On bundles of rank 3 computing Clifford indices

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To the memory of Masaki Maruyama

Abstract Let C be a smooth irreducible projective algebraic curve defined over the complex numbers. The notion of the Clifford index of C was extended a few years ago to semistable bundles of any rank. Recent work has been focused mainly on the rank-2 Clifford index, although interesting results have also been obtained for the case of rank 3. In this paper we extend this work, obtaining improved lower bounds for the rank-3 Clifford index. This allows the first computations of the rank-3 index in nontrivial cases and examples for which the rank-3 index is greater than the rank-2 index.

1. Introduction

Let C be a smooth irreducible projective algebraic curve defined over the complex numbers. The idea of generalizing the classical Clifford index $\mathrm{Cliff}(C)$ to higher-rank vector bundles was proposed some 20 years ago, but formal definitions and the development of a basic theory took place much more recently (see [11]). Since then, there have been major developments, in particular, the construction of curves for which the rank-2 Clifford index $\mathrm{Cliff}_2(C)$ is strictly less than $\mathrm{Cliff}(C)$ (see [7], [8], [13], [15], [16]), thus producing counterexamples to a conjecture of Mercat [18, Introduction]. A good deal is now known about bundles computing $\mathrm{Cliff}_2(C)$ (see [15]).

Examples are also known for $g=9$ and $g\geq 11$ for which the rank-3 Clifford index $\mathrm{Cliff}_3(C)$ is strictly smaller than $\mathrm{Cliff}(C)$ (see [10], [8]) and lower bounds for $\mathrm{Cliff}_3(C)$ had previously been established in [14]. However, with the exception of the case where $\mathrm{Cliff}(C)\leq 2$ (when $\mathrm{Cliff}_3(C)=\mathrm{Cliff}(C)$; see [11, Proposition 3.5]), no actual values of $\mathrm{Cliff}_3(C)$ are known. In the present paper, we improve the lower bounds of [14] in various circumstances. As a result, we are able to compute values of $\mathrm{Cliff}_3(C)$ in some cases and to give examples for which $\mathrm{Cliff}_3(C) > \mathrm{Cliff}_2(C)$, thus answering in the affirmative [10, Question 5.7].

Following definitions and some preliminary results in Section 2, we consider in Section 3 the curves of minimal rank-2 Clifford index constructed in [16]; these are good candidates for having $\mathrm{Cliff}_3(C) > \mathrm{Cliff}_2(C)$, and we prove in particular the following.

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THEOREM 3.9
If $16 \le g \le 24$, then there exists a curve C of genus g such that

$$\mathrm{Cliff}_3(C) > \mathrm{Cliff}_2(C).$$

This could hold also for other values of g (see Theorem 3.7 and Remark 3.10).

In Section 4, we establish the following improved lower bound for $\mathrm{Cliff}_3(C)$ when $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$.

THEOREM 4.1
Let C be a curve of genus $g \ge 7$ such that $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C) \ge 2$. Then

$$\mathrm{Cliff}_3(C) \ge \min\left\{\frac{d_9}{3} - 2, \frac{2\,\mathrm{Cliff}(C) + 2}{3}\right\}.$$

Moreover, if $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$, then any bundle computing $\mathrm{Cliff}_3(C)$ is stable.

For the definition of the gonalities d_r, see Section 2. These new bounds may appear to be a minor improvement on those of [14], but they are in some sense best possible in the light of current knowledge and have surprisingly strong consequences. In particular, in the course of proving Theorem 4.1, we are able to show that $\mathrm{Cliff}_3(C) = \frac{10}{3}$ for the general curve of genus 9 (see Proposition 4.8 and Corollary 4.9); to our knowledge, this is the first complete computation of $\mathrm{Cliff}_3(C)$ for any curve with $\mathrm{Cliff}(C) > 2$.

Section 5 is concerned with the case of plane curves, especially smooth plane curves. We note first that, if C is a smooth plane curve of degree $\delta \ge 6$, Theorem 4.1 implies that $\mathrm{Cliff}_3(C) \ge \frac{2\delta-6}{3}$ (see Proposition 5.1). The main result of this section identifies all possible bundles for which this lower bound could be attained.

THEOREM 5.6
If C is a smooth plane curve of degree $\delta \ge 7$ and $\mathrm{Cliff}_3(C) = \frac{2\delta-6}{3}$, then any bundle E computing $\mathrm{Cliff}_3(C)$ is stable and fits into an exact sequence

$$0 \to E_H \to E \to H \to 0,$$

and all sections of H lift to E. Moreover, such extensions exist if and only if $h^0(E_H \otimes E_H) \ge 10$.

Here H denotes the hyperplane bundle on C and E_H is defined by the evaluation sequence $0 \to E_H^* \to H^0(E) \otimes \mathcal{O}_C \to E \to 0$. For the normalization of a nodal plane curve, we prove a similar but more complicated result (Theorem 5.9).

In Section 6 we study curves with $\mathrm{Cliff}_3(C) = 3$. Our main result here is the following.

THEOREM 6.8
Let C be a curve of genus $g \ge 9$ with $\mathrm{Cliff}(C) = 3$. If $d_2 > 7$, and in particular if $g \ge 16$, then

$$\mathrm{Cliff}_3(C) = 3.$$

For all $g \geq 9$ there exist curves with these properties.

For curves with $\mathrm{Cliff}_3(C) = 3$ and $d_2 = 7$ (which can exist only for $7 \leq g \leq 15$) or with $g = 8$ and $d_2 = 8$, we have $\frac{8}{3} \leq \mathrm{Cliff}_3 \leq 3$, but we do not know the precise value of $\mathrm{Cliff}_3(C)$. We do however give a list of all bundles which could compute $\mathrm{Cliff}_3(C)$ if $\mathrm{Cliff}_3(C) = \frac{8}{3}$ (see Propositions 6.5, 6.6, 6.7). The problem is therefore reduced to that of determining whether any of these bundles exists.

In Section 7, we prove that, if $\mathrm{Cliff}_3(C) \leq \mathrm{Cliff}_2(C)$ and E computes $\mathrm{Cliff}_3(C)$, then the coherent system $(E, H^0(E))$ is α-semistable for all $\alpha > 0$; if in addition E is stable, then $(E, H^0(E))$ is α-stable for all $\alpha > 0$. (In fact we prove a result for rank n, Proposition 7.1, of which this is the case $n = 3$.) These results are of interest in connection with a conjecture of D. C. Butler [2, Conjecture 2].

Finally, Section 8 contains further comments and a discussion of open problems.

We suppose throughout that C is a smooth irreducible projective algebraic curve defined over $\mathbb{C}$ and denote the canonical bundle on C by K_C. For a vector bundle F on C, we denote the degree of F by d_F and its slope by $\mu(F) := \frac{d_F}{\mathrm{rk}\, F}$.

2. Definitions and preliminaries

We recall first the definition of $\mathrm{Cliff}_n(C)$. For any vector bundle E of rank n and degree d on C, we define

$$\gamma(E) := \frac{1}{n}\big(d - 2(h^0(E) - n)\big) = \mu(E) - 2\frac{h^0(E)}{n} + 2.$$

If C has genus $g \geq 4$, we then define, for any positive integer n,

$$\mathrm{Cliff}_n(C) := \min_E \left\{ \gamma(E) \;\middle|\; \begin{array}{c} E \text{ semistable of rank } n, \\ h^0(E) \geq 2n, \mu(E) \leq g-1 \end{array} \right\}$$

(this invariant is denoted in [11]–[15] by γ'_n). Note that $\mathrm{Cliff}_1(C) = \mathrm{Cliff}(C)$ is the usual Clifford index of the curve C. We say that E *contributes to* $\mathrm{Cliff}_n(C)$ if E is semistable of rank n with $h^0(E) \geq 2n$ and $\mu(E) \leq g - 1$. If in addition $\gamma(E) = \mathrm{Cliff}_n(C)$, we say that E *computes* $\mathrm{Cliff}_n(C)$. Moreover, as observed in [11, Proposition 3.3, Conjecture 9.3], the conjecture of [18] can be restated in a slightly weaker form as the following.

CONJECTURE

The rank-n Clifford index $\mathrm{Cliff}_n(C)$ *is equal to* $\mathrm{Cliff}(C)$.

In fact, for $n = 2$, this form of the conjecture is equivalent to the original (see [15, Proposition 2.7]).

LEMMA 2.1

The conjecture is valid in the following cases:

(i) $\mathrm{Cliff}(C) \le 2$,
(ii) $n = 2$ *and* $\mathrm{Cliff}(C) \le 4$.

Proof
See [11, Propositions 3.5, 3.8]. □

However, the conjecture is known to fail in many other cases (see [7], [8], [13], [15], [16]). For $n = 3$ it fails for the general curve of genus 9 or 11 (see [10]) and for curves of genus ≥ 12 contained in K3 surfaces (see [8, Corollary 1.6]). For $n = 2$ it is still conjectured to hold for the general curve of any genus (see [7]). Note that in any case

$$\mathrm{Cliff}_n(C) \le \mathrm{Cliff}(C) \tag{2.1}$$

(see [11, Lemma 2.2]), and for $n = 2$ we have the lower bound

$$\mathrm{Cliff}_2(C) \ge \min\Big\{\mathrm{Cliff}(C), \frac{\mathrm{Cliff}(C)}{2} + 2\Big\} \tag{2.2}$$

(see [11, Proposition 3.8]).

The *gonality sequence* $d_1, d_2, \ldots, d_r, \ldots$ of C is defined by

$$d_r := \min\big\{d_L \mid L \text{ a line bundle on } C \text{ with } h^0(L) \ge r + 1\big\}.$$

We have always $d_r < d_{r+1}$ and $d_{r+s} \le d_r + d_s$; in particular, $d_n \le nd_1$ for all n (see [11, Section 4]). We say that d_r *computes* $\mathrm{Cliff}(C)$ if $d_r \le g - 1$ and $d_r - 2r = \mathrm{Cliff}(C)$ and that C has *Clifford dimension* r if r is the smallest integer for which d_r computes $\mathrm{Cliff}(C)$. Note also (see [11, Lemma 4.6])

$$d_r \ge \min\big\{\mathrm{Cliff}(C) + 2r, g + r - 1\big\}. \tag{2.3}$$

We recall that $\mathrm{Cliff}(C) \le \left[\frac{g-1}{2}\right]$ with equality on the general curve of genus g. In fact equality holds on any *Petri curve*, that is, any curve for which the multiplication map

$$H^0(L) \otimes H^0(K_C \otimes L^*) \to H^0(K_C)$$

is injective for every line bundle L on C. Moreover,

$$d_r \le g + r - \left[\frac{g}{r+1}\right], \tag{2.4}$$

again with equality on any Petri curve.

In the following sections, we shall need a few basic results. The first is the lemma of Paranjape and Ramanan [21, Lemma 3.9], which can be stated as follows.

LEMMA 2.2
Let E be a bundle of rank n and degree d on C with $h^0(E) = n + s$ possessing no proper subbundle F with $h^0(F) > \mathrm{rk}\, F$. Then $d \ge d_{ns}$.

As a complement to this lemma in the case $n = 2$, we have (see [15, Lemma 2.6]) the following.

LEMMA 2.3

Suppose that F *is a semistable bundle of rank* 2 *and degree* $\leq 2g-2$ *which possesses a subbundle* M *with* $h^0(M) \geq 2$*. Then* $\gamma(F) \geq \mathrm{Cliff}(C)$*, with equality if and only if* $\gamma(M) = \gamma(F/M) = \mathrm{Cliff}(C)$ *and all sections of* F/M *lift to* F*.*

PROPOSITION 2.4

Suppose that either $\mathrm{Cliff}_3(C) < \mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ *or* $\mathrm{Cliff}_3(C) \leq \mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$*, and let* E *be a bundle computing* $\mathrm{Cliff}_3(C)$*. Then* E *is stable.*

Proof

Suppose that E is strictly semistable. Then E is S-equivalent to a bundle of the form $F \oplus L$, where $\mathrm{rk}\, F = 2$, $\mathrm{rk}\, L = 1$, and both bundles have the same slope as E. Moreover, $\gamma(E) \geq \gamma(F \oplus L)$.

Note that either F contributes to $\mathrm{Cliff}_2(C)$ or L contributes to $\mathrm{Cliff}(C)$. If both of these hold, then clearly $\gamma(F \oplus L) \geq \frac{2\,\mathrm{Cliff}_2(C)+\mathrm{Cliff}(C)}{3}$. If F does not contribute to $\mathrm{Cliff}_2(C)$, then $h^0(F) \leq 3$, so

$$\gamma(F) \geq \mu(F) - 1 = d_L - 1 > \gamma(L).$$

Since $\gamma(L) \geq \mathrm{Cliff}(C)$, it follows that $\gamma(F \oplus L) > \mathrm{Cliff}(C)$. Finally, suppose L does not contribute to $\mathrm{Cliff}(C)$. Then

$$\gamma(L) \geq d_L = \mu(F) > \gamma(F) \geq \mathrm{Cliff}_2(C),$$

so $\gamma(F \oplus L) > \mathrm{Cliff}_2(C)$. In all cases, we obtain the contradiction $\gamma(E) > \mathrm{Cliff}_3(C)$. $\square$

For the next result recall that, if L is a generated line bundle with $h^0(L) = 1 + u$, then the evaluation sequence

$$0 \to E_L^* \to H^0(L) \otimes \mathcal{O}_C \to L \to 0 \tag{2.5}$$

defines a vector bundle E_L of rank u and degree d_L.

LEMMA 2.5

If $u = 2$ *and* $d_L = d_2$ *in (2.5), then* E_L *is semistable. Moreover, if* $d_2 < 2d_1$*, then* E_L *is stable and* $h^0(E_L) = 3$*.*

Proof

See [11, Proposition 4.9, Theorem 4.15]. $\square$

PROPOSITION 2.6

Suppose that $3\,\mathrm{Cliff}(C) \geq 2d_2 - 6$*, and suppose that* $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$*. Let* F *be a stable bundle of rank* 2 *and degree* d_2 *with* $h^0(F) = 3$*, and let* L *be a line bundle of degree* d_2 *with* $h^0(L) = 3$*. Suppose further that*

$$0 \to F \to E \to L \to 0 \tag{2.6}$$

is a nontrivial extension with $h^0(E) = 6$*. Then* E *is semistable and generated. Moreover, extensions (2.6) with these properties exist if and only if* $h^0(F \otimes E_L) \geq 10$*.*

Proof
If F is not generated, then it possesses a subsheaf F' of rank 2 and degree $d_2 - 1$ such that $h^0(F') = 3$. Moreover, F' is semistable. This contradicts [11, Corollary 4.12].

Since also $h^0(F^*) = 0$, we have an exact sequence

$$0 \to F^* \to H^0(F)^* \otimes \mathcal{O}_C \to M \to 0,$$

where $M \simeq \det F$ has degree d_2 and $h^0(M) \geq 3$. Hence $h^0(M) = 3$ and $F \simeq E_M$. The semistability of E now follows as in [10, Proposition 3.5], noting that the inequality $3d_1 \geq 2d_2$ is weaker than $3\,\mathrm{Cliff}(C) \geq 2d_2 - 6$. Moreover, E is obviously generated.

For the last assertion note that a nontrivial extension (2.6) with $h^0(E) = 6$ corresponds to a nonzero element of the kernel of the natural map

$$H^1(L^* \otimes F) \to \mathrm{Hom}\big(H^0(L), H^1(F)\big) = H^0(L)^* \otimes H^1(F).$$

Now consider the sequence

$$0 \to L^* \otimes F \to H^0(L)^* \otimes F \to E_L \otimes F \to 0.$$

Since F is stable, $H^0(L^* \otimes F) = 0$. So we have an exact sequence

$$0 \to H^0(L)^* \otimes H^0(F) \to H^0(E_L \otimes F) \to H^1(L^* \otimes F) \to H^0(L)^* \otimes H^1(F).$$

Hence there exists a nontrivial extension (2.6) with $h^0(E) = 6$ if and only if $h^0(E_L \otimes F) > h^0(L) \cdot h^0(F) = 9$. □

3. Curves with minimal rank-2 Clifford index

In this section, we let C be a curve of genus $g \geq 11$ with

$$\mathrm{Cliff}(C) = \left[\frac{g-1}{2}\right] \quad \text{and} \quad \mathrm{Cliff}_2(C) = \frac{1}{2}\left[\frac{g-1}{2}\right] + 2. \tag{3.1}$$

Such curves exist by [16] and [7]. Note that (3.1) implies that $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$. By (2.2), $\mathrm{Cliff}_2(C)$ takes its minimum value for the given value of $\mathrm{Cliff}(C)$, so these curves are good candidates for obtaining values of $\mathrm{Cliff}_3(C)$ greater than $\mathrm{Cliff}_2(C)$. A further implication of (3.1) is that $d_4 \leq \mathrm{Cliff}(C) + 8$ (see [11, Theorem 5.2]). On the other hand, $d_4 \geq \mathrm{Cliff}(C) + 8$ for any curve of genus ≥ 8 by (2.3), so, for our curves, we have $d_4 = \mathrm{Cliff}(C) + 8$. This implies that C cannot be a Petri curve for $g \geq 12$. On the other hand, it is known that, for $g = 11$, C can be Petri (see [17, Theorem 1.5]).

We follow the arguments of [14].

PROPOSITION 3.1
Let E *be a semistable bundle of degree* d *computing* $\mathrm{Cliff}_3(C)$*. If* $g \geq 19$ *and*

$d < 2g - 2 + \frac{1}{2}[\frac{g-1}{2}]$, *then either*

$$\gamma(E) > \mathrm{Cliff}_2(C) \qquad \text{or} \qquad \gamma(E) \geq \frac{d_9}{3} - 2.$$

Moreover, if $\gamma(E) \leq \mathrm{Cliff}_2(C)$*, then* E *has no proper subbundle* F *with* $h^0(F) \geq \operatorname{rk} F + 1$.

Proof
By [14, Proposition 2.4] we have

$$\gamma(E) \geq \min\Big\{\frac{d_9}{3} - 2, \mathrm{Cliff}_2(C), \frac{2\,\mathrm{Cliff}(C)+1}{3}, \frac{1}{3}\big(2\,\mathrm{Cliff}(C) + 2g - d + 4\big)\Big\}.$$

Note that the bound $\mathrm{Cliff}_2(C)$ enters only in [14, formula (2.2)] and is a strict inequality. Moreover, the condition on d is necessary and sufficient for

$$\frac{1}{3}\big(2\,\mathrm{Cliff}(C) + 2g - d + 4\big) > \mathrm{Cliff}_2(C).$$

If $g \geq 23$, then

$$\frac{2\,\mathrm{Cliff}(C)+1}{3} > \mathrm{Cliff}_2(C),$$

and we are finished. For $g < 23$ we need to improve the bound $\frac{2\,\mathrm{Cliff}(C)+1}{3}$. The points where this enters in the proof of [14, Proposition 2.4] are [14, Lemma 2.2(i)] and [14, formula (2.3)]. (The inequality at the end of the proof of [14, Lemma 2.2] can be replaced by $\gamma(E) \geq \frac{4\,\mathrm{Cliff}(C)+2}{3}$, which is clearly greater than $\mathrm{Cliff}_2(C)$.)

For [14, Lemma 2.2(i)], we have

$$\gamma(E) \geq \frac{1}{3}\big(\mathrm{Cliff}(C) + d_6\big) - 2.$$

By (2.3),

$$d_6 \geq \min\Big\{\Big[\frac{g-1}{2}\Big] + 12, g+5\Big\} = \Big[\frac{g-1}{2}\Big] + 12$$

for $g \geq 12$. So

$$\gamma(E) \geq \frac{2}{3}\Big[\frac{g-1}{2}\Big] + 2 > \mathrm{Cliff}_2(C).$$

For [14, formula (2.3)], the estimate enters in two different cases. In the first case we have, for some integer $t \geq 1$,

$$\gamma(E) \geq \frac{2\,\mathrm{Cliff}(C)+2t}{3} \geq \frac{2\,\mathrm{Cliff}(C)+2}{3} > \mathrm{Cliff}_2(C) \tag{3.2}$$

for $g \geq 19$. In the second case we have

$$\begin{aligned}\gamma(E) &\geq \frac{2t+4g-4}{3} - \frac{d}{3}\\ &> \frac{2t+4g-4}{3} - \frac{2g-2+\frac{1}{2}[\frac{g-1}{2}]}{3}\\ &= \frac{2t+2g-2-\frac{1}{2}[\frac{g-1}{2}]}{3} > \mathrm{Cliff}_2(C).\end{aligned}$$

□

REMARK 3.2

For $g \le 18$ one can have $\gamma(E) \le \mathrm{Cliff}_2(C)$ in (3.2). Since $g \ge 11$, this can occur only if $t = 1$. If $15 \le g \le 18$, all the other inequalities in the proof of (3.2) must be equalities. In particular, $d_2 = d_{2t}$ computes $\mathrm{Cliff}(C)$. For $16 \le g \le 18$, one can check that all the hypotheses of [17, Theorem 9.1] hold except that we do not know whether the quadratic form in the statement of that theorem can take the value -1. However, this does not matter in view of [16, Corollaries 2.4, 2.6]. So [17, Theorem 9.2] applies, and a simple calculation shows that this gives $d_2 \ge \mathrm{Cliff}(C) + 5$, contradicting the assumption that d_2 computes $\mathrm{Cliff}(C)$. It follows that, for $16 \le g \le 18$, there exists a curve C satisfying (3.1) for which the conclusion of Proposition 3.1 holds.

PROPOSITION 3.3

Let E be a semistable bundle of degree d computing $\mathrm{Cliff}_3(C)$. *Suppose that E possesses a proper subbundle F of maximal slope with* $\mathrm{rk}\, F = 2$. *If*

$$d > \max\left\{\frac{3}{4}\left[\frac{g-1}{2}\right] + g + 12, \frac{9}{2}\left[\frac{g-1}{2}\right] - 2g + 30\right\} \tag{3.3}$$

and

$$d < 4g - \frac{3}{2}\left[\frac{g-1}{2}\right] - 12, \tag{3.4}$$

then

$$\gamma(E) > \mathrm{Cliff}_2(C).$$

Proof

We use the bound of [14, Lemma 3.2]. It is clear that

$$\frac{\mathrm{Cliff}(C) + 2\,\mathrm{Cliff}_2(C)}{3} > \mathrm{Cliff}_2(C).$$

Moreover, by simple computations,

$$\frac{\mathrm{Cliff}(C)}{3} + \frac{2d - 2g - 6}{9} > \mathrm{Cliff}_2(C) \Leftrightarrow d > \frac{3}{4}\left[\frac{g-1}{2}\right] + g + 12,$$

$$\frac{2\,\mathrm{Cliff}_2(C)}{3} + \frac{d}{9} > \frac{2\,\mathrm{Cliff}_2(C)}{3} + \frac{1}{12}\left[\frac{g-1}{2}\right] + \frac{g+12}{9} > \mathrm{Cliff}_2(C),$$

$$\frac{2\,\mathrm{Cliff}_2(C)}{3} + \frac{4g - d - 6}{9} > \mathrm{Cliff}_2(C) \Leftrightarrow d < 4g - \frac{3}{2}\left[\frac{g-1}{2}\right] - 12,$$

$$\frac{d + 2g - 12}{9} > \mathrm{Cliff}_2(C) \Leftrightarrow d > \frac{9}{2}\left[\frac{g-1}{2}\right] - 2g + 30. \qquad \square$$

REMARK 3.4

If $g = 32$ or $g \ge 34$, we have

$$4g - \frac{3}{2}\left[\frac{g-1}{2}\right] - 12 > 3g - 3.$$

Since we have always $d \le 3g - 3$, this means we can delete the inequality (3.4) in this case.

If $g < 34$ and $g \neq 32$, we can have

$$4g - \frac{3}{2}\left[\frac{g-1}{2}\right] - 12 \leq d \leq 3g - 3, \tag{3.5}$$

in which case

$$\frac{2\,\mathrm{Cliff}_2(C)}{3} + \frac{4g-d-6}{9} \leq \mathrm{Cliff}_2(C).$$

This allows the possibility of a bundle E computing $\mathrm{Cliff}_3(C)$ with $\gamma(E) \leq \mathrm{Cliff}_2(C)$ and sitting in an exact sequence

$$0 \to F \to E \to E/F \to 0$$

with F of maximal slope and rank 2 and $h^0(F) \geq 4, h^1(E/F) \leq 1$, and $d_{E/F} > g-1$. In the next proposition, we show that this still implies that $\gamma(E) > \mathrm{Cliff}_2(C)$ if $g \geq 16$.

PROPOSITION 3.5

Let E be a semistable bundle of degree d computing $\mathrm{Cliff}_3(C)$. Suppose that E possesses a proper subbundle of maximal slope with $\mathrm{rk}\,F = 2$, $h^0(F) \geq 4, h^1(E/F) \leq 1$, and $d_{E/F} > g-1$. If $g \geq 16$, then

$$\gamma(E) > \mathrm{Cliff}_2(C).$$

Proof

If $h^1(E/F) = 0$, then, using [20], we obtain

$$\gamma(E/F) = \gamma\big((E/F)^* \otimes K_C\big) = 2g - d_{E/F} \geq \frac{4g-d}{3}.$$

Since $\gamma(F) \geq \mathrm{Cliff}_2(C)$, this means that $\gamma(E) > \mathrm{Cliff}_2(C)$ provided $4g - d > 3\,\mathrm{Cliff}_2(C)$. A simple computation (using $d \leq 3g - 3$) shows that this holds for $g \geq 10$. So we can suppose $h^1(E/F) = 1$.

Suppose $\gamma(E) \leq \mathrm{Cliff}_2(C)$. As in the proof of [14, Lemma 3.2] we get $d_{E/F} \leq \frac{2g+d}{3}$ or, equivalently,

$$\gamma(E/F) \geq \frac{4g-d-6}{3}.$$

Now

$$\frac{2\gamma(F) + \gamma(E/F)}{3} \leq \gamma(E) \leq \frac{1}{2}\left[\frac{g-1}{2}\right] + 2. \tag{3.6}$$

So

$$2\gamma(F) \leq \frac{3}{2}\left[\frac{g-1}{2}\right] + 6 - \gamma(E/F) \leq \frac{3}{2}\left[\frac{g-1}{2}\right] + 6 - \frac{4g-d-6}{3}.$$

Hence

$$h^0(F) = 2 - \gamma(F) + \frac{d_F}{2} \geq -\frac{3}{4}\left[\frac{g-1}{2}\right] - 2 + \frac{2g+d}{6}.$$

If F possesses a line subbundle with $h^0 \geq 2$, then by Lemma 2.3, $\gamma(F) \geq \mathrm{Cliff}(C)$, which contradicts (3.6). So by Lemma 2.2,

$$d_F \geq d_t \quad \text{with } t = 2\big(h^0(F) - 2\big) \geq \frac{2g+d}{3} - \frac{3}{2}\left[\frac{g-1}{2}\right] - 8.$$

Since $d_t \geq \min\{\mathrm{Cliff}(C) + 2t, g + t - 1\}$ by (2.3), it suffices to show that

$$d_F < \frac{5g+d}{3} - \frac{3}{2}\left[\frac{g-1}{2}\right] - 9 \tag{3.7}$$

and

$$d_F < \frac{4g+2d}{3} - 2\left[\frac{g-1}{2}\right] - 16. \tag{3.8}$$

Since we are assuming that $\gamma(E) \leq \mathrm{Cliff}_2(C)$ and we know that $\gamma(F) \geq \mathrm{Cliff}_2(C)$, we must have $\gamma(E/F) \leq \mathrm{Cliff}_2(C)$; that is,

$$d_{E/F} \geq 2g - 4 - \frac{1}{2}\left[\frac{g-1}{2}\right],$$

and hence

$$d_F \leq d - 2g + 4 + \frac{1}{2}\left[\frac{g-1}{2}\right].$$

So for (3.7) it is enough to prove that

$$d - 2g + 4 + \frac{1}{2}\left[\frac{g-1}{2}\right] < \frac{5g+d}{3} - \frac{3}{2}\left[\frac{g-1}{2}\right] - 9.$$

Using $d \leq 3g-3$, it is sufficient to show that

$$-10g + 66 + 12\left[\frac{g-1}{2}\right] < 0,$$

which is valid for $g \geq 16$.

For (3.8) it is enough to prove that

$$d - 2g + 4 + \frac{1}{2}\left[\frac{g-1}{2}\right] < \frac{4g+2d}{3} - 2\left[\frac{g-1}{2}\right] - 16.$$

Again using $d \leq 3g-3$, it is sufficient to show that

$$-14g + 114 + 15\left[\frac{g-1}{2}\right] < 0,$$

which is valid for $g \geq 16$. □

PROPOSITION 3.6

Let E be a semistable bundle of degree d computing $\mathrm{Cliff}_3(C)$. Suppose that E possesses a proper subbundle L of maximal slope with $\mathrm{rk}\, L = 1$. If

$$d > g + \frac{3}{2}\left[\frac{g-1}{2}\right] + 6, \tag{3.9}$$

then

$$\gamma(E) > \mathrm{Cliff}_2(C).$$

Proof

We follow the proof of [14, Lemma 3.1]. Clearly

$$\frac{\mathrm{Cliff}(C)+2\,\mathrm{Cliff}_2(C)}{3} > \mathrm{Cliff}_2(C).$$

Moreover,

$$\frac{\mathrm{Cliff}(C)}{3} + \frac{2d-6}{9} > \mathrm{Cliff}_2(C)$$

under the assumption on d.

It remains to handle the case where $h^0(L) \leq 1$. In this case we have an exact sequence

$$0 \to L \to E \to Q \to 0$$

and, by [20],

$$\mu(Q) - \mu(L) \leq g.$$

Moreover, every line subbundle M of Q must have $d_M \leq d_L$. (Otherwise the pullback of M to E would have slope greater than d_L.) We can assume M has maximal slope as a subbundle of Q, so, again by [20],

$$\mu(Q/M) - \mu(M) \leq g.$$

In other words,

$$d - d_L - 2d_M \leq g.$$

It follows that

$$3d_L \geq d_L + 2d_M \geq d - g;$$

hence we can replace $\frac{d-2g}{3}$ in [14, formula (3.4)] by $\frac{d-g}{3}$. It therefore remains to prove that

$$\frac{2\,\mathrm{Cliff}_2(C)}{3} + \frac{d-g}{9} > \mathrm{Cliff}_2(C)$$

or, equivalently,

$$\frac{d-g}{9} > \frac{1}{3}\mathrm{Cliff}_2(C).$$

This is equivalent to $d > g + \frac{3}{2}[\frac{g-1}{2}] + 6$. □

Combining everything, we get the following theorem.

THEOREM 3.7

If $g \geq 16$, there exists a curve C satisfying (3.1) such that either

$$\mathrm{Cliff}_3(C) > \mathrm{Cliff}_2(C)$$

or there exists a semistable bundle E of degree $d < 2g-2+\frac{1}{2}[\frac{g-1}{2}]$ which possesses no proper subbundle F with $h^0(F) \geq \mathrm{rk}\,F + 1$ such that

$$\mathrm{Cliff}_2(C) \geq \gamma(E) \geq \frac{d_9}{3} - 2.$$

If $g \geq 19$, this holds for every curve C satisfying (3.1).

Proof
The theorem follows from Propositions 3.1, 3.3, 3.5, and 3.6 and Remarks 3.2 and 3.4. We need only to check that the lower bounds (3.3) and (3.9) for d are less than the upper bound of Proposition 3.1. □

LEMMA 3.8
If $14 \le g \le 24$, *then*

$$\frac{d_9}{3} - 2 > \mathrm{Cliff}_2(C). \tag{3.10}$$

Proof
By (2.3), we have

$$d_9 \ge \min\left\{ \left[\frac{g-1}{2}\right] + 18, g+8 \right\}.$$

The assertion follows from a simple computation. □

THEOREM 3.9
If $16 \le g \le 24$, *then there exists a curve* C *of genus* g *such that*

$$\mathrm{Cliff}_3(C) > \mathrm{Cliff}_2(C).$$

Proof
This follows at once from Theorem 3.7 and Lemma 3.8. □

REMARK 3.10
It is possible that (3.10) holds for other values of g, indeed, for all $g \ge 14$. If this is so, one can extend Theorem 3.9 accordingly.

4. An improved lower bound

In this section we shall improve the lower bound of [14, Theorem 4.1]. We have already remarked in the proof of [10, Theorem 4.6(ii)] that

$$\mathrm{Cliff}_3(C) \ge \min\left\{ \frac{d_9}{3} - 2, \frac{2\,\mathrm{Cliff}(C)+1}{3}, \frac{2\,\mathrm{Cliff}_2(C)+2}{3} \right\}.$$

For $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ this is an improvement. We consider here the case $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$. Note that this is true for $\mathrm{Cliff}(C) \le 4$ by Lemma 2.1, for all smooth plane curves (see [11, Proposition 8.1]) and for the general curve of genus ≤ 19 (see [7, Theorem 1.7] for the case $g \le 16$).

THEOREM 4.1
Let C *be a curve of genus* $g \ge 7$ *such that* $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C) \ge 2$. *Then*

$$\mathrm{Cliff}_3(C) \ge \min\left\{ \frac{d_9}{3} - 2, \frac{2\,\mathrm{Cliff}(C)+2}{3} \right\}.$$

Moreover, if $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$, *then any bundle computing* $\mathrm{Cliff}_3(C)$ *is stable.*

We may assume that $\mathrm{Cliff}(C) \geq 3$ by Lemma 2.1. We use the proofs in [14, Sections 2, 3] to make necessary improvements and proceed by a sequence of lemmas and propositions. We follow the argument of [14]. Suppose throughout that E is a bundle computing $\mathrm{Cliff}_3(C)$.

LEMMA 4.2

If E has a line subbundle F with $h^0(F) \geq 2$ and $d \leq 2g+6$, then

$$\gamma(E) \geq \frac{2\,\mathrm{Cliff}(C)+2}{3}.$$

Proof

By [14, Lemma 2.2] we know that

$$\gamma(E) \geq \min\Big\{\frac{2\,\mathrm{Cliff}(C)+1}{3}, \frac{1}{3}\big(4\,\mathrm{Cliff}(C)+2g+2-d\big)\Big\}.$$

We need first to improve the estimate in case (i) in the proof of this lemma. We have by (2.3)

$$d_6 \geq \min\big\{\mathrm{Cliff}(C)+12, g+5\big\} > \mathrm{Cliff}(C)+7.$$

So

$$\gamma(E) \geq \frac{\mathrm{Cliff}(C)}{3} + \frac{d_6}{3} - 2 > \frac{2\,\mathrm{Cliff}(C)+1}{3}.$$

It is therefore sufficient to show that

$$\frac{1}{3}\big(4\,\mathrm{Cliff}(C)+2g+2-d\big) \geq \frac{1}{3}\big(2\,\mathrm{Cliff}(C)+2\big).$$

This is true provided $\mathrm{Cliff}(C) \geq 3$ and $d \leq 2g+6$. $\square$

LEMMA 4.3

If E has a subbundle F of rank 2 with $h^0(F) \geq 3$ and no line subbundle with $h^0 \geq 2$, and $d \leq 2g+2$, then

$$\gamma(E) \geq \frac{2\,\mathrm{Cliff}(C)+2}{3}.$$

Proof

We use [14, Lemma 2.3]. We need only to note that the estimate [14, formula (2.3)] can be improved to give the required result. For this improvement, observe that

$$\gamma(E) \geq \frac{2t+g-1}{3} \geq \frac{2\,\mathrm{Cliff}(C)+2}{3}$$

since $t = h^0(F) - 2 \geq 1$. $\square$

LEMMA 4.4

Suppose that E has a proper subbundle of maximal slope and rank 1, and suppose *that* $d \geq 2g+4$. *Then*

$$\gamma(E) \geq \frac{2\,\mathrm{Cliff}(C)+2}{3}.$$

Proof
This is an immediate consequence of [14, Lemma 3.1], since $3\,\mathrm{Cliff}_3(C)$ is an integer. □

LEMMA 4.5
Suppose that $g=8$ *or* $g \geq 10$, *suppose that* E *has a proper subbundle of maximal slope and rank* 2, *and suppose that* $d \geq 2g+3$. *Then*

$$\gamma(E) \geq \frac{2\,\mathrm{Cliff}(C)+2}{3}.$$

Proof
We use [14, Lemma 3.2]. We need to check that

$$\frac{d+2g-12}{9} > \frac{2\,\mathrm{Cliff}(C)+1}{3}. \tag{4.1}$$

This holds for $d \geq 2g+3$ if $g=8$ or $g \geq 10$. □

PROPOSITION 4.6
Let C *be a curve of genus* $g=8$ *or* $g \geq 10$ *such that* $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C) \geq 3$, *and let* E *be a bundle computing* $\mathrm{Cliff}_3(C)$. *Then*

$$\gamma(E) \geq \min\left\{\frac{d_9}{3} - 2, \frac{2\,\mathrm{Cliff}(C)+2}{3}\right\}.$$

Proof
If E does not possess a proper subbundle F with $h^0(F) \geq \mathrm{rk}\,F + 1$, then

$$\gamma(E) \geq \frac{d_9}{3} - 2$$

by Lemma 2.2. So suppose E does have such a subbundle and

$$\gamma(E) \leq \frac{2\,\mathrm{Cliff}(C)+1}{3}.$$

This gives a contradiction by Lemmas 4.2 and 4.3 if $d \leq 2g+2$ and by Lemmas 4.4 and 4.5 if $d \geq 2g+4$.

If $d = 2g+3$, then Lemma 4.2 implies that E has no line subbundle with $h^0 \geq 2$. Let F be a subbundle of rank 2 with $h^0(F) \geq 3$. Then F possesses a line subbundle L with $h^0(L) = 1$, so $h^0(F/L) \geq 2$, and hence $d_{F/L} \geq d_1 > 2$. This implies $\mu(F) > 1$.

By Lemma 4.5 all proper subbundles of E of maximal slope are line bundles. Choose such a line bundle L, and consider the proof of Lemma 4.4, that is, of [14, Lemma 3.1]. In order to get $\gamma(E) = \frac{2\,\mathrm{Cliff}(C)+1}{3}$, we must have equality in [14, formula (3.4)], that is, $d_L = 1$. So L is not of maximal slope, a contradiction. □

The cases $g=7$ and $g=9$ require further arguments because (4.1) can fail.

PROPOSITION 4.7

Let C be a curve of genus $g = 7$ with $\mathrm{Cliff}(C) = 3$*, and let E be a bundle computing* $\mathrm{Cliff}_3(C)$*. Then*

$$\gamma(E) \geq \frac{8}{3}.$$

Proof

Recall that $\mathrm{Cliff}_2(C) = 3$ by [11, Lemma 2.1]. Moreover, $d_9 = 16$. So $\frac{d_9}{3} - 2 > 3$.

Note that $d \leq 3g - 3 = 18$. The proof of the theorem works for $d \leq 2g + 2 = 16$. So we are left with the cases $d = 17$ and 18.

If $d = 18$, [14, formula (2.4)] gives $\gamma(E) \geq \frac{2\,\mathrm{Cliff}_2(C)}{3} = 2$. Moreover, $\gamma(E) \geq \frac{8}{3}$ unless $h^0(E) = 9$, in which case $\gamma(E) = 2$. This contradicts [14, Proposition 3.3].

If $d = 17$, we can assume that E has no line subbundle with $h^0 \geq 2$ by Lemma 4.2. The only case in which we can have $\gamma(E) = \frac{2\,\mathrm{Cliff}(C)+1}{3}$ is when [14, formula (2.4)] is an equality. This implies that E fits into an exact sequence

$$0 \to F \to E \to E/F \to 0$$

with F of rank 2 and degree $d_2 = 7$ with $h^0(F) = 3, d_{E/F} = 10$, $h^0(E/F) = 5$, and all sections of E/F lift to E.

Since $h^0(E) = 8$, there exists a line subbundle $L \subset E$, $d_L \geq 2$ and $h^0(L) = 1$. This cannot be a subbundle of maximal slope, since this would require equality in [14, formula (3.4)], which means $d_L = 1$. So there exists a subbundle G of maximal slope with rank 2. If $d_G \geq 8$, then by the proof of [14, Lemma 3.2], $\gamma(G) \geq 3$ and also $\gamma(E/G) \geq 3$. So $\gamma(E) \geq 3$, a contradiction. Hence F is a subbundle of maximal slope.

Now $d_{E/L} = 17 - d_L$. If M is a subbundle of E/L, the pullback to E has degree $d_M + d_L \leq 7$. So

$$d_M \leq 7 - d_L < \frac{17 - d_L}{2},$$

and E/L is stable.

Note that $h^0(E/L) \geq 7$, so $h^1(E/L) \geq 7 + 12 - d_{E/L} \geq 4$. Hence either E/L or $K \otimes (E/L)^*$ contributes to $\mathrm{Cliff}_2(C)$. Since $\mathrm{Cliff}_2(C) = 3$, this gives $d_{E/L} - 2(h^0(E/L) - 2) \geq 6$, that is,

$$h^0(E/L) \leq \frac{d_{E/L}}{2} - 1 \leq \frac{13}{2},$$

a contradiction. □

PROPOSITION 4.8

Let C be a curve of genus $g = 9$ with $\mathrm{Cliff}(C) \geq 3$*, and let E be a bundle computing* $\mathrm{Cliff}_3(C)$*. Then*

- *either* $\mathrm{Cliff}(C) = 3$ *and* $\gamma(E) \geq \frac{8}{3}$*,*
- *or* $\mathrm{Cliff}(C) = 4$ *and* $\gamma(E) = \frac{10}{3}$*.*

Proof
Recall that $d_9 = 18$. So $\frac{d_9}{3} - 2 \geq \mathrm{Cliff}(C)$.

The only case we need to consider is $d = 2g + 3 = 21$. If $\mathrm{Cliff}(C) = 3$, then (4.1) holds and the proof goes through as for Proposition 4.6.

So suppose $\mathrm{Cliff}(C) = 4$. By Lemma 2.1, $\mathrm{Cliff}_2(C) = 4$. In this case $\gamma(E) \leq \frac{10}{3}$ by [10, Theorem 4.3], so we can assume that E has no line subbundle with $h^0 \geq 2$ by Lemma 4.2. However, we can have equality in [14, formula (2.4)]. Thus E fits into an exact sequence

$$0 \to F \to E \to E/F \to 0$$

with F of rank 2 and degree $d_2 = 8$ with $h^0(F) = 3$, $d_{E/F} = 13$, $h^0(E/F) = 6$, and all sections of E/F lift to E.

We argue similarly as in the proof of Proposition 4.7. Since $h^0(E) = 9$, there exists a line subbundle L with $d_L \geq 3$ and $h^0(L) = 1$. Again no line subbundle of E can be of maximal slope, and if G is a subbundle of rank 2 of maximal slope with $d_G \geq 9$, then the proof of [14, Lemma 3.2] gives $\gamma(G) \geq \frac{7}{2}$ and $\gamma_{E/G} \geq 4$. So

$$\gamma(E) \geq \frac{11}{3},$$

contradicting the fact that $\gamma(E) \leq \frac{10}{3}$. Hence F is a subbundle of maximal slope.

Note that $h^0(E/L) \geq 8$. Arguing similarly as above we get

$$h^0(E/L) \leq \frac{d_{E/L}}{2} - 2 \leq 7,$$

a contradiction. So we do not have equality in [14, formula (2.4)], which implies $\gamma(E) \geq \frac{10}{3}$. Since $\mathrm{Cliff}_3(C) \leq \frac{10}{3}$, this gives the result. □

As an immediate consequence we get the following corollary.

COROLLARY 4.9
Let C be a curve of genus $g = 9$ with $\mathrm{Cliff}(C) = 4$. *Then*

$$\mathrm{Cliff}_3(C) = \frac{10}{3}.$$

Proof of Theorem 4.1
The inequality for $\mathrm{Cliff}_3(C)$ is a consequence of Propositions 4.6, 4.7, and 4.8. The last assertion follows from Proposition 2.4. □

REMARK 4.10
Suppose that C is as in the statement of Theorem 4.1, and suppose further that $3\,\mathrm{Cliff}(C) \geq 2d_2 - 6$. Let L_1 and L_2 be line bundles on C of degree d_2 with $h^0(L_i) = 3$ for $i = 1, 2$. By Lemma 2.5, E_{L_1} and E_{L_2} are stable with $h^0 = 3$. By Proposition 2.6, all nontrivial extensions

$$0 \to E_{L_1} \to E \to L_2 \to 0$$

with $h^0(E)=6$ give semistable bundles E, and such bundles exist if and only if

$$h^0(E_{L_1}\otimes E_{L_2})\geq 10.$$

Moreover,

$$\gamma(E)=\frac{2d_2-6}{3}.$$

For the general curve of genus 9 or 11, these values are attained [10].

Note that, if in addition d_2 computes $\mathrm{Cliff}(C)$, then

$$\gamma(E)=\frac{2\,\mathrm{Cliff}(C)+2}{3}.$$

Smooth plane curves satisfy these conditions, and we have a more precise statement in the next section (Theorem 5.6). The normalizations of nodal plane curves with small numbers of nodes are also covered by this remark (see Theorem 5.9 and Remark 5.7).

REMARK 4.11

For a general curve C of genus g we have

$$d_9=g+9-\left[\frac{g}{10}\right].$$

So $\frac{d_9}{3}-2\geq\frac{2\,\mathrm{Cliff}(C)+2}{3}$ for $g\leq 30$. If $\mathrm{Cliff}_2(C)=\mathrm{Cliff}(C)$ for such curves, then

$$\mathrm{Cliff}_3(C)\geq\frac{2\,\mathrm{Cliff}(C)+2}{3}.$$

For instance, for a general curve of genus 10, we have $\mathrm{Cliff}_2(C)=\mathrm{Cliff}(C)=4$, so $\frac{10}{3}\leq\mathrm{Cliff}_3(C)\leq 4$. For a general curve of genus 11, we know that $\mathrm{Cliff}_2(C)=\mathrm{Cliff}(C)=5$ (see [8, Theorem 1.3]); so, using [10, Theorem 4.6], we obtain $4\leq\mathrm{Cliff}_3(C)\leq\frac{14}{3}$, which is an improvement on the known result $\frac{11}{3}\leq\mathrm{Cliff}(C)\leq\frac{14}{3}$.

5. Plane curves

To begin with, let C be a smooth plane curve of degree $\delta\geq 6$, and let H denote the hyperplane bundle on C. We know that

$$\mathrm{Cliff}_2(C)=\mathrm{Cliff}(C)=\delta-4$$

(see [11, Proposition 8.1]). We also know the values of all d_r by Noether's theorem. (A proof, which also works for any integral plane curve as claimed by Noether, was given by Hartshorne [9, Theorem 2.1].) In particular,

$$d_1=\delta-1,\qquad d_2=\delta,\qquad d_6=3\delta-3,\qquad d_9=3\delta.$$

Moreover, by the same theorem, H is the only line bundle of degree δ on C with $h^0(H)=3$ and also the only line bundle computing $\mathrm{Cliff}(C)$.

The following proposition is a consequence of Theorem 4.1.

PROPOSITION 5.1

Let C be a smooth plane curve of degree $\delta \geq 6$. Then

$$\operatorname{Cliff}_3(C) \geq \frac{2\delta - 6}{3}.$$

Proof

The result follows from Theorem 4.1, since

$$\frac{d_9}{3} - 2 = \delta - 2 > \frac{2\delta - 6}{3}. \tag{5.1}$$

□

Note that, if $\delta = 6$, we have equality in Proposition 5.1 since $\operatorname{Cliff}_3(C) = \operatorname{Cliff}(C) = 2$ by Lemma 2.1. So we can assume $\delta \geq 7$.

Suppose now that E is a bundle computing $\operatorname{Cliff}_3(C)$.

LEMMA 5.2

If $d \geq 2\delta + 6$ and E has a subbundle F of maximal slope with $\operatorname{rk} F = 1$, then

$$\gamma(E) > \frac{2\delta - 6}{3}.$$

Proof

We follow the proof of [14, Lemma 3.1]. First observe that

$$\frac{\operatorname{Cliff}(C)}{3} + \frac{2d - 6}{9} > \frac{2\delta - 6}{3}.$$

So we can use [14, formulas (3.4), (3.5)] obtaining

$$\gamma(E) \geq \frac{2\operatorname{Cliff}_2(C) + d_F}{3}.$$

To get $\gamma(E) = \frac{2\delta - 6}{3}$, this requires $d_F \leq 2$.

On the other hand, if $d \geq 2\delta + 6$ and $\gamma(E) = \frac{2\delta - 6}{3}$, then

$$h^0(E) = \frac{d - 3\gamma(E)}{2} + 3 = \frac{d}{2} - \delta + 6 \geq 9.$$

So E possesses a line subbundle of degree ≥ 3, a contradiction. □

LEMMA 5.3

If $d > g + \frac{3}{2}\delta$, then

$$\gamma(E) > \frac{2\delta - 6}{3}.$$

Proof

Note first that $d > g + \frac{3}{2}\delta$ implies that $d \geq 2\delta + 6$. By Lemma 5.2, we can therefore assume that every subbundle F of E of maximal slope has rank 2. We check now that all the numbers in the minimum of [14, Lemma 3.2] are $> \frac{2\delta - 6}{3}$. For the first number, this is immediate since $\operatorname{Cliff}_2(C) = \operatorname{Cliff}(C) = \delta - 4$. The second requires precisely the condition $d > g + \frac{3}{2}\delta$. The third needs only $d > 6$. For the fourth, we need $d < 4g - 12$, which is true since $d \leq 3g - 3$ and $g > 9$. Finally, for the fifth number, we need $d > 6\delta - 2g - 6$, which is easily seen to be true. $\square$

LEMMA 5.4

If E has a line subbundle with $h^0 \geq 2$ and $d < 2\delta - 8 + 2g$, then

$$\gamma(E) > \frac{2\delta - 6}{3}.$$

Proof

Since $d_6 = 3\delta - 3 > \delta + 4$, we see from the proof of [14, Lemma 2.2] that $\gamma(E) > \frac{2\delta-6}{3}$. $\square$

LEMMA 5.5

Suppose that E is a bundle computing $\mathrm{Cliff}_3(C) = \frac{2\delta-6}{3}$. If $d \leq 2g+1$, then E fits into a nontrivial exact sequence

$$0 \to E_H \to E \to L \to 0$$

where $L \simeq H$ or $\simeq H^{\delta-4}$ and all sections of L lift to E.

Proof

Since $2g + 1 < 2\delta - 8 + 2g$, it follows from (5.1) and Lemma 5.4 that E has a subbundle F of rank 2 with $h^0(F) \geq 3$ and no line subbundle with $h^0 \geq 2$.

We follow the proof of [14, Lemma 2.3]. In the case $d_{2t} < 2t + g - 1$ and $d_u < u + g - 1$, the only possibility is that all the inequalities are equalities. This gives $t = 1$ (hence $h^0(F) = 3$), $d_F = d_2 = \delta$, and $d_u = \delta - 4 + 2u$. For any line subbundle M of F we have $h^0(M) \leq 1$. So $h^0(F/M) \geq 2$. Hence $d_{F/M} \geq d_1 = \delta - 1$. So $d_M \leq 1$ and F is stable.

As in the proof of Proposition 2.6 we see that F is generated and has the form $F \simeq E_N$ for some line bundle N of degree d_2 with $h^0(N) = 3$. The only such bundle is H. Moreover, $L := E/E_H$ is a line bundle such that either L or $K_C \otimes L^*$ computes $\mathrm{Cliff}(C)$. It follows from Noether's theorem that either $L \simeq H$ or $L \simeq K_C \otimes H^* \simeq H^{\delta-4}$.

In the argument leading up to [14, formula (2.3)], we have the inequality $\frac{\gamma(E)}{2} \geq \frac{t}{3} + \frac{g-1}{6}$. This gives $\gamma(E) \geq \frac{g+1}{3}$, which implies that $\gamma(E) > \frac{2\delta-6}{3}$.

Finally, for [14, formula (2.4)], we obtain $\gamma(E) > \frac{2\delta-6}{3}$ provided $d \leq 2g+1$. $\square$

THEOREM 5.6

If C is a smooth plane curve of degree $\delta \geq 7$ and $\mathrm{Cliff}_3(C) = \frac{2\delta-6}{3}$, then any bundle E computing $\mathrm{Cliff}_3(C)$ is stable and fits into an exact sequence

$$0 \to E_H \to E \to H \to 0, \tag{5.2}$$

and all sections of H lift to E. Moreover, such extensions exist if and only if $h^0(E_H \otimes E_H) \geq 10$.

Proof
The stability of E follows from Proposition 2.4. Next we eliminate the possibility that $L \simeq H^{\delta-4}$ in Lemma 5.5. In this case $d = 2g-2$, and we can check that

$$2g-2 > g + \frac{3}{2}\delta$$

for $\delta \geq 7$. It follows from Lemma 5.3 that $\gamma(E) > \frac{2\delta-6}{3}$, a contradiction.

Since $2g+1 > g + \frac{3}{2}\delta$, Lemmas 5.3 and 5.5 cover all possibilities for d. This implies the existence of (5.2). The last assertion follows from Proposition 2.6. □

We now consider the case when C is the normalization of a plane curve Γ of degree δ whose only singularities are ν simple nodes. Since Noether's theorem applies to Γ rather than C, we cannot use it directly to obtain information about C. However, many relevant facts are known about C.

For our purposes, we shall assume that the nodes are in general position and that

$$1 \leq \nu \leq \frac{1}{2}(\delta^2 - 7\delta + 14). \tag{5.3}$$

Note that, since C has genus $g = \frac{1}{2}(\delta-1)(\delta-2) - \nu$, (5.3) is equivalent to

$$g \geq 2\delta - 6. \tag{5.4}$$

By [3] and [5, Corollary 2.3.1], we have $\mathrm{Cliff}(C) = \delta - 4$, and this is computed by both d_1 and d_2. Moreover, there are finitely many line bundles $H_1, \dots, H_\ell$ of degree $d_2 = \delta$ with $h^0(H_i) = 3$; in fact, this is true for $g \geq \frac{3}{2}\delta - 3$ (or equivalently $\nu \leq \frac{1}{2}(\delta^2 - 6\delta + 8)$) by [22, Section 4]. (For $g < \frac{3}{2}\delta - 3$, the result must fail since this is equivalent to the Brill–Noether number for line bundles of degree δ with 3 independent sections on C being positive.)

We shall make the additional assumption that

$$d_4 \geq 2\delta - 4; \tag{5.5}$$

it follows then by [11, Theorem 5.2] that

$$\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C) = \delta - 4. \tag{5.6}$$

REMARK 5.7
For $\delta \geq 7$, we certainly have $d_4 \geq \delta + 4$ by (2.3). So (5.5) is satisfied for $\delta = 7$ or 8. The formula (5.5) also holds for $\nu \leq 4$. To see this it is sufficient to show that any line bundle L of degree $2\delta - 5$ has $h^0(L) \leq 4$. For this we can write $\pi : C \to \Gamma$ for the normalization map and apply [9, Theorem 2.1] to the torsion-free sheaf $\pi_*(L)$ which has degree $2\delta - 5 + \nu \leq 2\delta - 1$. When $\nu \leq 3$, we obtain immediately $h^0(L) = h^0(\pi_*(L)) \leq 4$. If $\nu = 4$, we note that $\pi_*(L)$ is not of the required form for $h^0(\pi_*(L)) = 5$.

Before proceeding to our main result, we shall prove a lemma which we shall also need in Section 6.

LEMMA 5.8

Let C be a curve of genus 9 with $\mathrm{Cliff}(C) = 3$. Suppose that E is a semistable bundle of rank 3 and degree 24 with $h^0(E) \geq 6$. Then $\gamma(E) \geq 3$.

Proof

Since $\mathrm{Cliff}(C) \leq 4$, we have $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ by Lemma 2.1; moreover $d_9 \geq 17$ by (2.3). So, by Theorem 4.1, $\gamma(E) \geq \frac{8}{3}$. If $\gamma(E) = \frac{8}{3}$, then clearly $h^0(E) = \frac{d-3\gamma(E)}{2} + 3 = 11$, so E possesses a line subbundle of degree at least 3. If this is a subbundle of maximal slope, then, by [14, Lemma 3.1] and its proof (see in particular [14, formula (3.4)]), $\gamma(E) \geq 3$, a contradiction. So every subbundle F of E of maximal slope must have $\mathrm{rk}\, F = 2$.

We now consider the proof of [14, Lemma 3.2]. The first three numbers and the last number in the minimum are certainly at least 3. The fourth number, however, is $\frac{8}{3}$. We can have $\gamma(E) = \frac{8}{3}$ if and only if all inequalities leading up to this are equalities. This implies that

$$F \text{ computes } \mathrm{Cliff}_2(C), \quad h^1(E/F) = 1, \; d_{E/F} = 14.$$

So $d_F = 10$. Since E has no line subbundle of maximal slope, the maximal slope of a line subbundle of F is 4. So F has no line subbundle with $h^0 \geq 2$. By Lemma 2.2 this implies that $d_F \geq d_4$, and so ≥ 11 by (2.3). This is a contradiction. $\square$

THEOREM 5.9

Suppose that C is the normalization of a nodal plane curve of degree $\delta \geq 7$ with ν nodes in general position and that (5.3) holds. Suppose further that (5.5) holds and $g \geq 9$. Then

$$\mathrm{Cliff}_3(C) \geq \frac{2\delta - 6}{3}.$$

Moreover, if $\mathrm{Cliff}_3(C) = \frac{2\delta-6}{3}$, then any bundle E computing $\mathrm{Cliff}_3(C)$ is stable and fits into an exact sequence

$$0 \to E_{H_i} \to E \to L \to 0, \tag{5.7}$$

where $3 \leq h^0(L) \leq g + 4 - \delta$, $d_L = \delta - 6 + 2h^0(L)$, and all sections of L lift to E.

Proof

Note first, using (5.5), that $d_9 \geq d_4 + 5 \geq 2\delta + 1$; so (5.1) holds. Since (5.6) also holds, Proposition 5.1 is valid with the same proof as before; so $\mathrm{Cliff}_3(C) \geq \frac{2\delta-6}{3}$.

Suppose now that $\mathrm{Cliff}_3(C) = \frac{2\delta-6}{3}$ and that E is a bundle computing $\mathrm{Cliff}_3(C)$. The proof of Lemma 5.2 remains valid. For Lemma 5.3, we need first the fact that $d > g + \frac{3}{2}\delta$ implies $d \geq 2\delta + 6$. This follows from (5.4) for $\delta \geq 8$ and can easily be checked for $\delta = 7$ and $g \geq 9$. The condition $d < 4g - 12$ holds for $d \leq 3g - 3$ provided $g > 9$. For $g = 9$ (which requires $\delta = 7$ by (5.4)), the condition still holds for $d < 3g - 3$; the case $d = 3g - 3$ is covered by Lemma 5.8. Finally $d > 6\delta - 2g - 6$ holds for $g > 2\delta - 6$ since $d \geq 2\delta + 6$; when $g = 2\delta - 6$, the condition $d > g + \frac{3}{2}\delta$ implies $d > 6\delta - 2g - 6$ for $\delta \geq 8$.

For Lemma 5.4, the requirement is $d_6 > \delta + 4$, which follows from (2.3) and (5.4). It follows that every subbundle of maximal slope of E has rank 2, so we can apply Lemma 5.5. There is a minor change in the proof since $d_1 = \delta - 2$, which means that F can have a line subbundle of degree 2; however, this does not affect the argument. On the other hand, the hyperplane bundle H is no longer unique, and we do not know all the bundles computing $\mathrm{Cliff}(C)$, so we just obtain the form (5.7) for the exact sequence defining E.

The remaining problem is that Lemmas 5.3 and 5.5 may no longer cover all cases. In fact $d \leq g + \frac{3}{2}\delta$ implies $d \leq 2g + 2$ under our assumptions, but it is possible to have $d = 2g + 2$ for low values of δ. In this case, we need to reexamine [14, formula (2.4)]; if $d = 2g + 2$, we still require $t = 1$, and hence $d_F = d_2 = \delta$; the quotient line bundle $L = E/F$ no longer computes the Clifford index, but it is still the case that $\gamma(L) = \delta - 4$, giving a sequence of the form (5.7). The stability of E follows from Proposition 2.4, while the inequalities for $h^0(L)$ come from $h^0(E) \geq 6$ and $d \leq 2g + 2$. □

REMARK 5.10

The only case in which the possibility $d = 2g + 2$ needs to be included in (5.7) under the assumptions of the theorem is when $\delta = 8$, $g = 10$. For small numbers of nodes, other possibilities can be excluded; for example, when $\delta = 7$ and $\nu = 1$ (so $g = 14$), we have $2g - 2 > g + \frac{3}{2}\delta$. We can therefore assume $d \leq 2g - 4$ in (5.7), corresponding to $h^0(L) \leq 8$.

REMARK 5.11

The excluded case $\delta = 7$, $g = 8$ will be covered in Section 6 (Proposition 6.6), as will the case $\delta = 7$, $g = 7$ (hence $\nu = 8$). In the latter case, it is proved in [3] that $d_1 = 5$ but this does not imply that $\mathrm{Cliff}(C) = 3$ since there are infinitely many pencils on C with degree 5. Thus Theorem 5.9 does not apply, but a modified version does hold (see Proposition 6.5), perhaps under stronger generality conditions.

6. Curves with Clifford index three

Let C be a curve of genus g with $\mathrm{Cliff}(C) = 3$ and hence $g \geq 7$. We have $d_9 \geq 16$ for $g \geq 8$ from (2.3). For $g = 7, d_9 = 16$ by Riemann–Roch. By Theorem 4.1, we have

$$\frac{8}{3} \leq \mathrm{Cliff}_3(C) \leq 3.$$

Hence any bundle E computing $\mathrm{Cliff}_3(C)$ possesses a proper subbundle F with $h^0(F) \geq \mathrm{rk}\, F + 1$.

We now consider the possibility that $\gamma(E) = \frac{8}{3}$. Note that this can happen only if d is even. We suppose throughout that E is a bundle of degree d computing $\mathrm{Cliff}_3(C)$.

LEMMA 6.1

If E possesses a line subbundle with $h^0 \geq 2$ and $d \leq 2g+5$, then

$$\gamma(E) \geq 3.$$

Proof

Consider the proof of [14, Lemma 2.2]. Noting that $d_6 \geq 12$ by (2.3), we see that the only possibility for having $\gamma(E) < 3$ in the proof of [14, Lemma 2.2] is the inequality

$$\gamma(E) \geq \frac{1}{3}\big(4\,\mathrm{Cliff}(C) + 2g + 2 - d\big) = 4 + \frac{1}{3}(2g+2-d).$$

This gives $\gamma(E) \geq 3$. □

REMARK 6.2

Since we always have $d \leq 3g-3$, the assumption $d \leq 2g+5$ is redundant for $g=7$ and $g=8$.

LEMMA 6.3

Suppose that there exists an exact sequence

$$0 \to F \to E \to E/F \to 0 \tag{6.1}$$

with $\mathrm{rk}\, F = 2$ *and $h^0(F) \geq 3$, and suppose that E has no line subbundle with $h^0 \geq 2$. If $d \leq 2g+2$ and $\gamma(E) = \frac{8}{3}$, then*

$$d_F = d_2 = 7, \qquad h^0(F) = 3, \qquad h^0(E/F) \geq 3, \qquad d_{E/F} = 1 + 2h^0(E/F).$$

Moreover, all sections of E/F lift to E.

Proof

We follow the proof of [14, Lemma 2.3]. The first case to be considered is when $d_{2t} < 2t + g - 1$ and $d_u < u + g - 1$. Then we have $\gamma(E) = \frac{8}{3}$ only if $t = 1$ (hence $h^0(F) = 3$), $d_2 = d_F$, and $d_u = d_{E/F}$; moreover, $d_2 = \mathrm{Cliff}(C) + 4 = 7$ and $d_u = \mathrm{Cliff}(C) + 2u = 1 + 2h^0(E/F)$. Since $h^0(E) \geq 6$, we have also $h^0(E/F) \geq 3$. Moreover, $d = 10 + 2u$; since $\gamma(E) = \frac{8}{3}$, this gives $h^0(E) = 4 + u = h^0(F) + h^0(E/F)$. Hence all sections of E/F lift to E.

The case of [14, formula (2.3)] can give $\gamma(E) = \frac{8}{3}$ only if $t = 1$. In this case the hypothesis $d_{2t} \geq 2t + g - 1$ gives $d_2 \geq g + 1$, which is impossible. This leaves us with the case of [14, formula (2.4)]. If $d \leq 2g+1$, this gives $\gamma(E) \geq 3$. For $d = 2g+2$ we must have $t = 1, d_F = d_2 = 7, u = g - 4$, and $d_{E/F} = d_u = 2g - 5$. The result follows as in the first part of the proof. □

LEMMA 6.4

Suppose that there exists an exact sequence (6.1) with $\mathrm{rk}\, F = 2$ *and $h^0(F) \geq 3$ and that E has no line subbundle with $h^0 \geq 2$. If $d = 2g+4$ and $\gamma(E) = \frac{8}{3}$, then $h^0(E) = g+1, h^0(F) = 3$, and either*

- $d_F = d_2 = 7$, $d_{E/F} = 2g - 3$, $h^0(E/F) = g - 2$ *or* $g - 1$, *or*
- $d_F = 8$, $d_{E/F} = 2g - 4$, $h^0(E/F) = g - 2$.

Proof
Formula (2.4) of [14] gives

$$\gamma(E) \geq \frac{2\,\mathrm{Cliff}(C) + 6t - 6}{3}.$$

For $\gamma(E) = \frac{8}{3}$ we still need $t = 1$, so $h^0(F) = 3$, but it is now possible that $d_F = 8$.

We have $h^0(E) = g + 1$, since $\gamma(E) = \frac{8}{3}$. Hence $h^0(E/F) \geq g - 2$. The rest follows from Riemann–Roch. □

PROPOSITION 6.5
Let C be a curve of genus $g = 7$ with $\mathrm{Cliff}(C) = 3$, and suppose that $\mathrm{Cliff}_3(C) = \frac{8}{3}$. Then E is stable and fits into an exact sequence (6.1) with $h^0(F) = 3$. Moreover, one of the following holds:

- $d_F = 7$, $d_{E/F} = 7$, $h^0(E/F) = 3$, $h^0(E) = 6$,
- $d_F = 7$, $d_{E/F} = 9$, $h^0(E/F) = 4$, $h^0(E) = 7$,
- $d_F = 7$, $d_{E/F} = 11$, $h^0(E/F) = 5$ *or* 6, $h^0(E) = 8$,
- $d_F = 8$, $d_{E/F} = 10$, $h^0(E/F) = 5$, $h^0(E) = 8$.

Proof
Stability of E follows from Proposition 2.4. Since $d \leq 3g - 3$, the rest follows from Lemmas 6.1, 6.3, and 6.4. □

PROPOSITION 6.6
Let C be a curve of genus $g = 8$ with $\mathrm{Cliff}(C) = 3$, and suppose that $\mathrm{Cliff}_3(C) = \frac{8}{3}$. Then E is stable and fits into an exact sequence (6.1) with $h^0(F) = 3$. Moreover, one of the following holds:

- $d_F = 7$, $d_{E/F} = 7$, $h^0(E/F) = 3$, $h^0(E) = 6$,
- $d_F = 7$, $d_{E/F} = 9$, $h^0(E/F) = 4$, $h^0(E) = 7$,
- $d_F = 7$, $d_{E/F} = 11$, $h^0(E/F) = 5$, $h^0(E) = 8$,
- $d_F = 7$, $d_{E/F} = 13$, $h^0(E/F) = 6$ *or* 7, $h^0(E) = 9$,
- $d_F = 8$, $d_{E/F} = 12$, $h^0(E/F) = 6$, $h^0(E) = 9$.

For the general curve of genus 8 only the last possibility can occur.

Proof
The stability of E follows from Proposition 2.4. Since $d \leq 3g - 3$, the various possibilities for (6.1) follow from Lemmas 6.1, 6.3, and 6.4. For the last assertion note that the general curve of genus 8 has $d_2 = 8$. □

For $g \geq 9$ we need to consider the possibility that $d \geq 2g + 6$. For this we use the results of [14, Section 3].

PROPOSITION 6.7

Let C be a curve of genus $g \geq 9$ with $\mathrm{Cliff}(C) = 3$*, and suppose that* $\mathrm{Cliff}_3(C) = \frac{8}{3}$*. Then $d_2 = 7, 14 \leq d \leq 2g$, and E is stable and fits into an exact sequence (6.1) with*

$$\mathrm{rk}\, F = 2, \qquad d_F = 7, \qquad h^0(F) = 3, \qquad d_{E/F} = d - 7, \qquad h^0(E/F) = \frac{d-8}{2},$$

and all sections of E/F lift to E.

Proof

Once again stability follows from Proposition 2.4.

Suppose that E possesses a subbundle L of maximal slope of rank 1. The first and third numbers in the minimum of [14, Lemma 3.1] are clearly greater than $\frac{8}{3}$. (This requires only $d \geq 11$.) By [14, formula (3.4)], we see that the second number can be replaced by $\frac{2\,\mathrm{Cliff}_2(C)+d_L}{3}$, so we must have $d_L \leq 2$. It follows that E has no line subbundle with $h^0 \geq 2$. Hence E possesses a subbundle of rank 2 with $h^0 \geq 3$, which is stable and therefore of degree at least 7. This contradicts the definition of L, so every subbundle F of maximal slope has rank 2.

Let F be such a subbundle, and suppose $d \geq 2g + 2$. The first 3 numbers in the minimum of the statement of [14, Lemma 3.2] are greater than $\frac{8}{3}$. (The requirement for this is $d \geq g + 11$.) The fourth number is greater than $\frac{8}{3}$ if and only if $d < 4g - 12$. Since $d \leq 3g - 3$, this holds always if $g \geq 10$. For $g = 9$ the fourth number is greater than $\frac{8}{3}$ for $d < 3g - 3$. The remaining case $g = 9$, $d = 24$ is covered by Lemma 5.8. The last number is greater than $\frac{8}{3}$ if and only if $d > 36 - 2g$. This holds for $d \geq 2g + 2$ if $g \geq 9$.

We are left with the case $d \leq 2g$. The result now follows from Lemma 6.3. □

THEOREM 6.8

Let C be a curve of genus $g \geq 9$ with $\mathrm{Cliff}(C) = 3$*. If $d_2 > 7$, and in particular if $g \geq 16$, then*

$$\mathrm{Cliff}_3(C) = 3.$$

For all $g \geq 9$ there exist curves with these properties.

Proof

The first assertion follows from Proposition 6.7 once we know that $d_2 \geq 8$ whenever $g \geq 16$. In fact, if $d_2 = 7$, then C possesses as a plane model a septic. Hence $g \leq 15$.

For $9 \leq g \leq 15$ note that by the Hurwitz formula the family of curves with Clifford index 3 is of dimension $2g + 5$. On the other hand, the family of plane septics of genus g is of dimension $12 + g < 2g + 5$. This proves the final statement. □

COROLLARY 6.9

Let C be a smooth complete intersection of 2 cubics in $\mathbb{P}^3$. Then

$$\mathrm{Cliff}_3(C) = \mathrm{Cliff}(C) = 3.$$

Proof

It is known that C has Clifford dimension 3, genus 10, and $\mathrm{Cliff}(C) = 3$ (see [6]). In particular d_2 does not compute $\mathrm{Cliff}(C)$. So $d_2 > 7$. □

The curves of this corollary are the only curves of Clifford dimension ≥ 3 with $\mathrm{Cliff}(C) = 3$ (see [6]).

REMARK 6.10

Suppose that C is a curve of genus $g \geq 9$ with $\mathrm{Cliff}(C) = 3$ and $d_2 = 7$. Then C possesses as a plane model a septic. For $g = 15$ this model is smooth and Theorem 5.6 applies. In particular $\mathrm{Cliff}_3(C) = \frac{8}{3}$ if and only if $h^0(E_H \otimes E_H) \geq 10$.

If $9 \leq g \leq 14$, then the general curve of this type is the normalization of a nodal septic with nodes in general position, so Theorem 5.9 applies and gives a somewhat more precise result.

7. Coherent systems

Recall that a *coherent system of type* (n, d, k) on a curve C is a pair (E, V) where E is a vector bundle of rank n and degree d on C and V is a linear subspace of $H^0(E)$ of dimension k. For any $\alpha > 0$ we define the *α-slope* of (E, V) by

$$\mu_\alpha(E, V) := \frac{d}{n} + \alpha\frac{k}{n}.$$

The coherent system (E, V) is called *α-stable* (*α-semistable*) if, for all proper coherent subsystems (F, W) of (E, V),

$$\mu_\alpha(F, W) < (\leq)\, \mu_\alpha(E, V).$$

PROPOSITION 7.1

Suppose E computes $\mathrm{Cliff}_n(C)$ and $\mathrm{Cliff}_r(C) \geq \mathrm{Cliff}_n(C)$ for all $r \leq n$. Then $(E, H^0(E))$ is α-semistable for all $\alpha > 0$. If also E is stable, then $(E, H^0(E))$ is α-stable for all $\alpha > 0$.

Proof

Write $h^0(E) = n + s$ with $s \geq n$. If F is any subbundle of E, then $\mu(G) \leq \frac{d}{n}$ for any subbundle G of F. We need to show that

$$\frac{h^0(F)}{\mathrm{rk}\, F} \leq \frac{n+s}{n}.$$

If this is not true, then by [12, Lemma 2.1] we have

$$\gamma(E) = \frac{d-2s}{n} > \min\left\{\gamma(G) \;\middle|\; \begin{array}{l} G \text{ semistable}, \mathrm{rk}\, G \leq n, \\ \frac{d_G}{\mathrm{rk}\, G} \leq \frac{d}{n}, \frac{h^0(G)}{\mathrm{rk}\, G} \geq \frac{n+s}{n} \end{array}\right\}.$$

All such G contribute to $\mathrm{Cliff}_{\mathrm{rk}\,G}(C)$. Since $\mathrm{Cliff}_r(C) \geq \mathrm{Cliff}_n(C)$ for all $r \leq n$, we obtain $\gamma(E) > \mathrm{Cliff}_n(C)$, a contradiction. □

REMARK 7.2

In the case $n = 2$, the hypotheses of Proposition 7.1 hold. For $n = 3$, they reduce to $\mathrm{Cliff}_3(C) \leq \mathrm{Cliff}_2(C)$. We have seen in this paper that this hypothesis does not always apply.

REMARK 7.3

Under the same hypotheses as those of Proposition 7.1, it was proved in [12] that E is generated. We therefore have an evaluation sequence

$$0 \to M_E \to H^0(E) \otimes \mathcal{O}_C \to E \to 0. \tag{7.1}$$

A version of a conjecture of D. C. Butler [2] states that, for general stable E, the kernel M_E should be stable. Of course our bundles are not general, but it is still of interest to ask whether M_E is stable (or semistable) when the hypotheses of Proposition 7.1 hold. It has recently been noted by L. Brambila-Paz that the conclusion of the proposition is a necessary condition for the stability of M_E. For a line bundle L on a nonhyperelliptic curve C, it follows from [1, Theorem 1.3] that M_L is stable. (This has also been proved by E. Mistretta and L. Stoppino [19, Corollary 5.5].)

8. Further comments and open problems

There are several problems in connection with Section 3.

QUESTION 8.1

For curves of genus $g \geq 14$ satisfying (3.1), is it true that $\frac{d_9}{3} - 2 > \mathrm{Cliff}_2(C)$?

COMMENT

Note that by Lemma 3.8 the inequality holds for $14 \leq g \leq 24$. If the answer to the question is yes, then Theorem 3.9 holds for $g \geq 16$. The cases $g = 14$ and $g = 15$ require further investigation.

QUESTION 8.2

Can we extend Theorem 3.9 to values of g below 16?

QUESTION 8.3

On curves satisfying (3.1), can we determine $\mathrm{Cliff}_3(C)$ and identify bundles computing it? If so, do any of these bundles fail to be generated?

COMMENT

In connection with the last question, see Proposition 7.1 and Remark 7.3.

Moving on to Section 5, the following question looks interesting.

QUESTION 8.4

For the hyperplane bundle H on a (smooth) plane curve, is it true that $h^0(E_H \otimes E_H) \geq 10$?

COMMENT

It seems possible that the answer to this question is known. Note that for a smooth plane curve of degree $\delta \geq 7$, we have $\mathrm{Cliff}_3(C) = \frac{2\delta - 6}{3}$ if and only if the answer is yes.

QUESTION 8.5

If C is the normalization of a nodal plane curve Γ, under what conditions is it true that $d_4 \geq 2\delta - 4$?

QUESTION 8.6

For a curve C as in the previous question, under what conditions is it true that C possesses a unique line bundle H of degree δ with $h^0(H) = 3$?

COMMENT

We know this is true if $\nu = 0$. It is also true whenever every pencil on C is represented as a pencil of lines through a point of Γ. Under more restrictive conditions on ν than those given by (5.3), but without any assumptions of general position and allowing simple cusps as well as nodes, this is shown to be true in [4, Theorems 2.4 and 5.2].

Turning to Section 6, we can ask the following question.

QUESTION 8.7

For curves of Clifford index 3, can we determine when $\mathrm{Cliff}_3(C) = \frac{8}{3}$?

COMMENT

Any such curve must be one of the following:

- a smooth plane septic;
- a curve of genus g, $7 \leq g \leq 14$, which is representable by a singular plane septic;
- a curve of genus 8 with $d_2 = 8$.

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References

[1] L. Brambila-Paz and A. Ortega, "Brill–Noether bundles and coherent systems on special curves" in *Moduli Spaces and Vector Bundles*, London Math. Soc. Lecture Notes Ser. **359**, Cambridge Univ. Press, Cambridge, 2009, 456–472. MR 2537078.

[2] D. C. Butler, *Birational maps of moduli of Brill–Noether pairs*, preprint, arXiv:alg-geom/9705009

[3] M. Coppens, *The gonality of general smooth curves with a prescribed plane nodal model*, Math. Ann. **289** (1991), 89–93. MR 1087238. DOI 10.1007/BF01446560.

[4] M. Coppens and T. Kato, *The gonality of smooth curves with plane models*, Manuscripta Math. **70** (1990), 5–25. MR 1080899. DOI 10.1007/BF02568358.

[5] M. Coppens and G. Martens, *Secant spaces and Clifford's theorem*, Compositio Math. **78** (1991), 193–212. MR 1104787.

[6] D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer, *The Clifford dimension of a projective curve*, Compositio Math. **72** (1989), 173–204. MR 1030141.

[7] G. Farkas and A. Ortega, *The maximal rank conjecture and rank two Brill–Noether theory*, Pure Appl. Math. Q. **7** (2011), 1265–1296. MR 2918161.

[8] ———, *Higher rank Brill–Noether theory on sections of K3 surfaces*, Int. J. Math. **23**, published electronically 12 June 2012. DOI 10.1142/S0129167X/2500759.

[9] R. Hartshorne, *Generalized divisors on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. **26** (1986), 375–386. MR 0857224.

[10] H. Lange, V. Mercat, and P. E. Newstead, *On an example of Mukai*, Glasg. Math. J. **54** (2012), 261–271. MR 2911367. DOI 10.1017/S0017089511000577.

[11] H. Lange and P. E. Newstead, "Clifford indices for vector bundles on curves" in *Affine Flag Manifolds and Principal Bundles*, Trends Math., Birkhäuser, Basel, 2010, 165–202.

[12] ———, *Generation of vector bundles computing Clifford indices*, Arch. Math. (Basel) **94** (2010), 529–537. MR 2653669. DOI 10.1007/s00013-010-0126-0.

[13] ———, *Further examples of stable bundles of rank* 2 *with* 4 *sections*, Pure Appl. Math. Q. **7** (2011), 1517–1528. MR 2918172.

[14] ———, *Lower bounds for Clifford indices in rank three*, Math. Proc. Camb. Philos. Soc. **150** (2011), 23–33. MR 2739071. DOI 10.1017/S0305004110000502.

[15] ———, *Vector bundles of rank* 2 *computing Clifford indices*, to appear in Comm. in Algebra, preprint, arXiv:1012.0469v3 [mathAG]

[16] ———, "Bundles of rank 2 with small Clifford index on algebraic curves" in *Geometry and Arithmetic*, EMS Congress Reports, Eur. Math. Soc., Zurich, 2012, 267–281.

[17] M. Lelli-Chiesa, *Stability of rank-*3 *Lazarsfeld–Mukai bundles on arbitrary K3-surfaces*, preprint, arXiv:1112.2938v1 [math.AG]

[18] V. Mercat, *Clifford's theorem and higher rank vector bundles*, Int. J. Math. **13** (2002), 785–796. MR 1921510. DOI 10.1142/S0129167X02001484.

[19] E. C. Mistretta and L. Stoppino, *Linear series on curves: Stability and Clifford index*, Int. J. Math., published electronically 26 November 2012. DOI 10.1142/S0129167X12501212.

[20] S. Mukai and F. Sakai, *Maximal subbundles of vector bundles on a curve*, Manuscripta Math. **52** (1985), 251–256. MR 0790801. DOI 10.1007/BF01171494.

[21] K. Paranjape and S. Ramanan, "On the canonical ring of a curve" in *Algebraic Geometry and Commutative Algebra, Vol. II*, Kinokuniya, Tokyo, 1988, 503–516. MR 0977775.

[22] E. Sernesi, *On the existence of certain families of curves*, Invent. Math. **75** (1984), 25–57. MR 0728137. DOI 10.1007/BF01403088.

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