DIFFERENTIAL OPERATORS WITH THE POSITIVE MAXIMUM PROPERTY

BY

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1. Introduction

Let I be a fixed open interval, and consider a linear functional operator whose domain and range consist of real functions continuous in a subinterval of I. We say that f is in the local domain of Ω at the point $s \in I$, in symbols $f \in D(\Omega, s)$, if both f and Ωf are continuous in a neighborhood of s. Similarly, the global domain $D(\Omega, J)$ for an interval $J \subset I$ consists of the functions f such that both f and Ωf are continuous in J. We shall suppose that Ω is of *local character* in the following sense: if f vanishes identically in some neighborhood of s, then $f \in D(\Omega, s)$ and $\Omega f(s) = 0$. For such an operator the restriction of Ωf to an interval J depends only on the behavior of f in J.

We say that Ω has the *positive maximum property if one has* $\Omega f(s) \leqq 0$ *for each* $s \in I$ *and each* $f \in D(\Omega, s)$ *which attains a positive local maximum at* s. The classical differential operator defined by $af'' + bf' + cf$ with $a > 0$ has the positive maximum property if $c \leqq 0$. In view of the well-known role of such operators in many theories, we propose in this note to find a canonical form for the general operators of local character with the positive maximum property.

Let x be a continuous strictly increasing function in I, and m a strictly increasing right continuous function (not necessarily bounded). We shall view x as a scale, the increments of m as a measure on the Borel sets of I. The symbol $D_x f$ will be used indiscriminately for right and left derivatives provided they exist at each point and are continuous except for jumps. Differentiation with respect to m has the obvious meaning provided we agree to consider increments only for *closed* intervals. The operator $\Omega_0 = D_m D_x$ has been discussed in [1] as a natural generalization of the classical operator $aD_s^2 + bD_s$; it is characterized by the *strong* maximum property that $\Omega_0 f(x) \leqq 0$ for each x such that f attains a local maximum at x.

Operators of the form $\Omega_0 + c$ with $c \leqq 0$ have the positive maximum property, and we shall show that the most general operator with this property is of a similar, though slightly more intricate, form.

We parametrize I by x and consider an arbitrary *convex* continuous function ω of x; that is, $\omega > 0$ and $D_x \omega \uparrow$ throughout I. The operator A defined by

$$Af = \frac{1}{\omega} D_m \left\{ \omega^2 D_x \frac{f}{\omega} \right\} \tag{1.1}$$

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is a special case of the operators studied in [1]. We recall in particular that the representation (1.1) is *unique* in the sense that if

$$(1.2) \qquad \frac{1}{\omega} D_m \left(\omega^2 D_x \frac{f}{\omega}\right) = \frac{1}{\varphi} D_\mu \left(\varphi^2 D_\xi \frac{f}{\varphi}\right)$$

for all $f \in D(A)$, then $\xi = px + q$, $\mu = p^{-1}m +$ const., and $A\varphi = 0$. Conversely, if $\varphi > 0$ is an arbitrary solution of $A\varphi = 0$, then

$$(1.3) \qquad Af = \frac{1}{\varphi} D_m \left\{\varphi^2 D_x \frac{f}{\varphi}\right\}.$$

In other words, the canonical scale x and measure m are determined up to a trivial linear transformation, and the right side of (1.1) remains unchanged if ω is replaced by an arbitrary positive annihilator of A. The following lemma shows that operators of the form (1.1) represent a simple generalization of the operators $\Omega_0 +$ const.

LEMMA. *If $\varphi > 0$ and $A\varphi = 0$, then φ is convex, and*

$$(1.4) \qquad d\omega'/\omega = d\varphi'/\varphi \qquad (\omega' = D_x \omega).$$

Thus the measure γ defined by $d\gamma = d\omega'/\omega$ is independent of the choice of the annihilator ω. The operator A may be redefined by

$$(1.5) \qquad Af\cdot dm = df' - f\, d\gamma$$

in the sense that the integrals of the two sides are equal for each f in the domain of A. The derivative $D_x f = f'$ is continuous except for jumps, and these can occur only at the points of discontinuity of either m or γ.

Conversely, if γ is an arbitrary nonnegative measure on I, then there exists in I a two-parameter family of positive convex annihilators ω of (1.5), *and with each of them* (1.5) *is equivalent to* (1.1).

If ω is in the domain of $\Omega_0 = D_m Dx$, then (1.5) reduces to the simpler form $Af = D_m D_x - cf$ where $c \geqq 0$.

The main result of the present paper is contained in

THEOREM 1. *Let Ω have local character and the positive maximum property. Suppose that Ω nowhere degenerates into a first order operator.*[2] *Then there exists a uniquely determined operator A of the form* (1.1) *or* (1.5) *which is an extension of Ω, that is, $\Omega f = Af$ wherever Ωf is continuous.*

Every operator of the form (1.1) *or* (1.5) *has the positive maximum property.*

In [3] it is shown that operators of the form (1.5) with discontinuous measures m and γ occur naturally in the theory of the vibrating string. Examples will be found in that paper. The next theorem shows the role of

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[2] A differential operator A is of first order if there exists a scale parameter y such that $Af = bD_y f + cf$ for each $f \in D(A)$. For an intrinsic characterization see [2].

our operators for the diffusion with possible *destruction of masses*. Let $\mathbf{C}$ be the usual Banach space of continuous functions in I.

THEOREM 2.[3] *Let $\{T_t\}$ be a strongly continuous positivity-preserving semi-group of operators from $\mathbf{C}$ to $\mathbf{C}$ such that $T_t\mathbf{1} \leqq \mathbf{1}$ for $t > 0$, and suppose that its generator Ω defined by*

$$\Omega f(x) = \lim_{t\to 0} \frac{T_t f - f}{t}(x) \tag{1.6}$$

is of local character. Then Ω has the positive maximum property.

In other words, the most general diffusion process without creation of masses is generated by an operator of the form (1.5).

2. Proof of the lemma

Let A be defined by (1.1). If Af is continuous in J, the one-sided derivatives $D_x(f/\omega) = (f/\omega)'$ exist everywhere, and

$$Af = \frac{1}{\omega} D_m\{f'\omega - f\omega'\}. \tag{2.1}$$

On integration by parts

$$\int_\alpha^\beta Af\,dm = f' - f\frac{\omega'}{\omega}\Big|_\alpha^\beta + \int_\alpha^\beta \omega' \cdot d\,\frac{f}{\omega}, \tag{2.2}$$

and another integration by parts shows that (1.5) holds with $d\gamma = d\omega'/\omega$.

It follows that if $\varphi > 0$ and $A\varphi > 0$, then $D_m\varphi' \geqq 0$, so that φ' is non-decreasing, and hence φ is convex. On the other hand if

$$\varphi = \omega\int\frac{dx}{\omega^2} + \text{const.}\,\omega, \tag{2.3}$$

then $A\varphi = 0$, and conversely. Clearly the two arbitrary constants in (2.3) may be chosen so that at a prescribed point, φ will be positive and have a local minimum. Such φ can have no positive maximum, and therefore will be positive and convex throughout I. That with such φ the relations (1.3) and (1.1) are equivalent has been shown in [1] and follows easily on using integrations by parts. The remaining assertions of the lemma are now obvious.

3. Proof of Theorem 1

We assume that Ω is of local character, that it has the positive maximum property, and that it nowhere degenerates into a first order differential operator. Let the interval I be parametrized arbitrarily by s.

We prove first: suppose that in a subinterval $J \subset I$ we have $\Omega\phi = 0$,

[3] By means of this theorem the arguments of [1], Section 9, may be simplified considerably.

$\phi > 0$. *Then ϕ can have no strict maximum in the interior* (that is, the maximum of ϕ in J is assumed at a boundary point). Assume that ϕ has a strict maximum at the interior point s. Since Ω does not degenerate, we can find an $f \in D(\Omega, s)$ such that $\Omega f(s) > 0$, and therefore $\Omega f > 0$ in a neighborhood $N \subset J$ of s. It is possible to choose λ so large that the value of $f + \lambda\phi$ at s exceeds the value at either boundary point, and hence $f + \lambda\phi$ attains a local maximum at some interior point $s' \in N$. However, this contradicts the positive maximum property since $\Omega(f + \lambda\phi)(s') = \Omega f(s') > 0$. This proves the assertion.

The same argument shows that Ω has the weak maximum property: if $g \leqq 0$ in a neighborhood of s and $g(s) = 0$, then $\Omega g(s) \leqq 0$ for each $g \in D(\Omega, s)$. We know therefore from [2] that the domain of definition of Ω may be enlarged to include two functions ϕ, ψ such that $\Omega\omega = 0$ if and only if ω is a linear combination of ϕ and ψ. The functions ϕ and ψ are linearly independent in each subinterval of I. Furthermore, in every subinterval in which $\omega > 0$ the operator Ω may be represented in the form (1.1), x and m having the properties described in Section 1.

To two arbitrary points $\alpha < \beta$ choose a linear combination $\omega = p\phi + q\psi$ such that $\omega(\alpha) = \omega(\beta) = A > 0$. Then ω cannot have a strict positive maximum or a strict negative minimum at an interior point of (α, β), and hence we conclude that $0 \leqq \omega(s) \leqq A$ for $\alpha < s < \beta$. Furthermore, ω cannot decrease in the interval $s > \beta$, and cannot increase in the interval $s < \alpha$. Therefore $\omega \geqq 0$ everywhere, and a zero of ω is possible only in the interior of (α, β). The sum of two such functions will be strictly positive, and we conclude that *it is possible to choose two independent functions ϕ and ψ strictly positive throughout I such that $\Omega\phi = 0$ and $\Omega\psi = 0$.*

If for fixed λ the function $\omega = \psi - \lambda\phi$ had two zeros, it would somewhere in between attain a positive maximum or a negative minimum, which is impossible since $\Omega\omega = 0$. Therefore the ratio ψ/ϕ is strictly monotone, and without loss of generality we may assume that

$$\xi = \psi/\phi \tag{3.1}$$

is strictly increasing.

After these preliminaries we come to the main point of the proof. We introduce ξ as a new variable, and show that

As functions of ξ the reciprocals ϕ^{-1} and ψ^{-1} are concave.

In other words, it is asserted that for arbitrary points $\alpha < \beta$ and $\alpha < s < \beta$

$$\{\xi(s) - \xi(\alpha)\}\phi^{-1}(\beta) + \{\xi(\beta) - \xi(s)\}\phi^{-1}(\alpha) \leqq \{\xi(\beta) - \xi(\alpha)\}\phi^{-1}(s), \tag{3.2}$$

and similarly for ψ. To prove (3.2) define ω by

$$\begin{aligned} &\{\xi(\beta) - \xi(\alpha)\}\omega(s) \\ &\quad = \phi(s)\{\xi(s) - \xi(\alpha)\}\phi^{-1}(\beta) + \phi(s)\{\xi(\beta) - \xi(s)\}\phi^{-1}(\alpha). \end{aligned} \tag{3.3}$$

Then ω is a linear combination of ϕ and $\psi = \phi\xi$ such that $\omega(\alpha) = \omega(\beta) = 1$. As we have seen, this implies $\omega(s) \leqq 1$ for $\alpha < s < \beta$, which proves (3.2).

It has been shown in [2] that our Ω is of the form given in (1.1) where ω is an arbitrary positive annihilator of Ω. We choose $\omega = \phi$. The relation $\Omega\psi = 0$ is then equivalent to $\phi^2 D_x \xi = \lambda$ where $\lambda > 0$ is a constant. The concavity of ϕ^{-1} implies that a one-sided derivative $D_\xi \phi$ exists at all points, and we have

$$D_x \phi = D_\xi \phi \cdot D_x \xi = \lambda \phi^{-2} D_\xi \phi = -\lambda D_\xi \phi^{-1}. \tag{3.4}$$

Since the last term is increasing, we have proved that ϕ as a function of x is *convex*.

This proves the first part of Theorem 1. Given an arbitrary operator (1.5) and an f in its domain, $Af < 0$ together with $f > 0$ implies $D_m f' < 0$, so that f can have no positive maximum in an interval where $Af < 0$. Thus A has the positive maximum property.

4. Proof of Theorem 2

Let $f \in D(\Omega)$, and suppose that f attains a local positive maximum at a point s where $f(s) > 0$. Put

$$F = f - f(s)\mathbf{1}. \tag{4.1}$$

Then $F(0) = 0$ and $F \leqq \mathbf{0}$ in a neighborhood of s. Therefore $G = F \cup \mathbf{0}$ is identically zero in a neighborhood of s. In consequence of the local character of Ω we have therefore

$$0 = \Omega G(s) = \lim_{t\to 0} t^{-1}\{T_t G - G\}(s) = \lim_{t\to 0} t^{-1} T_t G(s). \tag{4.2}$$

On the other hand

$$T_t F \geqq T_t f - f(s)\mathbf{1}, \tag{4.3}$$

and therefore

$$\liminf_{t\to 0} t^{-1} T_t F(s) \geqq \lim_{t\to 0} t^{-1}\{T_t f - f\}(s) = \Omega f(s),$$

which proves the assertion.

References

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