The banks of the cohomology river

David Eisenbud and Frank-Olaf Schreyer

To the memory of Masaki Maruyama

Abstract We give sharp bounds on the vanishing of the cohomology of a tensor product of vector bundles on $\mathbb{P}^n$ in terms of the vanishing of the cohomology of the factors. For this purpose we introduce regularity indices generalizing the Castelnuovo–Mumford regularity.

As an application we give a sufficient condition for a vector bundle to have an unobstructed deformation theory that depends only on the cohomology table of the bundle. We construct complete families of bundles with such cohomology tables.

1. Introduction

If we know which cohomology groups of (all twists of) two vector bundles $\mathcal{F}, \mathcal{G}$ on $\mathbb{P}^n$ are zero and nonzero, what can we say about the cohomology of twists of $\mathcal{F} \otimes \mathcal{G}$? For example, one might naively suppose that if $H^i\mathcal{F}(a) \neq 0$ and $H^j\mathcal{G}(b) \neq 0$, and if $i+j \leq n$, then at least for some sheaves with the given vanishing pattern one might have $H^{i+j}\mathcal{F} \otimes \mathcal{G}(a+b) \neq 0$.

In this paper we will give sharp bounds on which cohomology groups of twists of $\mathcal{F} \otimes \mathcal{G}$ vanish, and we will see that they are much more restrictive than the naive idea above would suggest (see Example 1.3).

We were led to these bounds by a result from the Boij–Söderberg theory of cohomology tables of vector bundles: by [4, Theorems 0.5, 6.2] there is, for any vector bundle $\mathcal{F}$ on projective space, a uniquely defined homogeneous vector bundle (i.e., a direct sum of twists of Schur functors applied to the tangent bundle) that has, in characteristic zero, the same cohomology table as $\mathcal{F}$ up to a rational multiple. The inequalities we prove are sharp for these homogeneous bundles in characteristic zero, in a rather strong sense.

The inequalities we give strengthen those of Sidman [10] and Caviglia [2]. Those authors' work is based on the characterization of regularity in terms of *approximate free resolutions*—that is, free complexes that are resolutions away from some low-dimensional locus. (The idea of using such approximate resolutions seems to go back to the paper of Gruson, Lazarsfeld, and Peskine [9].)

Kyoto Journal of Mathematics, Vol. 53, No. 1 (2013), 131–144
DOI 10.1215/21562261-1966089,
Received October 4, 2011. Revised June 28, 2012. Accepted June 28, 2012.
2010 Mathematics Subject Classification: Primary 14F05; Secondary 13D07.
Eisenbud's work partially supported by National Science Foundation.

The improvement that gives us stronger bounds is the use of (approximate) free monads instead of resolutions. These ideas are described in Section 3.

One of the most interesting tensor products of two bundles is $\mathrm{End}(\mathcal{F}) = \mathcal{F}^* \otimes \mathcal{F}$, and one of its most interesting cohomology groups is

$$H^2(\mathcal{F}^* \otimes \mathcal{F}) = \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}),$$

the obstruction space for deformations of $\mathcal{F}$. The bounds on cohomology of a tensor product allow us to give an interesting sufficient condition under which this obstruction space is zero, so that the local deformation space of $\mathcal{F}$ is smooth. It turns out that in these unobstructed cases we can actually write down complete families of the bundles with the given cohomology that are irreducible smooth rational varieties. This application occupies Section 6.

The bounds will be given in terms of *regularity indices*, which we will now describe.

1.1. Regularity indices

Let $\mathbb{K}$ be a field, and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$. For $k = 0, \dots, n-1$ we define the *kth regularity index* of $\mathcal{F}$ to be

$$\mathrm{reg}^k \mathcal{F} := \inf\{m \mid H^j \mathcal{F}(m-j) = 0 \text{ for all } j > k\}$$

and the kth coregularity index to be

$$\mathrm{coreg}^k \mathcal{F} := \sup\{m \mid H^j \mathcal{F}(m-j) = 0 \text{ for all } j < n-k\}.$$

Thus $\mathrm{reg}^0 \mathcal{F} \le m$ if and only if $\mathcal{F}$ is m-regular in the classical sense of Castelnuovo and Mumford. Note that $\mathrm{reg}^k \mathcal{F}$ is always finite, but $\mathrm{coreg}^k \mathcal{F}$ may be $-\infty$. For any vector bundle $\mathcal{F}$ and any integer m we have $\mathrm{coreg}^m \mathcal{F} = -\mathrm{reg}^m \mathcal{F}^* - 1$, as one sees easily by duality. We note that $\mathrm{reg}^k \mathcal{F}$ is equal to the "cohomology range" $R_{k+1}(\mathcal{F})$ defined in [4, p. 883] and similarly for the coregularity.

The *cohomology table* of $\mathcal{F}$ is the collection of numbers

$$\gamma(\mathcal{F}) = \{h^i(\mathcal{F}(d)) := \dim_{\mathbb{K}} H^i(\mathcal{F}(d))\}.$$

We display it in a table with $h^i(\mathcal{F}(d))$ in the ith row (numbering from the bottom) and the $(i+d)$th column (numbering from left to right), and to simplify the picture we replace the elements that are zero by dots. As explained in our [3], the cohomology table is also the Betti table of the Tate resolution associated to $\mathcal{F}$. In Proposition 3.3 we use this idea to re-prove and generalize an important result about Castelnuovo–Mumford regularity. We show that if $\mathrm{reg}^k \mathcal{F} = m$, then $H^j \mathcal{F}(m'-j) = 0$ for all $j > k$ for every $m' > m$, and similarly for the coregularity.

For example, the cohomology table of the Horrocks–Mumford bundle $\mathcal{F}_{\mathrm{HM}}$ on $\mathbb{P}^4$ is the following:

```
4: 100 35  4  .  .  .  .  .  .  .   .
3:   .  2 10 10  5 .  .  .  .  .   .
2:   .  .  .  .  . 2  .  .  .  .   .
1:   .  .  .  .  . .  5 10 10  2   .
0:   .  .  .  .  . .  .  .  4 35 100
    -5 -4 -3 -2 -1 0  1  2  3  4   5
```

In this display, $\operatorname{reg}^m \mathcal{F}$ is the number of the leftmost column with only dots above row m, and $\operatorname{coreg}^m \mathcal{F}$ is the number of the rightmost column with only dots below row $n-m$. Thus, for example, $\operatorname{reg}^1 \mathcal{F}_{\mathrm{HM}} = 1$ and $\operatorname{coreg}^0 \mathcal{F}_{\mathrm{HM}} = -5$.

1.2. Banks of the cohomology river

We think of the nonzero values of the cohomology of $\mathcal{F}$ as forming the *cohomology river*, and the regularities and coregularities as defining its *banks*. Our first main result describes the banks of the cohomology river of a tensor product.

THEOREM 1.1

If $\mathcal{F}$ and $\mathcal{G}$ are vector bundles on $\mathbb{P}^n$, then

$$\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G}) \leq \min_{k+l=p} (\operatorname{reg}^k \mathcal{F} + \operatorname{reg}^\ell \mathcal{G})$$

and

$$\operatorname{coreg}^p(\mathcal{F} \otimes \mathcal{G}) \geq 1 + \max_{k+l=p} (\operatorname{coreg}^k \mathcal{F} + \operatorname{coreg}^\ell \mathcal{G}).$$

More generally, the inequality for $\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G})$ holds for any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ such that the support of $\operatorname{Tor}_1(\mathcal{F}, \mathcal{G})$ has dimension at most $p+2$, and the inequality for $\operatorname{coreg}^p(\mathcal{F} \otimes \mathcal{G})$ holds for any coherent sheaves $\mathcal{F}, \mathcal{G}$ such that the support of $\operatorname{Tor}_1(\mathcal{F}, \mathcal{G})$ has dimension at most 1.

Our second main result shows that Theorem 1.1 is sharp in a strong sense.

THEOREM 1.2

Given any pair of cohomology tables Φ, Γ of vector bundles on $\mathbb{P}^n$, there exists a pair of homogeneous vector bundles $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^n_{\mathbb{C}}$ whose cohomology tables are rational multiples of Φ and Γ, and such that equality holds for every p in the formulas for $\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G})$ and $\operatorname{coreg}^p(\mathcal{F} \otimes \mathcal{G})$ of Theorem 1.1.

The proofs are given in Section 4.

EXAMPLE 1.3

In fact, the formulas of Theorem 1.1 seem to be sharp rather often. For example, let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the projection defined by the symmetric functions as in [4]. Taking $\mathcal{F} = \pi_* \mathcal{O}(4, 1, -1)$ and $\mathcal{G} = \pi_* \mathcal{O}(3, -1, -2)$ we get bundles with cohomology tables

```
3: 70 24  .  .  .  .  .   .
2:  .  .  8  6 .  .  .    .
1:  .  .  .  . 4  .  .    .
0:  .  .  .  . . 18 56 120
   -4 -3 -2 -1 0  1  2    3
```

and

```
3: 168 84 30  .  .  . .  .
2:   .  .  . 12 12 6  .  .
1:   .  .  .  .  . .  .  .
0:   .  .  .  .  . . 12 42
    -4 -3 -2 -1  0 1  2  3
```

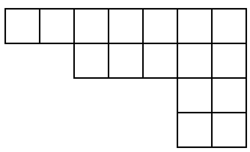
Computing the cohomology table of the tensor product in Macaulay2 (see [8]), we get

```
3: 624 216  72   8  .  .  .   .
2:   .  96 140 144 96 42  .   .
1:   .   .   .   . 18 36 48   .
0:   .   .   .   .  .  . 24 216
    -4  -3  -2  -1  0  1  2   3
```

Inspection shows that equality is achieved here for all the bounds of Theorem 1.1. Notice that we have, for example, $H^1\mathcal{F}(-1) = 4$ and $H^2\mathcal{G}(-1) = 6$ but that no sheaves with these vanishing patterns can have $H^3(\mathcal{F} \otimes \mathcal{G}(-2)) \neq 0$.

2. Boij–Söderberg theory for vector bundles

By a *homogeneous bundle* on $\mathbb{P}^n$ we mean the result of applying a Schur functor S_λ to the universal n-quotient bundle Q and then (possibly) tensoring with a line bundle. Here $\lambda = \lambda_{n-1}, \dots, \lambda_0$ is a partition with n parts; that is, the λ_i are integers such that $\lambda_{n-1} \geq \cdots \geq \lambda_0 \geq 0$. We choose our conventions so that $S_{m,0,\dots,0}Q$ is the mth symmetric power of Q, while $S_{1^m,0,\dots,0}Q$ is the mth exterior power of Q. We draw the Young diagram corresponding to λ by putting λ_i boxes in the ith row and right justifiying the picture; for example, the partition $(7,5,2,2,0,0)$ corresponds to the diagram

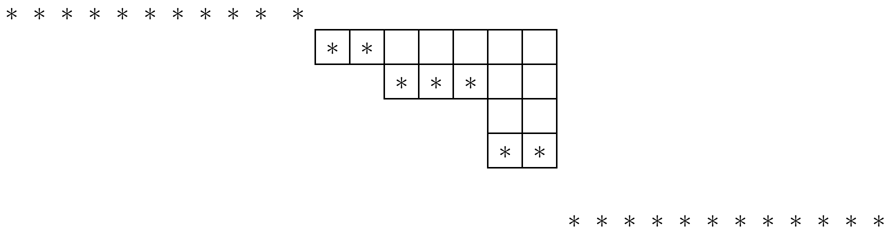


(where rows 0 and 1 have zero boxes!).

As Jerzy Weyman pointed out to us (see [6]), the vanishing part of Bott's theorem about homogeneous bundles in characteristic zero has a very simple statement in terms of cohomology tables.

THEOREM 2.1 (BOTT)

Let $\lambda_{n-1}, \dots, \lambda_0$ be a partition as above, and let Q be the universal rank n quotient bundle on $\mathbb{P}^n_{\mathbb{C}}$. The cohomology table of $S_\lambda(Q)$ has the form



where the nonzero entries of the table are exactly those marked by $*$, the top row of the Young diagram is row $n-1$, and the right-hand column of the Young diagram is column -1.

For example, we see from Theorem 2.1 that

$$\operatorname{reg}^k S_\lambda(Q) = -\lambda_k.$$

We partially order partitions componentwise. (In terms of Young diagrams this is the partial order by inclusion.) One of the main results of Boij–Söderberg theory for vector bundles can be thought of as associating to any vector bundle on projective space a homogeneous bundle with (in characteristic zero) the same cohomology table, up to a rational multiple. We restrict ourselves to 0-regular bundles for simplicity; of course we can apply the result to any bundle by first tensoring with a sufficiently positive line bundle. The following statement combines Theorems 0.5 and 6.2 of our paper [4].

THEOREM 2.2

The cohomology table of any bundle $\mathcal{F}$ with $\operatorname{reg}^0 \mathcal{F} \leq 0$ can be written uniquely as a positive rational linear combination of the (characteristic zero) cohomology tables of a sequence of homogeneous bundles corresponding to a chain (i.e., a totally ordered set in the componentwise order) of Young diagrams.

REMARK

One can use the Boij–Söderberg decomposition to bound the numbers in the cohomology table of the tensor product using just the knowledge of which entries of the cohomology table are zero and the Hilbert polynomial. But one might hope for a still stronger principle, asserting perhaps that if the cohomology tables Φ and Γ of bundles $\mathcal{F}$ and $\mathcal{G}$ have Boij–Söderberg decompositions

$$\Phi = \sum \alpha_i \Phi^i, \qquad \Gamma = \sum \beta_j \Gamma^j,$$

where Φ^i and Γ^j are the cohomology tables of the homogeneous bundles $S_{\phi^i}Q$ and $S_{\gamma^j}Q$, then each entry of the cohomology table of $\mathcal{F} \otimes \mathcal{G}$ would be bounded above by the sum over i, j of $\alpha_i \beta_j$ times the corresponding entry of the cohomology table of $S_{\phi^i}Q \otimes S_{\gamma^j}Q$. This is false, as Example 1.3 shows.

3. Linear monads and regularity indices

It is well known that the (Castelnuovo–Mumford) regularity $\operatorname{reg}^0 \mathcal{F}$ of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ can be characterized as the smallest integer m such that $\mathcal{F}(m)$ admits a *linear free resolution*, that is, such that there is a complex

$$\mathcal{M} : \cdots \to \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\beta_{t+1}} \to \mathcal{O}_{\mathbb{P}^n}(-t)^{\beta_t} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \to 0$$

with homology $\mathrm{H}^0(\mathcal{M}) = \mathcal{F}$ and no other homology. Our next result is a characterization of this sort for all the regularity and coregularity indices.

Recall that a *monad* $\mathcal{M}$ for a sheaf $\mathcal{F}$ is a finite complex of sheaves

$$\cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M}^1 \to \cdots$$

whose only homology is $\mathrm{H}^*(\mathcal{M}) = \mathrm{H}^0(\mathcal{M}) \cong \mathcal{F}$. The monad is called *linear* if $\mathcal{M}^i$ is a direct sum of copies of $\mathcal{O}(i)$ for each i.

PROPOSITION 3.1
If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then $\operatorname{reg}^k \mathcal{F}$ is the smallest integer m such that $\mathcal{F}(m)$ admits a linear monad $\mathcal{M}$ with $\mathcal{M}^\ell = 0$ for all $\ell > k$, and $\operatorname{coreg}^k \mathcal{F}$ is the supremum of the integers m such that $\mathcal{F}(m+1)$ admits a linear monad $\mathcal{M}$ with $\mathcal{M}^\ell = 0$ for all $\ell < -k$.

Note that for a coherent sheaf $\mathcal{F}$ we may have $\operatorname{coreg}^k \mathcal{F} = -\infty$; in this case no twist $\mathcal{F}(m)$ admits a linear monad with $\mathcal{M}^\ell = 0$ for all $\ell < -k$.

Proof
We prove the statement about reg^k, the case of coreg^k being similar.

Twisting by $-m$, the first statement will follow if we show that a coherent sheaf $\mathcal{F}$ admits a linear monad $\mathcal{M}$ with $\mathcal{M}^\ell = 0$ for all $\ell > k$ if and only if $\operatorname{reg}^k \mathcal{F} \leq 0$. The "only if" part follows from a standard argument in homological algebra. Here is a general version whose strength we will use later.

LEMMA 3.2
Let

$$\mathcal{M} : \cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M}^1 \to \cdots$$

be a complex of sheaves, and let $\mathcal{F}^i := \mathrm{H}^i \mathcal{M}$ be the homology of $\mathcal{M}$ at the ith term. If

$$H^{j-t}(\mathcal{M}^t) = 0 \quad \text{for all } t, \tag{1}$$

$$H^{j-t-1}(\mathcal{F}^t) = 0 \quad \text{for all } t > 0, \tag{2}$$

$$H^{j-t+1}(\mathcal{F}^t) = 0 \quad \text{for all } t < 0, \tag{3}$$

then $H^j(\mathcal{F}^0) = 0$.

Proof
Break $\mathcal{M}$ into the short exact sequences

$$0 \to \mathcal{Z}^i \to \mathcal{M}^i \to \mathcal{B}^{i+1} \to 0,$$
$$0 \to \mathcal{B}^i \to \mathcal{Z}^i \to \mathcal{F}^i \to 0$$

and chase the corresponding long exact sequences in cohomology. □

To complete the proof of Proposition 3.1 we must show that if $\operatorname{reg}^k \mathcal{F} \leq 0$, then $\mathcal{F}$ admits a linear monad with $\mathcal{M}^\ell = 0$ for all $\ell > k$. The object we need is the one mentioned in [3, Example 8.5] and is constructed using the Beilinson–Gel'fand–Gel'fand correspondence (BGG). Since the property we need was not spelled out there, we review the construction and add some details.

Set $W = H^0\mathcal{O}_{\mathbb{P}^n}(1)$, and let E be the exterior algebra $E = \Lambda W^*$. The cohomology table of a coherent sheaf $\mathcal{F}$, as we have presented it, is also the Betti table of the *Tate resolution* $\mathbb{T}(\mathcal{F})$, which is a minimal graded free exact complex over E. The terms of $\mathbb{T}(\mathcal{F})$ are

$$\mathbb{T}^e(\mathcal{F}) = \bigoplus_{i=0}^{n} \operatorname{Hom}_{\mathbb{K}}\big(E, H^i\mathcal{F}(e-i)\big)$$

where $H^i\mathcal{F}(e-i)$ is considered as a vector space concentrated in degree $e-i$. We consider the elements of W^* as having degree -1, so the E-module $\omega_E = \operatorname{Hom}_{\mathbb{K}}(E, K)$ is nonzero in degrees $n+1, n, \ldots, 0$, and $\operatorname{Hom}_{\mathbb{K}}(E, H^i\mathcal{F}(e-i))$ can be nonzero only in degrees $e-i+n+1, \ldots, e-i$.

Now suppose that $\operatorname{reg}^k \mathcal{F} \leq 0$; this means that $H^j\mathcal{F}(-j) = 0$ for $j > k$. Thus $\mathbb{T}^0\mathcal{F}$ is generated in degrees $\geq -k+n+1$, and it follows that the graded components of $\mathbb{T}^0\mathcal{F}$ are all zero below degree $-k$. This implies the same vanishing for the E-submodule $P = \ker(\mathbb{T}^0\mathcal{F} \to \mathbb{T}^1\mathcal{F})$.

To the E-module P the BGG correspondence associates a linear free complex $L(P)$ over S:

$$L(P) : \cdots \to S \otimes P_1 \xrightarrow{\partial} S \otimes P_0 \xrightarrow{\partial} S \otimes P_{-1} \longrightarrow \cdots.$$

The differential ∂ is defined to be multiplication by the element

$$\sum_{i=0}^{n} x_i \otimes e_i \in S \otimes E,$$

where $\{x_i\}$ and $\{e_i\}$ are dual bases of W and W^*. Since P_j is concentrated in degree j, the module $S \otimes P_j$ is a direct sum of copies of $S(-j)$.

It follows from BGG (see [3, Theorem 8.1]) with $\mathbb{T}' = \mathbb{T}(\mathcal{F})^{\geq 0}$ that the sheafification $\mathcal{M} := \widetilde{L(P)}$ of $L(P)$ is a monad for $\mathcal{F}$. The term $\mathcal{M}^\ell$ is equal to $\mathcal{O}_{\mathbb{P}^n}(\ell)^{\dim P_{-\ell}}$. The observation above that $P_j = 0$ for $j < -k$ implies that $\mathcal{M}^\ell = 0$ for $\ell > k$, as required. □

The correspondence between the Tate resolution and the cohomology table allows us to generalize an important fact about Castelnuovo–Mumford regularity.

PROPOSITION 3.3
If $\operatorname{reg}^k \mathcal{F} = m$, *then* $H^j\mathcal{F}(m' - j) = 0$ *for all* $j > k$ and $m' \geq m$. *Similarly, if* $\operatorname{coreg}^\ell \mathcal{F} = m$, *then* $H^j\mathcal{F}(m - j) = 0$ *for all* $j < n - k$ and $m' \leq m$.

Proof
The given conditions with $m' = m$ are simply the definitions of reg^k and coreg^k. If $H^j\mathcal{F}(m' - j) \neq 0$ for some $m' > m$ and $j > k$, then, because the Tate resolution is a minimal complex, no term of the resolution could map into the summand $H := \operatorname{Hom}_{\mathbb{K}}(E, H^j\mathcal{F}(m' - j))$, and it follows that this module would be a submodule of one of the syzygies in the resolution. Since H is an injective module over the exterior algebra, it would actually be a summand. However, H is also a free module over the exterior algebra, so this would contradict the minimality of the Tate resolution.

Since the dual of the Tate resolution is again exact and minimal, we can apply the same argument to the dual to get the corresponding statement about coregularity. □

4. Proof of Theorem 1.1

We begin by proving the regularity statement. Thus we suppose that $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on $\mathbb{P}^n$ with $\dim \operatorname{Tor}_1(\mathcal{F}, \mathcal{G}) \leq p + 2$. It suffices to show that if $p = k + \ell$, then $\operatorname{reg}^p(\mathcal{F} \otimes \mathcal{G}) \leq \operatorname{reg}^k \mathcal{F} + \operatorname{reg}^\ell \mathcal{G}$. Replacing $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}(-k)$ and $\mathcal{G}(-\ell)$, respectively, we may assume $\operatorname{reg}^k \mathcal{F} = \operatorname{reg}^\ell \mathcal{G} = 0$, and we must show that for each $j > p$ we have $H^j(\mathcal{F} \otimes \mathcal{G}(-j)) = 0$.

By Proposition 3.1, the sheaf $\mathcal{F}$ has a linear monad of the form

$$\mathcal{M} : \cdots \to \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M}^1 \to \cdots \to \mathcal{M}^k \to 0$$

where $\mathcal{M}^t$ is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^n}(t)$. Since the truncated complex

$$\mathcal{M}^+ : \mathcal{M}^0 \to \cdots \to \mathcal{M}^k \to 0$$

is locally split, we have

$$\ker(\mathcal{M}^0 \to \mathcal{M}^1) \otimes \mathcal{G} = \ker(\mathcal{M}^0 \otimes \mathcal{G} \to \mathcal{M}^1 \otimes \mathcal{G}),$$

and it follows that $\mathrm{H}^0(\mathcal{M} \otimes \mathcal{G}(-j)) = \mathcal{F} \otimes \mathcal{G}(-j)$.

We now apply Lemma 3.2 to the complex $\mathcal{M} \otimes \mathcal{G}(-j)$. Since the term $\mathcal{M}^t \otimes \mathcal{G}(-j)$ is a direct sum of copies of $\mathcal{G}(t - j)$, it suffices to show

$$H^{j-t}\mathcal{G}(t - j) = 0 \quad \text{for all } t \leq k, \tag{4}$$

$$H^{j-t-1}\big(\mathrm{H}^t(\mathcal{M} \otimes \mathcal{G}(-j))\big) = 0 \quad \text{for all } t > 0, \tag{5}$$

$$H^{j-t+1}\big(\mathrm{H}^t(\mathcal{M} \otimes \mathcal{G}(-j))\big) = 0 \quad \text{for all } t < 0. \tag{6}$$

Since $j > p = k + \ell$ and $-t \geq -k$ we have $j - t > \ell$, and (4) holds because $\operatorname{reg}^\ell \mathcal{G} = 0$.

To prove (5) we observe that $\mathrm{H}^t\mathcal{M} \otimes \mathcal{G}(-j) = 0$ for all $t > 0$ simply because $\mathcal{M}^+$ is locally split. It remains to prove (6). But for $t < 0$ and $j > p$ the number

$j-t+1 \geq p+3$, so it is enough to show that $\dim \mathrm{H}^t(\mathcal{M} \otimes \mathcal{G}(-j)) = \dim \mathrm{H}^t(\mathcal{M} \otimes \mathcal{G}) \leq p+2$.

The local splitting of $\mathcal{M}^+$ further implies that $\mathcal{Z}^0 := \ker(\mathcal{M}^0 \to \mathcal{M}^1)$ is a vector bundle, so

$$\mathcal{M}^- : \cdots \to \mathcal{M}^{-2} \to \mathcal{M}^{-1} \to \mathcal{Z}^0$$

is a locally free resolution of $\mathcal{F}$. Thus for $t < 0$,

$$\mathrm{H}^t(\mathcal{M} \otimes \mathcal{G}) = \mathrm{Tor}_{-t}(\mathcal{F}, \mathcal{G}).$$

By the rigidity of Tor (see [1]), our hypothesis that $\dim \mathrm{Tor}_1(\mathcal{F}, \mathcal{G}) \leq p+2$ implies $\dim \mathrm{Tor}_{-t}(\mathcal{F}, \mathcal{G}) \leq p+2$ for all $t < 0$, completing the proof of the bound for reg^k.

To prove the given bound on coregularity, we may assume that $\mathrm{coreg}^k \mathcal{F} = \mathrm{coreg}^\ell \mathcal{G} = 0$. Imitating the proof above, we work with a linear monad for $\mathcal{F}(1)$, and we need formula (6) for the case $j = 0$; that is, $H^u(\mathrm{Tor}_v(\mathcal{M} \otimes \mathcal{G})) = 0$ for $u \geq 2$ and $v \geq 1$. By the same argument as before, this will be true as long as $\dim \mathrm{Tor}_1(\mathcal{M} \otimes \mathcal{G}) \leq 1$, completing the proof.

5. Proof of Theorem 1.2

The statement for the coregularity follows from that for the regularity by duality, so we restrict ourselves to the regularity formulas.

We may shift $\mathcal{F}$ and $\mathcal{G}$ and assume without loss of generality that $\mathrm{reg}^0 \mathcal{F} = \mathrm{reg}^0 \mathcal{G} = 0$. By Theorem 2.2 we can write the cohomology table Φ of $\mathcal{F}$ as a sum of cohomology tables of homogeneous bundles in characteristic zero. Since reg^0 of a direct sum is the maximum of reg^0 of the summands, these homogeneous bundles must in fact have $\mathrm{reg}^0 \leq 0$; that is, they all have the form $S_\lambda Q$ for some partitions λ. Since $\mathrm{reg}^0 \mathcal{F} = 0$ we have at least one partition λ with with $\lambda_0 = 0$ occuring. Of course, similar statements hold for the cohomology table Γ of $\mathcal{G}$.

Multiplying Φ and Γ by sufficiently divisible integers, we may assume that the Boij–Söderberg decompositions have positive integral—not just rational—coefficients, so that they correspond to actual homogeneous bundles.

We will complete the proof of Theorem 1.2 by showing that if $\mathcal{F}$ is a direct sum of homogeneous bundles

$$\mathcal{F} = \bigoplus_{u=0}^{v} S_{\lambda^u} Q, \quad \text{with } \lambda^0 \leq \cdots \leq \lambda^v$$

and similarly for $\mathcal{G}$, then

$$(*) \quad \mathrm{reg}^p(\mathcal{F} \otimes \mathcal{G}) = \min_{k+l=p} (\mathrm{reg}^k \mathcal{F} + \mathrm{reg}^\ell \mathcal{G})$$

for every $0 \leq p \leq n-1$. Since the inequality $\leq$ is part of Theorem 1.1, it suffices to show that the left-hand side of $(*)$ is at least as large as the right-hand side.

Since the k-regularity index of $S_\lambda Q$ is $-\lambda_k$, the minimum on the right-hand side of $(*)$ is achieved by the minimal partition involved in the decomposition. On the other hand, the pth regularity index of a direct sum is the maximum of the pth regularity indices of the components, so it suffices to prove the inequality after replacing each of $\mathcal{F}$ and $\mathcal{G}$ by a single summand, corresponding to the minimal partitions in the two decompositions; that is, we may take $\mathcal{F} = S_\lambda Q$ and $\mathcal{G} = S_\mu Q$ for some partitions λ and μ.

We now have

$$\mathcal{F} \otimes \mathcal{G} = S_\lambda Q \otimes S_\mu Q = \bigoplus_u S_{\nu^u} Q$$

where the set of partitions ν^u (which may occur with multiplicity) is determined by the Littlewood–Richardson formula. Since the regularity indices are the negatives of the parts of the partition, this translates into the following result in representation theory.

PROPOSITION 5.1

Let V be an n-dimensional vector space over a field of characteristic zero, and let $0 \le p \le n-1$. There is a representation $S_\nu V$ appearing in $S_\lambda V \otimes S_\mu V$, such that $\nu_p \le \max_{k+l=p} \lambda_k + \mu_l$.

Proof

Let

$$\lambda' = \lambda_p, \lambda_{p-1}, \dots, \lambda_0,$$
$$\mu' = \mu_p, \mu_{p-1}, \dots, \mu_0$$

be the partitions obtained by truncating λ and μ. One sees from the Littlewood–Richardson formula as described, for example, in Fulton [7], that if a representation corresponding to the partition ν' occurs in $S_{\lambda'} V \otimes S_{\mu'} V$, then the representation corresponding to the partition

$$\nu = (\lambda_{n-1} + \mu_{n-1}, \dots, \lambda_{p+1} + \mu_{p+1},\ \nu'_p, \dots, \nu'_0)$$

occurs in $S_\lambda V \otimes S_\mu V$. Thus we may assume from the outset that $p = n-1$. If we set $g := \max_{k+l=n-1} \lambda_k + \mu_l$, then the termwise sum of λ with the sequence $(\mu_0, \dots, \mu_{n-1})$, which is the reverse of μ, is a sequence of numbers $\le g$. We want to show that in the product $S_\lambda V \otimes S_\mu V$ there occurs a representation $S_\nu V$ such that $\nu_{n-1} \le g$.

What we wish to prove can now be expressed as a statement about the intersection ring of the Grassmannian $\mathrm{Gr}(n, n+g)$ of n-planes in $\mathbb{C}^{n+g}$ as follows. Let

$$\mathcal{V} = (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n+g} = \mathbb{C}^{n+g})$$

be a complete flag in $\mathbb{C}^{n+g}$. We write $\Sigma_\lambda(\mathcal{V})$ for the Schubert cycle in $\mathrm{Gr}(n, n+g)$ defined by

$$\Sigma_\lambda(\mathcal{V}) = \left\{ W \in \mathrm{Gr}(n, n+g) \;\middle|\; \dim W \cap V_{g+i-\lambda_{n-i}} \ge i \text{ for } 1 \le i \le n-1 \right\},$$

and similarly for $\Sigma_\mu(\mathcal{V})$. The product of the classes $[\Sigma_\lambda(\mathcal{V})]$ and $[\Sigma_\mu(\mathcal{V})]$ in the intersection ring of $\mathrm{Gr}(n, n+g)$ is the sum (with multiplicity) of the classes of those Σ_ν such that $S_\nu V$ occurs in $S_\lambda V \otimes S_\mu V$ *and* $\nu_{n-1} \le g$. (This is explained, and the proof sketched, in [7, Section 9].) Thus our problem is to show that the intersection product $[\Sigma_\lambda(\mathcal{V})][\Sigma_\mu(\mathcal{V})]$ is nonzero. This well-known fact can be proved as follows.

Choose another flag

$$\mathcal{V}' = V_0' \subsetneq \cdots \subsetneq V_{n+g}'$$

in general position with respect to $\mathcal{V}$. With the evident definition of $\Sigma_\mu(\mathcal{V}')$, the product above can be computed as the class of the set-theoretic intersection

$$\Sigma_\lambda(\mathcal{V}) \cap \Sigma_\mu(\mathcal{V}').$$

(This follows, e.g., from Kleiman's transversality theorem.) Thus it suffices to show that this intersection is nonempty.

Since $\mathcal{V}$ and $\mathcal{V}'$ are generic, the subspaces $V_i \cap V_{n+g-i+1}'$ are all 1-dimensional. If e_i is a basis vector for this space, then the conditions $\lambda_i + \mu_{n-1-i} \leq g$ for $0 \leq i \leq n-1$ imply that

$$W \in \Sigma_\lambda(\mathcal{V}) \cap \Sigma_\mu(\mathcal{V}'),$$

where

$$W = \overline{e_{g+1-\lambda_{n-1}}, \ldots, e_{g+n-\lambda_0}}.$$

so the product of the classes of these Schubert cycles is nonzero, as required. □

6. Unobstructed families of vector bundles

Theorem 1.1 gives a criterion for the vanishing of the obstruction space $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$. In this section we describe the deformations of these unobstructed bundles. First we have the criterion.

COROLLARY 6.1

If $\mathcal{F}$ is a vector bundle on $\mathbb{P}^n$ with either $\mathrm{reg}^0 \mathcal{F} - \mathrm{coreg}^1 \mathcal{F} \leq 3$ or $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$, then the obstruction space $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ vanishes.

Proof

Since $\mathrm{reg}^k \mathcal{F}^* = -\mathrm{coreg}^k \mathcal{F} - 1$, the assumption gives

$$\mathrm{reg}^1(\mathcal{F} \otimes \mathcal{F}^*) \leq \min(\mathrm{reg}^0 \mathcal{F} + \mathrm{reg}^1 \mathcal{F}^*, \mathrm{reg}^1 \mathcal{F} + \mathrm{reg}^0 \mathcal{F}^*) \leq 3 - 1 = 2$$

by Theorem 1.1. Hence $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = H^2(\mathcal{F} \otimes \mathcal{F}^*) = 0$. □

Since replacing $\mathcal{F}$ by $\mathcal{F}^*$ interchanges the two assumptions, we will focus on the case $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$ in the following. To describe all bundles satisfying this assumption we will use Beilinson monad (see [3, Theorem 6.1]). Given a sheaf $\mathcal{F}$ on projective space, the Beilinson monad

$$\mathcal{B} : \cdots \to \mathcal{B}^{-1} \to \mathcal{B}^0 \to \mathcal{B}^1 \to \cdots$$

for $\mathcal{F}$ has terms

$$\mathcal{B}^e = \bigoplus_j H^j\big(\mathcal{F}(e-j)\big) \otimes \Omega^{j-e}(j-e).$$

$\mathcal{B}$ is obtained by applying the functor Ω to the Tate resolution $\mathbb{T}(\mathcal{F})$, where Ω is the additive functor that sends the E-module $\omega_E(i) = \mathrm{Hom}_K(E, K(i))$ to the sheaf of twisted i-forms $\Omega^i(i)$. The identification

$$\mathrm{Hom}\big(\Omega^i(i), \Omega^j(j)\big) = \Lambda^{i-j} W^* = \mathrm{Hom}_E\big(\omega_E(i), \omega_E(j)\big)$$

provides the maps.

THEOREM 6.2

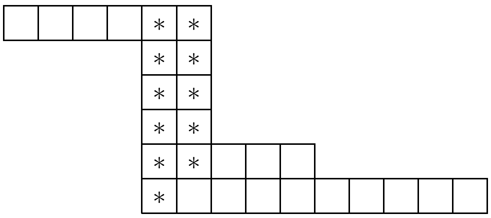
Let $\mathcal{F}$ be a vector bundle with $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$ *twisted such that* $\mathrm{reg}^1 \mathcal{F} = 2$. *Consider $A = \mathbb{T}^0\mathcal{F}$ and $B = \mathbb{T}^1\mathcal{F}$. There is nonempty Zariski open subset $U \subset$* $\mathrm{Hom}_E(A,B)$ *such that the kernel*

$$\mathcal{F}_\varphi = \ker\big(\Omega(\varphi) : \Omega A \to \Omega B\big)$$

is a vector bundle with the same Chern classes and rank as $\mathcal{F}$.

Proof

By assumption the cohomology of $\mathcal{F}$ is nonzero in the range indicated by the boxes in the following picture:



The Beilinson monad of $\mathcal{F}$ depends only the terms in the range indicated by a $*$, since all other terms are zero or are sent to zero. In particular, $\Omega(\mathbb{T}(\mathcal{F}))$ is a two-term complex

$$0 \to \Omega A \to \Omega B \to 0.$$

Since it is an open condition for φ to define a monad $0 \to \Omega A \to \Omega B \to 0$ and the set of φ is nonempty by the existence of $\mathcal{F}$, the result follows. □

Recall from [4] that a vector bundle $\mathcal{F}$ on $\mathbb{P}^n$ has *natural cohomology* in the sense of Hartshorne and Hirschowitz if, for each $d \in \mathbb{Z}$, at most one of the groups $H^i\mathcal{F}(d)$ is nonzero. The bundles $\mathcal{F}$ is called *supernatural* if, in addition, the polynomial function $\chi(\mathcal{F}(d))$ has n distinct integral roots.

COROLLARY 6.3

Let $\mathcal{F}$ be a vector bundle with $\mathrm{reg}^1 \mathcal{F} - \mathrm{coreg}^0 \mathcal{F} \leq 3$ *normalized (by tensoring with a line bundle) so that* $\mathrm{reg}^1 \mathcal{F} = 2$*, and assume that $\mathcal{F}$ has natural cohomology.*

Every vector bundle with natural cohomology with the same rank, Chern classes, regularity, and coregularity indices arises as

$$\mathcal{F}_\varphi = \ker\big(\Omega(\varphi) : \Omega A \to \Omega B\big)$$

for some $\varphi \in U$. In particular, these vector bundles form an irreducible unirational family.

Proof
The cohomology table of any of these bundles is determined by the Hilbert polynomial $d \mapsto \chi(\mathcal{F}(d))$, since for each twist at most two terms could be nonzero due to the narrow cohomology river, and because we have natural cohomology. So they all have the same cohomology table, and all arise from some $\varphi \in U$. $\square$

Note that these bundles are not necessarily stable. For example, we could have a direct sums of bundles with different slopes in these families. Thus we do not speak of a moduli space.

Corollary 6.3 does not settle the existence problem for such bundles. However, it provides a computational criterion. A bundle with the desired unobstructed natural cohomology table exists if and only if a general map $\varphi \in U$ yields such a bundle. Boij–Söderberg theory characterizes the cohomology tables that can occur, up to a rational multiple. Given an integral cohomology table γ satisfying the numerical condition of Corollary 6.3 such that some multiple of γ is the cohomology table of a bundle, we conjecture that there is a number $c_0(\gamma)$ such that $c\gamma$ is the cohomology table of a bundle if and only if $c \geq c_0(\gamma)$, as in the following example.

EXAMPLE 6.4
The table γ,

```
4: 56 15 . . .  .
3:  .  . 2 . .  .
2:  .  . . 1 .  .
1:  .  . . . .  .
0:  .  . . . 8 35
   -2 -1 0 1 2  3
```

"looks like" the cohomology table of a rank 4 vector bundle on $\mathbb{P}^4$, but it is not! This is because for any two 2-forms $(\eta_1, \eta_2) \in \bigoplus_1^2 \Lambda^2 W^* \subset E^2$ the kernel of the wedge product $\ker\big(\Lambda^2 W^* \to \bigoplus_1^2 \Lambda^4 W^*\big)$ is nonzero.

Indeed, $\eta_i \in \Lambda^2 V_i$ for some 4-dimensional subspace $V_i \subset W^*$, and the annihilator of η_i has codimension one in $\Lambda^2 V_i$. Thus the intersection $\operatorname{ann}(\eta_i) \cap \Lambda^2(V_1 \cap V_2)$ has codimension at most 1 in the 3-dimensional space $\Lambda^2(V_1 \cap V_2)$, and the intersection $\operatorname{ann}(\eta_1) \cap \operatorname{ann}(\eta_2)$ is at least 1-dimensional.

However, experiments with Macaulay2 (see [8]) convince us that every multiple $c\gamma$ with $c \geq 2$ does occur as the cohomology table of a bundle.

Acknowledgments. This paper reports on work done during a period of "Research in Pairs" at the Mathematische Forschungsinstitute Oberwolfach, August 1–14, 2011. We are grateful to the Institute for providing such a beautiful and peaceful setting, together with wonderful resources for research. We thank the referee, whose suggestions strengthened the result.

References

[1] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631–647. MR 0179211.

[2] G. Caviglia, *Bounds on the Castelnuovo–Mumford regularity of tensor products*, Proc. Amer. Math. Soc. **135** (2007), 1949–1957. MR 2299466. DOI 10.1090/S0002-9939-07-08222-6.

[3] D. Eisenbud, G. Fløystad, and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. **355** (2003), 4397–4426. MR 1990756. DOI 10.1090/S0002-9947-03-03291-4.

[4] D. Eisenbud and F.-O. Schreyer, *Betti numbers of graded modules and cohomology of vector bundles*, J. Amer. Math. Soc. **22** (2009), 859–888. MR 2505303. DOI 10.1090/S0894-0347-08-00620-6.

[5] ———, *Cohomology of coherent sheaves and series of supernatural bundles*, Eur. Math. Soc. (JEMS) **12** (2010), 703–722. MR 2639316. DOI 10.4171/JEMS/212.

[6] D. Eisenbud, F.-O. Schreyer, and J. Weyman, *Resultants and Chow forms via exterior syzygies*, J. Amer. Math. Soc. **16** (2003), 537–579. MR 1969204. DOI 10.1090/S0894-0347-03-00423-5.

[7] W. Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry*, London Math. Soc. Stud. Texts **35**, Cambridge Univ. Press, Cambridge, 1997. MR 1464693.

[8] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry, http://www.math.uiuc.edu/Macaulay2/

[9] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506. MR 0704401. DOI 10.1007/BF01398398.

[10] J. Sidman, *On the Castelnuovo–Mumford regularity of products of ideal sheaves*, Adv. Geom. **2** (2002), 219–229. MR 1924756. DOI 10.1515/advg.2002.010.

Eisenbud: Department of Mathematics, University of California, Berkeley, Berkeley California 94720; eisenbud@math.berkeley.edu

Schreyer: Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, D-66123 Saarbrücken, Germany; schreyer@math.uni-sb.de