FINITE HJELMSLEV PLANES

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1. Introduction

Klingenberg has carried out studies of Desarguesian and Pascalian Hjelmslev planes [4, 5]. In the present investigation we consider abstract Hjelmslev planes, abbreviated here as H-planes. Particular emphasis is placed on finite H-planes.

An H-plane π' is defined as a collection of points and lines together with an incidence relation subject to the following rules:

I. For each pair of distinct points there exists at least one line that passes through them.

II. For each pair of distinct lines there exists at least one point of intersection.

A pair of lines will be said to be neighbor in case they intersect in more than one point, and nonneighbor otherwise. This will be denoted by $l \bigcirc m$ and $l \varnothing m$, respectively. A similar definition and notation will be used for points.

III. If k, l, m, are concurrent lines such that $k \bigcirc l$ and $l \varnothing m$, then $k \varnothing m$.

IV. If $k \bigcirc l$ and $l \varnothing m$, then $km \bigcirc lm$, where km denotes an arbitrary point of intersection of the lines k and m.

V. If $P \bigcirc Q$ and $Q \varnothing R$, then $PR \bigcirc QR$.

VI. There exist points P_1, P_2, P_3, P_4 such that $P_i \varnothing P_j$ and $P_i P_k \varnothing P_i P_j$ whenever i, j, k are all different.

Klingenberg has shown that with each π' is associated a projective plane π as follows: The relation of neighbor is first shown to be an equivalence relation. The equivalence classes become the points and lines of π. A class of points $\mathfrak{A}$ is defined as incident on a class of lines $\mathfrak{B}$ if and only if there exists a point P in $\mathfrak{A}$ and a line k in $\mathfrak{B}$ such that P is incident on k.

For each finite H-plane π' we introduce two invariants s and t. Select any line k in π' and any point P on k, and suppose that there are s points on k not neighbor to P and t points on k neighbor to P. It turns out that s, t do not depend on the choice of P and k, that t divides s, and that $s \leqq t^2$.

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The total number of points, and also that of lines, is $s^2 + st + t^2$. A finite H-plane is called uniform in case there exists a line whose average number of intersections with all lines neighbor to it but distinct from it is an integer. In a uniform H-plane this number turns out to be independent of the choice of line. In fact it forces any pair of distinct neighbor lines to intersect in the maximum possible number of points, namely t, and also implies $s = t^2$. The incidence matrix of a uniform H-plane is a group-divisible, regular design with two associate classes. Some results on the nonexistence of finite H-planes are obtained. Finally a question raised by Klingenberg is answered: Namely there exist finite Desarguesian H-planes which are not Pascalian.

2. Invariants

We begin with the following:

THEOREM 1. *Let π' be a finite H-plane, k any line of π', and P any point on k, s the number of nonneighbor points of P on k, and t the number of neighbor points of P on k. Then* (i) *s, t are independent of the choice of P and k,* (ii) *the total number of points and also that of lines is $s^2 + st + t^2$,* (iii) *t divides s,* (iv) *π has order s/t, and* (v) *if $\pi \neq \pi'$, $s \leqq t^2$.*

Proof. The proof of part (i) is somewhat more complicated than the corresponding proof for projective planes, but nevertheless quite routine, and will be left to the reader. To establish (ii) we show first that the number of neighbor points of P in π' is t^2. There are t of them on k. Select R on k such that $R \not\bigcirc P$. If Q is any neighbor point of P not on k, then $PR \bigcirc QR$, because of III, so that any neighbor point of P lies on a neighbor line of k through R. There are t neighbor lines of k through R, and they all intersect in neighbor points of R, which are then nonneighbor points of P. Thus to get the total number of neighbor points of P, we just add up all the ones which lie on neighbor lines of k through R. Let m be such a line, $m \neq k$. Then let S be any point which is not neighbor to any of the points on m and k. Then SP intersects m in a unique point T. Since $m \bigcirc k$ and $k \not\bigcirc SPT$, we have $P \bigcirc T$ because of IV. Therefore m carries at least one neighbor point of P and consequently t neighbor points. It is now apparent that there are t^2 neighbor points of P in π'. To count up all the points, consider the $s + t$ lines of π' which pass through P. Each of them contributes s nonneighbor points of P, and there is no duplication. Thus there is a total of $(s + t)s + t^2 = s^2 + st + t^2$ points in π'. Because of duality the number of lines is the same. This establishes (ii). (iii) is a consequence of dividing up the $s + t$ points on a line into equivalence classes of neighbor points. Each class has t points, so that t divides $s + t$. This also shows that π has $s/t + 1$ points on a line and hence has order s/t. This establishes (iv). With each line k we associate the number $\lambda = \lambda(k)$, which is defined as the average number of points of intersection between k and all of its neighbor lines distinct from k. Adding up the number of intersections between k and all lines of π' we obtain the equation

$$(1) \qquad s + t + \lambda(t^2 - 1) + s(s + t) = (s + t)(s + t).$$

If $\pi \neq \pi'$, then $t \neq 1$. Solving (1) for λ one obtains

$$(2) \qquad \lambda = (s + t)/(t + 1).$$

Two lines intersect in at most t points, so that $\lambda \leqq t$. But this inequality is equivalent to (v). This completes the proof of the theorem.

DEFINITION. A finite H-plane π' will be called *uniform* in case there exists a line k in π' for which $\lambda(k)$ is an integer and π' is not a projective plane.

THEOREM 2. *Let π' be a finite, uniform H-plane. Then λ is independent of k. In fact* (i) $\lambda = t$, (ii) $s = t^2$, *and* (iii) *the incidence matrix A of π' is a group-divisible, regular design with two associate classes and parameters* $v = t^4 + t^3 + t^2$, $n = t^2$, $m = t^2 + t + 1$, $r = t^2 + t$, $k = t^2 + t$, $\lambda_2 = 1$, $\lambda_1 = t^2$.

Proof. Since $\lambda = (s + t)/(t + 1)$ is an integer and $s + t = t(s/t + 1)$, we must have $t + 1$ dividing $s/t + 1$, since t and $t + 1$ are relatively prime. But then $t + 1 \leqq s/t + 1$, so that $t^2 \leqq s$. This together with Theorem 1-(v) implies $s = t^2$, and consequently $\lambda = t$. This makes it obvious that every pair of neighbor lines intersects in the maximum number of points possible, namely t. This establishes (i) and (ii). The incidence matrix A of π' may be formed by writing points horizontally, lines vertically, listing neighbor points consecutively and neighbor lines consecutively. Whenever a line k_i passes through the point P_j, label $a_{ij} = 1$; otherwise $a_{ij} = 0$. Then A is a square matrix of dimension $t^4 + t^3 + t^2$. Also $B = AA^T = A^TA$ has square blocks of dimension t^2 along its main diagonal, each block having $t^2 + t$ along its main diagonal and t elsewhere, whereas the remainder of B consists entirely of ones. This suffices to establish (iii). For definitions and notation concerning group-divisible, regular designs, see for example [2]. This completes the proof.

In general, finite H-planes are not uniform. Some of the coordinate systems for Desarguesian H-planes introduced in Section 3, as well as some of the Pascalian H-planes of Klingenberg [4], serve to illustrate this point. The incidence matrix of such a plane has more than two associate classes.

3. Existence questions

One of the natural questions that comes up is, for what values of s and t do there exist H-planes? In all the known examples both s and t are powers of a fixed prime p. Since π is a projective plane of order s/t, then s/t must satisfy the Bruck-Ryser condition [3]. In addition we have of course the restrictions imposed by Theorem 1. In this section we give a special argument that rules out a few more values of s and t. However, much remains unanswered. Let us consider what happens if $t = 3$. Because of Theorem 1, either $s = 6$, or $s = 9$. The latter is actually possible, as will be seen in the next part where examples are given. However we can show that $s = 6$ is impossible. Since such an H-plane is not uniform, there must exist two neighbor lines k and m which intersect in exactly two points P and Q. Let R be the remaining point on k which is neighbor to P. Let S be a point on m not neighbor to P. Then RS is a neighbor line of k and m. However since RS and m intersect in S, P and Q cannot lie on RS, since they are not neighbor to S. Hence RS and k intersect in a single point, namely R. But this is clearly a contradiction, since RS and k are neighbor lines. A similar argument may be advanced to show that if t is larger than 2, then no lines can intersect in $t - 1$ points. We have shown that no H-plane with $t = 3$ and $s = 6$ is possible. On the other hand there certainly exists a projective plane of order 2. Another case which can be disposed of easily is the one where s is prime. For then t must equal 1, and we have just the projective planes of prime order, which are known to exist. If $s = 6$, then because of Theorem 1-(iii) and (v), either $t = 1$, or $t = 3$. We showed before that $t = 3$ is impossible, whereas the Bruck-Ryser condition rules out $t = 1$. Thus $s = 6$ is altogether impossible, no matter what t is. If $s = 8$, then either $t = 1$ or $t = 4$, and examples of both exist. Note that for $s = 8$ and $t = 4$ we have $\lambda = 2.4$. Neighbor lines can not intersect in 3 points, hence they intersect in either 2 or 4. Let a fixed line k intersect x neighbor lines in 2 points and y neighbor lines in 4 points. Then $2x + 4y = 36$ and $x + y = 15$. Solving we obtain $x = 12$ and $y = 3$.

We turn our attention now to some new examples of finite H-planes. In this connection we are able to answer a question raised by Klingenberg.

THEOREM 3. *There exist finite Desarguesian H-planes which are not Pascalian.*

We refer the reader for the definitions of Desarguesian and Pascalian to [4]. Actually this question is equivalent to a purely ring-theoretic one. Namely Klingenberg has shown that Desarguesian H-planes have coordinates from associative rings H, in which the divisors of zero form an ideal N, such that H/N is a division ring, and such that, for each n, n' in N, either there exists an element w in N such that $n = wn'$, or there exists an element x in N such that $xn = n'$; and either there exists an element y in N such that $n = n'y$, or there exists an element z in N such that $nz = n'$. We call such rings Desarguesian H-rings. Therefore the ring-theoretic equivalent of the original question is, do there exist finite Desarguesian H-rings which are not commutative? We answer this question in the affirmative by means of the following construction:

Let F be any Galois field, having p^n elements, and α any automorphism of F. We define H as the set of all couples (a, b), where a and b are elements of F. Addition and multiplication are defined by

$$(a, b) + (c, d) = (a + c, b + d),$$
$$(a, b)\cdot(c, d) = (ac, ad + bc^{\alpha}),$$

where c^{α} denotes the image of c under α. It is routine to verify that indeed H coordinatizes a Desarguesian H-plane. We note that H is a commutative ring if and only if α is the identity automorphism. Moreover the H-planes thus obtained are all uniform with $t = p^{n}$ and $s = p^{2n}$. It is easy to generalize this construction to Desarguesian H-rings which give rise to nonuniform planes. One uses triples, quadruples, etc. of elements in place of couples. In the case of triples we would have $s = p^{3n}$ and $t = p^{2n}$. The smallest incidence matrix of a uniform H-plane would correspond to the case $s = 4$, $t = 2$. The symmetric, group-divisible design this gives rise to is already in the literature [1]. It is coordinatized by the above H-ring with $p = 2$, $n = 1$. The next largest would be $s = 9$, $t = 3$, which gives rise to an incidence matrix of dimension 117.

References

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