Boundedness for fractional Hardy-type operator on variable-exponent Herz–Morrey spaces

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Abstract In this article, the fractional Hardy-type operator of variable order $\beta(x)$ is shown to be bounded from the variable-exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$ into the weighted space $M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n,\omega)$, where $\alpha(x)\in L^\infty(\mathbb{R}^n)$ is log-Hölder continuous both at the origin and at infinity, $\omega=(1+|x|)^{-\gamma(x)}$ with some $\gamma(x)>0$, and $1/q_1(x)-1/q_2(x)=\beta(x)/n$ when $q_1(x)$ is not necessarily constant at infinity.

1. Introduction

Let f be a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hardy operator is defined by

$$\mathscr{H}(f)(x):=\frac{1}{|x|^n}\int_{|t|<|x|}f(t)\,dt,\quad x\in\mathbb{R}^n\setminus\{0\}.$$

In 1995, Christ and Grafakos [3] obtained the result for the boundedness of $\mathscr{H}$ on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$ spaces, and they also found the exact operator norms of $\mathscr{H}$ on this space. In 2007, Fu et al. [12] gave the central bounded mean oscillation (BMO) estimates for commutators of n-dimensional fractional and Hardy operators. Recently, the first author [32], [29], [31], [35], [33], [34], [30] has also considered the boundedness for the Hardy operator and its commutator in (variable-exponent) Herz–Morrey spaces.

Lately, there has been an increase in the number of investigations related to both the theory of the variable-exponent function spaces and the operator theory in these spaces. This is caused by possible applications to models with nonstandard local growth (in elasticity theory, fluid mechanics, differential equations, and image processing; see, e.g., [24], [11], [2], [13], [25] and references therein) and is based on the breakthrough result on the boundedness of the Hardy–Littlewood

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maximal operator in these spaces (for more details see [19], [4]–[6], [9], [7], [10], [18], [20], [22], [23], and others).

We first define the n-dimensional fractional Hardy-type operators with variable order $\beta(x)$ as follows.

DEFINITION 1.1

Let f be a locally integrable function on $\mathbb{R}^n, 0 \le \beta(x) < n$. The n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ are defined by

$$\mathscr{H}_{\beta(\cdot)}(f)(x) := \frac{1}{|x|^{n-\beta(x)}} \int_{|t|<|x|} f(t)\,dt, \tag{1.0a}$$

$$\mathscr{H}^*_{\beta(\cdot)}(f)(x) := \int_{|t|\ge|x|} \frac{f(t)}{|t|^{n-\beta(x)}}\,dt, \tag{1.0b}$$

where $x \in \mathbb{R}^n \setminus \{0\}$.

Obviously, when $\beta(x) = 0$, $\mathscr{H}_{\beta(\cdot)}$ is just $\mathscr{H}$, and we denote by $\mathscr{H}^* := \mathscr{H}^*_{\beta(\cdot)} = \mathscr{H}^*_0$. When $\beta(x)$ is constant, $\mathscr{H}_{\beta(\cdot)}$ and $\mathscr{H}^*_{\beta(\cdot)}$ will become $\mathscr{H}_\beta$ and $\mathscr{H}^*_\beta$, respectively (see [12]).

The Riesz-type potential operator with variable order $\beta(x)$ is defined by

$$I_{\beta(\cdot)}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta(x)}}\,dy, \quad 0 < \beta(x) < n. \tag{1.1}$$

In 2004, Diening [8] proved Sobolev's theorem for the potential I_β on the whole space $\mathbb{R}^n$ assuming that $p(x)$ is constant at infinity ($p(x)$ is always constant outside some large ball) and satisfies the same logarithmic condition as in [26]. Another progress for unbounded domains is the result of Cruz-Uribe et al. [6] on the boundedness of the maximal operator in unbounded domains for exponents $p(x)$ satisfying the logarithmic smoothness condition both locally and at infinity.

Kokilashvili and Samko [17] proved a Sobolev-type theorem for the potential $I_{\beta(\cdot)}$ from the space $L^{p(\cdot)}(\mathbb{R}^n)$ into the weighted space $L^{q(\cdot)}_\omega(\mathbb{R}^n)$ with the power weight ω fixed to infinity, under the logarithmic condition for $p(x)$ satisfied locally and at infinity, not supposing that $p(x)$ is constant at infinity but assuming that $p(x)$ takes its minimal value at infinity.

In addition, the theory of function spaces with variable exponents has rapidly made progress in the past 20 years since some elementary properties were established by Kováčik and Rákosník [19].

In 2012, Almeida and Drihem [1] discussed the boundedness of a wide class of sublinear operators on Herz spaces $K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponents $\alpha(\cdot)$ and $q(\cdot)$. Meanwhile, they also established Hardy–Littlewood–Sobolev theorems for fractional integrals on variable Herz spaces. In 2013, Samko [28], [27] introduced a new Herz-type function space with variable exponent, where all three parameters are variable, and proved the boundedness of some sublinear operators (see also [15]). In 2014, Izuki and Noi [16] examined the duality and reflexivity of Herz spaces $K^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponents. Recently, Wu [30] considered the boundedness for fractional Hardy-type operators on Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ but fixed $\alpha \in \mathbb{R}$ and $p \in (0,\infty)$.

Motivated by the above results, we investigate mapping properties of the fractional Hardy-type operators $\mathscr{H}_{\beta(\cdot)}$ and $\mathscr{H}^*_{\beta(\cdot)}$ within the framework of the variable-exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$.

Throughout this article, we will denote by $|S|$ the Lebesgue measure and by χ_S the characteristic function for a measurable set $S \subset \mathbb{R}^n$. We let $B(x,r)$ be the ball centered at x and of radius r, and we let $B_0 = B(0,1)$. We denote by C a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $1 < q(x) < \infty$, we denote by $q'(x)$ its conjugate index, namely, $q'(x) = \frac{q(x)}{q(x)-1}$. For $A \sim D$, we mean that there is a constant $C > 0$ such that $C^{-1}D \le A \le CD$.

2. Preliminaries

In this section, we give the definition of Lebesgue and Herz–Morrey spaces with variable exponent, and give basic properties and useful lemmas.

2.1. Function spaces with variable exponent

Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first define variable-exponent Lebesgue spaces.

DEFINITION 2.1

Let $q(\cdot) : \Omega \to [1,\infty)$ be a measurable function.

(I) The Lebesgue space with variable exponent $L^{q(\cdot)}(\Omega)$ is defined by

$$L^{q(\cdot)}(\Omega) = \big\{ f \text{ is a measurable function} : F_q(f/\eta) < \infty \text{ for some constant } \eta > 0 \big\},$$

where $F_q(f) := \int_\Omega |f(x)|^{q(x)}\,dx$. The Lebesgue space $L^{q(\cdot)}(\Omega)$ is a Banach function space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf\Big\{ \eta > 0 : F_q(f/\eta) = \int_\Omega \Big(\frac{|f(x)|}{\eta} \Big)^{q(x)} dx \le 1 \Big\}.$$

(II) The space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega) = \big\{ f \text{ is measurable} : f \in L^{q(\cdot)}(\Omega_0) \text{ for all compact subsets } \Omega_0 \subset \Omega \big\}.$$

(III) The weighted Lebesgue space $L^{q(\cdot)}_\omega(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{q(\cdot)}_\omega(\Omega)} = \|\omega f\|_{L^{q(\cdot)}(\Omega)} < \infty.$$

Next we define some classes of variable-exponent functions. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} \left|f(y)\right| dy,$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

DEFINITION 2.2

Given a measurable function $q(\cdot)$ defined on $\mathbb{R}^n$, we write

$$q_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} q(x), \qquad q_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} q(x).$$

(I) $q'_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} q'(x) = \frac{q_+}{q_+-1}, q'_+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} q'(x) = \frac{q_-}{q_--1}$.

(II) Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot) : \mathbb{R}^n \to (1,\infty)$ such that

$$1 < q_- \le q(x) \le q_+ < \infty, \quad x \in \mathbb{R}^n.$$

(III) The set $\mathscr{B}(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying that the Hardy–Littlewood maximal operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

DEFINITION 2.3

Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$.

(I) The set $\mathscr{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ consists of all local log-Hölder continuous functions $\alpha(\cdot)$ satisfying

$$\left|\alpha(x) - \alpha(y)\right| \le \frac{-C}{\ln(|x-y|)}, \quad |x-y| \le 1/2,\ x,y \in \mathbb{R}^n.$$

(II) The set $\mathscr{C}^{\log}_0(\mathbb{R}^n)$ consists of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at the origin

$$\left|\alpha(x) - \alpha(0)\right| \le \frac{C}{\ln(e + \frac{1}{|x|})}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

(III) The set $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ consists of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at infinity

$$\left|\alpha(x) - \alpha_\infty\right| \le \frac{C_\infty}{\ln(e + |x|)}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $\alpha_\infty = \lim_{|x|\to\infty} \alpha(x)$.

(IV) Denote by $\mathscr{C}^{\log}(\mathbb{R}^n) := \mathscr{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions $\alpha(\cdot)$.

REMARK 1

The $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ condition is equivalent to the uniform continuity condition

$$\left|q(x) - q(y)\right| \le \frac{C}{\ln(e + |x|)}, \quad |y| \ge |x|, x, y \in \mathbb{R}^n.$$

The $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ condition was originally defined in this form in [6].

Next we define variable-exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

DEFINITION 2.4

Suppose that $0 \le \lambda < \infty, 0 < p < \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable-exponent Herz–Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}\left(\mathbb{R}^n \backslash \{0\}\right) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f \chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

Compare the variable Herz–Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with the variable Herz space (see [1]) $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$, where

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}\left(\mathbb{R}^n \backslash \{0\}\right) : \sum_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)} f \chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}.$$

Obviously, $M\dot{K}^{\alpha(\cdot),0}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$. When $\alpha(\cdot)$ is constant, we have $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ (see [30]). If both $\alpha(\cdot)$ and $q(\cdot)$ are constant and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$ are classical Herz spaces.

2.2. Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. We only describe the partial results we need.

PROPOSITION 2.1

Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

- (I) *If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then we have $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.*
- (II) *$q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ if and only if $q'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.*

Proposition 2.1(I) is independently due to Cruz-Uribe et al. [6] and Nekvinda [22], respectively. Proposition 2.1(II) belongs to Diening [9] (see also [5, Theorem 1.2]).

REMARK 2

Since

$$\left|q'(x) - q'(y)\right| \le \frac{|q(x) - q(y)|}{(q_- - 1)^2},$$

it follows at once that if $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then so does $q'(\cdot)$—that is, if the condition holds, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$ and $L^{q'(\cdot)}(\mathbb{R}^n)$. Furthermore, Diening has proved general results on Musielak–Orlicz spaces.

The order $\beta(x)$ of the fractional Hardy-type operators in Definition 1.1 is not assumed to be continuous. We assume that it is a measurable function on $\mathbb{R}^n$ satisfying the following assumptions:

$$\begin{aligned} \beta_0 &:= \operatorname{ess\,inf}_{x\in\mathbb{R}^n} \beta(x) > 0, \\ &\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)\beta(x) < n, \\ &\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(\infty)\beta(x) < n. \end{aligned} \tag{2.3}$$

In order to prove our main results, we need the Sobolev-type theorem for the space $\mathbb{R}^n$ which was proved in [17] for the exponents $p(x)$ not necessarily constant in a neighborhood of infinity, but with some extra power weight fixed to infinity and under the assumption that $p(x)$ takes its minimal value at infinity.

PROPOSITION 2.2

Suppose that $p(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$. *Let*

$$1 < p(\infty) \le p(x) \le p_+ < \infty, \tag{2.4}$$

and let $\beta(x)$ *meet condition (2.3). Then the following weighted Sobolev-type estimate is valid for the operator* $I_{\beta(\cdot)}$:

$$\left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n}$$

is the Sobolev exponent and

$$\gamma(x) = C_\infty \beta(x) \Big(1 - \frac{\beta(x)}{n}\Big) \le \frac{n}{4} C_\infty, \tag{2.5}$$

with C_∞ *being the Dini–Lipschitz constant from (2.2) in which* $\alpha(\cdot)$ *is replaced by* $p(\cdot)$.

REMARK 3

(i) If $\beta(x)$ satisfies the condition of type (2.2): $|\beta(x) - \beta_\infty| \le \frac{C_\infty}{\ln(e+|x|)}$ ($x \in \mathbb{R}^n$), then the weight $(1+|x|)^{-\gamma(x)}$ is equivalent to the weight $(1+|x|)^{-\gamma_\infty}$.

(ii) One can also treat operator (1.1) with $\beta(x)$ replaced by $\beta(y)$. In the case of potentials over bounded domains Ω, such potentials differ unessentially if the function $\beta(x)$ satisfies the smoothness logarithmic condition as (2.1), since

$$C_1 |x-y|^{n-\beta(y)} \le |x-y|^{n-\beta(x)} \le C_2 |x-y|^{n-\beta(y)}$$

in this case (see [26, p. 277]).

(iii) When $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, the assumption that $p(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)$ is equivalent to assuming that $1/p(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)$, since

$$\begin{aligned}\Big|\frac{p(x)-p(y)}{(p_+)^2}\Big| \le \Big|\frac{1}{p(x)}-\frac{1}{p(y)}\Big| &= \Big|\frac{p(x)-p(y)}{p(x)p(y)}\Big| \\ &\le \Big|\frac{p(x)-p(y)}{(p_-)^2}\Big|.\end{aligned}$$

Further, $1/p(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)$ implies that $1/q(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)$ as well.

The next proposition is the generalization of variable-exponent Herz spaces in [1], and it was used in [21].

PROPOSITION 2.3
Let $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, $p\in(0,\infty)$, and $\lambda\in[0,\infty)$. If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathscr{C}_0^{\log}(\mathbb{R}^n)\cap\mathscr{C}_\infty^{\log}(\mathbb{R}^n)$, then

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}\|2^{k\alpha(\cdot)}f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p}} \\ &\approx \max\Big\{\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p}}, \\ &\qquad \sup_{\substack{k_0\ge 0\\k_0\in\mathbb{Z}}}\Big(2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p}} \\ &\qquad + 2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p}}\Big)\Big\}.\end{aligned}$$

The next lemma is known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [19].

LEMMA 2.1 (GENERALIZED HÖLDER'S INEQUALITY)
Suppose that $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. Then for any $f\in L^{q(\cdot)}(\mathbb{R}^n)$ and any $g\in L^{q'(\cdot)}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n}\big|f(x)g(x)\big|\,dx\le C_q\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where $C_q=1+1/q_- - 1/q_+$.

The following lemma can be found in [14].

LEMMA 2.2
Let $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$.

(I) *Then there exist positive constants* $\delta \in (0,1)$ *and* $C > 0$ *such that*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta}$$

for all balls B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$.

(II) *Then there exists a positive constant* $C > 0$ *such that*

$$C^{-1} \le \frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C$$

for all balls B *in* $\mathbb{R}^n$.

REMARK 4

(i) If $q_1(\cdot), q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$, then we see that $q_1'(\cdot), q_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Hence, we can take positive constants $0 < \delta_1 < 1/(q_1')_+, 0 < \delta_2 < 1/(q_2)_+$ such that

$$\frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1}, \qquad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2} \tag{2.6}$$

hold for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$ (see [35], [14]).

(ii) On the other hand, Kopaliani [18] has proved the conclusion: if the exponent $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ equals a constant outside some large ball, then $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ if and only if $q(\cdot)$ satisfies the Muckenhoupt-type condition

$$\sup_{Q:\text{cube}} \frac{1}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} < \infty.$$

3. Main results and their proofs

Our main result can be stated as follows (for more details see [30]).

THEOREM 3.1

Suppose that $q_1(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ *satisfies condition (2.4), and* $\beta(x)$ *meets condition (2.3) in which* $p(\cdot)$ *is replaced by* $q_1(\cdot)$. *Define the variable exponent* $q_2(\cdot)$ *by*

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\beta(x)}{n}.$$

Let $0 < p_1 \le p_2 < \infty$, *let* $\lambda \ge 0$, *and let* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *be log-Hölder continuous both at the origin and at infinity, with* $\alpha(0) \le \alpha_\infty < \lambda + n\delta_1$, *where* $\delta_1 \in (0, 1/(q_1')_+)$ *is the constant appearing in (2.6). Then*

$$\big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x)$ *is defined as in (2.5), and* C_∞ *is the Dini–Lipschitz constant from (2.1) with* $q_1(\cdot)$ *instead of* $\alpha(\cdot)$.

Proof

For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$, if we denote $f_j := f \cdot \chi_j = f \cdot \chi_{A_j}$ for each $j \in \mathbb{Z}$, then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By (1.0a) and Lemma 2.1, we have

$$\begin{aligned}
&\big|\mathscr{H}_{\beta(\cdot)}(f)(x) \cdot \chi_k(x)\big| \\
&\quad \le \frac{1}{|x|^{n-\beta(x)}} \int_{B_k} \big|f(t)\big|\, dt \cdot \chi_k(x) \\
&\quad \le C 2^{-kn} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \cdot |x|^{\beta(x)} \chi_k(x).
\end{aligned} \tag{3.1}$$

For Proposition 2.2, we note that

$$\begin{aligned}
I_{\beta(\cdot)}(\chi_{B_k})(x) &\ge I_{\beta(\cdot)}(\chi_{B_k})(x) \cdot \chi_{B_k}(x) \\
&= \int_{B_k} \frac{1}{|x-y|^{n-\beta(x)}}\, dy \cdot \chi_{B_k}(x) \\
&\ge C|x|^{\beta(x)} \cdot \chi_{B_k}(x) \\
&\ge C|x|^{\beta(x)} \cdot \chi_k(x).
\end{aligned} \tag{3.2}$$

Using Proposition 2.2, Lemma 2.2, (2.6), (3.1), and (3.2), we have

$$\begin{aligned}
&\big\|(1+|x|)^{-\gamma(x)} \mathscr{H}_{\beta(\cdot)}(f) \cdot \chi_k(\cdot)\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C 2^{-kn} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times \big\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(\chi_{B_k})\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C 2^{-kn} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\
&\quad \le C \sum_{j=-\infty}^{k} 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.3}$$

Because of $0 < p_1/p_2 \le 1$, applying inequality

$$\Big(\sum_{i=-\infty}^{\infty} |a_i|\Big)^{p_1/p_2} \le \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2}, \tag{3.4}$$

and Proposition 2.3, we then have

$$
\begin{aligned}
&\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\big\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)}\\
&\le \max\Big\{\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big),\\
&\quad \sup_{\substack{k_0\ge 0\\k_0\in\mathbb{Z}}}\Big(2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)\\
&\quad + 2^{-k_0\lambda p_1}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1}\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)\Big)\Big\}\\
&= \max\{E_1, E_2+E_3\},
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big),\\
E_2 &= \sup_{\substack{k_0\ge 0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big),\\
E_3 &= \sup_{\substack{k_0\ge 0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1}\big\|\big(1+|x|\big)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big).
\end{aligned}
$$

To estimate E_1, E_2, and E_3, we need the following fact. By the condition of $\alpha(\cdot)$ and Proposition 2.3, we have the following cases.

Case 1 ($j<0$). We have

$$
\begin{aligned}
\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha(0)}\big(2^{j\alpha(0)p_1}\|f_j\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big)^{1/p_1}\\
&\le 2^{-j\alpha(0)}\Big(\sum_{i=-\infty}^{j} 2^{i\alpha(0)p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\\
&\le 2^{j(\lambda-\alpha(0))}\Big(2^{-j\lambda}\Big(\sum_{i=-\infty}^{j}\|2^{i\alpha(\cdot)}f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\Big)\\
&\le C2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.5}
$$

Case 2 ($j\ge 0$). We have

$$
\begin{aligned}
\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha_\infty}\big(2^{j\alpha_\infty p_1}\|f_j\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big)^{1/p_1}\\
&\le 2^{-j\alpha_\infty}\Big(\sum_{i=0}^{j} 2^{i\alpha_\infty p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}
\end{aligned}
\tag{3.6}
$$

$$\leq 2^{j(\lambda-\alpha_\infty)}\Big(2^{-j\lambda}\Big(\sum_{i=-\infty}^{j}\|2^{i\alpha(\cdot)}f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\Big)^{1/p_1}\Big)$$
$$\leq C2^{j(\lambda-\alpha_\infty)}\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}.$$

For E_1, noting that $j<0$, combining (3.3) and (3.5), and using $\alpha(0)\leq\alpha_\infty<\lambda+n\delta_1$, we have

$$E_1\leq C\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p_1}\Big(\sum_{j=-\infty}^{k}2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\Big)$$
$$\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}$$
$$\times\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p_1}\Big(\sum_{j=-\infty}^{k}2^{(j-k)n\delta_1}2^{j(\lambda-\alpha(0))}\Big)^{p_1}\Big)$$
$$\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\lambda p_1}\Big(\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_1+\lambda-\alpha(0))}\Big)^{p_1}\Big)$$
$$\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\lambda p_1}\Big)\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}.$$

We omit the estimate of E_2 since it is essentially similar to that of E_1.

Now we only simply estimate E_3. Noting that $j\geq 0$, combining (3.3) and (3.6), and using $\alpha(0)\leq\alpha_\infty<\lambda+n\delta_1$, we have

$$E_3\leq C\sup_{\substack{k_0\geq0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=0}^{k_0}2^{k\alpha_\infty p_1}\Big(\sum_{j=-\infty}^{k}2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\Big)$$
$$\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}\sup_{\substack{k_0\geq0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=0}^{k_0}2^{k\lambda p_1}\Big(\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_1+\lambda-\alpha_\infty)}\Big)^{p_1}\Big)$$
$$\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}\sup_{\substack{k_0\geq0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\Big(\sum_{k=0}^{k_0}2^{k\lambda p_1}\Big)\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}.$$

Combining all the estimates for E_i $(i=1,2,3)$ together, we complete the proof of Theorem 3.1. □

THEOREM 3.2

Let $\lambda,p_1,p_2,q_1(\cdot),q_2(\cdot),\beta(x),C_\infty$ be as in Theorem 3.1. Suppose that $\alpha\in L^\infty(\mathbb{R}^n)$ is log-Hölder continuous both at the origin and at infinity, and suppose that $\lambda-n\delta_2<\alpha(0)\leq\alpha_\infty$, where $\delta_2\in(0,1/(q_2)_+)$ is the constant appearing in (2.6). Then

$$\big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}^{*}(f)\big\|_{M\dot{K}_{p_2,q_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}.$$

Proof

This is similar to the proof of Theorem 3.1; therefore, we only give a simple proof.

For simplicity, for any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$, we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\cdot\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By (1.0b) and Lemma 2.1, we have

$$\begin{aligned}
&\left|\left(1+|x|\right)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)(x)\cdot\chi_k(x)\right| \\
&\le C\int_{\mathbb{R}^n\setminus B_k}\left|f(t)\right||x|^{\beta(x)-n}\,dt\cdot\left(1+|x|\right)^{-\gamma(x)}\chi_k(x) \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\left(1+|x|\right)^{-\gamma(x)}|\cdot|^{\beta(x)-n}\chi_j(\cdot)\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\cdot\chi_k(x).
\end{aligned} \tag{3.7}$$

Similar to (3.2), we give

$$I_{\beta(\cdot)}(\chi_{B_j})(x) \ge I_{\beta(\cdot)}(\chi_{B_j})(x)\cdot\chi_{B_j}(x) \ge C|x|^{\beta(x)}\cdot\chi_j(x). \tag{3.8}$$

Applying Proposition 2.2, Lemma 2.2, (2.6), (3.7), and (3.8), we obtain

$$\begin{aligned}
&\left\|\left(1+|x|\right)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\cdot\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\qquad\cdot 2^{-jn}\left\|\left(1+|x|\right)^{-\gamma(x)}I_{\beta(\cdot)}(\chi_{B_j})\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \\
&\le C\sum_{j=k+1}^{\infty}2^{(k-j)n\delta_2}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.9}$$

By (3.4) and Proposition 2.3, we have

$$\left\|\left(1+|x|\right)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\right\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \le \max\{E_1, E_2+E_3\},$$

where

$$E_1 = \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p_1}\left\|\left(1+|x|\right)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\cdot\chi_k\right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right),$$

$$E_2 = \sup_{\substack{k_0\ge 0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)p_1}\left\|\left(1+|x|\right)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\cdot\chi_k\right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right),$$

$$E_3 = \sup_{\substack{k_0\ge 0\\k_0\in\mathbb{Z}}}2^{-k_0\lambda p_1}\left(\sum_{k=0}^{k_0}2^{k\alpha_\infty p_1}\left\|\left(1+|x|\right)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\cdot\chi_k\right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right).$$

For E_1, E_2, and E_3, combining (3.5), (3.6), and (3.9) and using $\lambda - n\delta_2 < \alpha(0) \le \alpha_\infty$, we have

$$E_i \le C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}, \quad i = 1, 2, 3.$$

Combining all the estimates for E_i $(i = 1, 2, 3)$ together, we complete the proof of Theorem 3.2. □

In particular, when $\gamma(x) = 0$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are constant exponents, the main results above are proved by Zhang and Wu [34]. Let $\alpha(\cdot)$ be a constant exponent. Then the above results can be found in [30]. When $\lambda = 0$, our main results are also valid.

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References

[1] A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. **394** (2012), 781–795. MR 2927498. DOI 10.1016/j.jmaa.2012.04.043.

[2] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), 1383–1406. MR 2246061. DOI 10.1137/050624522.

[3] M. Christ and L. Grafakos, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), 1687–1693. MR 1239796. DOI 10.2307/2160978.

[4] D. Cruz-Uribe, L. Diening, and A. Fiorenza, *A new proof of the boundedness of maximal operators on variable Lebesgue spaces*, Boll. Unione Mat. Ital. (9) **2** (2009), 151–173. MR 2493649.

[5] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264. MR 2210118.

[6] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238. MR 1976842.

[7] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), 245–253. MR 2057643. DOI 10.7153/mia-07-27.

[8] ———, *Riesz potential and Sobolev embeddings on generalized Lebesgue spaces and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **268** (2004), 31–43. MR 2054530. DOI 10.1002/mana.200310157.

[9] ———, *Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces*, Bull. Sci. Math. **129** (2005), 657–700. MR 2166733. DOI 10.1016/j.bulsci.2003.10.003.

[10] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura, *Maximal functions in variable exponent spaces: Limiting cases of the exponent*, Ann. Acad. Sci. Fenn. Math. **34** (2009), 503–522. MR 2553809.

[11] L. Diening and M. Růžička, *Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. **563** (2003), 197–220. MR 2009242. DOI 10.1515/crll.2003.081.

[12] Z. Fu, Z. Liu, S. Lu, and H. Wang, *Characterization for commutators of n-dimensional fractional Hardy operators*, Sci. China (Ser. A) **50** (2007), 1418–1426. MR 2390459. DOI 10.1007/s11425-007-0094-4.

[13] P. Harjulehto, P. Hästö, V. Latvala, and O. Toivanen, *Critical variable exponent functionals in image restoration*, Appl. Math. Lett. **26** (2013), 56–60. MR 2971400. DOI 10.1016/j.aml.2012.03.032.

[14] M. Izuki, *Fractional integrals on Herz–Morrey spaces with variable exponent*, Hiroshima Math. J. **40** (2010), 343–355. MR 2766665.

[15] M. Izuki and T. Noi, *Boundedness of some integral operators and commutators on generalized Herz spaces with variable exponents*, OCAMI Preprint Ser. **11** (2011), no. 15.

[16] ———, *Duality of Besov, Triebel–Lizorkin and Herz spaces with variable exponents*, Rend. Circ. Mat. Palermo (2) **63** (2014), 221–245. MR 3232652. DOI 10.1007/s12215-014-0154-x.

[17] V. Kokilashvili and S. Samko, *On Sobolev theorem for Riesz-type potentials in the Lebesgue spaces with variable exponent*, Z. Anal. Anwend. **22** (2003), 899–910. MR 2036935. DOI 10.4171/ZAA/1178.

[18] T. S. Kopaliani, *Infimal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable L^p spaces*, Arch. Math. (Basel) **89** (2007), 185–192. MR 2341730. DOI 10.1007/s00013-007-2035-4.

[19] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618. MR 1134951.

[20] A. K. Lerner, *On some questions related to the maximal operator on variable L^p spaces*, Trans. Amer. Math. Soc. **362**, no. 8 (2010), 4229–4242. MR 2608404. DOI 10.1090/S0002-9947-10-05066-X.

[21] Y. Lu and Y. P. Zhu, *Boundedness of multilinear Calderón–Zygmund singular operators on Morrey–Herz spaces with variable exponents*, Acta Math. Sin. (Engl. Ser.) **30** (2014), 1180–1194. MR 3226766. DOI 10.1007/s10114-014-3410-2.

[22] A. Nekvinda, *Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$*, Math. Inequal. Appl. **7** (2004), 255–265. MR 2057644. DOI 10.7153/mia-07-28.

[23] L. Pick and M. Růžička, *An example of a space $L^{p(\cdot)}$ on which the Hardy–Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), 369–371. MR 1876258. DOI 10.1016/S0723-0869(01)80023-2.

[24] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math. **1748**, Springer, Berlin, 2000. MR 1810360. DOI 10.1007/BFb0104029.

[25] M. Sakharkar and M. Sarde, *Image denoising based on variable exponent using adaptive Perona Malik model*, Int. J. Eng. Sci. Invent. **3** (2014), 34–37.

[26] S. Samko, *Convolution and potential type operators in* $L^{p(x)}(\mathbb{R}^n)$, Integral Transforms Spec. Funct. **7** (1998), 261–284. MR 1775832. DOI 10.1080/10652469808819204.

[27] ———, *Erratum to "Variable exponent Herz spaces"*, Mediterr. J. Math. **10** (2013), 2027–2030. MR 3119349. DOI 10.1007/s00009-013-0335-4.

[28] ———, *Variable exponent Herz spaces*, Mediterr. J. Math. **10** (2013), 2007–2025. MR 3119348. DOI 10.1007/s00009-013-0285-x.

[29] J. L. Wu, *Boundedness of multilinear commutators of fractional Hardy operators*, Acta Math. Sci. Ser. A Chin. Ed. **31** (2011), 1055–1062. MR 2866772.

[30] ———, *Boundedness for fractional Hardy-type operator on Herz–Morrey spaces with variable exponent*, Bull. Korean Math. Soc. **51** (2014), 423–435. MR 3194387. DOI 10.4134/BKMS.2014.51.2.423.

[31] J. L. Wu and Q. G. Liu, *λ-central* BMO *estimates for higher order commutators of Hardy operators*, Commun. Math. Res. **30** (2014), 201–206. MR 3242292.

[32] J. L. Wu and J. M. Wang, *Boundedness of multilinear commutators of fractional Hardy operators*, Appl. Math. J. Chinese Univ. Ser. A **25** (2010), 115–121. MR 2650622.

[33] J. L. Wu and P. Zhang, *Boundedness of multilinear Hardy type operators on the product of Herz–Morrey spaces with variable exponent*, Appl. Math. J. Chinese Univ. Ser. A **28** (2013), 154–164. MR 3242412.

[34] P. Zhang and J. L. Wu, *Boundedness of fractional Hardy type operators on Herz–Morrey spaces with variable exponent*, Math. Pract. Theory **43** (2013), 247–254. MR 3114303.

[35] ———, *Boundedness of commutators of fractional Hardy operators on Herz–Morrey spaces with variable exponent*, Adv. Math (China) **43** (2014), 581–589. MR 3262223.

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