SOLUTION OF CERTAIN NONAUTONOMOUS DIFFERENTIAL SYSTEMS BY SERIES OF EXPONENTIAL FUNCTIONS

BY

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1. Introduction

In a recent paper [1] Wasow investigated systems of differential equations of the form

$$y' = f(y) + \sum g_k e^{i\omega_k x}. \tag{1.1}$$

Here y is an n-dimensional vector; y' denotes the derivative of y with respect to x; the g_k are constant vectors; the ω_k are real, not necessarily rationally independent numbers; the components of the vector $f(y)$ are assumed to be analytic functions of the components of y vanishing for $y = a$ and holomorphic in the neighborhood of a. The sum in (1.1) has $m < \infty$ terms.

Wasow constructs a solution of (1.1) of the form

$$y = a + \sum a_r e^{i\mu_r x}, \tag{1.2}$$

where the series converges uniformly and absolutely for $-\infty < x < \infty$ provided the coefficients g_k of (1.1) are sufficiently small. The numbers μ_r are linear combinations of the $\omega_1, \omega_2, \cdots, \omega_m$ with nonnegative integral coefficients, and the a_r are determined recursively by solving n^{th} order linear systems of equations. The individual terms of series (1.2) represent the theoretically and experimentally well-known combination harmonics in the response of system (1.1).

In this paper Wasow's results will be extended in several directions. The exponential polynomial of (1.1) will be replaced by a general exponential series

$$\sum_{k=1}^{\infty} g_k e^{i\omega_k x} \tag{1.3}$$

which includes the general almost periodic function and the general periodic function with absolutely convergent Fourier series. It will be shown that (1.2) is a real solution provided (1.1) is a real system. The general system

$$y' = g(x, y) \tag{1.4}$$

will be shown to have a solution of form (1.2) if the components of $g(x, y)$ are analytic functions of y, holomorphic for $\| y - a \| \leqq \rho$, with coefficients that are of the form (1.3).

In the next section it will be proved that if the linear homogeneous part of (1.1) has a solution of the form

$$a_0 e^{i(\mu_0/q)x}, \tag{1.5}$$

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where q is an integer $\geqq 2$, then (1.1) has infinitely many solutions of the form

$$y = a + a_0 e^{i(\mu_0/q)x} + \sum a_r e^{i(\mu_r/q)x}, \tag{1.6}$$

with $\| a_0 \|$ a prescribed arbitrary number sufficiently small. The a_r depend on a_0 and are determined recursively by solving n^{th} order linear systems of equations, and the μ_r are linear combinations of μ_0, $q\omega_1$, $q\omega_2$, $\cdots$ with nonnegative integral coefficients. In particular if the forcing term $\sum g_k e^{i\omega_k x}$ of (1.1) is a periodic function of period τ, then there are infinitely many solutions of period $q\tau$, including the so-called subharmonics.

Finally the question of the stability of the particular solution (1.2) is considered. It will be demonstrated that this solution is imbedded in a field of solutions $y(x, \alpha)$ all of which have expansions like (1.2), but with exponents that are linear combinations with nonnegative integral coefficients both of the ω_k appearing in the forcing term of (1.1) and of the eigenvalues $i\nu_1$, $i\nu_2$, $\cdots$, $i\nu_n$ of the matrix A which is the Jacobian of $f(y)$ at $y = a$. The particular solution (1.2) is then proved to be stable. This is true on the interval $-\infty < x < \infty$ if the ω_k and ν_j are real. When x is restricted to the interval $0 \leqq x < \infty$, then the ω_k and ν_j may be arbitrary complex numbers with nonnegative imaginary parts. If the imaginary parts of the ν_j are positive, the solution (1.2) is proved to be asymptotically stable.

An essential condition for the convergence of the expansions presented in this paper is that the eigenvalues $i\nu_j$ of the matrix A have a positive distance from the "compound spectrum" of heteronomous frequencies, that is, form the set of linear combinations of the ω_k with nonnegative integral coefficients. This condition (see (2.3), (4.11), (5.3), (6.9)) excludes the "small divisors" which occur in some expansions of periodic solutions due to integration of harmonics with small frequencies (for general reference see [4] with its extensive bibliography).

Two recent papers by G. I. Biryuk [2, 3] also deal with the existence of almost periodic solutions for systems of the form $y' = Ay + \varepsilon f(x, y)$ with a small parameter ε, $f(x, y)$ being Lipschitzian in y. Although a few results of this paper are implied by Biryuk's results, the major part are not, and the methods of approach are entirely different. The results of this article are also instrumental in constructing solutions of the form (1.2) for equations like (1.1) and (1.4) with nonanalytic $f(y)$, $g(x, y)$, as will be shown in a forthcoming paper.

2. General forcing terms

We assume $\sum g_k e^{i\omega_k x}$ is an infinite series with $\sum \| g_k \| < \infty$. The ω_k are real numbers, rationally independent or not, arranged in an arbitrary fashion. By putting $u = y - a$, system (1.1) takes the form

$$u' = \textstyle\sum_{k \geqq 1} g_k e^{i\omega_k x} + Au + h(u), \tag{2.1}$$

where A is a constant matrix and the components h_j $(j = 1, 2, \cdots, n)$ of the vector h possess expansions in powers of $u_1, u_2, \cdots, u_n$ without constant or linear terms, converging for

$$\| u \| \leqq \rho. \tag{2.2}$$

For any vector u with components $u_1, u_2, \cdots, u_n$ the symbol $\| u \|$ denotes the norm $\max_j \| u_j \|$. For a matrix B with components b_{ij} the symbol $\| B \|$ will denote the norm $\| B \| = \max_i \sum_{j=1}^{n} | b_{ij} |$. Then clearly

$$\| Bu \| \leqq \| B \| \cdot \| u \|.$$

Let $i\nu_j$ ($i = \sqrt{-1}$, $j = 1, 2, \cdots, n$) denote the eigenvalues of A (not necessarily distinct), M the set of numbers which are linear combinations of $\omega_1, \omega_2, \cdots$ with nonnegative integral coefficients. We make the assumption

$$\inf_{j=1,\cdots,n,\mu\in M} | \nu_j - \mu | = \delta > 0. \tag{2.3}$$

This assumption is satisfied in the two cases considered by Wasow [1]:

(a) None of the ν_j is real.

(b) The ω_k are rationally dependent (i.e., there exists some real number ω such that $\omega_k = n_k \omega$ ($k = 1, 2, \cdots$) where n_k is an integer) and $\nu_j \notin M$ ($j = 1, 2, \cdots, n$).

Another special case is

(c) The ω_k are positive, $\omega_k \geqq \omega > 0$, and $\nu_j \notin M$ ($j = 1, 2, \cdots, n$).

To construct a formal solution of (2.1) we arrange the set of all sequences $(n_1, n_2, \cdots)$ where the n_k are nonnegative integers, only a finite number of which are different from zero, as a sequence $N_1, N_2, \cdots$ in the following way. For $N = (n_1, n_2, \cdots)$ put

$$| N | = \sum k n_k, \tag{2.4}$$

and let $r < s$ if $| N_r | < | N_s |$. If $| N_r | = | N_s |$, then consider the first component that is not the same for the two sequences N_r, N_s, and let $r < s$ if the component of N_r is larger. Obviously, each sequence $(n_1, n_2, \cdots)$ appears as one and only one N_r. The particular sequences $(1, 0, 0, \cdots)$, $(0, 1, 0, 0, \cdots)$, $(0, 0, 1, 0, \cdots)$, $\cdots$ will have certain ordinals denoted as $(1), (2), (3), \cdots$, so that

$$\begin{aligned} N_{(1)} &= (1, 0, 0, 0, \cdots), \\ N_{(2)} &= (0, 1, 0, 0, \cdots), \\ N_{(3)} &= (0, 0, 1, 0, \cdots). \\ &\cdots\cdots \end{aligned} \tag{2.5}$$

$N_r + N_s$ is defined in the obvious way, as is also $kN_r + lN_s$, where k, l are nonnegative integers.

Let Ω stand for the sequence $(\omega_1, \omega_2, \cdots)$ and $N \cdot \Omega$ for the symbol $\sum n_k \omega_k$. Although this latter is a real number μ and will finally be so identified, in the construction of the formal solution to (2.1) the ω_k are considered as independent parameters, and $\sum n_k \omega_k$ is an element of the free additive semigroup generated by the $\omega_1, \omega_2, \cdots$. Multiplication of two exponential factors $e^{iN_r \cdot \Omega x}$, $e^{iN_s \cdot \Omega x}$ results in another exponential factor

$$e^{iN_r \cdot \Omega x} \cdot e^{iN_s \cdot \Omega x} = e^{iN_t \cdot \Omega x}, \tag{2.6}$$

where $N_t = N_r + N_s$. It follows readily from the ordering principle of the N's that $t \geqq r + s$.

We now construct a formal solution u of (2.1) of the form

$$u = \sum_{r \geqq 1} a_r \, e^{iN_r \cdot \Omega x} \tag{2.7}$$

in the same way as in Wasow's paper [1]. Inserting (2.7) into (2.1), expanding and rearranging according to the exponential factors $\exp (iN_s \cdot \Omega x)$ one obtains a recursive system of equations for the vectors a_r

$$(A - i\mu_r I)a_r = -g_r \qquad \text{if } r = (k) \tag{2.8a}$$

$$(A - i\mu_r I)a_r = h_r(a_1, a_2, \cdots, a_{r-1}) \qquad \text{if } r \neq (k) \tag{2.8b}$$

$$k = 1, 2, \cdots; \quad r = 1, 2, \cdots .$$

The components of the vector functions h_r are polynomials in the components of $a_1, a_2, \cdots, a_{r-1}$ without constant or linear terms. Because of assumption (2.3), the matrices $A - i\mu_r I$ are nonsingular, and as shown in [1]

$$c = \sup_{\mu \in M} \| (A - i\mu I)^{-1} \| < \infty. \tag{2.9}$$

Therefore,

$$\| a_r \| \leqq c \, \| g_r \| \qquad \text{if } r = (k), \tag{2.10a}$$

$$\| a_r \| \leqq c \, \| h_r(a_1, a_2, \cdots, a_{r-1}) \| \qquad \text{if } r \neq (k). \tag{2.10b}$$

Let $\hat{h}$ be a function dominating the h_j as defined in [1]. Then if

$$v = \sum_{r \geqq 1} b_r \, e^{iN_r \cdot \Omega x} \tag{2.11}$$

is substituted in $\hat{h}(v)$ and the resulting products are expanded and rearranged as above, one obtains

$$\hat{h}(v) = \sum_{r \geqq 1} \hat{h}_r(b_1, b_2, \cdots, b_{r-1}) e^{iN_r \cdot \Omega x}, \tag{2.12}$$

where

$$\hat{h}_r(b_1, b_2, \cdots, b_{r-1}) \geqq \| h_r(a_1, a_2, \cdots, a_{r-1}) \|, \qquad r = 1, 2, \cdots \tag{2.13}$$

if $b_k = \beta_k e_0$ and $\beta_k \geqq \| a_k \|$ $(k = 1, 2, \cdots, r - 1)$. Here e_0 denotes the vector

$$e_0 = (1, 1, \cdots, 1). \tag{2.14}$$

Then the vector $v = ve_0$, where v is defined by the equation

$$v - c\hat{h}(ve_0) = c\textstyle\sum_{r\geqq 1} \| g_r \| e^{i\omega_r x}, \tag{2.15}$$

will be shown to dominate u in the sense that

$$\| b_r \| \geqq \| a_r \|, \qquad r = 1, 2, \cdots . \tag{2.16}$$

To see this let (2.11) with $b_r = \beta_r e_0$ be substituted in (2.15). As before, one obtains recursive equations for the β_r:

$$\begin{aligned} \beta_r &= c \| g_r \| && \text{if} \quad r = (k), \\ \beta_r &= c\hat{h}_r(b_1, b_2, \cdots, b_{r-1}) && \text{if} \quad r \neq (k), \end{aligned} \tag{2.17}$$
$$k = 1, 2, \cdots; \qquad r = 1, 2, \cdots .$$

Thus, by (2.10), (2.13), and (2.17)

$$\| a_r \| \leqq c \| g_r \| = \beta_r = \| b_r \| \qquad \text{if} \quad r = (k) \tag{2.18a}$$

$$\begin{aligned} \| a_r \| &\leqq c \| h_r(a_1, a_2, \cdots, a_{r-1}) \| \\ &\leqq c\hat{h}_r(b_1, b_2, \cdots, b_{r-1}) = \beta_r = \| b_r \| \qquad \text{if} \quad r \neq (k) \end{aligned} \tag{2.18b}$$

provided $\| b_k \| \geqq \| a_k \|$ for $k = 1, 2, \cdots, r - 1$. Thus, (2.16) follows by induction.

To prove the convergence of the dominating series (2.11) consider (2.15) as an equation for v in terms of

$$\zeta = \textstyle\sum_{k\geqq 1} \| g_k \| z_k, \qquad z_k = e^{i\omega_k x}. \tag{2.19}$$

As in [1] it is immediately seen that there is a positive number σ_0 such that (2.15) has a unique solution v holomorphic in ζ for $| \zeta | \leqq \sigma_0$; this inequality will be satisfied if

$$c\textstyle\sum_{k\geqq 1} \| g_k \| \leqq \sigma_0 . \tag{2.20}$$

Thus, v is representable as a uniformly and absolutely convergent power series

$$v = \sum c_{r_1 r_2 \cdots r_k} z_1^{r_1} z_2^{r_2} \cdots z_k^{r_k}, \qquad k = 1, 2, \cdots; \quad r_1 + \cdots + r_k \geqq 1. \tag{2.21}$$

If the terms of series (2.21) are arranged according to the order of the sequences $(r_1, r_2, \cdots, r_k, 0, 0, \cdots)$ introduced above, the expansion

$$v = \textstyle\sum_{r\geqq 1} \beta_r e^{iN_r \cdot \Omega x} \tag{2.22}$$

is obtained. This proves the uniform and absolute convergence of series (2.11) and, therefore, of the dominated series (2.7) whenever condition (2.20) is satisfied.

It follows from (2.15) that there exists a positive number σ_1 such that $| v | \leqq \rho$ for $-\infty < x < \infty$ if

$$c\textstyle\sum_{k\geqq 1} \| g_k \| \leqq \sigma_1 . \tag{2.23}$$

Since $|v| = \|v\|$ and $\|u\| \leqq \|v\|$, it follows that the constructed series (2.7) satisfies equation (2.3) whenever conditions (2.20) and (2.23) are satisfied. From the construction it is then seen that (2.7) satisfies system (2.1). Thus the following theorem has been proved.

THEOREM 2.1. *Let the differential system*

$$y' = \sum_{k \geqq 1} g_k e^{i\omega_k x} + f(y)$$

on substitution of $y = a + u$ *take the form*

$$u' = \sum_{k \geqq 1} g_k e^{i\omega_k x} + Au + h(u),$$

where

$$h(u) = \sum_{k_1 + \cdots + k_n \geqq 2} h_{k_1 k_2 \cdots k_n} u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n}, \qquad \|u\| \leqq \rho,$$

and let condition (2.3) *be satisfied. Then the system has a solution which admits an expansion of the form*

$$y = a + \sum_{r \geqq 1} a_r e^{i\mu_r x}$$

converging absolutely and uniformly for $-\infty < x < \infty$ *provided*

$$\sum_{k \geqq 1} \|g_k\| \leqq \gamma,$$

where γ *is a positive number that depends only on* f. *The numbers* μ_r *are linear combinations of the numbers* ω_k *with nonnegative integral coefficients.*

For the special case where the ω_k are multiples of some real $\omega \neq 0$ one obtains

COROLLARY 2.1. *Let the differential system*

$$y' = \sum_{k=-\infty}^{\infty} g_k e^{ik\omega x} + f(y)$$

on substitution of $y = a + u$ *take the form*

$$u' = \sum_{k=-\infty}^{\infty} g_k e^{ik\omega x} + Au + h(u),$$

where

$$h(u) = \sum_{k_1 + \cdots + k_n \geqq 2} h_{k_1 k_2 \cdots k_n} u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n}, \qquad \|u\| \leqq \rho,$$

and assume none of the eigenvalues of A *equals* $ik\omega$ ($k = 0, \pm 1, \pm 2, \cdots$). *Then the system has a solution which admits an absolutely converging Fourier expansion*

$$y = a + \sum_{k=-\infty}^{\infty} a_k e^{ik\omega x}$$

provided

$$\sum_{k=-\infty}^{\infty} \|g_k\| \leqq \gamma,$$

where γ *is a positive number depending only on* f.

COROLLARY 2.2. *With* x *restricted to the interval* $0 \leqq x < \infty$, *Theorem* 2.1 *holds for arbitrary complex* ω_k *with* $\operatorname{Im} \omega_k \geqq 0$.

The proof is unchanged.

3. Real systems

Let (1.1) and the equivalent (2.1) be real systems. By this we mean a and A are real, $h(u)$ has real coefficients, and for each ω_k there exists $\omega_{k'} = -\omega_k$ such that $g_{k'} = \bar{g}_k$. We prove that the particular solution (2.8) of a real system constructed in Section 2 is real for $-\infty < x < \infty$.

Let the ω_k be so arranged that

$$\omega_{2k-1} = -\omega_{2k}, \qquad k = 1, 2, \cdots. \tag{3.1}$$

Then, by assumption,

$$g_{2k-1} = \bar{g}_{2k}, \qquad k = 1, 2, \cdots. \tag{3.2}$$

If N_r is the sequence $(n_1, n_2, n_3, n_4, \cdots)$ where the ordinal r is determined as in Section 2, let r' be the ordinal of the sequence $(n_2, n_1, n_4, n_3, \cdots)$. System (2.8a) now breaks up into

$$\begin{aligned} (A - i\mu_r I)a_r &= g_r, \\ (A + i\mu_r I)a_{r'} &= g_{r'} = \bar{g}_r, \end{aligned} \qquad r = (k), \quad k = 1, 2, \cdots, \tag{3.3}$$

where use is made of (3.1) and (3.2). It follows that

$$a_{r'} = \bar{a}_r, \qquad r = (k), \quad k = 1, 2, \cdots. \tag{3.4}$$

Similarly, (2.8b) breaks up into

$$\begin{aligned} (A - i\mu_r I)a_r &= h_r \\ (A + i\mu_r I)a_{r'} &= h_{r'} \end{aligned} \qquad r = (k), \quad k = 1, 2, \cdots. \tag{3.5}$$

Here h_r is a polynomial with real coefficients in $a_0, a_1, \cdots, a_s$, where $s < r$ and $s' < r$. Also it is easily seen that if

$$h_r = h_r(a_1, a_2, \cdots, a_s), \tag{3.6a}$$

then

$$h_{r'} = h_r(a_{1'}, a_{2'}, \cdots, a_{s'}). \tag{3.6b}$$

Thus, it follows from (3.5) and (3.6) that if $a_{k'} = \bar{a}_k$ for $k = 1, 2, \cdots, s$, then also $a_{r'} = \bar{a}_r$. Using (3.4) and induction gives

$$a_{r'} = \bar{a}_r, \qquad r = 1, 2, \cdots. \tag{3.7}$$

Thus it is proved that (2.8) is real for $-\infty < x < \infty$.

THEOREM 3.1. *If the system* (1.1) *is real, then the solution* (1.2) *is real for* $-\infty < x < \infty$.

4. General systems

We consider next the general system

$$y' = g(x, y). \tag{4.1}$$

We assume $g(x, y)$ is analytic in y for $-\infty < x < \infty$, holomorphic in some neighborhood of $y = a$. We put $u = y - a$ as before and rewrite (4.1) in terms of u as

$$u' = g(x) + A(x)u + h(x, u). \tag{4.2}$$

Here the vector $g(x) = g(x, a)$ is assumed to have an expansion

$$g(x) = \textstyle\sum_{k\geqq 1} g_k\, e^{i\omega_k x}, \tag{4.3}$$

where the ω_k are real numbers, rationally independent or not, and

$$\textstyle\sum_{k\geqq 1} \| g_k \| < \infty. \tag{4.4}$$

We also assume that the matrix $A(x)$, which is the Jacobian of $g(x, y)$ with respect to $y_1, y_2, \cdots, y_n$ evaluated at $y = a$, has an expansion

$$A(x) = A + \textstyle\sum_{k\geqq 1} A_k\, e^{i\omega_k x}, \tag{4.5}$$

with

$$\textstyle\sum_{k\geqq 1} \| A_k \| < \infty. \tag{4.6}$$

The fact that the constant matrix A is separated from the other terms in (4.5) does not rule out the possibility that one or more of the ω_k are equal to 0 and that the corresponding terms $A_k \exp(i\omega_k x)$ contribute to the constant term (the mean value) of $A(x)$.

The vector $h(x, u)$ can be expanded in a series of powers of $u_1, u_2, \cdots, u_n$ without constant or linear term. We put

$$h(x, u) = \sum_{k_1+\cdots+k_n\geqq 2} h_{k_1k_2\cdots k_n}(x)u_1^{k_1}u_2^{k_2}\cdots u_n^{k_n} \tag{4.7}$$

and assume that the vectors $h_{k_1k_2\cdots k_n}(x)$ possess expansions

$$h_{k_1k_2\cdots k_n}(x) = \textstyle\sum_{k\geqq 1} h_{k_1k_2\cdots k_nk}\, e^{i\omega_k x}, \tag{4.8}$$

with

$$\textstyle\sum_{k\geqq 1} \| h_{k_1k_2\cdots k_nk} \| = \eta_{k_1k_2\cdots k_n} < \infty \tag{4.9}$$

and

$$\sum_{k_1+\cdots+k_n\geqq 2} \eta_{k_1k_2\cdots k_n}\, \rho^{k_1+k_2+\cdots+k_n} = \eta < \infty \tag{4.10}$$

for some positive ρ.

These are the general assumptions which we assume to hold throughout this section and which will be considered as part of the definition of $g(x, y)$ in (4.1), and of $g(x)$, $h(x)$, $A(x, u)$ in (4.2). Besides, we impose the same condition as in Section 2 on the eigenvalues of the matrix A. If these are $i\nu_j$ $(j = 1, 2, \cdots, n)$ and M is the set of numbers which are linear combinations of $\omega_1, \omega_2, \cdots$ with nonnegative integral coefficients, we require that

$$\inf_{j=1,\cdots,n;\mu\in M} | \nu_j - \mu | = \delta > 0. \tag{4.11}$$

We now construct a formal solution u of (4.2) of the form

$$u = \textstyle\sum_{r \geqq 1} a_r \, e^{iN_r \cdot \Omega x}, \tag{4.12}$$

where N_r, Ω are as defined in Section 2. Inserting (4.12) into (4.2), expanding and rearranging according to the exponential factors $\exp(iN_s \cdot \Omega x)$, one obtains a recursive system of equations for the vectors a_r

$$(A - i\mu_r I)a_r = -g_r \qquad \text{if} \quad r = (k), \tag{4.13a}$$

$$(A - i\mu_r I)a_r = h_r(a_1, a_2, \cdots, a_{r-1}) \qquad \text{if} \quad r \neq (k), \tag{4.13b}$$

$$k = 1, 2, \cdots; \qquad r = 1, 2, \cdots.$$

The numbers μ_r and ordinals (k) are also as defined in Section 2. The components of the vector functions h_r are polynomials in the components of $a_1, a_2, \cdots, a_{r-1}$ without constant terms, with coefficients that are polynomials in the components of A_k and $\eta_{k_1 k_2 \cdots k_n k}$. As in Section 2 we conclude that equations (4.13) have a unique solution a_r and that

$$c = \sup_{\mu \in m} \| (A - i\mu I)^{-1} \| < \infty, \tag{4.14}$$

$$\| a_r \| \leqq c \| g_r \| \qquad \text{if} \quad r = (k), \tag{4.15a}$$

$$\| a_r \| \leqq c \| h_r(a_1, a_2, \cdots, a_{r-1}) \| \qquad \text{if} \quad r \neq (k), \tag{4.15b}$$

$$k = 1, 2, \cdots; \qquad r = 1, 2, \cdots.$$

In order to construct a dominating problem we introduce the scalar functions

$$\hat{a}(x) = \textstyle\sum_{k \geqq 1} \| A_k \| \, e^{i\omega_k x}, \tag{4.16}$$

$$\hat{h}_{k_1 k_2 \cdots k_n}(x) = \textstyle\sum_{k \geqq 1} \| h_{k_1 k_2 \cdots k_n k} \| \, e^{i\omega_k x}, \tag{4.17}$$

$$\begin{aligned} \hat{h}(x, u) = {} & \hat{a}(x)(u_1 + \cdots + u_n)/n \\ & + \sum_{k_1 + \cdots + k_n \geqq 2} \hat{h}_{k_1 k_2 \cdots k_n}(x) u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n}. \end{aligned} \tag{4.18}$$

By (4.10) the series in (4.18) converges in the domain $\| u \| \leqq \rho$, for $-\infty < x < \infty$. It clearly dominates, for each x, each component of the series of (4.7), which therefore also converges in the domain $\| u \| \leqq \rho$ for $-\infty < x < \infty$. It follows that the components of $g(x, y)$ are holomorphic in the region $\| y - a \| \leqq \rho$, for $-\infty < x < \infty$.

If

$$v = \textstyle\sum_{r \geqq 1} b_r \, e^{iN_r \cdot \Omega x} \tag{4.19}$$

is substituted in $\hat{h}(x, v)$ and the resulting products are expanded and rearranged as in Section 2, one obtains

$$\hat{h}(x, v(x)) = \textstyle\sum_{r \geqq 1} \hat{h}_r(b_1, \cdots, b_{r-1}) e^{iN_r \cdot \Omega x}. \tag{4.20}$$

If one recalls the definition of the h_r in (4.13b), one finds

$$\hat{h}_r(b_1, b_2, \cdots, b_{r-1}) \geqq \| h_r(a_1, a_2, \cdots, a_{r-1})\| \tag{4.21}$$

if $b_k = \beta_k e_0 = \beta_k(1, 1, \cdots, 1)$ and $\beta_k \geqq \| a_k \|$ $(k = 1, 2, \cdots, r-1)$.

Now let $v = ve_0$ where $v = v(x)$ is defined by the equation

$$v - c\hat{h}(x, ve_0) = c\textstyle\sum_{r\geq 1} \| g_r \| e^{i\omega_r x}. \tag{4.22}$$

If (4.19) with $b_r = \beta_r e_0$ is substituted in (4.22), one obtains the recursive equations

$$\beta_r = c \| g_r \| \qquad \text{if} \quad r = (k), \tag{4.23a}$$

$$\beta_r = c\hat{h}_r(b_1, b_2, \cdots, b_{r-1}) \qquad \text{if} \quad r \neq (k), \tag{4.23b}$$

$$k = 1, 2, \cdots; \qquad r = 1, 2, \cdots.$$

Thus, by (4.15), (4.21), and (4.25)

$$\| a_r \| \leqq c \| g_r \| = \beta_r = \| b_r \| \qquad \text{if} \quad r = (k) \tag{4.24a}$$

$$\begin{aligned} \| a_r \| &\leqq c \| h_r(a_1, a_2, \cdots, a_{r-1})\| \\ &\leqq c\hat{h}_r(b_1, b_2, \cdots, b_{r-1}) = \beta_r = \| b_r \| \qquad \text{if} \quad r \neq (k) \end{aligned} \tag{4.24b}$$

provided $\| b_k \| \geqq \| a_k \|$ for $k = 1, 2, \cdots, r-1$. It follows by induction that

$$\| b_r \| \geqq \| a_r \|, \qquad r = 1, 2, \cdots, \tag{4.25}$$

that is, $v(x)$ dominates $u(x)$.

To prove the convergence of series (4.19), consider (4.22) as an equation for v in terms of the infinitely many variables

$$z_k = e^{i\omega_k x}. \tag{4.26}$$

If we use (4.16), (4.17), (4.18), and (4.26), equation (4.22) becomes

$$\begin{aligned} \frac{v}{c} = \sum_{k\geq 1} \| g_k \| z_k + \Big(\sum_{k\geq 1} \| A_k \| z_k\Big) v \\ + \sum_{k_1+\cdots+k_n\geq 2} \Big(\sum_{k\geq 1} \| h_{k_1k_2\cdots k_nk} \| z_k\Big) v^{k_1+k_2+\cdots+k_n}. \end{aligned} \tag{4.27}$$

By (4.4), (4.6), and (4.10)

$$\Big| \sum_{k\geq 1} \| g_k \| z_k \Big| \leqq \sum_{k\geq 1} \| g_k \| < \infty, \tag{4.28a}$$

$$\Big| \sum_{k\geq 1} \| A_k \| z_k \Big| \leqq \sum_{k\geq 1} \| A_k \| < \infty, \tag{4.28b}$$

$$\sum_{k_1+\cdots+k_n=r} \Big| \sum_{k\geq 1} \| h_{k_1k_2\cdots k_nk} \| z_k \Big| \leqq \eta\rho^{-r}, \qquad r \geqq 2. \tag{4.28c}$$

If equation (4.27) is written as $F(v) = 0$, then $F(0) = 0$ and

$$F'(0) = c^{-1} - \textstyle\sum_{k\geq 1} \| A_k \| z_k. \tag{4.29}$$

Thus, if we impose the condition

$$\sum_{k\geqq 1} \| A_k \| < c^{-1}, \tag{4.30}$$

then, by (4.28b), $F'(0) \neq 0$. Using (4.28a) and (4.28c) one concludes that there exists a positive number σ_0 such that (4.27) has a unique solution representable as a uniformly and absolutely converging power series

$$v = \sum_{\substack{r_1+\cdots+r_m \geqq 1 \\ m \geqq 1}} c_{r_1 r_2 \cdots r_m} z_1^{r_1} z_2^{r_2} \cdots z_m^{r_m} \tag{4.31}$$

provided

$$c \sum_{k\geqq 1} \| g_k \| \leqq \sigma_0 . \tag{4.32}$$

If (4.26) is inserted in (4.31), products are expanded and the resulting terms are rearranged according to the order principle of Section 2, the expansion

$$v = \sum_{r\geqq 1} \beta_r \, e^{iN_r \cdot \Omega x} \tag{4.33}$$

defined by the recursive equations (4.23) is obtained. This proves the uniform and absolute convergence of series (4.19), hence also of the dominated series (4.12), whenever conditions (4.30) and (4.32) are satisfied.

Furthermore, it follows from (4.27) that there exists a positive number σ_1 such that $| v | \leqq \rho$ for $-\infty < x < \infty$ if

$$c \sum_{k\geqq 1} \| g_k \| \leqq \sigma_1 . \tag{4.34}$$

Since $\| u \| \leqq \| v \| = | v |$, it follows that the constructed series (4.12) satisfies equation (4.2) whenever conditions (4.30), (4.32), and (4.34) are satisfied. Thus, the following theorem has been proved.

THEOREM 4.1. *Let the differential system*

$$y' = g(x, y)$$

on substitution of $y = a + u$ *take the form*

$$u' = \sum_{k\geqq 1} g_k \, e^{i\omega_k x} + A(x)u + h(x, u)$$

with $A(x)$ *and* $h(x, u)$ *as in* (4.5) *and* (4.7), *and assume condition* (4.14) *is satisfied. Then the system has a solution which admits an expansion of the form*

$$y = a + \sum_{r\geqq 1} a_r \, e^{i\omega_r x}$$

converging absolutely and uniformly for $-\infty < x < \infty$ *provided*

$$\sum_{k\geqq 1} \| A_k \| < \sup_{\mu \in M} \|(A - i\mu I)^{-1} \|^{-1}$$

and

$$\sum_{k\geqq 1} \| g_k \| \leqq \gamma,$$

where γ *is a positive number depending only on* $A(x)$ *and* $h(x, u)$. *The numbers* ω_r, *which are the elements of the set* M, *are linear combinations of the numbers* ω_k *with nonnegative integral coefficients.*

For the special case where the ω_k are multiples of some real $\omega \neq 0$ one obtains

COROLLARY 4.1. *Let the differential system*

$$y' = g(x, y)$$

on substitution of $y = a + u$ *take the form*

$$u' = \sum_{k=-\infty}^{\infty} g_k\, e^{ik\omega x} + A(x)u + h(x, u),$$

where

$$A(x) = A + \sum_{k=-\infty}^{\infty} A_k\, e^{ik\omega x},$$

$$h(x, u) = \sum_{k_1+\cdots+k_n \geqq 2} \left(\sum_{k=-\infty}^{\infty} h_{k_1 k_2 \cdots k_n k}\, e^{ik\omega x} \right) u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n},$$

$$\sum_{k_1+\cdots+k_n \geqq 2} \left(\sum_{k=-\infty}^{\infty} \| h_{k_1 k_2 \cdots k_n k} \| \right) \rho^{k_1+k_2+\cdots+k_n} < \infty, \qquad \rho > 0,$$

and assume none of the eigenvalues of A *equals* $ik\omega$ $(k = 0, \pm 1, \pm 2, \cdots)$. *Then the system has a solution which admits an absolutely convergent Fourier expansion*

$$y = a + \sum_{k=-\infty}^{\infty} a_k\, e^{ik\omega x}$$

provided

$$\sum_{k=-\infty}^{\infty} \| A_k \| < \sup_k \| (A - ik\omega I)^{-1} \|^{-1}$$

and

$$\sum_{k=-\infty}^{\infty} \| g_k \| \leqq \gamma,$$

where γ *is a positive number depending only on* $A(x)$ *and* $h(x, u)$.

COROLLARY 4.2. *With* x *restricted to the interval* $0 \leqq x < \infty$ *Theorem* 4.1 *holds for arbitrary complex* ω_k, $\operatorname{Im} \omega_k \geqq 0$.

5. Subharmonics

For simplicity we return to equation (1.1) for y and (2.1) for u. The main assumption in this section is

$$q\nu_0 - \mu_0 = 0, \tag{5.1}$$

where q is an integer $\geqq 2$, $i\nu_0$ is one of the eigenvalues of A, and

$$\mu_0 = N_0 \cdot \Omega = \sum n_k^0\, \omega_k$$

with at least one of the finite number of positive integers n_k^0 not divisible by q. The solution will contain exponential terms of the form

$$\exp\, (i(N_r/q) \cdot \Omega x).$$

However, of all the sequences N_r $(r = 1, 2, \cdots)$ of Section 2, only those will be used in the solution which are of the form

$$N_r = (r_0\, n_1^0 + r_1\, q,\ r_0\, n_2^0 + r_2\, q, \cdots). \tag{5.2}$$

The set of these N_r with $(r_0, r_1, r_2, \cdots) \neq (1, 0, 0, \cdots)$ will be denoted by $\mathfrak{N}$ and the set of real numbers $\mu = N_r \cdot \Omega$, $N_r \in \mathfrak{N}$, will be denoted by $\mathfrak{M}$. Condition (2.3) is replaced by

$$\inf_{j=1,\cdots,n;\mu\in\mathfrak{M}} | \nu_j - \mu/q | = \delta > 0. \tag{5.3}$$

We construct a solution of equation (2.1) which admits an expansion of the form

$$u = a_0 e^{i(N_0/q)\cdot\Omega x} + \sum_{N_r\in\mathfrak{N}} a_r e^{i(N_r/q)\cdot\Omega x}. \tag{5.4}$$

If (5.4) is inserted in (2.1), products are expanded and the resulting terms are rearranged according to the exponential terms $\exp(i(N_s/q)\cdot\Omega x)$, one obtains the following system of equations for the vectors a_0, a_r:

$$(A - i(\mu_0/q)I)a_0 = 0, \tag{5.5a}$$

$$(A - i(\mu_r/q)I)a_r = h_r(a_0, a_1, \cdots, a_{r-1}) + f_r, \tag{5.5b}$$

where

$$f_r = -g_r \qquad \text{if } N_r = qN_{(k)}, \tag{5.6a}$$

$$f_r = 0 \qquad \text{if } N_r \neq qN_{(k)}, \tag{5.6b}$$

$$k = 1, 2, \cdots; \quad N_r \in \mathfrak{N}.$$

The components of the vector functions h_r are polynomials in the components of $a_0, a_1, \cdots, a_{r-1}$ without constant or linear terms. Equation (5.5a) results from the fact that $N_0 \notin \mathfrak{N}$. Because of assumption (5.1) this equation has a solution a_0 with arbitrarily prescribed $\alpha = \| a_0 \|$. The remaining a_r can be recursively determined from systems (5.5b), which are nonsingular on account of assumption (5.3). As before one concludes that

$$\sup_{\mu_r\in\mathfrak{M}} \|(A - i(\mu_r/q)I)^{-1}\| = c < \infty \tag{5.7}$$

and

$$\| a_r \| \leqq c(\| h_r(a_0, a_1, \cdots, a_{r-1})\| + \| f_r \|). \tag{5.8}$$

Let the function $\hat{h}$ be defined as in Section 2. Then if

$$v = b_0 e^{i(N_0/q)\cdot\Omega x} + \textstyle\sum_{N_r\in\mathfrak{N}} b_r e^{i(N_r/q)\cdot\Omega x} \tag{5.9}$$

is substituted in $\hat{h}(v)$ and the resulting products are expanded and rearranged as before, one obtains

$$\hat{h}(v) = \textstyle\sum_{N_r\in\mathfrak{N}} \hat{h}_r(b_0, b_1, \cdots, b_{r-1}) e^{i(N_r/q)\cdot\Omega x}, \tag{5.10}$$

where

$$\hat{h}_r(b_0, b_1, \cdots, b_{r-1}) \geqq \| h_r(a_0, a_1, \cdots, a_{r-1})\| \tag{5.11}$$

if $b_k = \beta_k e_0$ and $\beta_k \geqq \| a_k \|$ $(k = 0, 1, \cdots, r-1)$. We show that the vector $v = ve_0$ with the scalar v defined by the equation

$$v - c\hat{h}(v) = \alpha e^{i(\mu_0/q)x} + c\sum_{k\geq 1} \| g_k \| e^{i\omega_k x} \tag{5.12}$$

dominates the vector u in the sense that

$$\| b_0 \| \geqq \| a_0 \|, \qquad \| b_r \| \geqq \| a_r \|, \quad r \geqq 1. \tag{5.13}$$

To see this let (5.9) with $b_0 = \beta_0 e_0$, $b_r = \beta_r e_0$ be substituted in (5.12). Proceeding as before, one obtains recursive equations for the β_0, β_r

$$\begin{aligned} \beta_0 &= \alpha, \\ \beta_r &= c\hat{h}_r(b_0, b_1, \cdots, b_{r-1}) + c \| f_r \| . \end{aligned} \tag{5.14}$$

Thus, by (5.8), (5.11), and (5.14)

$$\| a_0 \| = \alpha = \beta_0 = \| b_0 \|, \tag{5.15a}$$

$$\begin{aligned} \| a_r \| &\leqq c(\| h_r(a_0, a_1, \cdots, a_{r-1})\| + \| f_r \|) \\ &\leqq c\hat{h}_r(b_0, b_1, \cdots, b_{r-1}) + c \| f_r \| \\ &= \beta_r = \| b_r \| \end{aligned} \tag{5.15b}$$

provided $\| b_k \| \geqq \| a_k \|$ for $k = 0, 1, \cdots, r-1$. Thus (5.13) is proved by induction.

To prove the convergence of the dominating series (5.9) consider (5.12) as an equation for v in terms of

$$z_0 = e^{i(\mu_0/q)x}, \tag{5.16a}$$

$$z_k = e^{i\omega_k x} = e^{iq(\mu(k)/q)x}, \qquad k = 1, 2, \cdots . \tag{5.16b}$$

One concludes as before that there exists a positive number σ_0 such that (5.12) has a unique solution v representable as a uniformly and absolutely converging power series

$$v = \sum_{\substack{r_0+r_1+\cdots+r_m\geqq 1 \\ m\geqq 1}} c_{r_0 r_1 \cdots r_m} z_0^{r_0} z_1^{r_0} \cdots z_m^{r_m} \tag{5.17}$$

provided

$$\alpha + c\sum_{k\geq 1} \| g_k \| \leqq \sigma_0 . \tag{5.18}$$

If (5.16) is inserted in (5.17), products are expanded and the resulting terms are properly arranged, the expansion

$$v = \beta_0 e^{i(N_0/q)\cdot\Omega x} + \sum_{N_r\in\mathfrak{N}} \beta_r e^{i(N_r/q)\cdot\Omega x} \tag{5.19}$$

defined by the recursive relations (5.14) is obtained. This proves the uniform and absolute convergence of series (5.9), hence also of the dominated series (5.4), whenever condition (5.18) is satisfied.

Furthermore, it follows from (5.12) that there exists a positive number σ_1 such that $|v| \leqq \rho$ for $-\infty < x < \infty$ if

$$\alpha + c\textstyle\sum_{k\geqq 1} \| g_k \| \leqq \sigma_1 . \tag{5.20}$$

Thus, the constructed series (5.4) satisfies equation (2.1) whenever conditions (5.18) and (5.20) are satisfied. The results are stated in the following theorem.

THEOREM 5.1. *Let the differential system*

$$y' = \textstyle\sum_{k\geqq 1} g_k e^{i\omega_k x} + f(y)$$

on substitution of $y = a + u$ *take the form*

$$u' = \textstyle\sum_{k\geqq 1} g_k e^{i\omega_k x} + Au + h(u),$$

where $h(u)$ *is as in Theorem* 2.1, *and let conditions* (5.1) *and* (5.3) *be satisfied. Then, for given* $\alpha \geqq 0$, *the system has a solution* y *which admits an expansion of the form*

$$y = a + a_0 e^{i(\mu_0/q)x} + \textstyle\sum_{N_r \in \mathfrak{N}} a_r e^{i(\mu_r/q)x}$$

with $\| a_0 \| = \alpha$, *converging absolutely and uniformly for* $-\infty < x < \infty$ *provided*

$$\alpha + c\textstyle\sum_{k\geqq 1} \| g_k \| \leqq \gamma,$$

where γ *is a positive number that depends only on* f. *The numbers* μ_r *are linear combinations of the numbers* μ_0, $q\omega_1$, $q\omega_2$, $\cdots$ *with nonnegative integral coefficients.*

In the special case where the ω_k are multiples of one real number $\omega \neq 0$, conditions (5.1), (5.3) will be satisfied if some eigenvalue of A is $i(p/q)\omega$ with p relatively prime to q and if otherwise no eigenvalue is equal to $i(mp/q + k)\omega$ ($m = 1, 2, \cdots ; k = 0, \pm 1, \pm 2, \cdots$). This leads to

COROLLARY 5.1. *Let the differential system*

$$y' = \textstyle\sum_{k=-\infty}^{\infty} g_k e^{ik\omega x} + f(y)$$

on substitution of $y = a + u$ *take the form*

$$u' = \textstyle\sum_{k=-\infty}^{\infty} g_k e^{ik\omega x} + Au + h(u)$$

where $h(u)$ *is as in Theorem* 2.1. *Assume the matrix* A *has an eigenvalue* $i(p/q)\omega$ *with* p *relatively prime to* q *and no other eigenvalue of the form* $i(mp/q + k)\omega$ ($m = 1, 2, \cdots ; k = 0, \pm 1, \pm 2, \cdots$). *Then, for* $\| a_0 \| = \alpha$ *given, the system has a solution admitting a Fourier expansion*

$$y = a + a_0 e^{i(p/q)\omega x} + \sum_{\substack{k=-\infty \\ m\geqq 1}}^{\infty} a_{k,m} e^{i(mp/q+k)\omega x}$$

with $\| a_0 \| = \alpha$, *converging absolutely and uniformly for* $-\infty < x < \infty$ *provided*

$$\alpha + c\textstyle\sum_{k=-\infty}^{\infty} \| g_k \| \leqq \gamma,$$

where γ *is a number that depends only on* f. *Whereas the forcing term has the period* $2\pi/\omega$, *the period of the solution is* $q \cdot 2\pi/\omega$.

6. Stability of solutions

In this section it will be proved that the particular solution found in Sections 2 and 4 is imbedded in a field of solutions each of which admits an expansion in an exponential series, and furthermore that the particular solution is uniformly stable.

Here we assume that x is restricted to the interval $0 \leqq x < \infty$. For simplicity the proofs are given for equation (1.1) rather than for equation (4.1). With slightly changed notation equation (1.1) is written as

$$y' = f(y) + \textstyle\sum_{k\geqq 1} g_k\, e^{\rho_k x}, \tag{6.1}$$

and we assume

$$\operatorname{Re} \rho_k \leqq 0, \quad k = 1, 2, \cdots, \qquad \textstyle\sum_{k\geqq 1} \| g_k \| < \infty. \tag{6.2}$$

With the substitution $y = a + u$ equation (6.1) becomes

$$u' = Au + h(u) + \textstyle\sum_{k\geqq 1} g_k\, e^{\rho_k x}, \tag{6.3}$$

and it is also assumed that the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A satisfy the condition

$$\operatorname{Re} \lambda_j \leqq 0, \qquad j = 1, 2, \cdots, n. \tag{6.4}$$

To simplify notation we put

$$\tau_k = \lambda_k, \qquad k = 1, 2, \cdots, n, \tag{6.5a}$$

$$ = \rho_{k-n}, \qquad k = n+1, n+2, \cdots. \tag{6.5b}$$

The sequences N_r ($r = 1, 2, \cdots$) are defined as in Section 2, likewise the $N_{(k)}$ ($k = 1, 2, \cdots$). $\mathfrak{T}$ is to denote the sequence

$$\mathfrak{T} = (\tau_1, \tau_2, \cdots) = (\lambda_1, \lambda_2, \cdots, \lambda_n, \rho_1, \rho_2, \cdots). \tag{6.6}$$

If $N = (n_1, n_2, \cdots)$, then $N \cdot \mathfrak{T} = \sum_{r\geqq 1} n_r \tau_r$, and the complex number $N_r \cdot \mathfrak{T}$ will be denoted as

$$\nu_r = N_r \cdot \mathfrak{T}, \qquad r = 1, 2, \cdots. \tag{6.7}$$

In particular,

$$\nu_{(k)} = N_{(k)} \cdot \mathfrak{T} = \lambda_k, \qquad k = 1, 2, \cdots, n, \tag{6.8a}$$

$$\phantom{\nu_{(k)}} = \rho_{k-n}, \qquad k = n+1, n+2, \cdots. \tag{6.8b}$$

The set of numbers ν_r ($r = 1, 2, \cdots; r \neq (k), k \leqq n$) will be denoted as $\mathfrak{N}$; it includes all the linear combinations of $\lambda_1, \lambda_2, \cdots, \lambda_n, \rho_1, \rho_2, \cdots$

with nonnegative integral coefficients except the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ themselves. Condition (2.3) is replaced by the more stringent condition

(6.9) $$\inf_{j=1,\cdots,n;\nu\in\mathfrak{N}} |\lambda_j - \nu| = \delta > 0.$$

We seek a solution of equation (6.3) of the form

(6.10) $$u = \sum_{r\geq 1} a_r e^{N_r \cdot \mathfrak{X} x}.$$

If we substitute (6.10) in (6.3) and proceed as in Section 2, we obtain the following equations for the a_r

(6.11a) $$(A - \lambda_k I)a_r = 0, \qquad r = (k), \quad k = 1, 2, \cdots, n,$$

(6.11b) $$(A - \rho_k I)a_r = -g_{k-n}, \qquad r = (k), \quad k = n+1, n+2, \cdots,$$

(6.11c) $$(A - \nu_r I)a_r = h_r(a_1, a_2, \cdots, a_{r-1}), \qquad r \neq (k), \quad k = 1, 2, \cdots.$$

The components of the vector function h_r are polynomials in the components of $a_1, a_2, \cdots, a_{r-1}$ without constant or linear terms.

Equations (6.11a) are satisfied if $a_{(k)}$ ($k = 1, 2, \cdots, n$) is taken to be

(6.12) $$a_{(k)} = \alpha_k e_k, \qquad k = 1, 2, \cdots, n,$$

where α_k is an arbitrary number and e_k is an eigenvector of A belonging to the eigenvalue λ_k, $\| e_k \| = 1$. The systems (6.11b), (6.11c) are nonsingular on account of assumption (6.9). After $a_{(1)}, a_{(2)}, \cdots, a_{(n)}$ have been chosen, the remaining a_r are determined recursively from equations (6.11b), (6.11c).

With the constant c defined as in Section 2,

(6.13) $$c = \sup_{\nu\in\mathfrak{N}} \|(A - \nu I)^{-1}\| < \infty,$$

one obtains from (6.11)

(6.14a) $$\| a_r \| \leqq c \| g_{k-n} \|, \qquad r = (k), \quad k = n+1, n+2, \cdots,$$

(6.14b) $$\| a_r \| \leqq c \| h_r(a_1, a_2, \cdots, a_{r-1})\|, \qquad r \neq (k), \quad k = 1, 2, \cdots.$$

Exactly as in Section 2 it is seen that the series

(6.15) $$v = \sum_{r\geq 1} b_r e^{N_r \cdot \mathfrak{X} x},$$

with $v = v e_0$ and v defined by the equation

(6.16) $$v - c\hat{h}(v) = \sum_{j=1}^{n} | \alpha_j | e^{\lambda_j x} + c \sum_{k\geq 1} \| g_k \| e^{\rho_k x},$$

dominates series (6.10) in the sense that

(6.17) $$\| a_r \| \leqq \| b_r \|, \qquad r = 1, 2, \cdots.$$

Considering (6.16) as an equation for v in terms of

(6.18) $$z_r = e^{\nu_r x}, \qquad r = 1, 2, \cdots;$$

one concludes as before that there exists a positive number σ such that

(6.19) $$v = \sum_{\substack{n_1+\cdots+n_m \geqq 1 \\ m \geqq 1}} c_{n_1 n_2 \cdots n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}$$

and $| v | \leqq \rho$ whenever

(6.20) $$\sum_{j=1}^{n} | \alpha_j | + c \sum_{k \geqq 1} \| g_k \| \leqq \sigma.$$

Thus, (6.15) and the dominated series (6.10) converge absolutely and uniformly for $0 \leqq x < \infty$ if condition (6.20) is satisfied. Moreover, (6.10) represents a solution of equation (6.3).

Let the family of solutions (6.10) with the arbitrary parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$ be denoted as $u(x, \alpha)$. We show that these functions form a field of solutions embedding the particular solution of Section 2. For this purpose, we denote the particular solution by u^* and write its expansion in the form

(6.21) $$u^* = \sum_{r \geqq 1} a_r^* e^{N_r \cdot \mathfrak{X} x}.$$

The coefficients a_r^* are then defined by the recursive equations

(6.22a) $$a_r^* = 0, \qquad r = (k), \quad k = 1, 2, \cdots, n,$$

(6.22b) $$(A - \rho_k I) a_r^* = -g_{k-n}, \qquad r = (k), \quad k = n + 1, n + 2 \cdots,$$

(6.22c) $$(A - \nu_r I) a_r^* = h_r(a_1^*, a_2^*, \cdots, a_{r-1}^*), \quad r \neq (k), \quad k = 1, 2, \cdots.$$

From these equations it follows that $u^*(x) = u(x, 0)$. Equation (6.16) for the dominating function v shows that the components of $u(x, \alpha)$ are holomorphic functions of $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the domain defined by (6.20) with coefficients that have expansions of the form (6.10) which converge absolutely and uniformly for $0 \leqq x < \infty$. Thus

(6.23) $$u(x, \alpha) = u^*(x) + \sum_{j=1}^{n} \alpha_j w_j(x) + w(x, \alpha).$$

The components of $w(x, \alpha)$ can be expanded as power series in $\alpha_1, \alpha_2, \cdots, \alpha_n$ and without constant or linear terms. The $w_j(x)$ can be expanded as

(6.24) $$w_j(x) = \sum_{r \geqq 1} a_{jr} e^{N_r \cdot \mathfrak{X} x},$$

and it follows from (6.11), (6.12), and (6.22) that

(6.25a) $$a_{jr} = \delta_{jk} e_k \qquad r = (k), \quad k = 1, 2, \cdots, n$$

(6.25b) $$a_{jr} = 0 \qquad r = (k), \quad k = n + 1, n + 2 \cdots$$

where δ_{jk} is the Kronecker symbol. The remaining a_{jr} ($r \neq (k)$, $k \geqq 1$) are polynomials in the variables $g_1, g_2, \cdots$ without constant terms. Thus, by (6.24) and (6.25)

(6.26) $$w_j(x) = e^{\lambda_j x} e_j + z_j(x, g),$$

and, by again using equation (6.16) for the dominating function v, it is readily seen that

$$\| z_j(x, g)\| \to 0 \quad \text{as} \quad \textstyle\sum_{k\geqq 1} \| g_k \| \to 0, \qquad j = 1, 2, \cdots, n. \tag{6.27}$$

Substituting (6.26) in (6.23) and putting $x = x_0 \geqq 0$ we have

$$u(x_0, \alpha) - u^*(x_0) = \textstyle\sum_{j=1}^{n} [e_j e^{\lambda_j x_0} + z_j(x_0, g)]\alpha_j + w(x_0, \alpha). \tag{6.28}$$

We now make the additional assumption that the eigenvectors $e_1, e_2, \cdots, e_n$ of A are linearly independent. Let κ be the norm of the inverse of the matrix whose rows are $e_1, e_2, \cdots, e_n$. By (6.27) one can find a positive number γ such that the vectors $e_j e^{\lambda_j x_0} + z_j(x_0, g)$ $(j = 1, 2, \cdots, n)$ are linearly independent and such that the inverse of the matrix $E(x_0, g)$ formed with these vectors as rows has a norm satisfying the inequality

$$\| E(x_0, g)^{-1} \| \leqq 2\kappa \tag{6.29}$$

provided

$$\textstyle\sum_{k\geqq 1} \| g_k \| \leqq \gamma. \tag{6.30}$$

Let now the initial value $u(x_0, \alpha)$ of the solution $u(x, \alpha)$ be given, say

$$u(x_0, \alpha) - u^*(x_0) = y_0. \tag{6.31}$$

Then (6.28) becomes an equation for the vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$:

$$E(x_0, g)\alpha + w(x_0, \alpha) = y_0. \tag{6.32}$$

Since the Jacobian of this equation with respect to $\alpha_1, \alpha_2, \cdots, \alpha_n$ at $\alpha = 0$ is $E(x_0, g)$, which by (6.29) is bounded away from zero, it follows that there exists a positive number η such that (6.32) has a unique solution α if

$$\| y_0 \| \leqq \eta. \tag{6.33}$$

Moreover $\| \alpha \| \to 0$ as $\| y_0 \| \to 0$, and $\| \alpha \| \to 0$ implies, in its turn,

$$\sup_{0\leqq x<\infty} \| u(x, \alpha) - u^*(x) \| \to 0, \tag{6.34}$$

as can be seen from (6.23). Thus, the solution $u^*(x)$ is stable. The results are summarized in the following theorem.

THEOREM 6.1. *Let the equation*

$$y' = f(y) + \textstyle\sum_{k\geqq 1} g_k e^{\rho_k x}, \qquad x \geqq 0,$$

take the form

$$u' = Au + h(u) + \textstyle\sum_{k\geqq 1} g_k e^{\rho_k x}$$

on substitution of $y = a + u$. *Let* $h(u)$ *be as in Theorem* 2.1, *and assume that the numbers* ρ_k *and the eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *of the matrix* A *have non-positive real parts. Also assume that there are* n *linearly independent eigenvec-*

tors of A and that condition (6.9) *is satisfied. Then there is a solution y^* admitting an expansion*

$$y^* = a + \sum_{r \geqq 1} a_r^* e^{\mu_r x}$$

which converges absolutely and uniformly for $0 \leqq x < \infty$ provided the condition

$$\sum_{k \geqq 1} \| g_k \| \leqq \gamma$$

is satisfied, where γ is a positive number depending only on the function f. The numbers μ_r are linear combinations of the ρ_k with nonnegative real coefficients. If γ is chosen sufficiently small, then there exists a positive number η such that for any x_0, $x_0 \geqq 0$ and arbitrary vector y_0 with

$$\| y_0 - y^*(x_0) \| \leqq \eta$$

there exists a solution $y = y(x; x_0, y_0)$ satisfying the initial condition

$$y(x_0 ; x_0, y_0) = y_0$$

and admitting an expansion

$$y(x; x_0, y_0) = a + \sum_{r \geqq 1} a_r e^{\nu_r x}$$

which converges absolutely and uniformly for $0 \leqq x < \infty$. The numbers ν_r are linear combinations of the λ_j and ρ_k with nonnegative real coefficients. For $y_0 = y^*(x_0)$ the solution $y(x; x_0, y_0)$ reduces to $y^*(x)$. Furthermore, the solution $y^*(x)$ is stable, and asymptotically stable if Re $\lambda_j < 0$ $(j = 1, 2, \cdots, n)$.

Proof of the asymptotic stability of $y^*(x)$. Let $\mathfrak{N}^*$ be the set of sequences N_r with the property that the first n components of N_r are zero. Then it follows from equations (6.22) that

$$u^*(x) = \sum_{r \in \mathfrak{N}^*} a_r e^{N_r \cdot \mathfrak{L} x}. \tag{6.35}$$

Thus,

$$u(x, \alpha) - u^*(x) = \sum_{r \notin \mathfrak{N}^*} a_r e^{N_r \cdot \mathfrak{L} x}. \tag{6.36}$$

If Re $\lambda_j < 0$ $(j = 1, 2, \cdots, n)$, then Re $(N_r \cdot \mathfrak{L}) < 0$ for $r \notin \mathfrak{N}$. Therefore

$$\lim_{x \to \infty} \| u(x, \alpha) - u^*(x) \| = 0. \tag{6.37}$$

It should be noted that if system (6.1) is real, that is, a and $f(y)$ are real, and if for each index k there exists an integer k' such that $g_{k'} = \bar{g}_k$ and $\rho_{k'} = \bar{\rho}_k$, then all the solutions $y(x; x_0, y_0)$ with real initial values y_0 are real.

A theorem analogous to Theorem 6.1 also holds for the general system $y' = g(x, y)$. The details are easily supplied by following the pattern of Section 4.

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