ON WEIERSTRASS PRODUCTS OF ZERO TYPE ON THE REAL AXIS

BY

J. P. KAHANE AND L. A. RUBEL

1. Introduction

Let \( \mathcal{W} \) be the class of even entire functions \( W(z) \) of exponential type, with real zeros only, and such that \( W(0) = 1 \). It follows readily from the Hadamard factorization theorem that \( \mathcal{W} \) is identical with the class of all Weierstrass products \( W(z) = \prod (1 - z^2/\lambda_n^2) \) with \( 0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \) and \( n/\lambda_n \) bounded. For a given function \( T(r) > 0 \), let \( \mathcal{W}_T \) be that subclass of \( \mathcal{W} \) consisting of those \( W \in \mathcal{W} \) for which \( |W(r)| = O(1) \exp(T(r)) \). If \( T(r) \) does not grow too fast as \( r \to \infty \) and \( W \in \mathcal{W}_T \), then (see (2.4)) the sequence \( \{\lambda_n\} \) must have density \( D \), and on each nonhorizontal ray \( z = re^{i\theta} \) through the origin, \( |W(z)| \) grows like \( \sin(vDz) \); and if \( W_1, W_2 \in \mathcal{W}_T \) and

\[
W(z) = W_1(z)W_2(z)
\]
is their product, then (see (2.6)) type \((W) = \text{type}(W_1) + \text{type}(W_2)\). The weakest known hypothesis on \( T \) that guarantees these conclusions is

\[
\int r^{-2}T(r) \, dr < \infty.
\]

Our main result says that if \( T \) violates this hypothesis, then the conclusions will no longer hold.

That the types need no longer add has particular significance for generalized harmonic analysis. Since a class \( \mathcal{W}_T \) corresponds to the collection of Fourier transforms of generalized distributions in a class \( \mathcal{T}_T \), multiplication in \( \mathcal{W}_T \) corresponding to convolution in \( \mathcal{T}_T \), and the type of \( W \in \mathcal{W}_T \) corresponding to the support of the corresponding \( F \in \mathcal{T}_T \), our main result shows, independently of the recent work of Roumieu [5], the impossibility of extending the "theorem of supports" to certain classes of generalized distributions.

This paper is essentially self-contained, but a knowledge of the general background material, as discussed, say, in Chapters I, II, and V of Boas's book [1] is probably indispensable.

2. Notation, history, and statements of results

With the Weierstrass product

\[
W(z) = \prod_{n=0}^{\infty} (1 - z^2/\lambda_n^2),
\]

\( 0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \), \( n/\lambda_n \) bounded,

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we associate the functions

\[ n(t) = \sum_{\lambda \geq t} 1, \quad D(t) = n(t)/t, \quad \bar{D}(t) = \frac{1}{t} \int_0^t D(u) \, du, \]

\[ h(\theta) = \limsup_{r \to \infty} r^{-1} \log |W(re^{i\theta})|, \quad \chi(\theta) = \liminf_{r \to \infty} r^{-1} \log |W(re^{i\theta})| \]

for \( 0 \leq \theta < 2\pi \).

In addition, we use the notation

\[ h = h(\pi/2) = \text{type } (W(z)), \]

\[ D^* = \limsup_{t \to \infty} D(t), \quad D_* = \liminf_{t \to \infty} D(t), \quad \bar{D}^* = \limsup \bar{D}(t). \]

We state some known results.

(2.2) \( h(0) = 0 \) if and only if \( h(\theta) = \pi D^* \sin \theta \) for all \( \theta \) [6, p. 428].

(2.3) If \( W(z) = W_1(z)W_2(z) \), then (trivially) \( h \geq \max (h_1, h_2) \).

(2.4) If

\[ \int_{r=0}^{\infty} r^{-2} \log^+ W(r) \, dr < \infty, \]

then \( D_* = D^* \) and \( h(\theta) = \chi(\theta) = \pi D^* \sin \theta \) for \( \theta \neq 0, \pi \) [3, p. 769].

(2.5) Corollary. If \( W(z) = W_1(z)W_2(z) \) and \( W_1(z) \) or \( W_2(z) \) satisfies (2.5), then \( h = h_1 + h_2 \).

Our main result, announced in [7], is that (2.3), (2.4), and (2.6) are essentially best possible. That the conclusion \( D_* = D^* \) of (2.4) is no longer valid if (2.5) is weakened to the condition \( h(0) = 0 \), is contained in [4, Theorem V].

Theorem. Let \( T(r) \) be a positive increasing function defined for \( r > r_0 \) with \( T(r)/r \) decreasing and \( T(r)/\log r \) increasing, and such that

\[ \int_{r=0}^{\infty} r^{-2}T(r) \, dr = \infty. \]

Then there exist, given any \( h_1, h_2 > 0 \), Weierstrass products (2.1), \( W_1(z) \) and \( W_2(z) \), whose types are \( h_1 \) and \( h_2 \) respectively, satisfying

\[ |W_i(r)| = O(1)e^{r^{\alpha}}, \quad i = 1, 2, \]

but such that if \( W(z) = W_1(z)W_2(z) \) is their product, then

\[ \text{type } (W) = \max (h_1, h_2). \]

In addition, for \( i = 1, 2 \), \( h_i = \pi D^*_i \), \( D_{-i} = 0 \), and \( \chi_i(\theta) = 0 \) for \( \theta \neq 0, \pi \).

Remarks. The conditions \( T(r)/r \downarrow \) and \( T(r)/\log r \uparrow \) are regularity conditions on \( T(r) \) and do not affect the convergence or divergence of the integral
in (2.7). It would be nice to eliminate these conditions, but we have not found a way to do this. The condition $T(r)/\log r \uparrow$ can be replaced, with certain changes in the proof, by any one of several somewhat related conditions of which three examples are

(i) $T(r)/\log (r/T(r)) \uparrow$,
(ii) $r^{1/2} \leq T(r) \leq r/\log r$,
(iii) the function $\tau(r)$, defined by $\tau(r) = T(r)/r$, is slowly oscillating in the sense that $\tau(ar)/\tau(r) \to 1$ as $r \to \infty$ for each positive $a$.

There is no difficulty in modifying the proof of the theorem to give a construction of an infinite set $W_j(z)$, $j = 1, 2, 3, \ldots$ of products (2.1) satisfying (2.8) such that

$$\prod_{j=1}^\infty W_j(z) = (\sin \pi z)/\pi z = \prod_{n=1}^\infty (1 - z^2/n^2),$$

but such that for each $W_j(z)$ and each product $W(z)$ of a finite number of the $W_j(z)$, we have $h_1 = h_2 = \cdots = h = \pi$. To do this, one need only replace the pair of functions $A_1$, $A_2$ of Section 4 by an infinite set having similar properties, and replace the constant $k$ there by a function $k(t)$ that decreases extremely slowly to 0 as $t \to \infty$.

The first two lemmas are interesting in themselves, and we state them here. Lemma 1 states that if $D(r)$ is slowly oscillating in the sense of (2.9), then for each $\theta \neq 0$, $\pi$, $|W(re^{i\theta})|$ imitates the behaviour of $D(r)$. Lemma 2 enables us to make the passage from continuous mass distributions to discrete ones. As a corollary of Lemma 1 it is easily seen that if (2.9) holds, then $h(\theta) = \pi D'$ $|\sin \theta|$ for $\theta \neq 0$, $\pi$, and by the well-known continuity of $h(\theta)$ that $h(0) = 0$, thus giving another proof of a result of Redheffer [4, Theorem II].

**Lemma 1.** If

$$\lim_{r \to \infty} [D(rt) - D(r)] = 0$$

(2.9)

uniformly for $t$ in any interval $0 < \varepsilon \leq t \leq 1/\varepsilon$,

then for $\theta \neq 0$, $\pi$

$$\log |W(re^{i\theta})| = \pi r D(r) |\sin \theta| + o(r).$$

**Lemma 2.** Suppose that $v(r)$ is a continuously differentiable function for $0 \leq r < \infty$, that $0 \leq v'(r) \leq q < \infty$, and that

$$v(r) \geq n(r) > v(r) - K$$

(2.10)

for some constant $K$ and all $r$.

Then

$$\log |W(r)| \leq \int_0^\infty \log \left| 1 - r^2/t^2 \right| v'(t) dt + O(\log r) \quad \text{as } r \to \infty.$$  

3. Proofs of Lemmas 1 and 2

**Proof of Lemma 1.** Write $\log W(re^{i\theta}) = \log \prod (1 - r^2e^{2i\theta}/\lambda_n^2) = \sum \log (1 - r^2 e^{2i\theta}/\lambda_n^2) = \int_\theta^\pi \log (1 - r^2 e^{2i\theta}/t^2) dn(t)$. For $\theta \neq 0$, $\pi$ we may
integrate by parts. The "integrated terms" drop out if the branch of the logarithm is conveniently chosen because $n/\lambda_n$ is bounded (see (2.1)), and we get, after a multiplicative change of variables,

$$\log W(re^{i\theta}) = \int_0^\infty \frac{2e^{2i\theta}}{e^{2i\theta} - t^2} D(rt) \, dt.$$  

Hence the familiar formula

(3.1) \[ \log |W(re^{i\theta})| = r \int_0^\infty P(t, \theta)D(rt) \, dt, \]

where

$$P(t, \theta) = \text{Re}\left\{\frac{2e^{2i\theta}}{e^{2i\theta} - t^2}\right\} = 2 \frac{1 - t^2 \cos 2\theta}{1 - 2t^2 \cos 2\theta + t^4}.$$

For each $\theta \neq 0, \pi$, $P(t, \theta)$ is a bounded and Lebesgue integrable function of $t$ on $(0, \infty)$, and it is well known that $\int_0^\infty P(t, \theta) \, dt = \pi |\sin \theta|$. Thus

$$\log |W(re^{i\theta})| - \pi r D(r) |\sin \theta| = r \int_0^\infty (D(rt) - D(r))P(t, \theta) \, dt.$$

By breaking the range of this last integral into three parts,

$$\int_0^\infty = \int_0^\epsilon + \int_{\epsilon}^{1/\epsilon} + \int_{1/\epsilon}^\infty,$$

it is easy to see that $\int_0^\infty (D(rt) - D(r))P(t, \theta) \, dt \to 0$ as $r \to \infty$ (but not uniformly in $\theta \neq 0, \pi$), and the lemma is proved.

**Remark.** The hypothesis (2.9) can be replaced by the following, apparently weaker, hypothesis:

(2.9') \lim_{r \to \infty} \{D(rt) - D(r)\} = 0 \quad \text{for each} \quad t \in (0, \infty),

since a frequently discovered result asserts that if (2.9') holds for a Lebesgue measurable function $D(r)$, then (2.9) actually holds. (The history of this result is too complicated for us to unravel here, and we give only the reference [2, 1.4].)

**Proof of Lemma 2.** For fixed $r$, we write, as in the proof of Lemma 1, $\log |W(r)| = \int_0^\infty L(t) \, dn(t)$, where $L(t) = \log |1 - r^2/t^2|$. We point out that $L(t)$ is Lebesgue integrable on $(0, \infty)$,

$$L(0+) = +\infty, \quad L(r-) = L(r+) = -\infty, \quad L(\infty) = 0,$$

and that $L(t)$ is decreasing and continuous in $(0, r)$ and increasing and continuous in $(r, \infty)$. We must compare

$$Y = \int_0^\infty L(t) \, dn(t) \quad \text{and} \quad Z = \int_0^\infty L(t) \, dv(t).$$

We will prove that $Y < Z + O(\log r)$ where $n(r)$ may be replaced by any increasing function $\mu(r)$ satisfying $\mu(0) = 0$ and $\mu(r) \geq \mu(r) > \nu(r) = K$ for
some constant $K$. We assume that $\nu'(t) \geq p > 0$. This involves no loss of generality since if we replace $\nu(t)$ by $\nu(t) + t$, and $\mu(t)$ by $\mu(t) + t$, we change $Z$ and $Y$ not at all because $\int_0^\infty L(t) \, dt = 0$. We may suppose without loss of generality that $\nu(0) = 0$ since suitably redefining $\nu$ on the interval $[0, 1]$ changes the value of the integral in the conclusion (2.11) only by $O(1)$. The additional $O(1)$ is negligible compared to $O(\log r)$, which is the discrepancy allowed in (2.11).

With each large $r$ we associate the numbers $r_1$ and $r_2$ such that

$$\nu(r_1) = \mu(r) = \nu(r_2) = K.$$

Since $\nu'(t) \geq p$, we will have $r - r_1 \leq r_2 - r_1 \leq K/p$. The following inequalities hold, as can be readily verified:

$$\int_0^r L(t) \, d\mu(t) \leq \int_0^{r_1} L(t) \, d\nu(t), \tag{3.2}$$

$$\int_r^\infty L(t) \, d\mu(t) \leq \int_{r_2}^\infty L(t) \, d\nu(t). \tag{3.3}$$

From these inequalities we deduce that $Y \leq Z + X$, where

$$X = - \int_{r_1}^{r_2} \log |1 - r^2/t^2| \, d\nu(t),$$

and we shall prove that $X \leq O(\log r)$. Clearly,

$$X \leq - \int_{r_1}^{r_2} \log \left| \frac{t - r}{t} \right| \, d\nu(t).$$

Since $r_2 - r_1 \leq K/p$ and $\nu'(t) \leq q$, we have

$$X \leq - q \int_{r_1}^{r_2} \log \left| \frac{t - r}{r} \right| \, dt \leq q(r_2 - r_1) \log r_2 - q \int_{r_1}^{r_2} \log |t - r| \, dt,$$

so that $X \leq (qK/p) \log (r + K/p) + 2q$.

4. Proof of the theorem

Let us first illustrate the method of proof with a simple example to show that one may have $h_1(0) = h_2(0) = 0$, but not $h = h_1 + h_2$. Put

$$n_1(r) = \left[ \int_0^r \{1 + \sin (\log \log t)\} \, dt \right],$$

$$n_2(r) = \left[ \int_0^r \{1 + \cos (\log \log t)\} \, dt \right],$$

and let $W_1(z)$ and $W_2(z)$ be the Weierstrass products (2.1) over the sets whose counting functions are $n_1(t)$ and $n_2(t)$, respectively. The slow oscillations imply (by Lemma 1 and the continuity of $h_1(\theta)$) that $h_1(0) = h_2(0) = 0$. Lemma 1 shows that $W_1(iy)$ behaves very much like
exp \{ \pi y (1 + \sin (\log \log y)) \}, and \( W_2(iy) \) like exp \{ \pi y (1 + \cos (\log \log y)) \} as \( y \to \infty \). But since sin and cos are out of phase, we get not
\[
h = 2\pi + 2\pi = 4\pi,
\]
but \( h = (2 + 2^{1/2})\pi \) instead.

Beginning now the proof of the theorem, we will suppose without loss of generality that \( T(r) \) is continuous and that \( \lim_{r \to \infty} T(r)/r = 0 \) because a function \( T(r) \) satisfying the hypotheses of the theorem certainly has a continuous minorant \( T^*(r) \) satisfying the hypotheses with \( \lim_{r \to \infty} T^*(r)/r = 0 \). Also, (2.7) implies that \( T(r)/\log r \to \infty \) since we have supposed that \( T(r)/\log r \uparrow \). We will not prove the "in addition" part of the theorem since it will be amply clear from the proof that each of the functions \( W_1(z), W_2(z) \) will satisfy the requirements of the second part. To construct these Weierstrass products \( W_1(z) \) and \( W_2(z) \), we take two functions \( A_1(t) \) and \( A_2(t) \) satisfying the following simple conditions:

\begin{enumerate}
  \item[(4.1)] \( A_1(t) \) and \( A_2(t) \) are nonnegative continuously differentiable periodic functions of period \( 2\pi \) for \( -\infty < t < \infty \).
  \item[(4.2)] \( A_1(t)A_2(t) = 0 \), i.e., \( A_1(t) \) vanishes where \( A_2(t) \) does not, and vice versa.
  \item[(4.3)] \( \max_t A_1(t) = h_1 \), \( \max_t A_2(t) = h_2 \).
\end{enumerate}

For example, we might choose
\[
A_1(t) = h_1 \max (\sin t, 0)^2 \quad \text{and} \quad A_2(t) = h_2 \min (\sin t, 0)^2.
\]

Now define \( \nu_i(t) \) (where, as throughout this section, \( i = 1, 2 \)) by
\[
\nu_i(t) = \int_0^t A_i(l(s)) \, ds,
\]
where \( l(s) \) is the continuous function defined by

\begin{align*}
  l'(H(t)) &= k \frac{\log t}{t} \quad \text{for} \quad t \geq t_0 = \max (r_0, e), \\
  l(t) &= k \frac{\log t_0}{t_0} t \quad \text{for} \quad 0 < t < H(t_0),
\end{align*}

where \( H(t) = T(t)/\log t \), and the constant \( k \) will be chosen later in a way that depends only on the choice of the functions \( A_1(t) \) and \( A_2(t) \).

Finally, we define \( W_i(z) \) by
\[
\log W_i(z) = \int_0^\infty \log (1 - z^2/t^2) \, dn_i(t),
\]
where \( n_i(t) = [\nu_i(t)] \).
Lemma 3. \( \lim_{r \to \infty} \{ A_i(l(rt)) - A_i(l(r)) \} = 0 \) uniformly for \( t \) in any interval \( 0 < \varepsilon \leq t \leq 1/\varepsilon \).

The proof follows from the estimate

\[
|A_i(l(rt)) - A_i(l(r))| \leq \| A_i \|_\infty \{ \max_{\xi \leq r} l'(\xi) \} r(1 - t)
\]

if \( t < 1 \), where \( \| \cdot \|_\infty \) denotes the supremum of the indicated function. There is a similar estimate if \( t > 1 \). But from (4.4), provided that \( r \geq H(t_0)/t \) (then \( H^{-1}(rt) \geq e \)), we have \( l'(\xi) = k(\log H^{-1}(\xi))/H^{-1}(\xi) \), and for such \( r \) we then have

\[
rl'(\xi) = kr \frac{\log H^{-1}(\xi)}{H^{-1}(rt)} \leq k \frac{\log H^{-1}(rt)}{H^{-1}(rt)} \leq k \frac{H(y) \log y}{y} = k \frac{T(y)}{y},
\]

where \( y = H^{-1}(rt) \). But \( (k/t)(T(y)/y) \to 0 \) uniformly for \( t \geq \varepsilon > 0 \) since \( T(y)/y \to 0 \) as \( y \to \infty \).

Lemma 4. \( D_i(r) = A_i(l(r)) + o(1) \) as \( r \to \infty \), and the hypothesis of Lemma 1 is satisfied by \( D_i(r) \).

We have to prove the first part, from which the second follows, by Lemma 3. The proof is immediate, on noticing that \( D_i(r) = r^{-1} \nu_i(r) + o(1) \), so that

\[
D_i(r) - A_i(l(r)) = \int_0^1 \{ A_i(l(rt)) - A_i(l(r)) \} dt + o(1),
\]

and by Lemma 3 the second member is \( o(1) \).

Lemma 5. \( l(r) \to \infty \) as \( r \to \infty \).

It is precisely at this point that the condition (2.7) enters the picture. We write

\[
l(H(r)) \geq \int_{t_0}^r l'(H(s)) dH(s).
\]

By (4.4) we may write this last integral as

\[
\int_{t_0}^r l'(H(s)) dH(s) = k \int_{t_0}^r \frac{\log s}{s} d\left( \frac{T(s)}{s} \right) = k \int_{t_0}^r \frac{\log s - 1}{s^2} \frac{T(s)}{\log s} ds + O(1)
\]

on integrating by parts. Since the divergence of the last integral is an easy consequence of (2.7), we are done.

From Lemma 4 and Lemma 1, we conclude that
\begin{align*}
\log |W_i(re^{i\theta})| &= \pi r A_i(l(r)) + o(r), \\
\text{and therefore that for } W &= W_1 W_2 \\
\log |W(re^{i\theta})| &= \pi r [A_1(l(r)) + A_2(l(r))] + o(r).
\end{align*}

Since, by Lemma 5, \( l(r) \to \infty \), it is clear that
\[
\text{type } (W_i) = h_i,
\]
and that because of (4.2) and (4.3)
\[
\text{type } (W) = \max (h_1, h_2).
\]

It remains only to verify that the \( W_i \) satisfy (2.8), which we now do. By Lemma 2, if we show that
\[
(4.5) \quad Z_i = \int_0^\infty |\log 1 - r^2/t^2| \, dv_i(t) \leq T(r)
\]
for large \( r \), we will be done except for the trivial enlargement of the \( O(1) \) of (2.8) to \( \exp (O(\log r)) \), that is, to a term of polynomial growth. We leave it to the reader to verify that by simply dropping a finite number of terms from each of the products (2.1) for \( W_i(z) \), the additional factors of polynomial growth are cancelled without affecting the other conditions.

To prove (4.5), write it as
\[
Z_i = - \int_0^\infty \varphi(t/r) t v''_i(t) \, dt,
\]
where
\[
\varphi(t) = \frac{1}{t} \int_0^t \log \left| 1 - \frac{1}{u^2} \right| \, du = \log \left| 1 - \frac{1}{t^2} \right| + \frac{1}{t} \log \left| \frac{1 + t}{1 - t} \right| \geq 0.
\]

Thus
\[
Z_i = \int_0^\infty - \varphi(t/r) t l''(t) A'_i(l(t)) \, dt = \int_0^H + \int_H^\infty,
\]
where \( H = H(r) = T(r)/\log r \) as before. Now
\[
\int_0^H - \varphi(t/r) t l''(t) A'_i(l(t)) \, dt \leq \| A'_i \|_\infty \| t l''(t) \|_\infty \int_0^H \varphi(t/r) \, dt.
\]

It is easy to verify that \( \int_0^H \varphi(t/r) \, dt \leq 3H \log (r/H) \leq 3T(r) \) and to show that \( \| t l''(t) \|_\infty \leq kT(t_0)/t_0 \), so that
\[
\int_0^H \leq kK_1 T(r),
\]
where \( K_1 \) is a constant that depends only on the choice of the functions \( A_i \).

Now for sufficiently large \( t \) the function \( t l''(t) \) is decreasing, and thus, for
large $r$, we have the estimate
\[
\int_{H}^{\infty} - \varphi(t/r) t' \varphi'(l(t)) \, dt \leq \| A'_i \|_{\infty} H' \int_{H}^{\infty} \varphi(t/r) \, dt.
\]
But $H' = k T(r)/r$ and $\int_{H}^{\infty} \varphi(t/r) \, dt \leq r \int_{0}^{\infty} \varphi(t) \, dt$. Hence
\[
\int_{H}^{\infty} \leq k K_2 T(r),
\]
where $K_2$ also depends only on the choice of the $A_i$.

Having chosen the $A_i$ then, we select $k$ so that $k(K_1 + K_2) < 1$ and conclude that $Z_i \leq T(r)$ for all sufficiently large $r$, and the theorem is proved.

BIBLIOGRAPHY


Université de Montpellier
Montpellier, France

University of Illinois
Urbana, Illinois