y^p))].$$

The preceding lemma applies since Y^p is an injective module, and hence $''E_1^{p,q} = 0$ if $q > 0$, and $''E_1^{p,0} = \mathrm{Hom}_R(B, Y^k)$. Therefore, $H^n(K) = {}''E_2^{n,0} = \mathrm{Ext}_R^n(B, A)$.

On the other hand, if K is filtered by $i + j$, then a spectral sequence is obtained in which

$$\begin{aligned} 'E_1^{p,q} &= H^q\left[\sum_{i+j=p} C^i(X, \mathbf{Hom}_R(P_j, Y^*))\right] \\ &= \sum_{i+j=p} C^i(X, \mathcal{H}^q(\mathbf{Hom}_R(P_j, Y^*))) \\ &= \sum_{i+j=p} C^i(X, \mathbf{Ext}_R^q(P_j, A)). \end{aligned}$$

But, by Proposition 1.1, $\mathbf{Ext}_R^q(P_j, A) = 0$ for $q > 0$ since P_j is weakly R-projective and R-coherent. Therefore $'E_1^{p,q} = 0$ if $q > 0$, and

$$'E_1^{p,0} = \sum_{i+j=p} C^i(X, \mathbf{Hom}_R(P_j, A)).$$

Hence $H^n(K) = {}'E_2^{n,0} = H^n(X, \mathbf{Hom}_R(P_*, A))$, which proves the first statement of the theorem. The second statement follows from the uniqueness theorem for differential sheaves on a paracompact Hausdorff space.

2. Standard complexes

2.1 *Sheaves of Augmented Rings.* Extensions of sheaves of algebras will be studied by use of the usual standard complexes. In this section it will be verified that these complexes yield resolutions to which Theorem 1.3 can be applied. As in [1], Ch. VIII, we shall utilize the unifying notion of a sheaf of augmented rings, and we shall try to employ the notations of [1] more or less consistently. Thus, a triple $(\Lambda, \varepsilon, Q)$ consisting of a sheaf Λ of rings (with units), a sheaf Q of left Λ-modules, and a Λ-epimorphism $\varepsilon : \Lambda \to Q$ will be called a sheaf of (left) augmented rings. If A is a sheaf of left Λ-modules, then $\mathrm{Ext}^n_\Lambda(Q, A)$ will be called the n^{th} cohomology group of the sheaf of augmented rings $(\Lambda, \varepsilon, Q)$ with coefficients in A.

2.2 *Sheaves of Associative Algebras.* Let Λ be a sheaf of associative algebras over a sheaf R of rings. We remark once and for all that all tensor products will be tensor products over R unless there is explicit indication to the contrary. Let $\Lambda^e = \Lambda \otimes \Lambda^*$, where Λ^* denotes the sheaf of opposite algebras; i.e., as a sheaf of R-modules, $\Lambda^* = \Lambda$, but $\lambda^*\mu^* = (\mu\lambda)^*$. If A is a sheaf of two-sided Λ-modules, then A may be regarded as a sheaf of left Λ^e-modules. The map $\rho : \Lambda^e \to \Lambda : \mu \otimes \gamma^* \to \mu\gamma$ determines a sheaf of augmented rings $(\Lambda^e, \rho, \Lambda)$, and thus the groups $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$ are defined for any sheaf A of two-sided Λ-modules. These groups will be denoted by $H^n(\Lambda, A)$ and will be called the (Hochschild) cohomology groups of Λ with coefficients in A.

LEMMA. *Let Γ and Λ be sheaves of associative R-algebras, and let A (resp., B) be a sheaf of left Γ-modules (resp., Λ-modules). If A is weakly Γ-projective and Γ-coherent, and if B is weakly Λ-projective and Λ-coherent, then $A \otimes B$ is weakly $\Gamma \otimes \Lambda$-projective and $\Gamma \otimes \Lambda$-coherent.*

Proof. Since $(A \otimes B)_x = A_x \otimes B_x$, it follows from [1], Ch. IX, Corollary 2.5 that $A \otimes B$ is weakly $\Gamma \otimes \Lambda$-projective. To prove that $A \otimes B$ is $\Gamma \otimes \Lambda$-coherent, suppose that

$$\Gamma^p \mid U \xrightarrow{\varphi} \Gamma^q \mid U \xrightarrow{\psi} A \mid U \to 0,$$

$$\Lambda^{p'} \mid U \xrightarrow{\varphi'} \Lambda^{q'} \mid U \xrightarrow{\psi'} B \mid U \to 0$$

are exact. Note that if r and s are any positive integers, then there exist mappings so that

$$\Gamma^{p+r+s} \mid U \to \Gamma^{q+r} \mid U \to A \mid U \to 0$$

is exact. Now, since the tensor product is a right exact functor, it follows that

$$[(\Gamma^p \otimes \Lambda^{q'}) \oplus (\Gamma^q \otimes \Lambda^{p'})] \mid U \xrightarrow{\varphi''} (\Gamma^q \otimes \Lambda^{q'}) \mid U \xrightarrow{\psi''} (A \otimes B) \mid U \to 0$$

is exact, where $\psi'' = \psi \otimes \psi'$ and $\varphi'' = (\varphi \otimes \mathrm{id}) \oplus (\mathrm{id} \otimes \varphi')$. Hence, if

$q' < q$, for example, and if r is large enough, then there is an exact sequence

$$(\Gamma \otimes \Lambda)^r \mid U \to (\Gamma \otimes \Lambda)^q \mid U \to (A \otimes B) \mid U \to 0$$

which proves that $A \otimes B$ is $\Gamma \otimes \Lambda$-coherent.

Let $S_n(\Lambda)$, $n \geqq -1$, denote the $(n+2)$-fold tensor product of Λ with itself, and let

$$d_n(\lambda_0, \cdots, \lambda_{n+1}) = \sum_{i=0}^{n} (-1)^i (\lambda_0, \cdots, \lambda_i \lambda_{i+1}, \cdots, \lambda_{n+1}).$$

If Λ is weakly R-projective and R-coherent, then by [1], Ch. IX, §6 and by the preceding lemma, $S_n(\Lambda)$, $n \geqq 0$, is a weakly Λ^e-projective and Λ^e-coherent resolution of Λ. If we write

$$S_n(\Lambda) = \Lambda \otimes \tilde{S}_n(\Lambda) \otimes \Lambda = \Lambda^e \otimes \tilde{S}_n(\Lambda),$$

where $\tilde{S}_n(\Lambda)$ denotes the n-fold tensor product of Λ with itself, then it follows from Theorem 1.3 that $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A) = H^n(\Lambda, A)$ may be calculated as the n^{th} cohomology group of X with coefficients in the differential sheaf

$$\mathbf{Hom}_{\Lambda^e}(S_*(\Lambda), A) = \mathbf{Hom}_R(\tilde{S}_*(\Lambda), A)$$

taken with the induced differential operator.

Let $\Lambda' = \mathrm{coker}(R \to \Lambda)$, let $\tilde{N}_n(\Lambda)$ denote the n-fold tensor product of Λ' with itself, and let $N_n(\Lambda) = \Lambda^e \otimes \tilde{N}_n(\Lambda)$. It is clear that if Λ' is also weakly R-projective and R-coherent, then $S_n(\Lambda)$ and $\tilde{S}_n(\Lambda)$ can be replaced by $N_n(\Lambda)$ and $\tilde{N}_n(\Lambda)$.

2.3 *Sheaves of Supplemented Algebras.* A sheaf Λ of R-algebras together with an R-algebra epimorphism $\varepsilon : \Lambda \to R$ is called a sheaf of supplemented algebras. The augmentation ideal $J = \ker \varepsilon$ is a direct summand, and so, as a sheaf of R-modules, $\Lambda = R + J$. Since $(\Lambda, \varepsilon, R)$ is a sheaf of augmented rings, the appropriate cohomology groups are $\mathrm{Ext}^n_\Lambda(R, A)$, where A is a sheaf of left Λ-modules.

On the other hand, given a sheaf A of left Λ-modules, a right action of Λ on A can be defined by

$$a\lambda = a(\varepsilon\lambda) = \varepsilon(\lambda)a.$$

If the resulting sheaf of two-sided Λ-modules is denoted by A_ε, then the Hochschild groups $H^n(\Lambda, A_\varepsilon) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, A_\varepsilon)$ can also be considered.

Proposition. *Let Λ be a weakly R-projective and R-coherent sheaf of supplemented R-algebras, and let P_* be a weakly Λ^e-projective and Λ^e-coherent resolution of Λ. Then $P_* \otimes_\Lambda R$ is a weakly Λ-projective and Λ-coherent resolution of R as a sheaf of left Λ-modules. In this case*

$$\mathrm{Ext}^n_\Lambda(R, A) = H^n(\Lambda, A_\varepsilon).$$

Proof. By [1], Ch. X, Theorem 2.1, $P_* \otimes_\Lambda R$ is a weakly Λ-projective resolution of R, and by Lemma 2.2, it is Λ-coherent. But, clearly,

$$\mathbf{Hom}_{\Lambda^e}(P_*, A_\varepsilon) = \mathbf{Hom}_\Lambda(\Lambda \otimes_{\Lambda^e} P_*, A_\varepsilon) = \mathrm{Hom}_\Lambda(P_* \otimes_\Lambda R, A)$$

by [1], Ch. X, Lemma 2.2. Hence

$$\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A_\varepsilon) = \mathrm{Ext}^n_\Lambda(R, A).$$

Note. In general, there is no induced cohomology map corresponding to a map of sheaves of augmented rings. However, the "mapping theorem" of [1], Ch. VIII, Theorem 3.1, which is used in the preceding proposition, is still valid since the only map of weakly projective resolutions which enters into the proof is the identity map.

The preceding proposition applies to the standard complexes of 2.2 and yields the complexes $S_n(\Lambda, \varepsilon) = S_n(\Lambda) \otimes_\Lambda R$ and $N_n(\Lambda, \varepsilon) = N_n(\Lambda) \otimes_\Lambda R$. In this case, $\Lambda' = J$, and therefore Λ' is weakly R-projective and R-coherent. Hence the complex

$$N_n(\Lambda) \otimes_\Lambda R = \Lambda \otimes \tilde{N}_n(\Lambda) \otimes \Lambda \otimes_\Lambda R = \Lambda \otimes \tilde{N}_n(\Lambda)$$

can be used. Thus the cohomology groups $\mathrm{Ext}^n_\Lambda(R, A)$ can be computed as the cohomology groups of X with coefficients in the differential sheaf

$$\mathbf{Hom}_\Lambda(N_*(\Lambda, \varepsilon), A) = \mathbf{Hom}_R(\tilde{N}_*(\Lambda), A).$$

2.4 *Sheaves of Lie Algebras.* Let L be a sheaf of Lie algebras over a sheaf of rings R. The universal enveloping sheaf $U(L)$ of L is the sheaf corresponding to the presheaf $W \to U[\Gamma(W, L \mid W)]$, where W is an open subset of X and where $U[\Gamma(W, L \mid W)]$ denotes the universal enveloping algebra of the Lie algebra $\Gamma(W, L \mid W)$. (See [1], Ch. XIII, §1.) It is clear that $U(L)_x = U(L_x)$, and hence the Poincaré-Birkhoff-Witt theorem remains true for sheaves of Lie algebras. Consequently, we shall assume from now on that L_x is R_x-free for each $x \in X$. If we assume in addition, as we must, that L is R-coherent, it follows that L is locally free; i.e., for each $x \in X$, there are a neighborhood W of x and an integer q such that $L \mid W \approx R^q \mid W$. If $E_*(L)$ denotes the sheaf of Grassmann algebras generated by L, then $E_*(L)$ is also locally free, and hence the standard complex $V_*(L) = U(L) \otimes E_*(L)$ is locally $U(L)$-free. By [1], Ch. XIII, §7, $V_*(L)$ is acyclic, since its restriction to each $x \in X$ is acyclic. Hence if we define

$$H^n(L, A) = \mathrm{Ext}^n_{U(L)}(R, A),$$

then $H^n(L, A)$ can be computed as the n^{th} cohomology group of X with coefficients in the differential sheaf

$$\mathbf{Hom}_{U(L)}(V_*(L), A) = \mathbf{Hom}_R(E_*(L), A).$$

Note that the complex $N_*(U(L), \varepsilon)$ cannot be used since $U(L)$ is never R-coherent.

3. Extensions of sheaves of algebras

3.1 *Locally Trivial Extensions of Sheaves of Modules.* If A' and A'' are sheaves of R-modules, then an extension of A'' by A' is an exact sequence of R-modules

$$0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0.$$

Two extensions are equivalent if there exists a commutative diagram

$$\begin{tikzcd} & & A \arrow[dd, "k"] \arrow[dr, "p"] & & \\ 0 \arrow[r] & A' \arrow[ur, "i"] \arrow[dr, "i^*"'] & & A'' \arrow[r] & 0. \\ & & A^* \arrow[ur, "p^*"'] & & \end{tikzcd}$$

It is well known that the set $E(A'', A')$ of equivalence classes of such extensions is in 1–1 correspondence with $\mathrm{Ext}^1_R(A'', A')$.

An extension of A'' by A' will be called locally trivial if there exists an open covering $\mathfrak{U} = \{U_\alpha\}$ of X such that for each U_α, the sequence

$$0 \to A' \mid U_\alpha \xrightarrow{i} A \mid U_\alpha \xrightarrow{p} A'' \mid U_\alpha \to 0$$

splits; i.e., if there exists an R-homomorphism $j_\alpha : A'' \mid U_\alpha \to A \mid U_\alpha$ such that $pj_\alpha =$ identity. In this case, $A \mid U_\alpha \approx (A' \oplus A'') \mid U_\alpha$. The set of equivalence classes of locally trivial extensions of A'' by A' will be denoted by $\mathrm{LTE}(A'', A')$.

PROPOSITION. (See Deheuvels, [2], §2.) $\mathrm{LTE}(A'', A')$ *is in* 1–1 *correspondence with the Čech cohomology group* $\check{H}^1(X, \mathbf{Hom}_R(A'', A'))$. *Furthermore, if X is paracompact Hausdorff, and if A'' is weakly R-projective and R-coherent, then every extension is locally trivial; i.e.,* $\mathrm{LTE}(A'', A') = E(A'', A')$.

Proof. A proof of the first part is given in [2]. We shall indicate the correspondence in some detail here in order to introduce some notation. Suppose

$$0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$$

is locally trivial extension with lifting homomorphisms $j_\alpha : A'' \mid U_\alpha \to A \mid U_\alpha$. In $U_{\alpha\beta} = U_\alpha \cap U_\beta$, let $h_{\alpha\beta} = j_\beta - j_\alpha$. Since $ph_{\alpha\beta} = 0$, it follows that $h_{\alpha\beta} : A'' \mid U_\alpha \to A' \mid U_\alpha$. Clearly, $\{h_{\alpha\beta}\}$ is a 1-cocycle in

$$\check{C}^1(U, \mathbf{Hom}_R(A'', A')).$$

Suppose two locally trivial extensions given by A and A^* are equivalent by a map $k : A \to A^*$. For simplicity, we shall assume that $A' = A \cap A^*$.

Then $k \mid A' = \mathrm{id}_{A'}$ (i.e., the identity map of A'). If $\{j_\alpha\}$ and $\{j^*_\alpha\}$ are the lifting homomorphisms, then

$$p^*(j^*_\alpha - kj_\alpha) = \mathrm{id}_{A''} - p^*kj_\alpha = \mathrm{id}_{A''} - pj_\alpha = 0,$$

and hence $(j^*_\alpha - kj_\alpha): A'' \mid U_\alpha \to A' \mid U_\alpha$. Furthermore, if $\{h_{\alpha\beta}\}$ and $\{h^*_{\alpha\beta}\}$ are the corresponding cocycles, then, since $kh_{\alpha\beta} = h_{\alpha\beta}$, it follows that

$$h^*_{\alpha\beta} - h_{\alpha\beta} = (j^*_\beta - k \circ j_\beta) - (j^*_\alpha - k \circ j_\alpha);$$

i.e., $\{h^*_{\alpha\beta}\} - \{h_{\alpha\beta}\} = \check{\delta}\{j^*_\alpha - kj_\alpha\}$.

Thus, equivalent extensions determine cohomologous cocycles.

Conversely, if $\{h_{\alpha\beta}\}$ is such a cocycle, let A be the sheaf which is the quotient of $\bigcup_\alpha (A' \oplus A'') \mid U_\alpha$ by the relation

$$(a', a'')_\alpha \sim (a' + h_{\alpha\beta}(a''), a'')_\beta \quad \text{for} \quad (a', a'') \in (A' \oplus A'') \mid U_{\alpha\beta}.$$

Then A determines an extension which realizes $\{h_{\alpha\beta}\}$. If $\{h^*_{\alpha\beta}\} - \{h_{\alpha\beta}\} = \check{\delta}\{\varphi_\alpha\}$, then the map

$$(a', a'')_\alpha \to (a' + \varphi_\alpha(a''), a'')_\alpha, \qquad (a', a'')_\alpha \in (A' \oplus A'') \mid U_\alpha$$

defines an equivalence $k: A \to A^*$. Thus equivalence classes of extensions are in 1–1 correspondence with cohomology classes.

To prove the second part of the theorem, write $\mathrm{Hom} = \Gamma \circ \mathbf{Hom}$ as a composite functor. By [4], Theorem 4.2.1, there is a spectral sequence such that part of the derived exact sequence of low degrees is

$$0 \to H^1(X, \mathbf{Hom}_R(A'', A')) \to \mathrm{Ext}^1_R(A'', A') \to H^0(X, \mathbf{Ext}^1_R(A'', A')) \to \cdots.$$

If A'' is weakly R-projective and R-coherent, then by 1.1, $\mathbf{Ext}^1_R(A'', A') = 0$, and hence $H^1(X, \mathbf{Hom}_R(A'', A'))$ is isomorphic to $\mathrm{Ext}^1_R(A'', A')$. If, in addition, X is paracompact Hausdorff, then

$$\check{H}^1(X, \mathbf{Hom}_R(A'', A')) \approx H^1(X, \mathbf{Hom}_R(A'', A')) \approx \mathrm{Ext}^1_R(A'', A').$$

It remains to be shown that the diagram

$$\begin{array}{ccc} \mathrm{LTE}(A'', A') & \longrightarrow & E(A'', A') \\ \downarrow & & \downarrow \\ \check{H}^1(X, \mathbf{Hom}_R(A'', A')) & \to & \mathrm{Ext}^1_R(A'', A') \end{array} \tag{1}$$

is commutative. This can be done as follows: If

$$0 \to A' \to A \to A'' \to 0 \tag{2}$$

is locally trivial, then

$$0 \to \mathbf{Hom}_R(A'', A') \to \mathbf{Hom}_R(A'', A) \to \mathbf{Hom}_R(A'', A'') \to 0$$

is an exact sequence of sheaves. Hence there is a diagram with exact rows

$$
(3)\qquad
\begin{tikzcd}[column sep=small]
 & & & \check{H}^1(X, \mathbf{Hom}_R(A'', A')) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & \mathrm{Hom}_R(A'', A) \arrow[r] & \mathrm{Hom}_R(A'', A'') \arrow[ur, "\check{\delta}"] \arrow[r, "\delta"] \arrow[dr, "d"'] & H^1(X, \mathbf{Hom}_R(A'', A')) \arrow[r] \arrow[d] & {} \\
 & & & \mathrm{Ext}^1_R(A'', A') \arrow[r] & {}
\end{tikzcd}
$$

where $\check{\delta}$, δ, and d are the coboundary operators in the various cohomology sequences. By [3], Part 2, §5.11, the upper half of diagram (3) is commutative. To prove that the lower half is commutative, let

$$0 \rightarrow {}'I^* \rightarrow I^* \rightarrow {}''I^* \rightarrow 0$$

be an injective resolution of (2). Then there is a commutative diagram of complexes

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & C^*(X, \mathbf{Hom}_R(A'', {}'I^*)) \arrow[r] & C^*(X, \mathbf{Hom}_R(A'', I^*)) \arrow[r] & C^*(X, \mathbf{Hom}_R(A'', {}''I^*)) \arrow[r] & 0 \\
0 \arrow[r] & C^*(X, \mathbf{Hom}_R(A'', {}'A)) \arrow[r] \arrow[u] & C^*(X, \mathbf{Hom}_R(A'', A)) \arrow[r] \arrow[u] & C^*(X, \mathbf{Hom}_R)A'', A'')) \arrow[r] \arrow[u] & 0.
\end{tikzcd}
$$

It follows immediately from this that the lower half of diagram (3) is commutative, where the vertical map is that given by the spectral sequence. Finally, we note that if $h \in \check{H}^1(X, \mathbf{Hom}_R(A'', A'))$ and $\bar{h} \in \mathrm{Ext}^1_R(A'', A')$ are the elements corresponding to the extension (2), and if $i \in \mathrm{Hom}_R(A'', A'')$ is the identity map, then $h = \check{\delta} i$ and $\bar{h} = di$. Therefore diagram (1) is commutative.

3.2 *Extensions of Sheaves of Associative Algebras.* Let Λ be a sheaf of associative R-algebras. We wish to classify extensions of the form

$$0 \rightarrow A \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \rightarrow 0$$

where Γ is a sheaf of associative R-algebras, A is a sheaf of two-sided Λ-modules with trivial multiplication, and p and i are R-algebra homomorphisms such that if $\lambda = p(\gamma)$, then $i(\lambda a) = \gamma i(a)$ and $i(a\lambda) = i(a)\gamma$ for all $a \in A$. For a fixed sheaf A of two-sided Λ-modules, $F(\Lambda, A)$ will denote the set of equivalence classes of such extensions with respect to the usual equivalence relation.

The following notations will be used: If A is a sheaf of two-sided Λ-modules, then A^{Λ} will denote the subsheaf consisting of elements $a \in A$ such that $\lambda a - a\lambda = 0$ for all $\lambda \in \Lambda$. $H^n[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)]$ will denote the cohomology groups of the complex $\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)$ with respect to the differential operator described in 2.2. The cycles and boundaries of this complex will be denoted respectively by $Z^n[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)]$ and $B^n[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)]$. It is clear that $H^0[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)] = \Gamma(X, A^\Lambda)$.

THEOREM. *If Λ is weakly R-projective and R-coherent, and if X is paracompact Hausdorff, then there are two exact sequences*

$$\begin{aligned}(1)\quad & 0 \to H^1[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)] \to \mathrm{Ext}^1_{\Lambda^e}(\Lambda, A) \to \check{H}^1(X, A) \\ & \qquad \to F(\Lambda, A) \to \mathrm{Ext}^2_{\Lambda^e}(\Lambda, A) \to \check{H}^2(X, A) \to \cdots,\end{aligned}$$

$$\begin{aligned}(2)\quad & 0 \to H^2[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)] \to F(\Lambda, A) \\ & \qquad \to \check{H}^1(X, \mathbf{Hom}_R(\Lambda, A)) \to \cdots.\end{aligned}$$

Proof. It will be shown that associative algebra extensions correspond to cohomology classes of a certain subcomplex of the bicomplex K given by

$$K^{i,j} = \check{C}^i(X, \mathbf{Hom}_R(\tilde{S}_j(\Lambda), A)).$$

In this bicomplex, δ will denote the coboundary operator induced from that of the standard complex $\mathbf{Hom}_R(\tilde{S}_*(\Lambda), A)$, and $\check{\delta}$ will denote the Čech coboundary operator. The total differential operator in $K^{i,j}$ is $(-1)^{j+1}\check{\delta} + \delta$. We shall regard K as being filtered by the second degree and define $F^pK = \sum_{j \geqq p} K^{i,j}$. In the resulting spectral sequence

$$E_0^{p,*} = F^pK/F^{p+1}K = \check{C}^*(X, \mathbf{Hom}_R(\tilde{S}_p(\Lambda), A))$$

with differential operator $(-1)^q\check{\delta}$, and hence

$$E_1^{p,*} = H[F^pK/F^{p+1}K] = \check{H}^*(X, \mathbf{Hom}_R(\tilde{S}_p(\Lambda), A)).$$

Now, suppose $0 \to A \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0$ is an extension. Under the hypotheses of the theorem it follows from Proposition 3.1 that this extension is locally trivial as an extension of sheaves of R-modules. Hence, on a sufficiently fine covering $\mathfrak{U}$ of X, there exist lifting homomorphisms $j_\alpha : \Lambda \mid U_\alpha \to \Gamma \mid U_\alpha$. Let $h_{\alpha\beta} = j_\beta - j_\alpha$ be the corresponding cocycle in $\check{C}^1(\mathfrak{U}, \mathbf{Hom}_R(\Lambda, A))$, and let m denote ambiguously the multiplications in A, Γ, and Λ. Note that m regarded as multiplication in A is identically zero. If we consider A as a subsheaf of Γ, then, as in [1], Ch. XIV, §2, the composition $f_\alpha : (\Lambda \otimes \Lambda) \mid U_\alpha \to A \mid U_\alpha$ determined by

$$(1)\quad (\Lambda \otimes \Lambda) \mid U_\alpha \xrightarrow{j_\alpha \otimes j_\alpha} (\Gamma \otimes \Gamma) \mid U_\alpha \xrightarrow{m} \Gamma \mid U_\alpha \xrightarrow{q_\alpha} A \mid U_\alpha,$$

where $q_\alpha = \mathrm{id}_\Gamma - j_\alpha p$, characterizes the multiplication in $(\Gamma \otimes \Gamma) \mid U_\alpha$ completely. We shall consider $\{f_\alpha\}$ as representing a $(0, 2)$-cochain in K.

As usual, the fact that $\{f_\alpha\}$ corresponds to an associative multiplication is equivalent to $\delta\{f_\alpha\} = 0$. Furthermore, if we write $j_\alpha = j_\beta - h_{\alpha\beta}$ and note that $q_\alpha = q_\beta + h_{\alpha\beta}\, p$ and that $pm = m(p \otimes p)$, then substitution in (1) yields immediately that $\check{\delta}\{f_\alpha\} = \delta\{h_{\alpha\beta}\}$. Thus $\{f_\alpha\} \oplus \{h_{\alpha\beta}\}$ is a cocycle of F^1K of total degree 2. Conversely, given such a cocycle the "collation" process of the preceding paragraph produces a locally trivial R-module extension from $\{h_{\alpha\beta}\}$. It is clear that the cocycle requirement is just the requirement that the multiplication determined by $\{f_\alpha\}$ be compatible with the identifications. Since it is obvious that cohomologous cocycles in F^1K yield equivalent extensions, it remains to be shown that equivalent extensions determine cohomologous cocycles.

Suppose the diagram

$$\begin{tikzcd} & \Gamma \arrow[dd, "k"] \arrow[dr, "p"] & \\ 0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "i^*"'] & & \Lambda \arrow[r] & 0 \\ & \Gamma^* \arrow[ur, "p^*"'] & \end{tikzcd}$$

is commutative. We know from 3.1 that $\{h^*_{\alpha\beta}\} - \{h_{\alpha\beta}\} = \check{\delta}\{j^*_\alpha - kj_\alpha\}$, and we wish to show that $\{f^*_\alpha\} - \{f_\alpha\} = \delta\{j^*_\alpha - kj_\alpha\}$. As before, we shall assume that $A = \Gamma \cap \Gamma^*$ so that $k \mid A = \mathrm{id}_A$. There is a diagram

$$\begin{tikzcd} (\Lambda \otimes \Lambda) \mid U_\alpha \arrow[r, "j_\alpha \otimes j_\alpha"] \arrow[d, "\mathrm{id}"] & (\Gamma \otimes \Gamma) \mid U_\alpha \arrow[r, "m"] \arrow[d, "k \otimes k"] & \Gamma \mid U_\alpha \arrow[r, "q"] \arrow[d, "k"] & A \mid U_\alpha \arrow[d, "\mathrm{id}"] \\ (\Lambda \otimes \Lambda) \mid U_\alpha \arrow[r, "j^*_\alpha \otimes j^*_\alpha"] & (\Gamma^* \otimes \Gamma^*) \mid U_\alpha \arrow[r, "m^*"] & \Gamma^* \mid U_\alpha \arrow[r, "q^*"] & A \mid U_\alpha \end{tikzcd}$$

where the composition of the top row is f_α and that of the bottom row is f^*_α. Since k is an algebra homomorphism, the middle square is commutative. Now, let $s_\alpha = q^*m^*(k \otimes k)(j_\alpha \otimes j_\alpha)$, and let $b_\alpha = j^*_\alpha - kj_\alpha$. Then, by substitution,

$$s_\alpha(\lambda_1, \lambda_2) = f^*_\alpha(\lambda_1, \lambda_2) - \lambda_1 b_\alpha(\lambda_2) - b_\alpha(\lambda_1)\lambda_2 .$$

Furthermore, let $t_\alpha = q^*km(j_\alpha \otimes j_\alpha)$, and write

$$q^* k\, \mathrm{id}_\Gamma = (\mathrm{id}_{\Gamma^*} - j^*_\alpha p^*)k(j_\alpha p + q_\alpha).$$

Then it is easy to see that

$$t_\alpha(\lambda_1, \lambda_2) = f_\alpha(\lambda_1, \lambda_2) - b_\alpha(\lambda_1 \cdot \lambda_2).$$

Since, by commutativity, $s_\alpha = t_\alpha$, it follows that

$$\{f^*_\alpha\} - \{f_\alpha\} = \delta\{b_\alpha\} = \delta\{j^*_\alpha - kj_\alpha\}.$$

Therefore $F(\Lambda, A)$ is in 1–1 correspondence with $H^2(F^1K)$.

Now, by 2.2, $H^n(F^0K) = H^n(K) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$. Furthermore, the cohomology groups of F^1K in low dimensions are $H^0(F^1K) = 0$, $H^1(F^1K) = Z^1[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)$, and $H^2(F^1K) = F(\Lambda, A)$. Since $E_0^{0,*} = \check{C}^*(X, \mathbf{Hom}_R(\tilde{S}_0(\Lambda), A)) = \check{C}^*(X, \mathbf{Hom}_R(R, A))$, the cohomology sequence associated with the exact sequence of complexes

$$0 \to F^1K \to F^0K \to E_0^0 \to 0$$

starts with the terms

$$0 \to \Gamma(X, A^{\Lambda}) \to \Gamma(X, A) \xrightarrow{\delta} Z^1[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)] \to \mathrm{Ext}^1_{\Lambda^e}(\Lambda, A) \to \cdots.$$

But $\delta(\Gamma(X, A)) = B^1[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)]$, and hence we get the first exact sequence stated in the theorem.

The second exact sequence is derived in exactly the same manner from the cohomology sequence associated with the exact sequence

$$0 \to F^2K \to F^1K \to E_0^1 \to 0.$$

COROLLARY 1. *If* $\check{H}^1(X, A) = 0$, *then* $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, A)$ *is isomorphic to the group of derivations of* Λ *in* A *modulo the subgroup of inner derivations.*

Proof. By the standard argument, $H^1[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)]$ is isomorphic to this group, and hence the result follows from the first sequence.

COROLLARY 2. *If* $\check{H}^1(X, A) = \check{H}^2(X, A) = 0$, *then* $\mathrm{Ext}^2_{\Lambda^e}(\Lambda, A) = F(\Lambda, A)$.

COROLLARY 3. *The set of globally trivial extensions of* Λ *by* A, *i.e., extensions which split over* X *as sheaves of* R*-modules, is in 1–1 correspondence with* $H^2[\mathrm{Hom}_R(\tilde{S}_*(\Lambda), A)]$.

Proof. This is an immediate consequence of the second sequence, since the map

$$F(\Lambda, A) \to \check{H}^1(X, \mathbf{Hom}_R(\Lambda, A))$$

assigns to each algebra extension its underlying locally trivial module extension.

COROLLARY 4. *If* $\check{H}^1(X, \mathbf{Hom}_R(\Lambda, A)) = 0$, *then every extension is globally trivial as a sheaf of* R*-modules.*

COROLLARY 5. *If* $H^3(F^2K) = 0$, *then every locally trivial* R*-module extension of* Λ *by* A *can be given the structure of a sheaf of associative* R*-algebras.*

Proof. $H^3(F^2K)$ is the next term in the second exact sequence. Hence, if it is zero, it follows that the map

$$F(\Lambda, A) \to \check{H}^1(X, \mathbf{Hom}_R(\Lambda, A))$$

is an epimorphism.

Remark. The hypothesis of Corollary 5 is not very useful since it is not clear what $H^3(F^2K)$ is. It seems likely, though, that this group will also contain the obstructions to the existence of a multiplication in the case of a kernel A with nontrivial multiplication. We note also that if

$$\check{H}^1(X, \mathbf{Hom}_R(\Lambda \otimes \Lambda, A)) = 0,$$

then every locally trivial R-module extension of Λ by A can be given the structure of a sheaf of R-algebras, but it can no longer be guaranteed that the multiplication is associative.

3.3 *Extensions of Sheaves of Supplemented Algebras.* Let Λ be a sheaf of supplemented R-algebras with $\varepsilon : \Lambda \to R$ as augmentation map and $J = \ker \varepsilon$ as augmentation ideal. We wish to classify extensions of the form

$$0 \to A_\varepsilon \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0$$

where Γ is a sheaf of supplemented R-algebras with augmentation $\varepsilon p : \Gamma \to R$, A is a sheaf of left Λ-modules with A_ε the corresponding sheaf of two-sided Λ-modules constructed by defining $a\lambda = a(\varepsilon(\lambda)) = \varepsilon(\lambda)a$, the product in A_ε being identically zero, and p and i are R-algebra homomorphisms such that if $\lambda = p(\gamma)$, then $i(\lambda a) = \gamma i(a)$ and $i(a)\gamma = i(a(\varepsilon\lambda))$ for all $a \in A$. The set $F(\Lambda, A_\varepsilon)$ satisfies the relations given in Theorem 3.2. However, exactly as in [1], Ch. XIV, §3, the situation can be improved by using the complex

$$\bar{K}^{i,j} = \check{C}^i(X, \mathbf{Hom}_R(\tilde{N}_j(\Lambda), A)).$$

$F(\Lambda, A_\varepsilon)$ still corresponds to $H^2(F^1\bar{K})$ since the liftings j_α may be chosen so that $j_\alpha(1) = 1$ and, consequently, $h_{\alpha\beta} \mid R = 0$. Hence $\{h_{\alpha\beta}\}$ is a cocycle of $\check{C}^1(U, \mathbf{Hom}_R(J, A))$. It follows then that $\{f_\alpha\}$ may be regarded as a cochain in $\check{C}^0(U, \mathbf{Hom}_R(J \otimes J, A))$.

THEOREM. *If Λ is a weakly R-projective and R-coherent sheaf of supplemented R-algebras, and if X is paracompact Hausdorff, then* $H^n(F^1\bar{K}) = \mathrm{Ext}_\Lambda^{n-1}(J, A)$. *Furthermore, the first exact sequence of Theorem* 3.2 *is the cohomology sequence corresponding to the exact sequence of sheaves*

$$0 \to J \to \Lambda \to R \to 0$$

except that the connecting homomorphisms $\mathrm{Ext}_\Lambda^n(J, A) \to \mathrm{Ext}_\Lambda^{n+1}(R, A)$ *are multiplied by* $(-1)^{n+1}$.

Proof. The complex $N_*(\Lambda, \varepsilon) = \Lambda \otimes \tilde{N}_*(\Lambda)$ is a weakly Λ-projective and Λ-coherent resolution of R. The last few terms are

$$\cdots \to \Lambda \otimes J \xrightarrow{d_1} \Lambda \xrightarrow{\varepsilon} R \to 0.$$

By exactness, the image of d_1 is J. Hence the complex $\Lambda \otimes \tilde{N}_j(\Lambda)$, $j \geqq 1$, is a weakly Λ-projective and Λ-coherent resolution of J. Therefore, the cohomology groups of $F^1\bar{K}$ are just the cohomology groups of X with coefficients in the differential sheaf $\mathbf{Hom}_R(\tilde{N}_j(\Lambda), A)$, $j \geqq 1$, the dimension being shifted by $+1$. By Theorem 1.3 these groups are $\mathrm{Ext}_\Lambda^n(J, A)$. The correspondence of the cohomology sequences is an immediate corollary of Theorem 7.1 of [1], Ch. V.

COROLLARY. $F(\Lambda, A) = \mathrm{Ext}_\Lambda^1(J, A)$.

Remark. What has been shown here is that the analogue of Proposition 3.3 of [1], Ch. XIV still is true except that the map $\mathrm{Ext}_\Lambda^1(I(\Lambda), C) \xrightarrow{\vartheta} \mathrm{Ext}_\Lambda^2(K, C)$ (in the terminology of [1]) is no longer an isomorphism.

3.4 *Extensions of Sheaves of Lie Algebras.* Let L be a locally free, finitely generated sheaf of Lie algebras over R, and let $U(L)$ denote the universal enveloping sheaf of L. Further, let $\Sigma(L, A)$ denote the set of equivalence classes of extensions of L by an abelian kernel A, i.e., of exact sequences.

$$0 \to A \xrightarrow{i} M \xrightarrow{p} L \to 0$$

where M is a sheaf of Lie algebras and A is a sheaf of left L-modules with trivial bracket operations such that $xa = [y, a]$ whenever $p(y) = x \in L$. Then the discussion of [1], Ch. XIV, §5 shows that every extension of $U(L)$ considered as a sheaf of supplemented algebras determines an extension of L. This discussion shows also that equivalent extensions of $U(L)$ yield equivalent extensions of L, and that if two extensions of $U(L)$ determine equivalent extensions of L, then they are equivalent. Thus, it remains to be shown that the map $F(U(L), A) \to \Sigma(L, A)$ is onto. To do this we must assume that X is paracompact Hausdorff. Then every extension of L is locally trivial and is represented by a cocycle $\{f_\alpha\} \oplus \{h_{\alpha\beta}\}$ of total degree 2 in the subcomplex $F^1(\bar{\bar{K}})$ of the complex

$$\bar{\bar{K}}^{i,j} = \check{C}^i(X, \mathbf{Hom}_R(E_j(L), A)).$$

Since $V_*(L) = U(L) \otimes E_*(L)$ may be regarded as a direct summand of $N_*(U(L), \varepsilon)$, $\{f_\alpha\} \oplus \{h_{\alpha\beta}\}$ can be extended to a cocycle $\{\bar{f}_\alpha\} \oplus \{\bar{h}_{\alpha\beta}\}$ of $N_*(U(L), \varepsilon)$ which is zero on the complement of $V_*(L)$. It follows easily that $\{\bar{f}_\alpha\} + \{\bar{h}_{\alpha\beta}\}$ yields a locally trivial extension of $U(L)$ which induces the given extension of L. Furthermore, it is clear that if $\bar{U}(L)$ denotes the kernel of the augmentation map $\varepsilon: U(L) \to R$, then, exactly as in the preceding paragraph, the complex $U(L) \otimes E_j(L)$, $j \geqq 1$, is a locally $U(L)$-free resolution of $\bar{U}(L)$. Therefore, we have the following theorem.

THEOREM. *If L is locally free and X is paracompact Hausdorff, then every extension of $U(L)$ is locally trivial, and there is a 1–1 correspondence $F(U(L), A) \to \Sigma(L, A)$. Furthermore,*

$$\Sigma(L, A) = \mathrm{Ext}^1_{U(L)}(\bar{U}(L), A).$$

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THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY
COLUMBIA UNIVERSITY
NEW YORK, NEW YORK