BEHAVIOR OF INTEGRAL GROUP REPRESENTATIONS UNDER GROUND RING EXTENSION

BY

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1. Let $K$ be an algebraic number field, and let $R$ be a subring of $K$ containing 1 and having quotient field $K$. Of primary interest will be the cases

(i) $R = K$,
(ii) $R = \text{alg. int. } \{K\}$, the ring of all algebraic integers in $K$.
(iii) $R = \text{valuation ring of a discrete valuation of } K$.

Given a finite group $G$, we denote by $RG$ its group ring over $R$. By an $RG$-module we shall mean a left $RG$-module which as $R$-module is finitely generated and torsion-free, and upon which the identity element of $G$ acts as identity operator. Each $RG$-module $M$ is contained in a uniquely determined smallest $KG$-module

$$K \otimes_R M,$$

hereafter denoted by $KM$. For a pair $M$, $N$ of $RG$-modules, we write

$$M \sim_R N$$

to denote the fact that $M \cong N$ as $RG$-modules. The notation

$$M \sim_K N$$

shall mean that $KM \cong KN$ as $KG$-modules.

Now let $K'$ be an algebraic number field containing $K$, and let $R'$ be a subring of $K'$ which contains 1 and has quotient field $K'$. Suppose further that $R'$ is a finitely generated $R$-module such that

$$R' \cap K = R.$$

Each $RG$-module $M$ then determines an $R'G$-module denoted by $R'M$, given by

$$R'M = R' \otimes_R M.$$ 

If $M$, $N$ are a pair of $RG$-modules, we write $M \sim_{R'} N$ if $R'M \cong R'N$ as $R'G$-modules. Surely

$$M \sim_{R'} N \Rightarrow M \sim_{R'} N.$$

The reverse implication is false, as we shall see. We propose to investigate more closely the connection between $R$- and $R'$-equivalence.

As a first step we may quote without proof a well-known result [9, page 70] which is a consequence of the Krull-Schmidt theorem for $KG$-modules.

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Theorem 1. Let $M, N$ be $K$G-modules, and let $K'$ be an extension field of $K$. Then

$$M \sim_{K'} N \Rightarrow M \sim_{K} N.$$  

Remark. This result is valid for any pair of fields $K \subset K'$, even for those of nonzero characteristic.

Corollary. If $M, N$ are $RG$-modules, then

$$M \sim_{R'} N \Rightarrow M \sim_{K} N.$$  

2. An $RG$-module $M$ is called irreducible if it contains no nonzero submodule of smaller $R$-rank. As is known [10], $M$ is irreducible if and only if $KM$ is irreducible as $K$G-module. Call $M$ absolutely irreducible if for every field $K' \supset K$, the module $K'M$ is irreducible as $K'G$-module. Repeated use will be made of the following result [9, page 52]:

$M$ is absolutely irreducible if and only if every $K$G-endomorphism of $KM$ is given by a scalar multiplication

$$x \rightarrow ax,$$

for some $a \in K$.

As a first result, we prove

Theorem 2. Let $R$ be a principal ideal ring, and let $M, N$ be a pair of absolutely irreducible $RG$-modules. Then

$$M \sim_{R'} N \Rightarrow M \sim_{R} N.$$  

Proof. The preceding corollary shows that $M \sim_{K} N$. After replacing $N$ by some new $RG$-module which is $RG$-isomorphic to $N$, we may in fact assume that $M \supset N$.

The isomorphism $R'M \cong R'N$ can be extended to an isomorphism $K'M \cong K'N$. As a consequence of the absolute irreducibility of $M$, and the fact that $K'M = K'N$, this latter isomorphism must be given by a scalar multiplication. Consequently there exists a scalar $\alpha \in K'$ such that

$$R'N = \alpha \cdot R'M.$$  

Since $R$ is a principal ideal ring, we may find an $R$-basis $\{m_1, \ldots, m_k\}$ of $M$, and nonzero elements $a_1, \ldots, a_k \in R$, such that

$$M = Rm_1 \oplus \cdots \oplus Rm_k,$$

$$N = Ra_1 m_1 \oplus \cdots \oplus Ra_k m_k.$$  

Then

$$R'M = \sum R'm_i, \quad R'N = \sum R'a_i m_i = \sum R'\alpha m_i.$$  

Let $u(R')$ be the group of units of $R'$, and $u(R)$ that of $R$. Then (4)
implies the existence of $\beta_1, \cdots, \beta_k \in u(R')$ such that
\[ a_i = \beta_i \alpha, \quad 1 \leq i \leq k. \]
Therefore
\[ a_i/a_1 = \beta_i/\beta_1 \in u(R'), \]
and so
\[ b_i = a_i/a_1 \in u(R') \cap K = u(R). \]
Therefore
\[ N = \sum Ra_i m_i = a_1 \sum Rb_i m_i = a_1 M, \]
which shows that $N, M$ are $R$-equivalent, Q.E.D.

We next give an example to show that the result stated in Theorem 2 need not hold when $R$ is not a principal ideal ring. Set
\[ \sigma = \text{alg. int. } \{K\}, \quad \sigma' = \text{alg. int. } \{K'\}, \]
where $\sigma$ is not a principal ideal ring. It is possible to choose $K'$ so that for each ideal $a$ in $\sigma$, the induced ideal $\sigma'a$ in $\sigma'$ is principal (see [4]). Now let $M$ be any absolutely irreducible $\sigma G$-module, a nonprincipal ideal in $\sigma$, and set $N = aM$. Then $M, N$ cannot be $\sigma$-equivalent, since by the above remarks the isomorphism $M \cong N$ would imply that $N = aM$ for some $a \in K$. On the other hand,
\[ \sigma'N = \sigma'aM = \alpha'\sigma'M \]
for some $\alpha' \in K'$, and so $M, N$ are $\sigma'$-equivalent.

If $M, N$ are $\sigma G$-modules, we say that $M, N$ are in the same genus (notation: $M \vee N$) if $RM \cong RN$ for each valuation ring $R$ of a discrete valuation of $K$ (see [5, 6]).

**Corollary.** Let $M, N$ be absolutely irreducible $\sigma G$-modules. Then
\[ M \sim_{\sigma'} N \Rightarrow M \vee N. \]

**Proof.** Let $R$ be a valuation ring of a discrete valuation $\phi$ of $K$, and let $\phi'$ be an extension of $\phi$ to $K'$, with valuation ring $R'$. Then $R$ is a principal ideal ring, and so
\[ M \sim_{\phi'} N \Rightarrow M \sim_{R'} N \Rightarrow M \sim_{R} N \]
by Theorem 2, Q.E.D.

Maranda [5] showed that a pair of absolutely irreducible $\sigma G$-modules $M, N$ are in the same genus if and only if $M \cong aN$ for some $\sigma$-ideal $a$ in $K$. But then $\sigma'M \cong \sigma'aN$, so $M, N$ are $\sigma'$-equivalent if and only if $\sigma'a$ is a principal ideal in $K'$. Thus, the converse of the above corollary holds if and only if every ideal in $\sigma$ induces a principal ideal in $\sigma'$.

**3.** Throughout this section let $R$ be the valuation ring of a discrete valuation $\phi$ of $K$, with unique maximal ideal $P$, and residue class field $\overline{K} = R/P$. Let $\phi'$ be an extension of $\phi$ to $K'$, with valuation ring $R'$, maximal ideal $P'$,
residue class field $\bar{K}' = R'/P'$. We shall give some sufficient conditions for the validity of the implication:

\begin{equation}
M \sim_{R'} N \implies M \sim_{R} N,
\end{equation}

where $M, N$ denote $RG$-modules.

**Theorem 3.** If the group order $(G:1)$ is a unit in $R$, then (5) is valid.

**Proof.** Use Theorem 1, together with the result [5] that if $(G:1)$ is a unit in $R$, then $M \sim_{R} N$ if and only if $M \sim_{K} N$.

**Theorem 4.** If $\bar{K}' = \bar{K}$, then (5) holds.

**Proof.** Since $R, R'$ are principal ideal rings, we may use matrix terminology. Let $M, N$ be $R$-representations of $G$ such that $M \sim_{R'} N$. Set

\[
C = \{ X \text{ over } R : M(g)X = XN(g), g \in G \},
\]

\[
C' = \{ X \text{ over } R' : M(g)X = XN(g), g \in G \}.
\]

Since $C$ is a finitely generated torsion-free $R$-module, we may choose an $R$-basis $\{ X_1, \cdots, X_n \}$ of $C$. It is easily verified that this is also an $R'$-basis of $C'$.

The hypothesis $M \sim_{R'} N$ is equivalent to the statement that there exist elements $\alpha_1, \cdots, \alpha_n \in R'$ such that

\[
\alpha_1 X_1 + \cdots + \alpha_n X_n
\]

is unimodular over $R'$, that is, has entries in $R'$ and satisfies

\[
| \alpha_1 X_1 + \cdots + \alpha_n X_n | \in u(R') \quad \text{(the group of units of } R').
\]

Since $\bar{K}' = \bar{K}$, we may choose $\alpha_1, \cdots, \alpha_n \in R$ such that

\[
a_i \equiv \alpha_i \pmod{P'},
\]

1 \leq i \leq n.

In that case,

\[
a_1 X_1 + \cdots + a_n X_n \in C,
\]

and is unimodular over $R$. Therefore $M \sim_{R} N$, Q.E.D.

In particular, suppose that $K'$ is an *Eisenstein extension* of $K$ relative to the valuation $\phi$, that is, suppose that $K' = K(\alpha)$ where

\[
\text{Irr } (\alpha, K) = x^m + b_1 x^{m-1} + \cdots + b_m
\]

with $b_1, \cdots, b_m \in P$, $b_m \in P^2$ (see [3]). In this case $\phi$ is uniquely extendable to $K'$, and $\bar{K}' = \bar{K}$, so that (5) is true. We shall apply this later on.

Let us call a matrix of the form

\[
\begin{bmatrix}
1 \\
\vdots \\
* \\
\vdots \\
1
\end{bmatrix}
\]
a translation; by such a notation, we mean to imply that the elements below
the main diagonal are all zero. If \( M, N \) are \( R \)-representations of \( G \), we write
\( M \cong N \) to indicate that \( M, N \) can be intertwined by a translation matrix.

On the other hand, suppose that

\[
M = \begin{bmatrix}
M_1 & * & \\
\vdots & \ddots & * \\
M_k & & \\
\end{bmatrix}, \quad N = \begin{bmatrix}
M_1 & * & \\
\vdots & \ddots & * \\
M_k & & \\
\end{bmatrix}
\]

are a pair of \( R \)-representations of \( G \) in which the \( \{M_i\} \) are distinct (that is,
not \( K \)-equivalent) and absolutely irreducible. If \( M, N \) can be intertwined
by a matrix \( X \) over \( R \) of the form

\[
X = \begin{bmatrix}
a_1 I & \\
\vdots & \ddots & * \\
a_k I & \\
\end{bmatrix},
\]

in which \( a_i \in \text{u}(R) \), the group of units of \( R \), then we shall say that \( M, N \) are
\( i \)-intertwinable. Call \( M, N \) everywhere intertwinable if for each \( i, 1 \leq i \leq k \),
\( M, N \) are \( i \)-intertwinable. Clearly if \( M, N \) are \( i \)-intertwinable, and if\(^2\)

\[
M \cong M', \quad N \cong N',
\]

then also \( M', N' \) are \( i \)-intertwinable.

**Lemma.** Let \( M, N \) be given by (6), and suppose the \( \{M_i\} \) distinct and abso-
lutely irreducible. Suppose that \( M, N \) are everywhere intertwinable, and further
that they are intertwined by a matrix \( X \) given by (7) for which

\[
a_1, \ldots, a_r \in \text{u}(R), \quad a_{r+1}, \ldots, a_k \in \text{u}(R).
\]

Then

\[
M \cong \begin{bmatrix}
M_1 & * & 0 \\
& \ddots & * \\
& & M_r \\
& & & M_{r+1} \\
& & & & \ddots & * \\
& & & & & M_k
\end{bmatrix}, \quad N \cong \begin{bmatrix}
M_1 & * & 0 \\
& \ddots & * \\
& & M_r \\
& & & M_{r+1} \\
& & & & \ddots & * \\
& & & & & M_k
\end{bmatrix}.
\]

**Proof.** Use induction on \( r \). The result is trivial when \( r = 0 \), so assume
\( r \geq 1 \), and write

\[
M = \begin{bmatrix}
M_1 & * & * \\
M' & \Delta & * \\
M'' & & \\
\end{bmatrix}, \quad N = \begin{bmatrix}
M_1 & * & * \\
N' & \Delta & * \\
N'' & & \\
\end{bmatrix},
\]

\(^2\) We use \( ^tM \) to denote the transpose of \( M \); thus, \( M' \) is just another representation in
this context.
where

\[
M' = \begin{bmatrix} M_2 & \cdots & \ast \\ \vdots & \ddots & \vdots \\ M_r & \cdots & \ast \end{bmatrix}, \quad M'' = \begin{bmatrix} M_{r+1} & \cdots & \ast \\ \vdots & \ddots & \vdots \\ \ast & \cdots & \ast \end{bmatrix}, \quad \text{(submatrices of } M)\text{),}
\]

\[
N' = \begin{bmatrix} M_2 & \cdots & \ast \\ \vdots & \ddots & \vdots \\ M_r & \cdots & \ast \end{bmatrix}, \quad N'' = \begin{bmatrix} M_{r+1} & \cdots & \ast \\ \vdots & \ddots & \vdots \\ \ast & \cdots & \ast \end{bmatrix}, \quad \text{(submatrices of } N)\text{).}
\]

Then also

\[
\begin{bmatrix} M' & \Delta \\ M'' \end{bmatrix}, \quad \begin{bmatrix} N' & \Delta \\ N'' \end{bmatrix}
\]

are everywhere intertwining, and furthermore are intertwined by

\[
\begin{bmatrix} a_1 I \\ \vdots \\ a_k I \end{bmatrix},
\]

a submatrix of \( X \). It follows from the induction hypothesis that by transforming \( M, N \) by suitable translation matrices, we can make \( \Delta = \Delta = 0 \). The new \( M, N \) will still be everywhere intertwining, and also intertwined by a new \( X \) for which (8) still holds.

Let us write

\[
M = \begin{bmatrix} M_1 & \ast & \Lambda_{r+1} & \cdots & \Lambda_k \\ \Lambda_{r+1} & \cdots & \ast & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \ast & \cdots & \ast & \ast & \ast \\ M' & 0 & \cdots & \cdots & M'' \end{bmatrix}, \quad N = \begin{bmatrix} M_1 & \ast & \Delta_{r+1} & \cdots & \Delta_k \\ \Delta_{r+1} & \cdots & \ast & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \ast & \cdots & \ast & \ast & \ast \\ N' & 0 & \cdots & \cdots & N'' \end{bmatrix},
\]

\[
X = \begin{bmatrix} a_1 I & \ast & T_{r+1} & \cdots & T_k \\ \ast & \vdots & \ast & \vdots & \vdots \\ X' & \cdots & \vdots & \ddots & \vdots \\ \ast & \cdots & \ast & \cdots & \ast \\ X'' & \ast & \cdots & \cdots & a_k I \end{bmatrix}, \quad X'' = \begin{bmatrix} a_{r+1} I & \ast \\ \ast & \vdots \\ \ast & \ast \\ \ast & \vdots \\ a_k I \end{bmatrix}.
\]

Then

\[
\begin{bmatrix} M' & 0 \\ M'' \end{bmatrix} \begin{bmatrix} X' & T \\ X'' \end{bmatrix} = \begin{bmatrix} X' & T \\ X'' \end{bmatrix} \begin{bmatrix} N' & 0 \\ N'' \end{bmatrix},
\]

whence \( M'T = TN'' \). Since \( M', N'' \) have no common irreducible constituent, we conclude that \( T = 0 \).

It now follows that

(10) \[
\begin{bmatrix} M_1 & \Lambda_{r+1} \\ M_{r+1} & \ast \end{bmatrix}, \quad \begin{bmatrix} M_1 & \Delta_{r+1} \\ M_{r+1} & \ast \end{bmatrix}
\]

are \( R \)-representations intertwined by
This implies that
\[ M_1 T_{r+1} + a_{r+1} A_{r+1} = a_1 \Delta_{r+1} + T_{r+1} M_{r+1}, \]
and hence (since \( a_{r+1} \in u(R) \)),
\begin{align*}
(12) \quad \Delta_{r+1} & = b \Delta_{r+1} + M_1 U - UM_{r+1}, \\
& \quad b = a_{r+1}^{-1} a_1 \in u(R),
\end{align*}
for some \( U \) over \( R \). On the other hand, the hypothesis that \( M, N \) are 1-inter-twinable guarantees the existence of a matrix of the form (11) which intertwines the representations given in (10), but for which the element playing the role of \( a_1 \) is a unit in \( R \). Therefore we also have
\[ \Delta_{r+1} = c \Delta_{r+1} + M_1 V - VM_{r+1} \]
for some \( c \in R \) and some \( V \) over \( R \). Combining (12) and (13), we obtain
\[ (1 - bc) \Delta_{r+1} = M_1 W - WM_{r+1} \]
for some \( W \) over \( R \). Since \( (1 - bc) \in u(R) \), we conclude that
\[ \Delta_{r+1} = M_1 Y - YM_{r+1} \]
for some \( Y \) over \( R \). Hence by a translation transformation of \( M \), we can make \( \Delta_{r+1} = 0 \). From (13) it follows that we can also make \( \Delta_{r+1} = 0 \) by a translation transformation of \( N \). For this new \( M, N \) we must have \( T_{r+1} = 0 \).

But now we observe that
\[ \begin{bmatrix} M_1 & \Delta_{r+2} \\ M_{r+2} & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M_1 & \Delta_{r+2} \\ M_{r+2} & \end{bmatrix} \]
are representations intertwined by
\[ \begin{bmatrix} a_1 I & T_{r+2} \\ a_{r+2} I & \end{bmatrix}. \]
The above type of argument shows that we can make \( \Delta_{r+2} = \Delta_{r+2} = 0 \), and therefore also \( T_{r+2} \) must be 0. By continuing this process, we establish the validity of (9), Q.E.D.

We may now prove one of the main results of this paper.

**Theorem 5.** Let \( M, N \) be \( RG \)-modules which are \( R' \)-equivalent, and suppose that the irreducible constituents of \( KM \) (which coincide with those of \( KN \)) are distinct from one another and are absolutely irreducible. Then also \( M, N \) are \( R \)-equivalent.

**Proof.** Again use matrix terminology, and proceed by induction on the number \( k \) of irreducible constituents of \( KM \). The result for \( k = 1 \) follows from Theorem 2; suppose it known up to \( k - 1 \), and let \( KM \) have \( k \) distinct absolutely irreducible constituents. There will be no confusion from our
using $M$ to denote both the module and the $R$-representation it affords. The $R$-representations of $G$ afforded by the $RG$-modules $M, N$ may be taken to be of the form

\begin{equation}
M = \begin{bmatrix} M_1 & \cdots & \ast \\ \vdots & \ddots & \vdots \\ M_k & \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & \cdots & \ast \\ \vdots & \ddots & \vdots \\ & & N_k \end{bmatrix},
\end{equation}

where the $\{M_i\}$ and $\{N_i\}$ are absolutely irreducible, and where

\begin{equation}
M_i \sim_K N_i, \quad M_i \sim_K M_j, \quad j \neq i, \quad 1 \leq i \leq k.
\end{equation}

Since $M, N$ are $R'$-equivalent, they are intertwined by a matrix $X'$ unimodular over $R'$. From (15) we find readily (see [6]) that $X'$ has the form

\begin{equation}
X' = \begin{bmatrix} X'_1 & \cdots & \ast \\ \vdots & \ddots & \vdots \\ X'_k & \end{bmatrix},
\end{equation}

and necessarily each $X'_i$ is also unimodular over $R'$. But we have then

\begin{equation}
M_i X'_i = X'_i N_i, \quad 1 \leq i \leq k,
\end{equation}

so that $M_i, N_i$ are $R'$-equivalent for each $i$. By the induction hypothesis it follows that for each $i, 1 \leq i \leq k, M_i$ and $N_i$ are $R$-equivalent. Consequently for each $i$ there exists a matrix $Y_i$ unimodular over $R$ which intertwines $M_i$ and $N_i$. Setting $Y = \text{diag} (Y_1, \cdots, Y_k)$, we deduce that

\begin{equation}
N \sim_R Y N Y^{-1} = \begin{bmatrix} M_1 & \cdots & \ast \\ \vdots & \ddots & \vdots \\ & & M_k \end{bmatrix} \quad \text{(say)}.
\end{equation}

Replacing $N$ by $YNY^{-1}$, we may henceforth assume that $N_1 = M_1, \cdots, N_k = M_k$, that is, that $M, N$ are given by (6).

From the $R'$-equivalence of $M, N$ it follows that they are intertwined by a unimodular matrix $X'$ over $R'$, given by (16). Since now $M_i = N_i$, and $M_i$ is absolutely irreducible, (17) implies that each $X'_i$ is a scalar matrix, so that we may write

\begin{equation}
X' = \begin{bmatrix} \alpha_1 I & \cdots & \ast \\ \vdots & \ddots & \vdots \\ & \alpha_k I \end{bmatrix}, \quad \alpha_1, \cdots, \alpha_k \in u(R').
\end{equation}

Let us now set

\begin{equation}
R' = R \beta_1 \oplus \cdots \oplus R \beta_n, \quad \beta_1 = 1, \quad n = (K':K).
\end{equation}

\footnote{This really follows from [10].}
Then we may write

\[ X' = \sum_{\nu=1}^{n} X^{(\nu)} \beta_{\nu}, \]

we note that

\[
X^{(\nu)} = \begin{bmatrix}
a^{(\nu)}_1 I \\
\ddots \\
1 & \ddots \\
& a^{(\nu)}_k I
\end{bmatrix}, \quad 1 \leq \nu \leq n,
\]

where

\[
\alpha_i = \sum_{\nu} a^{(\nu)}_i \beta_{\nu}, \quad a^{(\nu)}_i \in R.
\]

Let us fix \( i, 1 \leq i \leq k \). Then \( \alpha_i \in u(R') \), and so by (19) at least one of \( a^{(1)}_i, \ldots, a^{(n)}_i \) is a unit in \( R \). Since each \( X^{(\nu)} \) intertwines \( M \) and \( N \), and since \( a^{(\nu)}_i \) occurs in the \( i \)th diagonal block of \( X^{(\nu)} \), we may conclude that \( M, N \) are \( i \)-intertwinable. This shows then that if \( M, N \) given by (6) are \( R' \)-equivalent, they must be everywhere intertwinable.

Since \( M, N \) are \( 1 \)-intertwinable, there exists an \( X \) (over \( R \)) given by (7) which intertwines \( M \) and \( N \), and for which \( a_1 \in u(R) \). If also \( a_2, \ldots, a_k \in u(R) \), then \( X \) is unimodular over \( R \), and so \( M, N \) are \( R \)-equivalent. For the remainder of the proof we may therefore suppose that not all of \( a_2, \ldots, a_k \) are units in \( R \). Let us write

\[
a_1, \ldots, a_q \in u(R), \quad a_{q+1}, \ldots, a_r \notin u(R), \quad a_{r+1}, \ldots, a_k \in u(R), \ldots.
\]

Partition \( X \) accordingly, say

\[
X = \begin{bmatrix} Y_1 & \cdots & \ast \\ \ast & \ddots & \ast \\ Y_k & \cdots & \ast \end{bmatrix}, \quad Y_1 = \begin{bmatrix} X_1 & \cdots & \ast \\ \ast & \ddots & \ast \\ X_q & \cdots & \ast \end{bmatrix}, \quad Y_2 = \begin{bmatrix} \cdots & \ast & \ast \\ \cdots & \ddots & \ast \\ \cdots & \ast & \cdots \end{bmatrix}, \ldots
\]

Correspondingly partition \( M, N \), say

\[
M = \begin{bmatrix} M_1 & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & M_2 & \Lambda_{23} \\ \cdots & \cdots & \ddots \end{bmatrix}, \quad N = \begin{bmatrix} \bar{N}_1 & \Delta_{12} & \Delta_{13} \\ \Delta_{22} & \bar{N}_2 & \Delta_{23} \\ \cdots & \cdots & \ddots \end{bmatrix},
\]

where

\[
\bar{M}_1 = \begin{bmatrix} M_1 & \cdots & \ast \\ \ast & \ddots & \ast \\ \ast & \cdots & M_q \end{bmatrix}, \quad \bar{N}_1 = \begin{bmatrix} M_1 & \cdots & \ast \\ \ast & \ddots & \ast \\ \ast & \cdots & M_q \end{bmatrix}, \ldots
\]

By repeated use of the lemma, we may transform \( M, N \) by translations so as to make successively

\[
\Lambda_{12} = \Delta_{12} = 0, \quad \Lambda_{23} = \Delta_{23} = 0, \quad \cdots, \quad \Lambda_{t-1,t} = \Delta_{t-1,t} = 0.
\]
Such transformations do not affect the diagonal blocks of $X$, nor the $R'$-equiva-
lence of $M, N$. We may therefore assume for the remainder of the proof that
(21) holds. But in that case we see from (20) that
\[
\begin{bmatrix}
\tilde{M}_1 & \Lambda_{14} \\
\tilde{M}_4 & \Lambda_{14}
\end{bmatrix}, \quad
\begin{bmatrix}
\tilde{N}_1 & \Delta_{14} \\
\tilde{N}_4 & \Delta_{14}
\end{bmatrix}
\]
are $R$-representations of $G$, and again we may apply the lemma to conclude
that $M, N$ may be further transformed by translation matrices so as to make
$\Lambda_{14} = \Delta_{14} = 0$, and so on. Continuing in this way, we find that
\[
M \approx M' = \begin{bmatrix}
\tilde{M}_1 & \Omega \\
\cdot & \cdot \\
\tilde{M}_t & 
\end{bmatrix}, \quad
N \approx N' = \begin{bmatrix}
\tilde{N}_1 & \Sigma \\
\cdot & \cdot \\
\tilde{N}_t & 
\end{bmatrix},
\]
where $\Omega_{ij} = \Sigma_{ij} = 0$ whenever the diagonal entries of $X$ associated with
$\tilde{M}_i$ are units, those with $\tilde{M}_j$ nonunits, or vice versa. But we may then find a
permutation matrix $F$ such that
\[
FM'F^{-1} = \begin{bmatrix}
M^* & 0 \\
M^{**}
\end{bmatrix}, \quad
FN'F^{-1} = \begin{bmatrix}
N^* & 0 \\
N^{**}
\end{bmatrix},
\]
where
\[
M^* = \begin{bmatrix}
\tilde{M}_1 & * \\
\cdot & \cdot \\
\tilde{M}_3 & *
\end{bmatrix}, \quad
M^{**} = \begin{bmatrix}
\tilde{M}_2 & * \\
\cdot & \cdot \\
\tilde{M}_4 & *
\end{bmatrix},
\]
\[
N^* = \begin{bmatrix}
\tilde{N}_1 & * \\
\cdot & \cdot \\
\tilde{N}_3 & *
\end{bmatrix}, \quad
N^{**} = \begin{bmatrix}
\tilde{N}_2 & * \\
\cdot & \cdot \\
\tilde{N}_4 & *
\end{bmatrix}.
\]
We now have
(22) $M \sim_R \begin{bmatrix} M^* & 0 \\ M^{**} \end{bmatrix}$, \quad $N \sim_R \begin{bmatrix} N^* & 0 \\ N^{**} \end{bmatrix}$,
and so (since $M \sim_{R'} N$),
\[
\begin{bmatrix} M^* & 0 \\ M^{**} \end{bmatrix} \sim_{R'} \begin{bmatrix} N^* & 0 \\ N^{**} \end{bmatrix}.
\]
Since $M^*, M^{**}$ have no common irreducible constituents, this latter equiva-
lence implies that $M^* \sim_{R'} N^*$, $M^{**} \sim_{R'} N^{**}$.

We may (at last) use the induction hypothesis to conclude from this that
\[
M^* \sim_R N^*, \quad M^{**} \sim_R N^{**}.
\]
This, together with (22), implies that $M, N$ are $R$-equivalent. Thus the theorem is proved.

4. We shall apply the preceding result to the case of $p$-groups.

**Theorem 6.** Let $G$ be a $p$-group, where $p$ is an odd prime. Let $R$ be the ring of $p$-integral elements of the rational field $Q$. Suppose that $K'$ is an algebraic number field, and $R'$ any valuation ring of $K'$ such that $R' \supseteq R$. Then for any pair of irreducible $RG$-modules $M, N$ we have

$$M \sim_{R'} N \Rightarrow M \sim_R N.$$

**Proof.** Set $(G:1) = p^m$, $m > 1$, and let $\zeta$ be a primitive $(p^m)^{th}$ root of 1 over $Q$. Let $M, N$ be $R'$-equivalent irreducible $RG$-modules. As a first step, let us set $K_1 = K'((\zeta))$, and let $R_1$ be a valuation ring of $K_1$ such that $R_1 \supseteq R'$. Then since $M \sim_{R'} N \Rightarrow M \sim_{R_1} N$, we may now restrict our attention to $K_1, R_1$ instead of $K', R'$.

Next we note that

$$f(x) = \text{Irr}(\zeta, Q) = x^{p^m-1(p-1)} + x^{p^m-1(p-2)} + \cdots + x^{p^m-1} + 1,$$

and that $f(x + 1)$ is an Eisenstein polynomial at the prime $p$. If we set $K_0 = Q((\zeta))$, it follows that $K_0$ contains a uniquely determined valuation ring $R_0$ such that $R_0 \supseteq R_1$, and further that the residue class fields corresponding to $R_0, R$ coincide. We may therefore conclude from Theorem 4 that

$$M \sim_{R_0} N \Rightarrow M \sim_R N.$$

The proof will be complete as soon as we establish

$$M \sim_{R_1} N \Rightarrow M \sim_{R_0} N.$$

This is a consequence of Theorem 5, however, as we now proceed to demonstrate. The modules $R_0 M, R_0 N$ are (in general) no longer irreducible. Since $K_0$ is an absolute splitting field for $G$ (see [1]), the irreducible constituents of $K_0 M$ and $K_0 N$ are all absolutely irreducible. The multiplicity with which any absolutely irreducible constituent of $K_0 M$ occurs is precisely the Schur index of that constituent relative to the rational field (see [7]). On the other hand, for $p$-groups ($p$ odd) it is known [2, 8] that this Schur index is 1. Hence the irreducible constituents of $R_0 M$ and $R_0 N$ are distinct and absolutely irreducible. We may therefore apply Theorem 5, and obtain

$$R_1 M \cong R_1 N \Rightarrow R_0 M \cong R_0 N,$$

so that (25) is proved, Q.E.D.

The referee has kindly pointed out that the preceding theorem is also valid for the more general case in which $R$ is a valuation ring of an algebraic number field $K$ such that $R$ lies over the ring of $p$-integral elements of the rational
field. Indeed, the above proof requires only a minor modification for the more general case.

5. We conclude by listing a number of open questions.

A. If $R \subset R'$ are valuation rings, does (5) hold without any restrictive hypotheses?

B. Using the notation of Section 2, under what conditions does $\sigma'M \triangleright \sigma'N$ imply $M \triangleright N$, where $M$ and $N$ are $\sigma G$-modules?

C. If $\sigma$ is a principal ideal ring, does $\sigma'$-equivalence imply $\sigma$-equivalence?

It may be of interest to mention yet one more special case in which additional information may be obtained. Suppose that $M$ and $N$ are projective $RG$-modules, where $R$ is the valuation ring of a discrete valuation of $K$. (For example, $M$ and $N$ might be direct summands of $RG$.) Then it is known\(^4\) that $M \sim_R N$ if and only if $M \sim_K N$. Using Theorem 1 and its corollary, we conclude that (5) holds in this case.

In particular, if $M$ and $N$ are projective $\sigma G$-modules, then $\sigma'M \triangleright \sigma'N$ surely implies that $M$ and $N$ are $K'$-equivalent, and hence by the above discussion that $M \triangleright N$.

Added in proof. In a recently completed paper [11], Zassenhaus and the author have shown that (5) holds without any restrictive hypotheses, assuming still that $R$ and $R'$ are valuation rings as in Section 3. This settles questions A and B, but C is still open.

References


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\(^4\) R. G. Swan, Induced representation and projective modules, University of Chicago, mimeographed notes, 1959, Corollary 6.4.