A NEW PROOF OF THE CONVERGENCE THEOREM FOR δ-SUBHARMONIC FUNCTIONS

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Convergence theorems play a fundamental role in the development of the theory of subharmonic functions. The simplest such theorem asserts that the limit of a decreasing sequence of functions subharmonic on a common region (connected open set) is either subharmonic or identically $-\infty$. For an increasing sequence of subharmonic functions the situation is more complicated. It was observed first of all by T. Radó [9, p. 22] that if a sequence of this sort is locally bounded above, then the limit function is almost subharmonic (that is to say, coincides almost everywhere with a function subharmonic on the region). Later, Brelot succeeded in showing [4] that the exceptional set is not only of Lebesgue measure zero but is, in fact, of interior capacity zero. Utilizing the energy norm, Cartan [6] then further refined this result to replace "interior capacity" by "exterior capacity". However, with the advent of the theorem of Choquet [7] to the effect that all Borel sets are capacitable, the latter form of the convergence theorem is immediate from the earlier version given by Brelot.

We shall be concerned here with the extension of the convergence theorem to δ-subharmonic functions, that is, to functions representable as differences of subharmonic functions on a region Ω of k-dimensional Euclidean space. For simplicity, the actual calculations will be carried out only for the case of the classical δ-subharmonic functions on plane regions. It is clear, however, that the same techniques yield corresponding results for wider classes of functions and regions. For example, the classical δ-subharmonic functions can be replaced by functions similarly related to more general potential kernels, and the plane region Ω can be replaced by a region in an arbitrary Green's space.

A weak form of the convergence theorem was first established in [2, pp. 345–346], and we state it here for reference.

THEOREM 1. *Let $\{w_n\}$ be a sequence of functions almost δ-subharmonic on Ω. If the sequence of total variations of the corresponding mass distributions is bounded, and if $\{w_n\}$ converges in the mean locally to a function w on Ω, then*

(i) *w is almost δ-subharmonic on Ω, and*

(ii) *the sequence $\{m_n\}$ of mass distributions for the functions w_n converges weakly*[1] *to the mass distribution m for w.*

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[1] Weak convergence is used here in the sense of Definition 7 of [2, p. 345].

By means of the theory of functions of potential type, this result was later extended [3, Theorem 34] to the following analogue of the Brelot-Cartan theorem.

THEOREM 2. *Let $\{w_n\}$ be a sequence of quasi δ-subharmonic functions on Ω, and let ϕ and ψ be functions on Ω locally summable with respect to all mass distributions admitting continuous potentials.*[2] *Further, let the mass distributions for the functions w_n ($n = 1, 2, \cdots$) have uniformly bounded total variations, and let*

$$\phi \leqq w_n \leqq \psi \qquad (n = 1, 2, \cdots). \tag{1}$$

If $\{w_n\}$ converges pointwise quasi everywhere on Ω to a function w, then

(i) *w is quasi δ-subharmonic on Ω, and*

(ii) *the sequence $\{m_n\}$ of mass distributions for the functions w_n converges weakly*[1] *to the mass distribution m for w.*

We recall here that "quasi everywhere" means "except for a set of capacity zero" and that a quasi δ-subharmonic function is defined as a function coinciding quasi everywhere with a δ-subharmonic function.

Inasmuch as the proof given originally [3, pp. 544–545] makes use of the theory of functions of potential type (developed specifically for the plane), the validity of the theorem for higher-dimensional spaces is not altogether obvious. What we propose to do in the present note is to modify our original proof so as to remove this criticism. However, the main lines of the proof are preserved intact; we use weak convergence of the sequence of mass distributions (as already proved in Theorem 1), together with the Choquet theorem and the fact that every set of positive interior capacity admits a distribution of the unit mass giving rise to a continuous potential.[3] The changes that do appear in the present proof result from the observation that the role originally played by functions of potential type can just as well be taken over by Green's potentials. Also, avoidance of the dependence on functions of potential type makes the proof more elementary. The details are as follows.

Under the hypothesis of Theorem 2 we can conclude from Theorem 1 that $\{w_n\}$ converges almost everywhere to a function w^* δ-subharmonic on Ω and that the corresponding sequence $\{m_n\}$ of mass distributions converges weakly to the mass distribution m for w^*. Let us suppose that w and w^* differ on a set of positive (exterior) capacity. In view of the fact that w and w^* are

[2] All δ-subharmonic functions have this property, and for most of the applications ϕ and ψ can be taken as δ-subharmonic.

[3] This appears as Lemma 3 of [3, p. 543], the author having been unaware that the same result had already been established by de la Vallée Poussin [8, pp. 84–85]. In turn, the idea of basing a proof of the convergence theorem on this lemma, as exploited in the author's thesis [3], has since been rediscovered for the subharmonic case by Brelot and Choquet [5] and utilized by Anger [1]. Brelot and Choquet have also made the important observation that the lemma in question follows directly from Lusin's theorem.

Baire functions, this set is a Borel set and therefore has positive interior capacity. A routine argument shows that $\pm(w^* - w)$ must exceed some $\varepsilon > 0$ on a compact set K of positive capacity. Moreover, there is no loss of generality in supposing that K lies inside some open disc ω having closure in Ω and that

$$w^* - w > \varepsilon \quad \text{on } K. \tag{2}$$

Now, we employ the Riesz decomposition theorem to obtain on ω

$$w_n = W_n + h_n \quad \text{and} \quad w^* = W + h, \tag{3}$$

where W_n and W are the Green's potentials of m_n and m, respectively, confined to ω, and h_n and h are harmonic on ω. Denoting by α_r the areal mean operator over discs with closure in ω, we have $\alpha_r w_n = \alpha_r W_n + h_n$ and $\alpha_r w^* = \alpha_r W + h$. From condition (1) in conjunction with the Lebesgue convergence theorem we conclude that $\lim_{n\to\infty} \alpha_r w_n = \alpha_r w^*$. Also, weak convergence of the sequence of mass distributions yields $\lim_{n\to\infty} \alpha_r W_n = \alpha_r W$,[4] so that

$$h_n \to h \quad \text{on } \omega.$$

Since K has positive capacity, there exists a distribution q of the unit mass on K having a continuous Green's potential U on ω. Invoking the weak convergence of $\{m_n\}$ to m and noting that U is a continuous function vanishing on the boundary of ω, we have

$$\int W_n\, dq = \int U\, dm_n \to \int U\, dm = \int W\, dq.$$

A corresponding convergence property holds for the harmonic functions figuring in (3), as is evident from the Lebesgue convergence theorem and the fact that each h_n lies between the harmonic functions defined on ω by the Poisson integrals of ϕ and ψ. There results

$$\int w_n\, dq \to \int w^*\, dq.$$

Appealing once more to the Lebesgue convergence theorem, we see from (1) and (2) that

$$\varepsilon \leqq \int (w^* - w)\, dq = \lim_{n\to\infty} \int w_n\, dq - \int w\, dq = 0.$$

This contradiction completes the proof.

[4] To see this, let p_r be the uniform distribution of total mass $1/\pi r^2$ on the disc of radius r over which the areal mean is taken. Since the Green's potential A_r of p_r is a continuous function vanishing on the boundary of ω, there follows

$$\alpha_r W_n = \int W_n\, dp_r = \int A_r\, dm_n \to \int A_r\, dm = \int W\, dp_r = \alpha_r W.$$

An important observation here is that whenever $\{w_n\}$ is a sequence of potentials, condition (1) can clearly be replaced by the following weaker condition: for all mass distributions λ lying on compact subsets of Ω and giving rise to continuous potentials

$$\lim_{n\to\infty} \int w_n \, d\lambda = \int w \, d\lambda. \tag{4}$$

Mass distributions λ of this sort occupy an important position in the theory of functions of potential type, and Anger [1] has employed them systematically to formulate a new approach to certain problems in potential theory. In particular, this approach has led to a theorem [1, Satz 15] analogous to Theorem 2 but for Ω a compact set and $\{w_n\}$ a sequence of potentials.

We remark further that the present demonstration of Theorem 2 does not entirely supplant that of [3], in view of the additional information which the latter provides for functions of potential type.

As mentioned in [3, p. 545], the Brelot-Cartan theorem for sequences of subharmonic functions appears as a direct consequence of Theorem 2. Moreover, even for the case of subharmonic functions Theorem 2 is considerably stronger than the Brelot-Cartan theorem, since it applies to sequences which may fail to be monotone. In fact, essentially the same argument as that used in proving Corollary 34.1 of [3] shows that the mass restriction is automatically fulfilled locally in the subharmonic case, and we state the result explicitly as

THEOREM 3. *Let $\{u_n\}$ be a sequence of quasi subharmonic functions on Ω, and let ϕ and ψ be functions on Ω locally summable with respect to all mass distributions admitting continuous potentials.*[2] *If*

$$\phi \leqq u_n \leqq \psi \qquad (n = 1, 2, \cdots) \tag{5}$$

and $\{u_n\}$ converges pointwise quasi everywhere on Ω to a function u, then

(i) *u is quasi subharmonic on Ω, and*

(ii) *for every bounded region Ω^* with closure in Ω the sequence $\{m_n\}$ of mass distributions for the functions u_n converges weakly*[1] *on Ω^* to the mass distribution m for u.*

Again we note that if $\{u_n\}$ is a sequence of potentials, then condition (5) can be replaced by the weaker condition corresponding to (4): for all mass distributions λ lying on compact subsets of Ω and giving rise to continuous potentials

$$\lim_{n\to\infty} \int u_n \, d\lambda = \int u \, d\lambda. \tag{6}$$

Let us denote by $S_r(z)$ the closed disc of radius r about z, and by $\mu_r\, v(z)$ the integral mean of a suitable function v over the circumference of $S_r(z)$. The proof of Theorem 3 is based on the following auxiliary result.

LEMMA. *Let $\{u_n\}$ be a sequence of subharmonic functions on Ω, and let $\{m_n\}$ be the corresponding sequence of negative mass distributions. If the sequence $\{\mu_r u_n(z)\}$ is bounded for each disc $S_r(z) \subset \Omega$, then the sequence $\{m_n(K)\}$ is bounded for each compact subset K of Ω.*[5]

Proof. By the usual covering theorems there is no loss of generality in supposing that $K = S_R(0)$ and that $S_{R'}(0)$ lies in Ω for some $R' > R$. Formulas due to F. Riesz (see [9, pp. 35–36]) then show that the total variation on $S_R(0)$ of m_n can be computed as $\partial^+ \mu_r u_n(0)/\partial \log r \,|_{r=R}$. We recall also that $\mu_r u_n(0)$ is an increasing convex function of $\log r$ on $[0, R']$. Since the given hypothesis yields constants M and M' such that

$$M \leqq \mu_R u_n(0) \leqq \mu_{R'} u_n(0) \leqq M' \qquad (n = 1, 2, \cdots),$$

it is easily seen that $\{\partial^+ \mu_r u_n(0)/\partial \log r \,|_{r=R}\}$ is bounded, and the lemma is established.

The lemma, together with condition (5) (or the weaker condition (6) in the case of potentials), permits us to apply Theorem 2 to the sequence $\{u_n\}$ on any bounded region Ω^* with closure in Ω. From conclusion (ii) it is then obvious that m is a negative mass distribution, so that the limit function u must be quasi subharmonic. This completes the proof of Theorem 3.

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[5] For simplicity the lemma has been stated in terms of a sequence, but it obviously remains valid for an arbitrary family of subharmonic functions.