A DIMENSION THEOREM FOR SAMPLE FUNCTIONS OF STABLE PROCESSES

BY

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1. Introduction

The main theorem of this paper concerns the Hausdorff-Besicovitch dimension of the range of the sample functions of a stable process in R_N. Results of this sort for the symmetric stable processes were obtained earlier by McKean [6], [7] and by us [1]. The symmetric stable processes are subordinate to Brownian motion, a fact that we found useful in [1]; but there seems to be no similar relationship for the general stable processes, so a different approach is necessary.

2. Preliminaries

If F is a stable probability distribution on R_N and φ is its N-dimensional characteristic function, then either F is a (possibly degenerate) N-dimensional normal distribution, or else

$$\log \varphi(y) = i(a, y) - \lambda \mid y \mid^{\alpha} \int_{S_N} w_{\alpha}(y, \theta)\mu(d\theta) \tag{1}$$

for some a in R_N, $\lambda > 0$, $0 < \alpha < 2$, μ a probability measure on the surface of the unit sphere S_N in R_N. In this formula θ denotes a variable point on S_N, and the function w_α is defined by

$$w_{\alpha}(y, \theta) = [1 - i \operatorname{sgn} (y/\mid y \mid, \theta) \tan \tfrac{1}{2}\pi\alpha]\cdot\mid (y/\mid y \mid, \theta) \mid^{\alpha}$$

if $\alpha \neq 1$, and

$$w_1(y, \theta) = \mid (y/\mid y \mid, \theta) \mid + (2i/\pi)(y/\mid y \mid, \theta) \log \mid (y, \theta) \mid.$$

The correct interpretation of this if $y = 0$ or if $(y, \theta) = 0$ is obvious. The number α is called the index of the stable distribution. Formula (1) is due to Lévy [5]. If $\alpha < 2$, then φ is integrable, so any stable distribution of index $\alpha < 2$ has a bounded continuous density. From now on we will consider only the nonnormal stable distributions.

If F is stable of index α, then for every $k > 0$

$$F(\{x: \mid x \mid > r\})/F(\{x: \mid x \mid > kr\}) \to k^{\alpha} \quad \text{as} \quad r \to \infty.$$

This is a consequence of Theorem 4.2 of [8], and it implies that if $p > 0$, then

$$\int_{R_N} \mid x \mid^{p} F(dx) < \infty$$

if and only if $p < \alpha$.

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Let $\{X(t);\, t \geqq 0\}$ be a stable process in R_N of index $\alpha < 2$ defined over some basic probability space Ω of points ω; that is, a process with stationary independent increments, such that for each $t > s$ the characteristic function of $X(t) - X(s)$ is $e^{(t-s)\log\varphi(y)}$ with φ given by (1). We assume that $X(0) = 0$ and that in (1) we have $a = 0$ and $\lambda = 1$. It follows that if $\alpha \neq 1$ and r and t are positive, then $r^{1/\alpha}X(t)$ has the same distribution as $X(rt)$. In the case $\alpha = 1$, $rX(t)$ has the same distribution as $X(tr) + t(r \log r)\, a$, where a is the point in R_N with coordinates

$$a_j = \int_{S_N} \frac{2}{\pi}\, \theta_j\, \mu(d\theta).$$

We will assume that the process has been normalized to have right-continuous sample functions.

Now let β be a positive real number, and E a subset of R_N. For each $\varepsilon > 0$ set $\Lambda_\varepsilon^\beta(E) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^\beta$ where $\{E_i\,;\, i \geqq 1\}$ is a cover of E by subsets of R_N all of diameter not exceeding ε, and the infimum is taken over all such covers. We would get the same number if we restricted the E_i's to be open sets or closed sets or, in the case of the real line, closed intervals. Let $\Lambda^\beta(E) = \lim_{\varepsilon\to 0} \Lambda_\varepsilon^\beta(E)$. Λ^β is called the Hausdorff β-dimensional outer measure on R_N. It is a metric outer measure, and so the Borel sets are always measurable. If E is a Borel set with $\Lambda^\beta(E) = M \leqq \infty$, and if $0 < h < M$, then there is a closed set F contained in E such that $\Lambda^\beta(F) = h$. This fact, actually for analytic E, is proved by Davies in [2], and it implies that Λ^β restricted to the Borel sets is inner regular. In general Λ^β is not outer regular. It is also true that

$$\sup\{\beta\colon \Lambda^\beta(E) = \infty\} = \inf\{\beta\colon \Lambda^\beta(E) = 0\}.$$

This common value is called the Hausdorff-Besicovitch dimension of E, and is denoted by dim E.

We need two more facts. First of all, a Borel subset E of R_N is said to have positive β-capacity ($C_\beta(E) > 0$) if there is a probability measure m, concentrated on E, such that

$$\int_E \int_E |\, x - y\,|^{-\beta}\, m(dx)m(dx) < \infty. \tag{2}$$

A theorem of Frostman [3, p. 86] states that if E is closed and $\Lambda^\beta(E) > 0$, then $C_\beta(E) > 0$. Secondly we need the following fact which is implicit in [7]: If f is a measurable function from [0, 1] to R_N and E is a Borel subset of [0, 1], and if there is probability measure m on [0, 1] with $m(E) = 1$ such that

$$\int_E \int_E |\, f(s) - f(t)\,|^{-\beta}\, m(ds)m(dt) < \infty,$$

then $\Lambda^\beta[f(E)] > 0$.

3. Dimension theorem

In what follows, E will be a Borel subset of $[0, 1]$ of dimension γ, and α will be the index of our process $\{X(t); t \geqq 0\}$ taking values in R_N. *If* $N = 1$, *we will always assume* $\alpha\gamma \leqq 1$. We denote by $X(E, \omega)$ the range of the function $X(t, \omega)$ as t varies over E. We will almost always delete the ω in expressions involving the sample functions. Our theorem is that if $\dim E = \gamma$, then $\dim X(E, \omega) = \alpha\gamma$ for almost all ω. We proceed in steps.

(i) $P\{\dim X(E) \geqq \alpha\gamma\} = 1$.

Proof. Assume $\alpha \neq 1$. Let β be positive and strictly less than $\alpha\gamma$, but otherwise arbitrary. Then $\beta/\alpha < \gamma$, so $\Lambda^{\beta/\alpha}(E) = \infty$, and according to Davies' theorem there is a closed set F contained in E such that $\Lambda^{\beta/\alpha}(F) > 0$. Then $C_{\beta/\alpha}(F) > 0$ by Frostman's theorem. Let m be a probability measure concentrated on F such that (2) holds with β replaced by β/α. Now

$$\mathcal{E}| X(t) - X(s) |^{-\beta} = \mathcal{E}| X(t - s) |^{-\beta}$$
$$= | t - s |^{-\beta/\alpha}\mathcal{E}| X(1) |^{-\beta} = c| t - s |^{-\beta/\alpha}$$

with $0 < c < \infty$ (recall that $\alpha\gamma \leqq 1$ if $N = 1$ and that $X(1)$ has a continuous density). Integrating this relation over $F \times F$ with respect to $m \times m$ and using Fubini's theorem, we find that

$$\int_F \int_F | X(t, \omega) - X(s, \omega) |^{-\beta} m(dt)m(ds) < \infty \tag{3}$$

for almost all ω. Then as noted above, $P\{\Lambda^{\beta}(X(F)) > 0\} = 1$ and so $P\{\Lambda^{\beta}(X(E)) > 0\} = 1$. The necessary modification of this argument in case $\alpha = 1$ is obvious. Since $\beta < \alpha\gamma$ was arbitrary, the proof is complete.

(ii) *If* $\gamma < 1$, *then* $P\{\dim X(E) \leqq \alpha\gamma\} = 1$.

Proof. Assume $\alpha \neq 1$. Choose $\beta > \gamma$ with $\beta\alpha < \alpha$, but β otherwise arbitrary. For each n let $\{E_{in}; i \geqq 1\}$ be a cover of E by closed intervals such that $\sum_{i=1}^{\infty} (\operatorname{diam} E_{in})^{\beta} \rightarrow 0$ as $n \rightarrow \infty$. This can be done since $\Lambda^{\beta}(E) = 0$. Now for each n, $\{X(E_{in}, \omega); i \geqq 1\}$ is a cover of $X(E, \omega)$, and moreover $[\operatorname{diam} X(E_{in})]^{\beta\alpha}$ is distributed as

$$(\operatorname{diam} E_{in})^{\beta} [\operatorname{diam} X([0, 1])]^{\beta\alpha}.$$

Assuming for the moment that $\mathcal{E}(\operatorname{diam} X([0, 1]))^{\alpha\beta} < \infty$, we have

$$\mathcal{E}\sum_{i=1}^{\infty} [\operatorname{diam} X(E_{in})]^{\beta\alpha} = \sum_{i=1}^{\infty} (\operatorname{diam} E_{in})^{\beta}\mathcal{E}[\operatorname{diam} X([0, 1])]^{\beta\alpha}. \tag{4}$$

The right side of (4) goes to 0 as $n \rightarrow \infty$, and so for a subsequence of n's approaching ∞ (which is all we need) $\sum_{i=1}^{\infty} [\operatorname{diam} X(E_{in}, \omega)]^{\beta\alpha} \rightarrow 0$ for almost all ω. Since β was arbitrary, this implies $P\{\dim X(E) \leqq \alpha\gamma\} = 1$. Concerning the finiteness of the expected value above: pick a number M such that for every $t \leqq 1$, $P\{| X(t) - X(1) | \geqq M\} \leqq \frac{1}{2}$. This can be done since almost all sample functions of our process are bounded on bounded t-intervals. A standard argument then shows that for every $\lambda > M$

$$P\{\sup_{t \leqq 1} | X(t) | \geqq 2\lambda\} \leqq 2\, P\{| X(1) | \geqq \lambda\},$$

and so for all $\lambda > M$

$$P\{\operatorname{diam} X[0, 1] \geqq 4\lambda\} \leqq 2\, P\{| X(1) | \geqq \lambda\}.$$

We observed in Section 2 that $\mathcal{E}| X(1) |^{\beta\alpha} < \infty$, and so the expected value in question is finite. Again, the necessary modification of the proof if $\alpha = 1$ is easily found, and we omit the details.

(iii) *If* $\gamma = 1$, *then* $P\{\dim\ X(E) \leqq \alpha\} = 1$.

Proof. We may as well assume $E = [0, 1]$. We first remark that if $\alpha \leqq 1$, then an argument involving the variation of the sample functions, as used in [7], gives the result, and if $N = 1$, these are the only values of α worth considering. But for the other cases, this argument is not available. We proceed with the proof in general.

First assume $\alpha \neq 1$. Choose $\beta > 1$, but otherwise arbitrary, and for each $\varepsilon > 0$ define as follows:

$$T_{1\varepsilon} = \inf \{t > 0: | X(t) | > \varepsilon^{1/\alpha}\},$$

$$T_{k+1,\varepsilon} = \inf \{t > 0: | X(t + T_{1\varepsilon} + \cdots + T_{k\varepsilon}) - X(T_{1\varepsilon} + \cdots + T_{k\varepsilon}) | > \varepsilon^{1/\alpha}\}$$

for all $k \geqq 1$. Our process has right-continuous paths and stationary independent increments, and so it follows from the extended Markov property of such processes (see [4, Sections 1–3]) that $T_{1\varepsilon}$, $T_{2\varepsilon} \cdots$ is a sequence of mutually independent and identically distributed random variables. Now

$$\begin{aligned} P\{T_{1\varepsilon} < a\} &= P\{\sup_{t<a} | X(t) | > \varepsilon^{1/\alpha}\} \\ &= P\{\sup_{t<a} \varepsilon^{-1/\alpha}| X(t) | > 1\} = P\{\sup_{t<a} | X(t\varepsilon^{-1}) | > 1\} \\ &= P\{\sup_{t<a\varepsilon^{-1}} | X(t) | > 1\} = P\{T_{11} < a\varepsilon^{-1}\}, \end{aligned}$$

so $T_{k\varepsilon}$ has the same distribution as εT_{k1} (T_{k1} is defined as above with $\varepsilon = 1$). Let N_ε be the smallest value of n such that $T_{1\varepsilon} + \cdots + T_{n\varepsilon} > 1$. If $S(0, \varepsilon)$ denotes the solid closed sphere with center at 0 and radius $\varepsilon^{1/\alpha}$, and $S(k, \varepsilon)$ denotes a similar sphere with center at $X(T_{1\varepsilon} + \cdots + T_{k\varepsilon})$, then $S(0, \varepsilon), \cdots, S(N_\varepsilon - 1, \varepsilon)$ is a cover of $X[0, 1]$ by sets of diameter $2\varepsilon^{1/\alpha}$, and

$$\textstyle\sum_{k=0}^{N_\varepsilon - 1} (\operatorname{diam} S(k, \varepsilon))^{\alpha\beta} = 2^{\alpha\beta}\varepsilon^{\beta}N_\varepsilon .$$

Given any $x > 0$

$$\begin{aligned} P\{\varepsilon^{\beta}N_\varepsilon \leqq x\} &= P\{T_{1\varepsilon} + \cdots + T_{[x\varepsilon^{-\beta}],\varepsilon} > 1\} \\ &= P\{\varepsilon T_{11} + \cdots + \varepsilon T_{[x\varepsilon^{-\beta}],1} > 1\}. \end{aligned}$$

If we write ε as $x^{1/\beta}/k^{1/\beta}$, this probability is

$$P\{x^{1/\beta}(T_{11} + \cdots + T_{k1})/k^{1/\beta} > 1\},$$

and, since $\beta > 1$, by the law of large numbers this probability approaches 1 as $k \to \infty$ $(\varepsilon \to 0)$. We have shown then that $\varepsilon^{\beta} N_\varepsilon \to 0$ in probability as $\varepsilon \to 0$. Hence a subsequence approaches 0 with probability 1, and thus $P\{\Lambda^{\beta\alpha}(X[0, 1]) = 0\} = 1$. Since $\beta > 1$ was arbitrary, the proof is complete, at least if $\alpha \neq 1$.

We will indicate the changes required if $\alpha = 1$. Assume now $\alpha = 1$. We observed earlier that for each positive r and t, $rX(t)$ has the same distribution as $X(rt) + tr \log r \cdot a$ where a is a point in R_N. Moreover the process $\{X(rt) + tr \log r \cdot a;\ t \geqq 0\}$ has stationary independent increments and hence is probabilistically the same as the process $\{rX(t);\ t \geqq 0\}$. Given any $\beta > 1$, pick $\delta > 0$ but such that $\beta - \beta\delta > 1$. Now $\varepsilon^{\delta} |\log \varepsilon| \to 0$ as $\varepsilon \to 0$, and hence there is an $\varepsilon_0 > 0$ such that $|\log \varepsilon|\,|a| + 1 < \varepsilon^{-\delta}$ for all $\varepsilon \leqq \varepsilon_0$. Given any $\varepsilon > 0$ let

$$T_{1\varepsilon} = \inf\{t > 0: |X(t)| > \varepsilon^{1-\delta}\},$$

and define $T_{2\varepsilon}, T_{3\varepsilon}, \cdots$ inductively as we did above. Then $T_{1\varepsilon}, T_{2\varepsilon}, \cdots$ are independent and identically distributed. Now given any $\varepsilon \leqq \varepsilon_0$ (above) and $c \leqq \varepsilon$ we have

$$\begin{aligned} P\{T_{1\varepsilon} < c\} &= P\{\sup_{t<c} |X(t)| > \varepsilon^{1-\delta}\} \\ &= P\{\sup_{t<c} \varepsilon^{-1}|X(t)| > \varepsilon^{-\delta}\} \\ &= P\{\sup_{t<c} |X(t\varepsilon^{-1}) - t\varepsilon^{-1} \log \varepsilon \cdot a| > \varepsilon^{-\delta}\} \\ &= P\{\sup_{r<c\varepsilon^{-1}} |X(r) - r \log \varepsilon \cdot a| > \varepsilon^{-\delta}\}. \end{aligned}$$

Since $c/\varepsilon \leqq 1$ and $\varepsilon \leqq \varepsilon_0$, it follows that $r|\log \varepsilon|\,|a| + 1 < \varepsilon^{-\delta}$, and so the last displayed expression above does not exceed

$$P\{\sup_{r<c\varepsilon^{-1}} |X(r)| > 1\} = P\{T_{11} < c\varepsilon^{-1}\}.$$

Let

$$\begin{aligned} R_k &= T_{k1} \quad \text{if} \quad T_{k1} \leqq 1, \\ &= 1 \quad\ \ \text{if} \quad T_{k1} > 1. \end{aligned}$$

Then $R_1, R_2, \cdots$ are independent and identically distributed, and for each $\varepsilon > 0$ and each x, $P\{\varepsilon R_k \leqq x\} \geqq P\{T_{k\varepsilon} \leqq x\}$. From here the proof proceeds as in the case $\alpha \neq 1$. We let N_ε denote the smallest n for which

$$T_{1\varepsilon} + \cdots + T_{n\varepsilon} > 1,$$

cover $X[0, 1]$ with N_ε closed spheres each of diameter $2\varepsilon^{1-\delta}$, and thus get a cover by sets, the sum of whose diameters raised to the β power is $2^{\beta}\varepsilon^{\beta-\beta\delta}N_\varepsilon$. Then for any $x > 0$

$$\begin{aligned} P\{\varepsilon^{\beta-\beta\delta}N_\varepsilon \leqq x\} &= P\{T_{1\varepsilon} + \cdots + T_{[x\varepsilon^{\beta\delta-\beta}],\varepsilon} > 1\} \\ &\geqq P\{\varepsilon R_1 + \cdots + \varepsilon R_{[x\varepsilon^{\beta\delta-\beta}]} > 1\}, \end{aligned}$$

and since $\beta - \beta\delta > 1$, this probability approaches 1 as $\varepsilon \to 0$. Thus the proof is complete. Let us summarize the results of this section.

THEOREM. *If* $\dim E = \gamma$, *then* $P\{\dim X(E) = \alpha\gamma\} = 1$.

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