THE COHOMOLOGY THEORY OF A PAIR OF GROUPS[1]

BY
F. HAIMO AND S. MAC LANE

1. Introduction

In a series of papers by S. Eilenberg and S. Mac Lane [4], [5] and by S. Mac Lane [12], the cohomology theory of groups has been expounded in such a way that the group extension problem is recast in homological terms. (See also [9].) In particular, those authors were able to show that group extensions can be related to appropriate 2-cohomology classes in the abelian case, while in the non-abelian case the possibility of extension depends upon a certain obstruction, a 3-cocycle, becoming a coboundary. Let us suppose that we are given an abelian group A with two groups of operators, B_1 and B_2, where each operator from B_2 commutes with each operator from B_1. As in R. Baer [2], one can set up cochains, cocycles, coboundaries, and cohomology classes (herein referred to with the prefix *bi*, as in *bicocycle*) for this pair of groups B_1, B_2 with coefficients in A. In §2, using resolutions, we show that the various bicohomology groups $\mathfrak{H}^{(n)}(B_1, B_2; A)$ of the pair B_1, B_2 are isomorphic to the corresponding cohomology groups of the direct sum $B = B_1 \oplus B_2$. In fact, we can find a specific map $\mathfrak{F}$ over the identity automorphism on the group of integers Z from the tensor product of the standard projective resolutions of Z as a left $Z(B_1)$-module and as a left $Z(B_2)$-module to the standard projective resolution of Z as a left $Z(B)$-module. In §3, we consider an extension G of A by B, letting ω be the corresponding epimorphism from G to B. Then the subgroups $G_k = \omega^{-1}B_k$ extend A by B_l $(l \neq k)$ and have the property that each operator b_k (from B_k) on A extends to an automorphism of G_l which induces the identity automorphism on $G_l/A \cong B_l$ so that, as elements in A, (where $u_k(b_k)$ represents b_k in G_k),

$$[u_1(b_1)]^{-1}b_2[u_1(b_1)] + [u_2(b_2)]^{-1}b_1[u_2(b_2)] = 0.$$

Such pairs of extensions, G_1, G_2 of A by B_1, B_2, are called *coherent*. Conversely, given such a pair of coherent extensions, we can find, using the map $\mathfrak{F}$, an extension G of A by B with epimorphism ω from G to B such that each $G_k = \omega^{-1}B_k$. The set of coherent pairs of extensions of A by B_1, B_2 can be made into a group $\mathfrak{T}(B_1, B_2; A)$ which is an epimorphic image of $\mathfrak{H}^{(2)}(B_1, B_2; A)$ where the kernel is the inverse image of the coherent pair of splitting extensions. We map both $\mathfrak{H}^{(2)}$ and $\mathfrak{T}$ above into $\mathfrak{H}^{(2)}(B_1, A) \oplus \mathfrak{H}^{(2)}(B_2, A)$, forming part of an exact diagram.

In §4, we show that the group of autoequivalences of G over A by B (the

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stability group S of the chain $G \triangleright A \triangleright (0)$ with quotients B and A) can be extended, via the inner automorphisms of G, by B under certain mild restrictions. For a given coherent pair of extensions G_1, G_2 of A by B_1, B_2 corresponding to some $\mathfrak{t} \in \mathfrak{T}(B_1, B_2; A)$, where the B_k act effectively on A, let S_k be the stability group of the chain $G_k \triangleright A \triangleright (0)$ (with quotients B_k and A). We show that there exists a coherent pair of subgroups of the automorphism group of G_k which extend S_k by B_1, B_2. Further, the element $\mathfrak{t}' \in \mathfrak{T}(B_1, B_2; S_k)$ which corresponds to the latter pair of extensions is the map of $\mathfrak{t}$ under the homomorphism $\eta_k^{\#}$ from $\mathfrak{T}(A)$ to $\mathfrak{T}(S_k)$ induced by the function which carries each $a \in A$ onto the principal crossed character in $\mathfrak{Z}^{(1)}(B_k, A)$ generated by a. It is shown that ker $\eta_k^{\#}$ is included in the set of pairs of coherent extensions, at least one of which is splitting.

Reduction theorems follow readily from the classical results: the cup-product reduction theorem has an immediate analogue; but, at lowest dimension, it is $\mathfrak{T}$, not $\mathfrak{H}^{(2)}$ which, in our case, has the more natural reduction in terms of operator homomorphisms. We could, of course, develop a theory of biobstructions and of B_1-B_2-bikernels (like Q-kernels) for the non-abelian case. But it soon becomes clear that our results are implicit in the classical ones [5], [9] so that we need say no more in this direction.

The automorphism group of G is to be denoted by $\mathfrak{A}(G)$; the inner automorphism group of G, by $\mathfrak{J}(G)$; the subgroup of the latter, each element of which has a generator in a subgroup H of G, by $\mathfrak{J}(H, G)$; the center of G, by $\mathfrak{z}(G)$; the centralizer of a subgroup H in G, by $\mathfrak{z}(H, G)$. For $x, y \in G$, $\langle x \rangle y$ is to be xyx^{-1}, so that $\langle x \rangle_G = \langle x \rangle \in \mathfrak{J}(G)$, the inner automorphism of G with generator x. By $A \lhd B$, we mean that A is a normal subgroup of B, while $A \subset B$, the ordinary inclusion, does not exclude equality or normality. For a mapping α on a group A with subgroup B, $\alpha \mid B$ or $\alpha \mid_B$ is to mean α restricted to B. For an abelian group A with a group of operators B, we let $\mathfrak{C}^{(k)}$, $\mathfrak{Z}^{(k)}$, $\mathfrak{B}^{(k)}$, and $\mathfrak{H}^{(k)}(B, A)$ be the groups of k-dimensional cochains, cocycles, coboundaries, and cohomology classes of B with coefficients in A [4]. To say that B operates on A means that we are considering a particular $\phi \in \mathrm{Hom}(B, \mathfrak{A}(A))$. Should ϕ be a monomorphism, we say that B operates *effectively* on A. Although we strive to use group-theoretic rather than homological language whenever possible, homological formulations are often convenient if not indispensable. (See [3] for homological notions and locutions.)

2. The bicohomology groups

Let B_1, B_2 be a pair of groups, and let A be a left $Z(B_1)$-$Z(B_2)$-bimodule, (where Z is the ring of integers) [2, p. 22]; that is, there are homomorphisms υ_k from the group rings $Z(B_k)$ to the endomorphism ring of the abelian group A in such a way that each of Im υ_1 and Im υ_2 is in the centralizer of the other. The set $\mathfrak{C}^{(n_1, n_2)}(B_1, B_2; A)$ of all functions on n_1 arguments from B_1, n_2 from B_2, to A is likewise a left $Z(B_1)$-$Z(B_2)$-bimodule under addition of functions and is isomorphic to the groups of cochains $\mathfrak{C}^{(n_1)}(B_1, \mathfrak{C}^{(n_2)}(B_2, A)) \cong$

$\mathfrak{C}^{(n_2)}(B_2, \mathfrak{C}^{(n_1)}(B_1, A))$. Let us define the group of p-bicochains by $\mathfrak{C}^{(p)}(B_1, B_2; A) = \sum\oplus \mathfrak{C}^{(n_1,n_2)}(B_1, B_2; A)$, where $n_1 + n_2 = p \geqq 1$. The elements of $\mathfrak{C}^{(p)}$ consist of all $(p+1)$-tuples $\{f_{k,p-k}\}$, $k = 0, 1, \cdots, p$, where $f_{k,p-k} \in \mathfrak{C}^{(k,p-k)}$. We may wish, "by abuse of language," to consider $f_{k,p-k}$ as an element in $\mathfrak{C}^{(k)}(B_1, \mathfrak{C}^{(p-k)}(B_2, A))$ or in $\mathfrak{C}^{(p-k)}(B_2, \mathfrak{C}^{(k)}(B_1, A))$. Let $\delta_i^{(k,m)}$ be the usual coboundary operator on $\mathfrak{C}^{(k)}(B_i, \mathfrak{C}^{(m)}(B_j, A))$, where i, j is the set 1, 2 in some order. In what follows, we shall abbreviate this operator to δ_i, though it should be kept in mind that one has a different graded module $\mathfrak{C}(B_i, \mathfrak{C}^{(m)}(B_j, A))$ for each m and consequently a distinct differentiation δ_i on each graded module. We define [2] a differentiation δ on $\mathfrak{C}$ by specifying a mapping $\delta^{(p)}$ on $\mathfrak{C}^{(p)}$ to $\mathfrak{C}^{(p+1)}$ by

$$\begin{aligned}(2.1)\qquad &\delta^{(p)}(f_{p,0}, f_{p-1,1}, \cdots, f_{k,p-k}, \cdots, f_{0,p})\\ &\quad = (\delta_1 f_{p,0}, \cdots, \delta_1 f_{k-1,p-k+1} + (-1)^k \delta_2 f_{k,p-k}, \cdots, \delta_2 f_{0,p}).\end{aligned}$$

We can let $\mathfrak{C}^{(0)} = A$ and define $\delta^{(0)}$ by $\delta^{(0)}(a) = (\delta_1 a, \delta_2 a) \in \mathfrak{C}^{(1)}$. One can show that $\delta^{(p+1)}\delta^{(p)} = 0$, the trivial map, so that δ, the set of all the $\delta^{(p)}$, is indeed a differentiation on $\mathfrak{C}$. The bicocycles are defined as members of the kernels of the $\delta^{(p)}$'s, the bicoboundaries as members of the images. For completeness, we take the 0-dimensional bicoboundaries to be trivial. For $n \geqq 0$, we form the bicohomology groups $\mathfrak{H}^{(n)}(B_1, B_2; A)$, the group of n-bicocycles $\mathfrak{Z}^{(n)}(B_1, B_2; A)$ modulo the group of n-bicoboundaries $\mathfrak{B}^{(n)}(B_1, B_2; A)$.

Let $X_n^{(k)} = \otimes_{n+1} Z(B_k)$ be the tensor product of $n+1$ copies of the group ring $Z(B_k)$. The group of integers is itself a left $Z(B_k)$-module; for if $u \in Z(B_k)$, if $m \in Z$, and if ε_k is the unit augmentation [3, p. 189] we let

$$um = \varepsilon_k(u)m.$$

The $X_n^{(k)}$ are free left $Z(B_k)$-modules, and the negative complex $X^{(k)}$ given by the sequence,

$$\cdots \to X_n^{(k)} \to X_{n-1}^{(k)} \to \cdots \to X_1^{(k)} \to X_0^{(k)} \to Z \to (0),$$

with appropriately defined differentiations ∂_i and contracting homotopies, is the standard projective resolution of Z as a left $Z(B_k)$-module [3, p. 174 ff, p. 189], $k = 1, 2$. Now take the tensor product of the two resolutions to obtain a sequence of left $Z(B_1)\otimes Z(B_2)$-modules $Y_n = \sum\oplus (X_u^{(1)} \otimes X_v^{(2)})$, where $u, v, n = u + v \geqq 0$. Since $Z(B_1) \otimes Z(B_2) \cong Z(B_1 \oplus B_2)$ under a ring isomorphism, each Y_n is a left $Z(B)$-module where $B = B_1 \oplus B_2$. The standard differentiation on this tensor product is given [3, p. 64] by

$$(2.2)\qquad \partial(x_u^{(1)} \otimes x_v^{(2)}) = \partial_1(x_u^{(1)}) \otimes x_v^{(2)} + (-1)^u x_u^{(1)} \otimes \partial_2 x_v^{(2)},$$

where $x_q^{(k)} \in X_q^{(k)}$. The augmentation ε turns out to be $\varepsilon_1 \otimes \varepsilon_2$ [3, p. 214]. We should observe that $Z \otimes Z = Z$, a left $Z(B)$-module, so that we are working with a resolution Y of Z. It is well known that the contracting homotopies of the resolution

$$Y\colon \cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_0 \to Z \to (0)$$

can be constructed from the differentiations, the contracting homotopies, and the augmentations of $X^{(1)}$ and of $X^{(2)}$ [3, p. 214]. This means that Y is a projective resolution of Z. We construct the $W_n = \mathrm{Hom}_{Z(B)}(Y_n, A)$ and the homomorphisms $\delta = \mathrm{Hom}\ \partial$ from W_n to W_{n+1}, differentiation operators, computing the $\mathfrak{H}^{(n)}(B, A)$ from the sequence

$$W_0 \xrightarrow{\delta} W_1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} W_n \xrightarrow{\delta} \cdots$$

of "cochains" [3, p. 20, p. 175]. Each $f \in \mathrm{Hom}(Y_n, A) = W_n$ is completely determined by the values which it assumes on the generators of each $X_p^{(1)} \otimes X_{n-p}^{(2)}$, $n \geqq 1$. We can use the nonhomogeneous form of the standard complexes $X^{(1)}$ and $X^{(2)}$ [3, pp. 189–190] to obtain free generators for each $X_q^{(k)}$. We can treat the leftmost component of such a generator as an operator, so that the free generators of $X_q^{(k)}$ are just q-tuples of elements of B_k. That is, each $f \in W_n$ determines a set of functions $\{f_{k,n-k}\}$, $k = 0, 1, \cdots, n$, where $f_{k,n-k} \in \mathfrak{C}^{(k)}(B_1, \mathfrak{C}^{(n-k)}(B_2, A))$ corresponds, under the isomorphism from $\mathfrak{C}^{(k,n-k)}(B_1, B_2; A)$, to f restricted to $X_k^{(1)} \otimes X_{n-k}^{(2)}$. Conversely, such a set determines an $f \in W_n$, where f is defined on $X_k^{(1)} \otimes X_{n-k}^{(2)}$ by $f_{k,n-k}$ (recalling the "abuse of language" above). We set $\rho(f) = \{f_{k,n-k}\}$ and observe that ρ commutes with δ and is an isomorphism of W_n onto $\mathfrak{C}^{(n)}(B_1, B_2; A)$. Hence,

THEOREM 2.3. *For each nonnegative integer n,*

$$\mathfrak{H}^{(n)}(B_1, B_2; A) \cong \mathfrak{H}^{(n)}(B_1 \oplus B_2, A).$$

Instead of the standard projective resolutions of Z as a left $Z(B_k)$-module, we could have used the normalized standard complex [3, p. 186, p. 190], $X_N^{(k)}$. The tensor product of the two resolutions $X_N^{(k)}(k = 1, 2)$ is again normalized, since the tensor product elements $x \otimes 0$ and $0 \otimes x$ are both 0. The bicohomology groups of the pair B_1, B_2 can thus be computed using only normal bicochains; that is, each component $f_{k,n-k}$ of such a cochain takes on the value 0 whenever any one of its first k arguments is the unity of B_1 or whenever any one of its latter $n - k$ arguments is the unity of B_2.

We shall now find a specific map $\mathfrak{F}$ over the identity automorphism on Z from the complex Y to the standard projective resolution of Z as a left $Z(B)$-module. First, suppose that the Λ_k $(k = 1, 2)$ are two Z-projective, supplemented Z-algebras with augmentations $\varepsilon_k : \Lambda_k \to Z$ [3, Chapter IX, §1, and Chapter X, §§1, 2]. Form the supplemented, standard normalized complexes $N(\Lambda_k, \varepsilon_k)$ [3, p. 186] from the $N_n(\Lambda_k, \varepsilon_k) = \Lambda_k \otimes \tilde{N}_n(\Lambda_k)$ where $\tilde{N}_0(\Lambda_k) = Z$ and where, for $n > 0$, $\tilde{N}_n(\Lambda_k) = \otimes_n \mathrm{Coker}(Z \to \Lambda_k)$ [3, p. 176]. The n-cells of the complex can be written $\lambda_0^{(k)}[\lambda_1^{(k)}, \cdots, \lambda_n^{(k)}]$, and differentiation is given [3, p. 186] by

$$\partial_k \lambda_0^{(k)} = 0, \tag{2.3.0}$$

$$\partial_k \lambda_0^{(k)}[\lambda_1^{(k)}] = \lambda_0^{(k)}\lambda_1^{(k)} - \lambda_0^{(k)}\varepsilon_k(\lambda_1^{(k)}), \tag{2.3.1}$$

$$\begin{aligned}(2.3.n)\quad \partial_k\, \lambda_0^{(k)}[\lambda_1^{(k)}, \lambda_2^{(k)}, \cdots, \lambda_n^{(k)}] &= \lambda_0^{(k)}\lambda_1^{(k)}[\lambda_2^{(k)}, \cdots, \lambda_n^{(k)}] \\ &+ \textstyle\sum_{0<q<n} (-1)^q[\lambda_1^{(k)}, \cdots, \lambda_q^{(k)}\lambda_{q+1}^{(k)}, \cdots, \lambda_n^{(k)}] \\ &+ (-1)^n[\lambda_1^{(k)}, \lambda_2^{(k)}, \cdots, \lambda_{n-1}^{(k)}]\varepsilon_k(\lambda_n^{(k)}).\end{aligned}$$

We now define a function $\mathfrak{F}$ on $N(\Lambda_1 \otimes \Lambda_2, \varepsilon_1 \otimes \varepsilon_2)$ to $N(\Lambda_1, \varepsilon_1) \otimes N(\Lambda_2, \varepsilon_2)$ by specifying

$$(2.4.0)\qquad \mathfrak{F}_0(\lambda_0^{(1)} \otimes \lambda_0^{(2)}) = \lambda_0^{(1)} \otimes \lambda_0^{(2)},$$

$$\begin{aligned}&\mathfrak{F}_n(\lambda_0^{(1)} \otimes \lambda_0^{(2)})[\lambda_1^{(1)} \otimes \lambda_1^{(2)}, \cdots, \lambda_n^{(1)} \otimes \lambda_n^{(2)}] \\ (2.4.n)\qquad &= \textstyle\sum_{0\leq k\leq n} (-1)^{k(n-k)}\lambda_0^{(1)}\lambda_1^{(1)} \cdots \lambda_k^{(1)}[\lambda_{k+1}^{(1)}, \cdots, \lambda_n^{(1)}] \\ &\quad \otimes \varepsilon_2(\lambda_{k+1}^{(2)}\lambda_{k+2}^{(2)} \cdots \lambda_n^{(2)})\lambda_0^{(2)}[\lambda_1^{(2)}, \cdots, \lambda_k^{(2)}].\end{aligned}$$

The function $\mathfrak{F}$ is an operator modification of the Eilenberg-Zilber mapping [6, p. 59, (4.2)], [7]. A tedious calculation shows that $\partial\mathfrak{F}_n = \mathfrak{F}_{n-1}\partial$, $n = 1, 2, \cdots$, while $\mathfrak{F}$ is readily shown to be a map over the identity map on Z to Z.

We now specify that $\Lambda_k = Z(B_k)$, $k = 1, 2$, taking each ε_k to be the unit augmentation. To obtain the bicochains we form

$$\mathrm{Hom}(N(Z(B_1), \varepsilon_1) \otimes N(Z(B_2), \varepsilon_2), A)$$

and

$$\mathrm{Hom}(N(Z(B_1) \otimes Z(B_2), \varepsilon_1 \otimes \varepsilon_2), A).$$

It is not difficult to see that the former is just the same right complex W with the same $\delta = \mathrm{Hom}\,\partial$ as was obtained above (in the normalized case, of course). In particular, for elements $b_{kj} \epsilon B_k$,

$$\begin{aligned}&\mathfrak{F}_2(b_{10} \otimes b_{20})[b_{11} \otimes b_{21}, b_{12} \otimes b_{22}] \\ (2.5.1)\qquad &= b_{10}[b_{11}, b_{12}] \otimes b_{20} - b_{10}\, b_{11}[b_{12}] \otimes b_{20}[b_{21}] \\ &\quad + b_{10}\, b_{11}\, b_{12} \otimes b_{20}[b_{21}, b_{22}].\end{aligned}$$

Since $\mathfrak{F}$ is a map over the identity mapping ι, it induces a map of n-cocycles U_n on $N(Z(B_1), \varepsilon_1) \otimes N(Z(B_2), \varepsilon_2)$ to n-cocycles $V_n = U_n\mathfrak{F}_n$ on $N(Z(B_1) \otimes Z(B_2), \varepsilon_1 \otimes \varepsilon_2)$. Specifically, a 2-cocycle $U_2 = (w_1, r, w_2)$ on the former complex is determined by the three functions

w_1 on $N_2(Z(B_1), \varepsilon_1) \otimes N_0(Z(B_2), \varepsilon_2)$ with values $w_1(b_{11}, b_{12}) \epsilon A$,

r on $N_1(Z(B_1), \varepsilon_1) \otimes N_1(Z(B_2), \varepsilon_2)$ with values $r(b_{12}, b_{21}) \epsilon A$,

w_2 on $N_0(Z(B_1), \varepsilon_1) \otimes N_2(Z(B_2), \varepsilon_2)$ with values $w_2(b_{21}, b_{22}) \epsilon A$,

where, from (2.1), $\delta_k w_k = 0$ and $\delta_i r = (-1)^i\delta_j w_i$, $i \neq j$, $i, j, k = 1, 2$. From (2.5.1), we have

$$\begin{aligned}(2.5.2)\qquad V_2\,[b_{11} \otimes b_{21}, b_{12} \otimes b_{22}] &= U_2\,\mathfrak{F}_2\,[\cdots] \\ &= w_1(b_{11}, b_{12}) - b_{11}\, r(b_{12}, b_{21}) + b_{11}\, b_{12}\, w_2(b_{21}, b_{22}).\end{aligned}$$

We shall later need the fact that the rightmost member of (2.5.2) is the value of an element w of $\mathfrak{Z}^{(2)}(B_1 \oplus B_2, A)$ (since ρ commutes with δ).

3. Coherent pairs of extensions

Let G extend an abelian group A by $B = B_1 \oplus B_2$. There is an epimorphism ω on G to B such that the sequence $(0) \to A \to G \xrightarrow{\omega} B \to (1)$ is exact. By this extension, A becomes a left $Z(B_1)$-$Z(B_2)$-bimodule. For $k = 1, 2$, let $G_k = \omega^{-1}B_k$ be the complete inverse images in G under ω of the B_k. Choose functions u_k which yield representatives $u_k(b_k)$ in G_k of the $b_k \in B_k$. For $i \neq j$, each $b_i \in B_i$ operates on G_j via

$$(3.1.1) \qquad b_i\, g_j = u_i(b_i) g_j (u_i(b_i))^{-1},$$

where $g_j \in G_j$. Further, the operator b_i on G_j extends the operator b_i on A and induces the identity automorphism on the quotient G_j/A which is isomorphic to B_j. Thus we can define functions r_i on $B_1 \times B_2$ to A by

$$(3.1.2) \qquad r_i(b_1, b_2) = (b_i\, u_j(b_j))(u_j(b_j))^{-1}.$$

It follows that

$$(3.1.3) \qquad r_i(b_1, b_2) = [u_i(b_i), u_j(b_j)],$$

so that the properties of commutators lead at once to

$$(3.1.4) \qquad r_1 + r_2 = 0.$$

If p_k is the projection epimorphism of B onto B_k, one sees that $\ker p_i\, \omega = G_j$ where $j \neq i$, so that if G extends A by B, then G extends G_i by B_j while G_i extends A by B_i, whence we have a double two-stage extension of A by B_1 and B_2.

Conversely, suppose that A is a left $Z(B_1)$-$Z(B_2)$-bimodule and that the G_k are extensions of A by the B_k with normalized factor systems w_k: each G_k can be faithfully represented as a group of ordered pairs (a, b_k), $a \in A$, $b_k \in B_k$, with multiplication given [9] by

$$(3.2) \qquad (a_1, b_{k1})(a_2, b_{k2}) = (a_1 + b_{k1}(a_2) + w_k(b_{k1}, b_{k2}), b_{k1} b_{k2}).$$

The mapping $a \to (a, 1)$ is a monomorphism. Let us further suppose that each operator b_i on A can be extended to an operator on G_j $(i \neq j)$ which induces the identity automorphism on B_j. Call such an extension of the operator b_i a *complementary extending automorphism.* It follows that there exist functions $r_i \in \mathfrak{C}^{(1,1)}(B_1, B_2; A)$ associated with the particular complementary extending automorphisms b_i such that, for the mappings on the coset representatives $(0, b_k)$ of B_k in G_k,

$$(3.3) \qquad b_i(0, b_j) = (r_i(b_1, b_2), b_j), \qquad i, j = 1, 2, \quad i \neq j.$$

Applying b_i to the product $(0, b_{j1})(0, b_{j2})$ and simplifying, one has

$$\delta_i w_j = \delta_j r_i, \qquad i, j = 1, 2, \quad i \neq j. \tag{3.4}$$

Conversely, if one can find a pair of functions r_k which are solutions of (3.4) where the factor systems w_i are given, then each operator b_i has a complementary extending automorphism on G_j.

In G_k, let a second set of coset representatives of B_k be given by the $(c_k(b_k), b_k)$, where $c_k \in \mathfrak{C}^{(1)}(B_k, A)$. Each G_k can now [9] be represented by ordered pairs $[a, b_k] = (a + c_k(b_k), b_k)$, where w_k is to be replaced by $w_k + \delta_k c_k$. A brief calculation shows that $b_i[0, b_j] = [r_i'(b_1, b_2), b_j]$ where

$$r_i' = r_i + \delta_i c_j, \qquad i \neq j, \tag{3.5.1}$$

whence

$$r_1' + r_2' = r_1 + r_2 + \delta_1 c_2 + \delta_2 c_1. \tag{3.5.2}$$

If there are two complementary extending automorphisms $b_i^{(k)}$, $k = 1, 2$' for b_i, then they differ from each other by an autoequivalence of G_j over A by B_j, that is [8], by an automorphism s belonging to the stability group of the chain $G_j \triangleright A \triangleright (0)$, an automorphism which specializes to the identity automorphism on A and induces the identity automorphism on B_j; and conversely, if s is a member of the stability group S_j of the chain above, and if b_i stands for any complementary extending automorphism of the operator b_i on A, then sb_i is also a complementary extending automorphism of the operator b_i. It is well known that S_j is isomorphic to the group of crossed characters of B_j into A, that is, to $\mathfrak{Z}^{(1)}(B_j, A)$ [1], [9, p. 130]. Let s_j with values $s_j(b_i)$ be any function on B_i to S_j. Then

$$s_j(b_i)(0, b_j) = (d_j(b_i)(b_j), b_j), \qquad d_j(b_i) \in \mathfrak{Z}^{(1)}(B_j, A), \tag{3.6.1}$$

so that, for the most general complementary extending automorphism for b_i, $s_j(b_i)b_i$,

$$s_j(b_i)b_i(0, b_j) = (r_i(b_1, b_2) + d_j(b_i)(b_j), b_j). \tag{3.6.2}$$

We see that $d_j \in \mathfrak{C}^{(1)}(B_i, \mathfrak{Z}^{(1)}(B_j, A))$, defining a function

$$d_j^* \in \mathfrak{C}^{(1,1)}(B_1, B_2; A)$$

by $d_j^*(b_1, b_2) = d_j(b_i)(b_j)$. If we put $r_i'' = r_i + d_j^*$, we see that (3.4) holds with r_i replaced by r_i''.

For functions $z_i \in \mathfrak{C}^{(1)}(B_j, \mathfrak{Z}^{(1)}(B_i, A))$, let z_i^* be defined by

$$z_i^*(b_1, b_2) = z_i(b_j)(b_i).$$

Suppose that, for a pair of extensions G_1, G_2 of A by B_1, B_2, each of the operators b_k $(k = 1, 2)$ has a complementary extending automorphism in at least one way. In this case, we call the G_k *a complementary pair of extensions of A by the B_k*. Let us assume, in addition, that the sum $r_1 + r_2$ can be rewritten as $z_1^* + z_2^*$ for suitable $z_i \in \mathfrak{C}^{(1)}(B_j, \mathfrak{Z}^{(1)}(B_i, A))$.

By (3.5.2) and by the definition of the r_i'', we see that a change of coset representatives and/or a change of complementary extending automorphisms does not alter the property (P) of the sum $r_1 + r_2$ that it decompose into a sum of two "partial cocycles", $\overset{*}{z_1} + \overset{*}{z_2}$, so that (P) is a property of the pair of extensions G_1, G_2 of A by B_1, B_2, not of the particular coset representatives of the elements of the B_k in the G_k, or of the particular complementary extending automorphisms of the operators b_k. A complementary pair of extensions of A by the B_k is said to be *coherent* if property (P) holds. A complementary pair of extensions is coherent if and only if appropriate changes in the complementary extending automorphisms make $r_1 + r_2 = 0$; for if one has coherence, $r_1 + r_2 = \overset{*}{z_1} + \overset{*}{z_2}$, and the modifying factors $s_j(b_i)$ can always be chosen in such a way that the corresponding function d_j is just $-z_j$, $j = 1, 2$. By (3.1.4), the pair of subgroups $\omega^{-1}B_k$ of G is a coherent pair of extensions of A by the B_k, in the example discussed at the beginning of this section.

Suppose, now, that G_1, G_2 is a coherent pair of extensions of A by the B_1, B_2. Choose coset representative selection functions u_k on B_k to G_k, factor sets w_k, and complementary extending automorphisms so that $r = r_1 = -r_2$. It follows from (3.4) that $\delta_i r = (-1)^i \delta_j w_i$ $(j = 1,2)$. We may then construct an extension G of A by B with factor set w from (2.5.2) by forming all ordered triples (a, b_1, b_2) with multiplication rule

$$
\begin{aligned}
&(a_1, b_{11}, b_{21})(a_2, b_{12}, b_{22}) \\
&\quad = (a_1 + b_{11} b_{21} a_2 + w(b_{11}, b_{21}, b_{12}, b_{22}), b_{11} b_{12}, b_{21} b_{22}) \\
&\quad = (a_1 + b_{11} b_{21} a_2 + w_1(b_{11}, b_{12}) - b_{11} r(b_{12}, b_{21}) \\
&\qquad\qquad + b_{11} b_{12} w_2(b_{21}, b_{22}), b_{11} b_{12}, b_{21} b_{22}).
\end{aligned} \tag{3.7}
$$

Describe the natural epimorphism ω on G to B by $\omega(a, b_1, b_2) = (b_1, b_2)$, so that the sequence $(0) \to A \to G \xrightarrow{\omega} B \to (1)$ is exact. The subgroup $\overset{*}{G_1} = \omega^{-1}B_1$ of G is the set of all $(a, b_1, 1)$, and $\overset{*}{G_1}$ extends A by B_1. Since $w(b_{11}, 1, b_{12}, 1)$ reduces to $w_1(b_{11}, b_{12})$, $\overset{*}{G_1}$ is isomorphic to G_1 under the map Φ_1 which carries $(a, b_1, 1)$ onto $au_1(b_1)$. A direct computation employing (3.7) allows us to assert that b_2 operates on $\overset{*}{G_1}$ via

$$
\begin{aligned}
b_2(a, b_1, 1) &= (0, 1, b_2)(a, b_1, 1)(0, 1, b_2)^{-1} \\
&= (b_2 a - r(b_1, b_2), b_1, 1),
\end{aligned} \tag{3.7.1}
$$

where the leftmost component of the rightmost member is the general element of A. Therefore, the operator b_2 on A extends to an automorphism b_2 on $\overset{*}{G_1}$ which induces the identity automorphism on $\overset{*}{G_1}/A$. However, in G_1,

$$
b_2(au_1(b_1)) = (b_2 a - r(b_1, b_2))u_1(b_1), \tag{3.7.2}
$$

so that $b_2 \Phi_1 = \Phi_1 b_2$. Likewise, the subgroup $\overset{*}{G_2} = \omega^{-1}B_2$ consists of all $(a, 1, b_2) \in G$; and since $w(1, b_{21}, 1, b_{22})$ reduces to $w_2(b_{21}, b_{22})$, the map Φ_2 which carries $(a, 1, b_2)$ onto $au_2(b_2) \in G_2$ is an isomorphism on $\overset{*}{G_2}$ onto G_2.

Again, b_1 operates on G_2^* via

$$(3.7.3)\qquad \begin{aligned} b_1(a, 1, b_2) &= (0, b_1, 1)(a, 1, b_2)(0, b_1, 1)^{-1} \\ &= (b_1 a + r(b_1, b_2), 1, b_2), \end{aligned}$$

so that the operator b_1 on A extends to an automorphism b_1 on G_2^* which induces the identity automorphism on G_2^*/A. Moreover, in G_2,

$$(3.7.4)\qquad b_1(au_2(b_2)) = (b_1 a + r(b_1, b_2))u_2(b_2),$$

so that $b_1 \Phi_2 = \Phi_2 b_1$. If p_k, as before, is the projection epimorphism from B to B_k, one readily obtains $\ker p_i \omega = G_j$. We summarize in

THEOREM 3.8. *Let G be an extension of an abelian group A by $B = B_1 \oplus B_2$, expressed in the form of the exact sequence*

$$(3.8.0)\qquad (0) \to A \to G \xrightarrow{\omega} B \to (1).$$

Then the $\omega^{-1}B_k$ $(k = 1, 2)$ are a coherent pair of extensions of A by the B_k where the operators on $\omega^{-1}B_i$ by B_j $(j \neq i)$ are induced by inner automorphisms of G generated by coset representatives of B_j in $\omega^{-1}B_j$ and where G extends each $\omega^{-1}B_i$ by B_j. Conversely, if A is a left $Z(B_1)$-$Z(B_2)$-bimodule and if the G_k are a coherent pair of extensions of A by the B_k with the sequences

$$(3.8.k)\qquad (0) \to A \to G_k \xrightarrow{\omega_k} B_k \to (1)$$

exact, then (1) *there exists an extension G of A by B where the sequence* (3.8.0) *is exact,* (2) *there exists a pair of operator isomorphisms Φ_k on the $\omega^{-1}B_k$ onto the G_k in the sense that $b_i \Phi_j = \Phi_j b_i$ for every operator b_i from B_i, and* (3) $\omega_k \Phi_k = \omega \mid \omega^{-1}B_k$.

A coherent pair of extensions G_1, G_2 of A by B_1, B_2 is completely determined by a quadruple of functions $[w_1, r_1, r_2, w_2]$ where the w_k are cocycles, where the sum of the r_k decomposes into the sum of two partial cocycles and where (3.4) holds. Call such quadruples *standard*. A change of coset representatives and of complementary extending automorphisms replaces the standard quadruple above by a new standard quadruple

$$[w_1 + \delta_1 c_1, r_1 + \delta_1 c_2 + d_2^*, r_2 + \delta_2 c_1 + d_1^*, w_2 + \delta_2 c_2]$$

where $c_k \in \mathfrak{C}^{(1)}(B_k, A)$ and $d_j \in \mathfrak{C}^{(1)}(B_i, \mathfrak{Z}^{(1)}(B_j, A))$. Let us say that two standard quadruples are *equivalent* if, under componentwise addition, they differ by a quadruple

$$[\delta_1 c_1, \delta_1 c_2 + d_2^*, \delta_2 c_1 + d_1^*, \delta_2 c_2],$$

a standard quadruple which we shall call *trivial*. It is clear that the standard quadruples are thus partitioned into equivalence classes. Define an addition on the equivalence classes by adding a pair of representatives, component by component, and forming the equivalence class of the standard quadruple which is their sum. It is clear that addition is independent of the representatives chosen for the summands and that, under this addition, the set of quadruple classes is an abelian group $\mathfrak{T}(B_1, B_2; A)$ with the class of trivial quadruples as the zero element. The set of quadruple classes is in one-to-one correspondence with the set of pairs of coherent extensions of A by the B_k, so that one may look upon $\mathfrak{T}$ as *the group of coherent pair extensions of A by the pair B_1, B_2*. The zero element of $\mathfrak{T}$ corresponds to the pair of those splitting extensions of A by the B_k which are associated with the given pair of homomorphisms $\phi^{(k)}$ which carry the B_k into $\mathfrak{A}(A)$. This pair of splitting extensions is always coherent for all left $Z(B_1)$-$Z(B_2)$-bimodules A, so that $\mathfrak{T}$ is never vacuous, though it may be trivial (e.g., $\mathfrak{T}(F_1, F_2; A) = (0)$ if the F_k are free).

On $\mathfrak{H}^{(2)}(B_1, B_2; A)$ to $\mathfrak{H}^{(2)}(B_k, A)$ there is a homomorphism θ_k given by $\theta_k[(w_1, r, w_2) + \mathfrak{B}^{(2)}(B_1, B_2; A)] = w_k + \mathfrak{B}^{(2)}(B_k, A)$, a mapping which is independent of the particular bicocycle which represents its cohomology class. Let $\mathfrak{S}(B_1, B_2; A)$ be $\ker \theta_1 \cap \ker \theta_2$, which consists of all

$$(0, d^*, 0) + \mathfrak{B}^{(2)}(B_1, B_2; A)$$

where d^* is any member of $\mathfrak{C}^{(1,1)}(B_1, B_2; A)$ for which $\delta_k d^* = 0$, $k = 1, 2$. There is a homomorphism θ on $\mathfrak{H}^{(2)}(B_1, B_2; A)$ to $\mathfrak{H}^{(2)}(B_1, A) \oplus \mathfrak{H}^{(2)}(B_2, A)$ defined by $\theta(\mathfrak{h}) = \theta_1(\mathfrak{h}), \theta_2(\mathfrak{h}))$ for every $\mathfrak{h} \in \mathfrak{H}^{(2)}(B_1, B_2; A)$. It is clear that $\ker \theta = \mathfrak{S}$. Likewise, there is a monomorphism Δ on $\mathfrak{T}$ into $\mathfrak{H}^{(2)}(B_1, A) \oplus \mathfrak{H}^{(2)}(B_2, A)$ given by

$$\Delta\{[w_1, r_1, r_2, w_2]\} = (w_1 + \mathfrak{B}^{(2)}(B_1, A), w_2 + \mathfrak{B}^{(2)}(B_2, A)).$$

It is immediate that Δ is independent of coset representatives, as is Λ defined by

$$\Lambda[(w_1, r, w_2) + \mathfrak{B}^{(2)}(B_1, B_2; A)] = \{[w_1, r, -r, w_2]\}.$$

In fact, $\Lambda \in \operatorname{Hom}(\mathfrak{H}^{(2)}(B_1, B_2; A), \mathfrak{T})$ and is an epimorphism since each class of standard quadruples has at least one member of the form $[w_1, r, -r, w_2]$. If the 2-bicocycle (w_1, r, w_2) represents a bicohomology class in $\ker \Lambda$, then there exist 1-cochains c_i with coefficients in A and 1-cochains d_j with coefficients which are crossed characters such that $\delta_i c_i = w_i$ and

$$r = \delta_1 c_2 + d_2^* = -\delta_2 c_1 - d_1^*.$$

That is, (w_1, r, w_2) is cohomologous to $(0, d^*, 0)$ where

$$d^* = d_2^* + \delta_2 c_1 = -d_1^* + \delta_1 c_2.$$

Since $\delta_j d_j^* = 0$,

$$(w_1, r, w_2) + \mathfrak{B}^{(2)}(B_1, B_2; A) \in \mathfrak{S}(B_1, B_2; A).$$

Conversely, Λ carries each element of $\mathfrak{S}$ onto the trivial class of quadruples, so that $\ker \Lambda = \mathfrak{S}$. We summarize in

THEOREM 3.9. *The commutative diagram below has exact rows and columns:*

$$\begin{tikzcd}
 & & (0) \arrow[d] & (0) \arrow[d] & \\
(0) \arrow[r] & \mathfrak{S}(B_1, B_2; A) \arrow[r] \arrow[d, "\iota"] & \mathfrak{H}^{(2)}(B_1, B_2; A) \arrow[r, "\Lambda"] \arrow[d, "\iota"] & \mathfrak{T}(B_1, B_2; A) \arrow[r] \arrow[d, "\Delta"] & (0) \\
(0) \arrow[r] & \mathfrak{S}(B_1, B_2; A) \arrow[r] & \mathfrak{H}^{(2)}(B_1, B_2; A) \arrow[r, "\theta"] \arrow[d] & \mathfrak{H}^{(2)}(B_1, A) \oplus \mathfrak{H}^{(2)}(B_2, A). & \\
 & & (0) & &
\end{tikzcd}$$

4. Coherent pairs of extensions of stability groups

Let G be an extension of the abelian group A by the group B, and let S be the stability group of the chain $G \rhd A \rhd (0)$ with quotients B and A. Not only is A a left $Z(B)$-module, but S can also be turned into one as follows: First, there is an isomorphism τ, let us call it *the canonical isomorphism*, on $\mathfrak{Z}^{(1)}(B, A)$ onto S such that, if $\mathfrak{z} \in \mathfrak{Z}^{(1)}(B, A)$, then $\tau(\mathfrak{z})$ carries $(0, b) \in G$ onto $(\mathfrak{z}(b), b)$, where G has a representation as a group of ordered pairs (a, b), $a \in A$, $b \in B$, as in §3. If we let $b \in B$ operate on $\mathfrak{Z}^{(1)}(B, A)$ by

$$(b\mathfrak{z})(x) = \mathfrak{z}(xb) - \mathfrak{z}(b) = b\mathfrak{z}(b^{-1}xb) \tag{4.0.1}$$

for every $\mathfrak{z} \in \mathfrak{Z}^{(1)}(B, A)$ and for every $x \in B$, then $\mathfrak{Z}^{(1)}(B, A)$ is turned into a left $Z(B)$-module. Then the operator b can be carried over to work on S in the form

$$bs = \tau(b\tau^{-1}(s)) = \langle(0, b)\rangle_G \, s\langle(0, b)\rangle_G^{-1} \tag{4.0.2}$$

for all $s \in S$. There is an operator homomorphism $\chi: a \rightarrow \chi_a$ on A to $\mathfrak{Z}^{(1)}(B, A)$ where χ_a is the principal crossed character on B to A given by

$$\chi_a(b) = a - ba. \tag{4.0.3}$$

The combined map $\eta = \tau\chi$ on A to S induces a map η^* on $\mathfrak{H}^{(2)}(B, A)$ to $\mathfrak{H}^{(2)}(B, S)$ which can also be viewed as induced by the map η' on $\mathfrak{Z}^{(2)}(B, A)$ to $\mathfrak{Z}^{(2)}(B, S)$ which is induced directly by η. Observe that $\eta(a) = \langle a \rangle_G$.

LEMMA 4.1. *Suppose, for an abelian group A, that the sequence*

$$(0) \rightarrow A \rightarrow G \xrightarrow{\omega} B \rightarrow (1)$$

is exact, where the extension G of A by B corresponds to some $\mathfrak{h} \in \mathfrak{H}^{(2)}(B, A)$. Let S be the stability group of the chain $G \rhd A \rhd (0)$ (with quotients B and A). Suppose, further, that $\mathfrak{J}(G) \cap S \subset \mathfrak{J}(A, G)$ and that $\mathfrak{Z}(G) \subset A$. Then the group $M = \{S, \mathfrak{J}(G)\}$ of automorphisms of G extends S by B where the extension belongs to $(\tau\chi)^*(\mathfrak{h}) \in \mathfrak{H}^{(2)}(B, S)$.

Proof. One readily verifies that $S \lhd M$, so that each element of M can be represented in the form $\langle g\rangle s$, $g \in G$, $s \in S$. If $\langle g_1\rangle s_1 = \langle g_2\rangle s_2$, then, for $g' = g_2^{-1}g_1$, $\langle g'\rangle \in S$. We can map M onto B via $\Omega(\langle g\rangle s) = \omega(g)$; for if g is replaced by $gg'\mathfrak{z}$ where $\langle g'\rangle \in S$ and $\mathfrak{z} \in \mathrm{Z}(G)$, then $\omega(z) = 1$ since $\mathrm{Z}(G) \subset A = \ker \omega$; while $\langle g'\rangle \in S \cap \mathfrak{J}(G) \subset \mathfrak{J}(A, G)$ implies that $g' = a\mathfrak{z}'$, $\mathfrak{z}' \in \mathrm{Z}$, whence

$$\omega(g') = \omega(a)\, \omega(\mathfrak{z}') = 1.$$

This shows that Ω is uniquely defined. Further, since $\omega(g) = 1$ if and only if $g \in A$, the fact that $\mathfrak{J}(A, G) \subset \mathfrak{J}(G) \cap S \subset S$ implies that $\ker \Omega = S$.

Observe that if the homomorphism ϕ on B to $\mathfrak{A}(A)$ determined by the extension G is a monomorphism (that is, if B operates effectively on A), then $\mathfrak{J}(G) \cap S$ can be determined as follows: giving G its representation by ordered pairs (a, b), suppose that $\langle (a, b)\rangle \in S$. That is, for every $a' \in A$, $\langle (a, b)\rangle (a', 1) = (a', 1)$. But since ϕ is a monomorphism, $b = 1$, so that $\langle (a, b)\rangle = \langle (a, 1)\rangle \in \mathfrak{J}(A, G)$. A similar proof shows that also $\mathrm{Z}(G) \subset A$. We have

COROLLARY 4.1.0. *If B operates effectively on A, the conditions of the lemma are met.*

Let us suppose that G_1, G_2 is a coherent pair of extensions of A by B_1, B_2, where if $(0) \to A \to G \xrightarrow{\omega} B_1 \oplus B_2 \to (1)$ is exact, we can take $G_k = \omega^{-1}B_k$. Let us assume that this pair of coherent extensions corresponds to the element $((w_1, r, w_2) + \mathfrak{B}^{(2)}(B_1, B_2; A)) + \mathfrak{S}(B_1, B_2; A) \in \mathfrak{T}(B_1, B_2; A)$. Let S_k be the stability group of the chain $G_k \rhd A \rhd (0)$ (with quotients B_k and A). We already know (4.0.1) that each S_k is a left $Z(B_k)$-module. For $j \neq i$, one can turn $\mathfrak{Z}^{(1)}(B_i, A)$ into a left $Z(B_j)$-module by putting

$$(4.1.1) \qquad (b_j\, \mathfrak{z}_i)(x) = b_j(\mathfrak{z}_i(x))$$

for every $\mathfrak{z}_i \in \mathfrak{Z}^{(1)}(B_i, A)$ and $x \in B_i$. We can carry the operator b_j over to an operator on S_i by setting

$$(4.1.2) \qquad b_j\, s_i = \tau_i\, b_j\, \tau_i^{-1}(s_i) = \langle b_j\rangle_{\mathfrak{A}(G_i)}\, s_i,$$

where τ_i is the canonical isomorphism on $\mathfrak{Z}^{(1)}(B_i, A)$ onto S_i and where $\langle b_j\rangle_{\mathfrak{A}(G_i)}$ is the inner automorphism on $\mathfrak{A}(G_i)$ induced by the complementary extending automorphism $b_j \in \mathfrak{A}(G_i)$ of the operator b_j on A. Moreover, b_1 and b_2 commute over the S_k so that the latter are left $Z(B_1)$-$Z(B_2)$-bimodules. It is not difficult to show that the mapping $\chi^{(k)}: a \to \chi_a^{(k)}$ where

$$\chi_a^{(k)}(b_k) = a - b_k\, a,$$

on A to $\mathfrak{Z}^{(1)}(B_k, A)$ is an operator homomorphism (with respect both to B_1 and to B_2), and so is the combined map $\eta_k = \tau_k\, \chi^{(k)}$ on A to S_k which induces the homomorphism η_k' on $\mathfrak{Z}^{(2)}(B_1, B_2; A)$ to $\mathfrak{Z}^{(2)}(B_1, B_2; S_k)$ given by

(4.1.3) $\eta_k'(w_1, r, w_2) = (\langle(w_1, 1, 1)\rangle_G|_{G_k}, \langle(r, 1, 1)\rangle_G|_{G_k}, \langle(w_2, 1, 1)\rangle_G|_{G_k})$.

On the right, $(w_1, 1, 1) = (w_1(b_{11}, b_{12}), 1, 1)$ stands for an element of G, with notation of (3.7).

Consider, now, $M_k = \{S_k, \mathfrak{J}(G) \mid G_k\}$. One readily shows that $S_k \lhd M_k$, so that the elements of M_k can be written in the form $\langle g\rangle_G|_{G_k} s_k$, where $s_k \in S_k$. Let us assume that $\mathrm{Z}(G_k, G) \subset A$ and that $S_k \cap (\mathfrak{J}(G)| G_k) \subset \mathfrak{J}(A, G_k)$. One then sees that the map Ω_k on M_k onto $B = B_1 \oplus B_2$ given by

$$\Omega_k\langle g\rangle_G|_{G_k} s_k = \omega(g)$$

is independent of coset representatives and is thus an epimorphism. Since the kernel turns out to be S_k, one has the exact sequence

$$(0) \to S_k \to M_k \xrightarrow{\Omega_k} B \to (1).$$

By Theorem 3.8, the groups of automorphisms $\Omega_k^{-1}B_1$ and $\Omega_k^{-1}B_2$ are a coherent pair of extensions of S_k by B_1, B_2 within $\mathfrak{A}(G_k)$. Each $\Omega_k^{-1}B_k$ is the group of automorphisms $\{S_k, \mathfrak{J}(G_k)\} \subset \mathfrak{A}(G_k)$, while for $j \neq i$ each $\Omega_i^{-1}B_j$ is the group of automorphisms $\{S_i, \mathfrak{J}(G_j, G)| G_i\} \subset \mathfrak{A}(G_i)$. If b_1 is given the representative $\langle(0, b_1, 1)\rangle_G|_{G_k}$ in $\Omega_k^{-1}B_1$, and if b_2 is given the representative $\langle(0, 1, b_2)\rangle_G|_{G_k}$ in $\Omega_k^{-1}B_2$, a routine calculation shows that the pair of extensions $\Omega_k^{-1}B_1$ and $\Omega_k^{-1}B_2$ of S_k by B_1, B_2 belongs to the element of $\mathfrak{Z}^{(2)}(B_1, B_2; S_k)$ which is on the right of (4.1.3). We have established

THEOREM 4.2. *Suppose, for an abelian group A, that the sequence*

$$(0) \to A \to G \xrightarrow{\omega} B_1 \oplus B_2 \to (1)$$

is exact. Let the coherent pair of extensions $\omega^{-1}B_1$, $\omega^{-1}B_2$ of A by B_1, B_2 correspond to an element of $\mathfrak{T}(B_1, B_2; A)$ which has as representative the 2-bicocycle $\mathfrak{z} \in \mathfrak{Z}^{(2)}(B_1, B_2; A)$. Suppose that $\mathrm{Z}(\omega^{-1}B_k, G) \subset A$ and that

$$S_k \cap (\mathfrak{J}(G) \mid \omega^{-1}B_k) \subset \mathfrak{J}(A, \omega^{-1}B_k),$$

where S_k is the group of autoequivalences of $\omega^{-1}B_k$ over A by B_k. Then S_k has a coherent pair of extensions by B_1, B_2, a pair of subgroups of automorphisms of $\omega^{-1}B_k$, namely $\{S_k, \mathfrak{J}(\omega^{-1}B_k)\}$ and $\{S_k, \mathfrak{J}(\omega^{-1}B_l, G) \mid \omega^{-1}B_k\}$, $l \neq k$, corresponding to the bicocycle $\eta_k'(\mathfrak{z}) \in \mathfrak{Z}^{(2)}(B_1, B_2; S_k)$.

Suppose now that $B_1 \oplus B_2$ operates effectively on A; i.e., that $\phi: B_1 \oplus B_2 \to \mathfrak{A}(A)$ is a monomorphism. In the extension G with these operators ϕ this means that to each g with $\omega g \neq 1$ in $B_1 \oplus B_2$ there is an $a \in A$ with $\langle g\rangle a \neq a$. This states that $\mathrm{Z}(A, G) \subset A$; a fortiori, $\mathrm{Z}(\omega^{-1}B_k, G) \subset A$. Furthermore, as in the proof of Corollary 4.10, $\langle g\rangle_G|_{\omega^{-1}B_k} \in S_k$ if and only if $g \in A$. This proves

COROLLARY 4.2.1. *If $B_1 \oplus B_2$ operates effectively on A, the conditions of the theorem hold.*

It is not hard to show that ker η_k' consists of all $(w_1, r, w_2) \in \mathfrak{Z}^{(2)}(B_1, B_2; A)$ for which the values assumed by w_1, r, and w_2 are fixed by all operators b_k from B_k, $k = 1, 2$. Let $\mathfrak{Q}_k(B_1, B_2; A)$ be the subgroup of $\mathfrak{T}(B_1, B_2; A)$ of coherent pairs of extensions G_1, G_2 of A by B_1, B_2, where for this fixed index k, G_k is a splitting extension of A by B_k. One readily proves that $\mathfrak{Q}_1 \cap \mathfrak{Q}_2 = (0)$, so that there is a monomorphism from $\mathfrak{Q} = \mathfrak{Q}_1 \oplus \mathfrak{Q}_2$ to $\mathfrak{T}$. Let $\mathfrak{W}(B_1, B_2; A) = \Lambda^{-1}(\mathfrak{Q})$, the complete inverse image in $\mathfrak{H}^{(2)}(B_1, B_2; A)$ of $\mathfrak{Q}$, where by abuse of language the latter is considered as a subgroup of $\mathfrak{T}$. We see that $\mathfrak{W} \supset \mathfrak{S}$, and a not very involved argument, using the fact that $\mathfrak{W}$ consists of precisely those bicohomology classes with bicocycle representatives (w_1, r, w_2) where r splits into the sum of two partial cocycles $r_1 + r_2$, $\delta_k r_k = 0$, allows us to conclude that $\mathfrak{W} \supset \ker \eta_k^*$, $k = 1, 2$. It turns out that ker η_i^* consists of those cohomology classes in $\mathfrak{W}$ with bicocycle representatives $(w_1, r_1 + r_2, w_2)$ where

$$\chi_{w_i}^{(i)} = (-1)^i \delta_i t, \tag{4.3.1}$$

$$\chi_{r_j}^{(i)} = \delta_j t, \tag{4.3.2}$$

($i \neq j$ for some suitable $t \in \mathfrak{C}^{(1)}(B_i, \mathfrak{Z}^{(1)}(B_i, A))$. It can be shown that if one bicocycle in a bicohomology class has components which obey equations like (4.3.1)–(4.3.2), then all cohomologous bicocycles likewise have such components. If $\mathfrak{H}^{(0)}(B_j, A)$ is trivial, then (4.3.2) suffices to characterize ker η_i^*. In any event, (4.3.2) characterizes a subgroup $\mathfrak{V}_i(B_1, B_2; A)$ of $\mathfrak{W}(B_1, B_2; A)$. Since η_k^* carries $\mathfrak{S}(B_1, B_2; A)$ into $\mathfrak{S}(B_1, B_2; S_k)$, η_k^* induces a homomorphism $\eta_k^{\#}$ on $\mathfrak{T}(B_1, B_2; A)$ to $\mathfrak{T}(B_1, B_2; S_k)$. One can show that

$$\begin{aligned} \Lambda(\ker \eta_k' + \mathfrak{B}^{(2)}(B_1, B_2; A)) &\subset \Lambda(\ker \eta_k^*) \\ &\subset \Lambda(\mathfrak{V}_k(B_1, B_2; A)) \subset \ker \eta_k^{\#} \subset \mathfrak{Q}. \end{aligned} \tag{4.3.3}$$

5. Reduction theorems

Let A be a left $Z(B_1)$-$Z(B_2)$-bimodule, and let $B_i \cong F_i/R_i$, where F_i is free with natural map ψ_i on F_i onto B_i with kernel R_i [4, p. 73ff], [9, p. 131ff]. For coset representatives $f_i(b_i)$ of B_i in F_i, construct corresponding normalized factor sets n_i from $B_i \times B_i$ to R_i. The free group F_i operates on A in standard fashion [9, loc. cit.] via

$$f_i a = \psi_i(f_i) a \tag{5.0.1}$$

$a \in A, f_i \in F_i$; and on R_i via

$$f_i r_i = \langle f_i \rangle |_{R_i} r_i, \tag{5.0.2}$$

$r_i \in R_i$. The group $\mathrm{Ophom}(R_i, A; F_i)$ of F_i-operator homomorphisms of R_i into A is the subgroup of all $\alpha \in \mathrm{Hom}(R_i, A)$ for which, on R_i,

$$\alpha\langle f_i \rangle = f_i \alpha. \tag{5.0.3}$$

Each member of $\mathfrak{Z}^{(1)}(F_i, A)$ induces by restriction a member of $\text{Ophom}(R_i, A; F_i)$, and such induced members constitute a subgroup of Ophom which we denote by $\text{Crophom}(R_i, A; F_i)$. The classical result [4, p. 73ff] is that $\mathfrak{H}^{(2)}(B_i, A) \cong \text{Ophom}(R_i, A; F_i)/\text{Crophom}(R_i, A; F_i)$.

Similarly, define a subset $\text{Biophom}(R_1, R_2; A; F_1, F_2)$ of

$$\text{Ophom}(R_1, A; F_1) \oplus \text{Ophom}(R_2, A; F_2)$$

as the set of all ordered pairs $[\zeta_1, \zeta_2]$, $\zeta_k \in \text{Ophom}(R_k, A; F_k)$, for which there exists at least one function $y \in \mathfrak{C}^{(1,1)}(B_1, B_2; A)$ with

$$(\zeta_1 n_1, y, \zeta_2 n_2) \in \mathfrak{Z}^{(2)}(B_1, B_2; A).$$

(We recall that the classical theory [9, p. 131ff] yields $\delta_i \zeta_i n_i = 0$ for all $\zeta_i \in \text{Ophom}(R_i, A; F_i)$.) Under componentwise addition, Biophom is an abelian group. Next, we distinguish a significant subgroup thereof, Bicrophom: If $\zeta_k \in \text{Crophom}(R_k, A; F_k)$, the classical theory asserts that there is a $u_k \in \mathfrak{C}^{(1)}(B_k, A)$ with $\zeta_k n_k = \delta_k u_k$. The fact that

$$\delta(u_1, u_2) = (\delta_1 u_1, \delta_1 u_2 - \delta_2 u_1, \delta_2 u_2)$$

shows that $y = \delta_1 u_2 - \delta_2 u_1$ suffices to place $[\zeta_1, \zeta_2]$ in Biophom. Hence $\text{Bicrophom}(R_1, R_2; A; F_1, F_2)$, which is defined as

$$\text{Crophom}(R_1, A; F_1) \oplus \text{Crophom}(R_2, A; F_2),$$

is a subgroup of Biophom. By methods based on the proof of the classical result, we can establish

THEOREM 5.1. *Let A be a left $Z(B_1)$-$Z(B_2)$-bimodule where each $B_k \cong F_k/R_k$, F_k free. Then the following sequence is exact:*

$$(0) \to \text{Bicrophom}(R_1, R_2; A; F_1, F_2) \to \text{Biophom}(R_1, R_2; A; F_1, F_2) \to \mathfrak{T}(B_1, B_2; A) \to (0).$$

We could, of course use the same method to reduce $\mathfrak{H}^{(2)}(B_1, B_2; A)$, but the resulting lack of elegance of the reduction makes it clear that $\mathfrak{T}$ is the natural object to reduce.

From the standard "cup-product reduction theorem" [4], [10], it is possible to prove

THEOREM 5.2. *Let A be a left $Z(B_1)$-$Z(B_2)$-bimodule where each $B_k \cong F_k/R_k$, F_k free. Let σ be the obvious epimorphism from $F_1 * F_2$ (the free product) to $F_1/R_1 \oplus F_2/R_2$. Then, for $n > 0$,*

$$\begin{aligned}\mathfrak{H}^{(n+2)}(F_1/R_1, F_2/R_2; A) &\cong \mathfrak{H}^{(n)}(F_1/R_1, F_2/R_2; \text{Hom}(R_1, A) \oplus \text{Hom}(R_2, A)) \\ &\cong \mathfrak{H}^{(n)}(F_1/R_1, F_2/R_2; \text{Hom}(\ker \sigma, A)).\end{aligned}$$

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Washington University
St. Louis, Missouri
University of Chicago
Chicago, Illinois