Uniform large deviations for multivalued stochastic differential equations with Poisson jumps

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Abstract Based on a variational representation for functionals of a general Poisson random measure plus an independent infinite-dimensional Brownian motion developed by Budhiraja, Dupuis, and Maroulas, the Freidlin-Wentzell large deviation principle is established for multivalued stochastic differential equations with Poisson jumps in this paper.

1. Introduction

Consider the following multivalued stochastic differential equation (MSDE) with Poisson jumps:

$$\begin{cases} dX(t) \in -A(X(t))\,dt + b(X(t))\,dt + \sigma(X(t))\,dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X(t-),y)\tilde{N}(dt,dy), \\ X(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{1}$$

where A is a multivalued maximal monotone operator (see Definition 2.1) on $\mathbb{R}^d$, $b:\mathbb{R}^d \to \mathbb{R}^d$, $\sigma:\mathbb{R}^d \to \mathbb{R}^d \times l^2$, l^2 denotes the usual sequence Hilbert space and $\gamma:\mathbb{R}^d \times \mathbb{Y} \to \mathbb{R}^d$ are measurable functions, W is a sequence of independent standard Brownian motions, N is a Poisson random measure defined on $(\Omega,\mathcal{F},P)$ with intensity measure ν and $\tilde{N}(dt,dy) := N(dt,dy) - \nu(dy)\,dt$ is the compensated measure, N and W are independent, and $\mathbb{Y}$ is a locally compact Polish space.

When $\gamma \equiv 0$, (1) becomes the following continuous MSDE:

$$\begin{cases} dX(t) \in -A(X(t))\,dt + b(X(t))\,dt + \sigma(X(t))\,dW(t), \\ X(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{2}$$

which has been studied by Cépa [7]. In the same paper Cépa observed that when $A = \partial I_D$ is the subdifferential operator of the indicator function of a convex and closed domain D with nonempty interior, (2) is equivalent to the following

Kyoto Journal of Mathematics, Vol. 51, No. 3 (2011), 535–559
DOI 10.1215/21562261-1299891,
Received June 14, 2010. Revised November 4, 2010. Accepted November 8, 2010.
2010 Mathematics Subject Classification: Primary 60H10; Secondary 60J75, 60F10.
Author's work supported by National Science Foundation of China grant 10871215 and Doctoral Academic Newcomer Award (2010).

stochastic differential equation with reflecting boundary:

$$\begin{cases} dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t) - dK(t), \\ X(0) = x_0 \in \bar{D}. \end{cases}$$

After that, there are some works on the properties of solutions to (2) (see, e.g., [14], [15]).

When $\gamma \neq 0$, (1) is an MSDE with jumps which has been studied in a previous work [13]. There we proved the existence of a unique solution in the sense of Definition 2.3 below, under an additional assumption that $D(A) = \mathbb{R}^d$. Later in [17], we relaxed this additional assumption to (**H**3) in Section 3. In particular, by an argument similar to Cépa's, it is trivial to prove that when $A = \partial I_D$, where D is a convex and closed domain with nonempty interior, (1) is equivalent to a stochastic differential equation (SDE) with reflecting boundary:

$$\begin{cases} dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X(t-), y)\tilde{N}(dt, dy) - dK(t), \\ X(0) = x_0 \in \bar{D}, \end{cases} \tag{3}$$

which has been investigated in [12] for the case $\mathbb{Y} = \mathbb{R}^m \backslash \{0\}$.

On the other hand, Ren, Xu, and Zhang [15] recently proved the Freidlin-Wentzell large deviation principle for MSDEs like (2) by using the weak convergence method developed by Dupuis and Ellis [9]. Their idea is based on some variational representations about the Laplace transform of bounded continuous functionals of finite- and infinite-dimensional Brownian motions (see [3], [4]), which leads to the equivalence between the Laplace principle and the large deviation principle. Later in [18], these representations are generalized to abstract Wiener spaces. In this respect, there exist other works on large deviations for stochastic equations driven by Brownian motions (see, e.g., [16]).

Compared with the discretization method, the main advantage of the weak convergence method is that some exponential probability estimates and the discretization arguments can be avoided. But it had not been applied to diffusion processes with jumps until recently; a variational representation for functionals of a Poisson random measure satisfying certain conditions has been obtained in [19] through a Clark-Ocone formula. More recently, Budhiraja, Dupuis, and Maroulas [6] have extended the result to bounded measurable functionals of a general Poisson random measure plus an independent infinite-dimensional Brownian motion under weaker conditions. Later, Maroulas [11] put forward a uniform large deviation.

In this paper we want to use the new representation obtained in [6] to establish a uniform large deviation principle for (1). More precisely, consider the following perturbed equation:

$$\begin{cases} dX^\epsilon(t) \in -A(X^\epsilon(t))\,dt + b(X^\epsilon(t))\,dt + \sqrt{\epsilon}\sigma(X^\epsilon(t))\,dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X^\epsilon(t-), y)(\epsilon N^{\epsilon^{-1}}(dt, dy) - \nu(dy)\,dt) \\ X^\epsilon(0) = x_0 \in \overline{D(A)}, \quad \epsilon \in (0, 1]. \end{cases} \tag{4}$$

Denote the solution as (X^ϵ, K^ϵ). Our aim is to establish a uniform large deviation principle for the law of X^ϵ in $\mathbb{D} := \mathcal{D}([0,T] \times \overline{D(A)}; \overline{D(A)})$. Here $\mathcal{D}([0,T] \times \overline{D(A)}; \overline{D(A)})$ denotes the space of càdlàg functions from $[0,T] \times \overline{D(A)}$ to $\overline{D(A)}$. To this aim, we consider the corresponding controlled equation

$$\begin{cases} dX^{\epsilon,u_\epsilon}(t) \in -A(X^{\epsilon,u_\epsilon}(t))\,dt + b(X^{\epsilon,u_\epsilon}(t))\,dt + \sigma(X^{\epsilon,u_\epsilon}(t))\psi_\epsilon(t)\,dt \\ \qquad\qquad + \sqrt{\epsilon}\sigma(X^{\epsilon,u_\epsilon}(t))\,dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X^{\epsilon,u_\epsilon}(t-),y)(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(dt,dy) - \nu(dy)\,dt); \\ X^{\epsilon,u_\epsilon}(0) = x_0 \in \overline{D(A)}, \quad \epsilon \in (0,1], \end{cases}$$

where $u_\epsilon = (\psi_\epsilon, \varphi_\epsilon)$ is defined as in Section 3. Compared with SDEs considered in [6], the main difficulty lies in proving the tightness of X^{ϵ,u_ϵ} in $\mathbb{D}$ due to the existence of the operator A. To overcome this, we switch to another method by applying the relative entropy and convergence in probability as our tools.

The paper is organized as follows. In Section 2, some notions and notations about multivalued SDEs and the Laplace principle are presented. We state our main result and give a detailed proof in Section 3. Finally, applications to SDEs and SDEs with reflecting boundaries are given in Section 4.

Throughout the paper, c and C with or without indexes are constants whose values may change from line to line.

2. Preliminaries

First of all, we present some notations on multivalued maximal monotone operator A. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space.

DEFINITION 2.1

By a multivalued operator on $\mathbb{R}^d$, we mean an operator A from $\mathbb{R}^d$ to $2^{\mathbb{R}^d}$. We set

$$D(A) := \{x \in \mathbb{R}^d : A(x) \neq \emptyset\},$$
$$\mathrm{Gr}(A) := \{(x,y) \in \mathbb{R}^{2d} : x \in D(A), y \in A(x)\}.$$

(1) A multivalued operator A is called *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall (x_1,y_1),(x_2,y_2) \in \mathrm{Gr}(A).$$

(2) A monotone operator A is called *maximal monotone* if and only if

$$(x_1,y_1) \in \mathrm{Gr}(A) \Leftrightarrow \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall (x_2,y_2) \in \mathrm{Gr}(A).$$

We collect here some facts about the maximal monotone operator which we use in the paper. For more details, we refer the reader to [7].

PROPOSITION 2.2

Let A be a maximal monotone operator on $\mathbb{R}^d$. Then we have the following.

(i) $\mathrm{Int}(D(A))$ *and* $\overline{D(A)}$ *are convex subsets of* $\mathbb{R}^d$, *and* $\mathrm{Int}(D(A)) = \mathrm{Int}(\overline{D(A)})$.

(ii) *For each* $x \in D(A)$, $A(x)$ *is a closed and convex subset of* $\mathbb{R}^d$. *Let* $A^\circ(x) := \mathrm{proj}_{A(x)}(0)$ *be the minimal section of* A, *where* proj_D *is designated as the projection on every closed and convex subset* D *on* $\mathbb{R}^d$ *and* $\mathrm{proj}_\emptyset(0) = \infty$. *Then*

$$x \in D(A) \Leftrightarrow |A^\circ(x)| < +\infty.$$

Now we give the definition and some properties of solutions to (2).

DEFINITION 2.3

A pair of processes (X, K) is called a *strong solution* of (1) if X, K are $(\mathcal{F}_t)$-adapted processes satisfying

(i) X is càdlàg, $X_0 = x_0$, and $X(t) \in \overline{D(A)}$ for every $t \geq 0$;

(ii) K is continuous, $K_0 = 0$, and the total variation $|K|_T^0 < \infty$ almost surely for any $0 < T < \infty$;

(iii) $X(t) = x_0 + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s) + \int_0^t \int_{\mathbb{Y}} \gamma(X(s-), y)\tilde{N}(ds, dy) - K(t)$, $0 \leq t < \infty$, almost surely;

(iv) for any càdlàg and $(\bar{\mathcal{F}}_t)$-adapted functions (α, β) with

$$\big(\alpha(t), \beta(t)\big) \in \mathrm{Gr}(A), \quad \forall t \in [0, +\infty),$$

the measure $\langle X(t) - \alpha(t), \mathrm{d}K(t) - \beta(t)\,dt\rangle \geq 0$ almost surely.

The two lemmas below are on the solutions. For the proofs, we refer the reader to [7].

LEMMA 2.4

Let (X, K) *and* (X', K') *be two pairs of processes satisfying (i), (ii), and (iv) of the above definition. Then*

$$\langle X(t) - X'(t), dK(t) - dK'(t)\rangle \geq 0.$$

LEMMA 2.5

Suppose that $\mathrm{Int}(D(A)) \neq \emptyset$. *Then there exists an* $a \in \mathbb{R}^d$ *and* $r > 0$, $\mu \geq 0$ *such that for any pair* (X, K) *satisfying Definition 2.3,*

$$\int_s^t \langle X(v) - a, dK(v)\rangle \geq r|K|_t^s - \mu \int_s^t |X(v) - a|\,dv - r\mu(t - s),$$

where $|K|_t^s$ *denotes the total variation of* K *on* $[s, t]$.

Now we recall some notations and results on the large deviation principle from [6]. Let $\mathbb{Y}$ be a locally compact Polish space, and denote by $\mathcal{M}_R(\mathbb{Y})$ the space of Radon measures on it. Endow $\mathcal{M}_R(\mathbb{Y})$ with the weakest topology such that for every $f \in \mathcal{C}_c(\mathbb{Y})$,

$$\mathcal{M}_R(\mathbb{Y}) \ni \nu \to \int_{\mathbb{Y}} f(y)\nu(dy)$$

is continuous. This topology can be metricized so that $\mathcal{M}_R(\mathbb{Y})$ is a Polish space and convergence in the metric is equivalent to weak convergence on each compact subset of $\mathbb{Y}$ (see [8, Chapter VIII]).

Fix $0 < T < \infty$. Let $\mathbb{Y}_T := [0,T] \times \mathbb{Y}$, and let $\mathbb{Y}_T^\infty := \mathbb{Y}_T \times [0,\infty)$. Note that $\mathbb{R}^\infty$ endowed with the topology of coordinate-wise convergence is a Polish space. Denote by $\mathbb{W}$ the Polish space $\mathcal{C}([0,T];\mathbb{R}^\infty)$, and let $\mathbb{V} := \mathbb{W} \times \mathcal{M}_R(\mathbb{Y}_T)$ and $\bar{\mathbb{V}} := \mathbb{W} \times \mathcal{M}_R(\mathbb{Y}_T^\infty)$.

Define canonical maps $N : \mathbb{V} \to \mathcal{M}_R(\mathbb{Y}_T)$ and $\beta_i : \mathbb{V} \to \mathbb{W}$ as

$$N(w,m)([0,t] \times A) := m([0,t] \times A), \quad 0 \le t \le T,\ A \in \mathcal{B}(\mathbb{Y})$$

$$\beta_i(w,m) := w_i, \quad i \ge 1.$$

Let $\beta := (\beta_i)_{i=1}^\infty$. Similarly, we can define maps $\bar{N} : \bar{\mathbb{V}} \to \mathcal{M}_R(\mathbb{Y}_T^\infty)$ and β on $\bar{\mathbb{V}}$.

Fix $\nu \in \mathcal{M}_R(\mathbb{Y})$. Define $\mathcal{G}_t := \sigma\{N([0,s] \times A), \beta_i(s); 0 \le s \le t, A \in \mathcal{B}(\mathbb{Y})\}$. For $\theta > 0$, denote by P_θ the unique probability measure on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$ under which

(1) $(\beta_i)_{i=1}^\infty$ is a family of standard Brownian motions;

(2) N is a Poisson random measure with intensity measure $\theta\nu_T$, where $\nu_T = \lambda_T \otimes \nu$, λ_T is the Lebesgue measure on $[0,T]$;

(3) for every $i \ge 1$ and $A \in \mathcal{B}(\mathbb{Y})$, $\{\beta_i\}$ and $\{N([0,t] \times A)\}$ are independent of each other.

Define $(\bar{P}, \{\overline{\mathcal{G}}_t\})$ on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}))$ analogously by replacing $(N, \theta\nu_T)$ with $(\bar{N}, \bar{\nu}_T)$, where $\bar{\nu}_T = \lambda_T \otimes \nu \times \lambda_\infty$ and where λ_∞ is the Lebesgue measure on $[0,\infty)$. Let $\bar{\mathcal{F}}_t$ be the $\bar{P}$-completion of $\bar{\mathcal{G}}_t$.

Let $\bar{\mathcal{P}}$ be the predictable σ-field on $[0,T] \times \bar{\mathbb{V}}$ with respect to $\{\bar{\mathcal{F}}_t, 0 \le t \le T\}$, and

$$\bar{\mathcal{A}}_1 := \Big\{\psi = (\psi_i)_{i=1}^\infty; \psi_i \in \bar{\mathcal{P}} \backslash \mathcal{B}(\mathbb{R}), \int_0^T \|\psi(s)\|_{l^2}^2\, ds < \infty \text{ almost surely } \bar{P}\Big\},$$

$$\bar{\mathcal{A}}_2 := \big\{\varphi : \mathbb{Y}_T \times \bar{\mathbb{V}} \to [0,\infty);\ \varphi \in \bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{Y}) \backslash \mathcal{B}[0,\infty)\big\},$$

$$\mathcal{U} := \bar{\mathcal{A}}_1 \times \bar{\mathcal{A}}_2.$$

DEFINITION 2.6

For $\varphi \in \bar{\mathcal{A}}_2$, define a counting process N^φ on $\mathbb{Y}_T$: for $t \le T$, $A \in \mathcal{B}(\mathbb{Y})$,

$$N^\varphi([0,t] \times A) := \int_{[0,t] \times A} \int_{(0,\infty)} \mathbb{1}_{[0,\varphi(s,y)]}(r) \bar{N}(ds, dy, dr).$$

N^φ is called a *controlled random measure*, with φ selecting the intensity measure for the points at location y and time s.

Before proceeding further, we give the definition and a property of relative entropy (see [3]).

DEFINITION 2.7

Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathcal{P}(\Omega)$ be the set of probability measures defined on it. For $\theta \in \mathcal{P}(\Omega)$, the relative entropy function $R(\cdot\|\theta)$ is the mapping from $\mathcal{P}(\Omega)$ given by

$$R(\mu\|\theta) := \int_\Omega \Big(\log \frac{d\mu}{d\theta}(\omega)\Big)\mu(d\omega),$$

whenever $\mu \ll \theta$ and $\log \frac{d\mu}{d\theta}$ is μ-integrable. Otherwise, set $R(\mu\|\theta) = \infty$.

PROPOSITION 2.8

Let $(\Omega, \mathcal{F})$ be a measurable space with Ω a Polish space and $\mathcal{F}$ the corresponding Borel σ-field. Let θ be a probability measure defined on it, and let $f : \Omega \to \mathbb{R}$ be a bounded Borel measurable function. Suppose that $\{\mu_n\}$ is a sequence of probability measures in $\mathcal{P}(\Omega)$ satisfying that $\sup_{n\in\mathbb{N}} R(\mu_n\|\theta) \le \alpha < \infty$ and μ_n converges weakly to a probability measure μ. Then

$$\lim_{n\to\infty} \int_\Omega f\, d\mu_n = \int_\Omega f\, d\mu.$$

Now for $u = (\psi, \varphi) \in \mathcal{U}$, define

$$L_T(u) := L_{1,T}(\psi) + L_{2,T}(\varphi), \quad L_{1,T}(\psi) := \frac{1}{2}\int_0^T \|\psi(s)\|_{l^2}^2\, ds$$

$$L_{2,T}(\varphi) := \int_{\mathbb{Y}_T} \big(\varphi(s,y)\log\varphi(s,y) - \varphi(s,y) + 1\big)\nu(dy)\, ds.$$

For $M \in \mathbb{N}$, define

$$S_{1,M} := \big\{g \in L^2([0,T]; l^2); L_{1,T}(g) \le M\big\}$$

$$S_{2,M} := \big\{h : \mathbb{Y}_T \to [0,\infty); L_{2,T}(h) \le M\big\}.$$

Let $S_M := S_{1,M} \times S_{2,M}$ be endowed with the product topology, let $S := \bigcup_{M\ge 1} S_M$, and let $\mathcal{U}_M := \{u \in \mathcal{U} : u(\omega) \in S_M, \bar{P}\text{-almost everywhere}\}$.

REMARK 2.9

$S_{1,M}$ endowed with the weak topology is a Polish space. Every $h \in S_{2,M}$ can be identified with a measure $\nu_T^h \in \mathcal{M}_R(\mathbb{Y}_T)$ defined by

$$\nu_T^h(B) = \int_B h(s,y)\nu_T(ds, dy), \quad B \in \mathcal{B}(\mathbb{Y}_T).$$

Since convergence in the metric of $\mathcal{M}_R(\mathbb{Y}_T)$ is equivalent to weak convergence on each compact subset of $\mathbb{Y}_T$, $\{\nu_T^h, h \in S_{2,M}\}$ is a compact subset of $\mathcal{M}_R(\mathbb{Y}_T)$, and with this identification $S_{2,M}$ is compact.

Proof of Proposition 2.8

Since $L^2([0,T]; l^2)$ is reflexive and since any unit closed ball in a reflexive space is compact, $S_{1,M}$ is compact.

For $h_n \in S_{2,M}$, we have

$$\nu_T^{h_n}([0,T] \times \mathbb{Y}) = \int_{\mathbb{Y}_T} h_n \nu(dy)\, ds \le M.$$

By [8, Theorem VIII.5], $\{\nu : \nu(\mathbb{Y}_T) \le C\}$ is compact. Therefore there exists a subsequence (still denoted by n) such that $\nu_T^{h_n} \xrightarrow{w} \mu$ and $\mu(\mathbb{Y}_T) \le M$. Moreover, $\nu_T^{h_n}(\mathbb{Y}_T) \to \mu(\mathbb{Y}_T)$. Hence there exists a subsequence n_k such that

$$0 < \frac{1}{2}\mu(\mathbb{Y}_T) < \nu_T^{h_{n_k}}(\mathbb{Y}_T) \le M.$$

Set

$$dP_n := \frac{\nu_T^{h_n}(ds,dy)}{\nu_T^{h_n}([0,T]\times\mathbb{Y})}, \qquad dP := \frac{\nu(dy)\, ds}{\nu_T([0,T]\times\mathbb{Y})}, \qquad dQ := \frac{\mu(ds,dy)}{\mu([0,T]\times\mathbb{Y})}.$$

Then P_n, P, and Q are probability measures on $[0,T]\times\mathbb{Y}$ and

$$\begin{aligned}
&\sup_k R(P_{n_k}\|P) \\
&\quad = \int \log \frac{dP_{n_k}}{dP}\, dP_{n_k} \\
&\quad = \frac{1}{\nu_T^{h_{n_k}}(\mathbb{Y}_T)} \int \big(\log h_{n_k} + \log \nu_T(\mathbb{Y}_T) - \log \nu_T^{h_{n_k}}(\mathbb{Y}_T)\big) h_{n_k} \nu(dy)\, ds \\
&\quad \le C(M) < \infty.
\end{aligned}$$

By Proposition 2.8, for any bounded measurable function f,

$$\int f\, dP_{n_k} \to \int f\, dQ,$$

which implies that $P_{n_k} \to Q$, and furthermore that $\nu_T^{h_{n_k}} \to \mu$. Moreover, for each k, $\nu_T^{h_{n_k}}$ is absolutely continuous with respect to ν_T. Therefore by [10, Theorem 8.24],

$$h_{n_k} = \frac{d\nu_T^{h_{n_k}}}{d\nu_T} \to \frac{d\mu}{d\nu_T}, \quad \nu_T\text{-a.e.}$$

Set $h := \frac{d\mu}{d\nu_T}$. Since $g(x) := x\log x - x + 1$ is continuous, by Fatou's lemma,

$$\int_{\mathbb{Y}_T} g(h)\nu(dy)\, ds = \int_{\mathbb{Y}_T} \lim_n g(h_n)\nu(dy)\, ds \le \lim_n \int_{\mathbb{Y}_T} g(h_n)\nu(dy)\, ds \le M,$$

thus $h \in S_{2,M}$, and the proof is complete. □

The following two lemmas are taken from [6], and we give a detailed proof here.

LEMMA 2.10

Let $\{\Gamma_n\}$ be an increasing sequence of compact subsets of $\mathbb{Y}$ such that $\mathbb{Y} = \bigcup_{n=1}^{n} \Gamma_n$. Let $\bar{\mathcal{A}}_{2,b} = \bigcup_n \bar{\mathcal{A}}_{2,b}^n$, where

$$\bar{\mathcal{A}}_{2,b}^n := \Big\{\varphi \in \bar{\mathcal{A}}_2 : \frac{1}{n} \le \varphi(t,y) \le n \textit{ for } y \in \Gamma_n \textit{ and } \varphi = 1 \textit{ otherwise}\Big\}.$$

Let $\varphi \in \bar{\mathcal{A}}_{2,b}$. *Then*

$$\begin{aligned}\mathcal{E}_t(\varphi) := \exp\Big\{ & \int_{[0,t]\times\mathbb{Y}} \log\varphi(s,y) N_c^1(ds,dy) \\ & + \int_{[0,t]\times\mathbb{Y}} \big(\log\varphi(s,y) - \varphi(s,y) + 1\big)\nu(dy)\,ds\Big\} \\ = \exp\Big\{ & \int_{[0,t]\times\mathbb{Y}\times[0,1]} \log\varphi(s,y)\bar{N}(ds,dy,dr) \\ & + \int_{[0,t]\times\mathbb{Y}\times[0,1]} \big(-\varphi(s,y)+1\big)\bar{\nu}_T(ds,dy,dr)\Big\}\end{aligned}$$

is an $\{\bar{\mathcal{F}}_t\}$*-martingale. Here* $N_c^1(ds,dy) := N^1(ds,dy) - \nu(dy)\,ds$*, and* N^1 *is defined as in Definition 2.6. Define a new probability measure* Q^φ *by*

$$Q^\varphi(B) = \int_B \mathcal{E}_T(\varphi) d\bar{P}, \quad B \in \mathcal{B}\big(\mathcal{M}_R(\mathbb{Y}_T^\infty)\big).$$

Then for any $(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{Y}^\infty)\backslash\mathcal{B}(\mathbb{R}))$ *measurable function* ϑ *mapping* $\mathbb{Y}_T^\infty \times \mathcal{M}_R(\mathbb{Y}_T^\infty) \to \mathbb{R}$ *such that* ϑ *is bounded and for some compact subset* $\Lambda \subset \mathbb{Y} \times [0,\infty)$*,* $\vartheta(t,y,r) = 0$ *when* $(y,r) \in \Lambda^c$*,*

$$\begin{aligned}& E^{Q^\varphi} \int \vartheta(s,y,r)\bar{N}(ds,dy,dr) \\ & = E^{Q^\varphi} \int \vartheta(s,y,r)[\varphi(s,y)\mathbb{1}_{(0,1]}(r) + \mathbb{1}_{(1,\infty)}(r)]\bar{\nu}_T(ds,dy,dr).\end{aligned}$$

$\bar{N}$ *is thus a random counting measure under* Q^φ *with compensator* $[\varphi\mathbb{1}_{(0,1]}(r) + \mathbb{1}_{(1,\infty)}(r)]\bar{\nu}_T(ds,dy,dr)$*.*

Proof

By applying Itō's formula we get

$$\begin{aligned}\mathcal{E}_t(\varphi) = 1 & + \int_0^t \int_{\mathbb{Y}} \mathcal{E}_s(\varphi)\big(\log\varphi(s,y) - \varphi(s,y) + 1\big)\nu(dy)\,ds \\ & + \int_0^t \int_{\mathbb{Y}} \mathcal{E}_{s-}(\varphi)\big(\exp(\log\varphi(s-,y)) - 1\big)N_c^1(ds,dy) \\ & + \int_0^t \int_{\mathbb{Y}} \mathcal{E}_s(\varphi)\big(\exp(\log\varphi(s,y)) - 1 - \log\varphi(s,y)\big)\nu(dy)\,ds \\ = 1 & + \int_0^t \int_{\mathbb{Y}} \mathcal{E}_{s-}(\varphi)\big(\varphi(s-,y) - 1\big)N_c^1(ds,dy).\end{aligned} \tag{5}$$

Note that φ is positive and bounded. It then follows from [1, Chapter 5] that $\mathcal{E}_t$ is a martingale.

Now let $Z(t) := \int \vartheta(s,y,r)(\bar{N}(ds,dy,dr) - \bar{\nu}_T(ds,dy,dr))$. Then Z is a martingale under $\bar{P}$ as well. By Girsanov-Meyer's theorem,

$$Z(t) - \int_0^t \mathcal{E}_{s-}(\varphi)^{-1} d\langle Z, \mathcal{E}(\varphi)\rangle(s)$$

is a Q^φ-martingale. Hence by (5),

$$E^{Q^\varphi}(Z(t)) = E^{Q^\varphi}\int_0^t \mathcal{E}_{s-}(\varphi)^{-1}\, d\langle Z, \mathcal{E}(\varphi)\rangle(s)$$
$$= E^{Q^\varphi}\int \vartheta(s,y,r)\big(\varphi(s,y)-1\big)\mathbb{1}_{(0,1]}(r)\bar{\nu}_T(ds,dy,dr).$$

Therefore

$$E^{Q^\varphi}\int \vartheta(s,y,r)\bar{N}(ds,dy,dr)$$
$$= E^{Q^\varphi}\int \vartheta(s,y,r)\big(\varphi(s,y)-1\big)\mathbb{1}_{(0,1]}(r)\bar{\nu}_T(ds,dy,dr)$$
$$+ E\int \vartheta(s,y,r)\bar{\nu}_T(ds,dy,dr)$$
$$= E^{Q^\varphi}\int \vartheta(s,y,r)\big(\varphi(s,y)\mathbb{1}_{(0,1]}(r)+\mathbb{1}_{(1,+\infty)}(r)\big)\bar{\nu}_T(ds,dy,dr). \qquad \square$$

Analogously, we can prove the following.

LEMMA 2.11

Let $u := (\psi,\varphi) \in \mathcal{U}$ such that $\varphi \in \bar{\mathcal{A}}_{2,b}$ and $\psi \in \bar{\mathcal{A}}_{1,b} := \bigcup_n \bar{\mathcal{A}}^n_{1,b}$, where

$$\bar{\mathcal{A}}^n_{1,b} := \big\{\psi \in \bar{\mathcal{A}}_1 : \|\psi\|_{l^2} \le n\big\}.$$

Set $\bar{\mathcal{E}}_t(u) := \mathcal{E}_t(\varphi)\tilde{\mathcal{E}}_t(\psi)$, $0 \le t \le T$, where $\mathcal{E}_t(\varphi)$ is the same as above and

$$\tilde{\mathcal{E}}_t(\psi) := \exp\Big\{\sum_{i=1}^\infty \int_0^t \psi_i(s)\, d\beta_i(s) - \frac{1}{2}\int_0^t \|\psi(s)\|^2_{l^2}\, ds\Big\}.$$

Then $\bar{\mathcal{E}}_t(u)$ is an $\{\bar{\mathcal{F}}_t\}$-martingale. Define a probability measure Q^u by

$$Q^u(B) = \int_B \bar{\mathcal{E}}_T(u)\, d\bar{P}, \quad B \in \mathcal{B}(\bar{\mathbb{V}}).$$

Then for any bounded $(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{Y}^\infty)\backslash\mathcal{B}(\mathbb{R}))$ measurable function ϑ for $\vartheta(t,y,r) = 0$ when (y,r) is not in some compact subset Λ,

$$E^{Q^u}\int \vartheta(s,y,r)\bar{N}(ds,dy,dr)$$
$$= E^{Q^u}\int \vartheta(s,y,r)\big(\varphi(s,y)\mathbb{1}_{(0,1]}(r)+\mathbb{1}_{(1,\infty)}(r)\big)\bar{\nu}_T(ds,dy,dr),$$

and $\{\beta_i(t) - \int_0^t \psi_i(s)\, ds, 0 \le t \le T\}$ is a sequence of standard Brownian motions under the probability Q^u.

Now we are in a proposition to present the Laplace principle taken from [6] and [11]. Let $\mathbb{D}$ and $\mathbb{D}_0$ be Polish spaces. Suppose that $\{X^{\epsilon,x}\}$ is a family of random variables taking values in $\mathbb{D}$.

DEFINITION 2.12

A function I from $\mathbb{D}$ to $[0,\infty]$ is called a *good rate function* if for every $N<\infty$, $\{f\in\mathbb{D}:I(f)\le N\}$ is a compact subset of $\mathbb{D}$. A family of rate functions I_x on $\mathbb{D}$ parameterized by $x\in\mathbb{D}_0$ is said to have compact level sets on compacts if for all compact subsets $K\subset\mathbb{D}_0$ and for each $N<\infty$, $\Lambda_{N,K}:=\bigcup_{x\in K}\{f\in\mathbb{D}:I_x(f)\le N\}$ is a compact subset of $\mathbb{D}$.

REMARK 2.13

A function having compact level set is automatically lower semicontinuous. Thus for any fixed $x\in\mathbb{D}_0$, $I_x(f)$ is a lower semicontinuous function of f (see [9, Chapter 1]).

DEFINITION 2.14 (UNIFORM LAPLACE PRINCIPLE)

Let I_x be a family of rate functions on $\mathbb{D}$ and assume that it has compact level sets on compacts. The family $\{X^{\epsilon,x}\}$ is said to *satisfy* the Laplace principle on $\mathbb{D}$ with rate function I_x, uniformly on compacts, if for all compact subsets $K\subset\mathbb{D}_0$ and all bounded continuous functions F mapping $\mathbb{D}$ into $\mathbb{R}$,

$$\lim_{\epsilon\to 0}\sup_{x\in K}\Big|\epsilon\log\bar{E}_x\Big(\exp\Big\{-\frac{1}{\epsilon}F(X^{\epsilon,x})\Big\}\Big)+\inf_{f\in\mathbb{D}}\big\{F(f)+I_x(f)\big\}\Big|=0.$$

Suppose that $\mathcal{G}^\epsilon$ is a measurable map from $\mathbb{D}_0\times\mathbb{V}\to\mathbb{D}$ and that there exists a measurable map $\mathcal{G}^0:\mathbb{D}_0\times\mathbb{V}\to\mathbb{D}$ such that

($\mathbf{C}1$) for every $M\in\mathbb{N}$, if a family $\{u_\epsilon:=(\psi_\epsilon,\varphi_\epsilon),\epsilon\in(0,1)\}\subset\mathcal{U}_M$ converges in distribution to $u:=(\psi,\varphi)\in\mathcal{U}_M$ and $\mathbb{D}_0\ni x_\epsilon\to x\in\mathbb{D}_0$ as $\epsilon\to 0$, then

$$\mathcal{G}^\epsilon\Big(x_\epsilon,\sqrt{\epsilon}\beta+\int_0^\cdot\psi_\epsilon(s)\,ds,\epsilon N^{\epsilon^{-1}\varphi_\epsilon}\Big)\Rightarrow\mathcal{G}^0\Big(x,\int_0^\cdot\psi(s)\,ds,\nu_T^\varphi\Big);$$

($\mathbf{C}2$) for every $M\in\mathbb{N}$ and compact subset $K\subset\mathbb{D}_0$,

$$\Big\{\mathcal{G}^0\Big(x,\int_0^\cdot g(s)\,ds,\nu_T^h\Big),x\in K,(g,h)\in S_M\Big\}$$

is a compact subset of $\mathbb{D}$.

The following result is from [11, Theorem 4.4] (see also [6, Theorem 4.2]).

THEOREM 2.15

Let $X^{\epsilon,x}:=\mathcal{G}^\epsilon(x,\sqrt{\epsilon}\beta,\epsilon N^{\epsilon^{-1}})$, and let ($\mathbf{C}1$)–($\mathbf{C}2$) hold. For $x\in\mathbb{D}_0$ and $f\in\mathbb{D}$, define

$$I_x(f):=\inf_{\{(g,h)\in S:f=\mathcal{G}^0(x,\int_0^\cdot g(s)\,ds,\nu_T^h)\}}\big\{L_{1,T}(g)+L_{2,T}(h)\big\}.$$

Suppose that $x\to I_x$ defined above is a lower semicontinuous function from $\mathbb{D}_0$ to $[0,\infty]$. Then I_x is a rate function having compact level sets on compacts. Furthermore, $\{X^{\epsilon,x},\epsilon\in(0,1)\}$ satisfies the uniform Laplace principle in $\mathbb{D}$ with rate function I_x. Moreover, $X^{\epsilon,x}$ satisfies a uniform large deviation principle; that is, if we denote the law of $X^{\epsilon,x}$ by μ_ϵ, then for any $B\in\mathcal{B}(\mathbb{D})$ and any *compact subset* $K \subset \mathbb{D}_0$, $x \in K$,

$$-\inf_{f \in B^\circ} I_x(f) \le \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(B) \le \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(B) \le -\inf_{f \in \bar{B}} I_x(f).$$

3. Main result and the proof

We restrict our discussion on the finite interval $[0,T]$. Throughout this paper we assume that the following conditions hold:

($\mathbf{H}1$) $D(A)$ has nonempty interior;
($\mathbf{H}2$) b, σ, and γ are bounded, and for any $x_1, x_2 \in \mathbb{R}^d$, $\forall y \in \mathbb{Y}$,

$$|b(x_1) - b(x_2)| + \|\sigma(x_1) - \sigma(x_2)\|_{l^2} + |\gamma(x_1,y) - \gamma(x_2,y)| \le C_1 |x_1 - x_2|;$$

($\mathbf{H}3$) for every $x \in \overline{D(A)}$, $x + \gamma(x,y) \in \overline{D(A)}$, $\forall y \in \mathbb{Y}$;
($\mathbf{H}4$) for some compact subset $\Gamma \subset \mathbb{Y}$, $\gamma(x,y) = 0$ for all $(x,y) \in \mathbb{R}^d \times \Gamma^c$.

Consider the following perturbed equation of (1):

$$\begin{cases} dX^\epsilon(t) \in -A(X^\epsilon(t))\,dt + b(X^\epsilon(t))\,dt + \sqrt{\epsilon}\sigma(X^\epsilon(t))\,dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X^\epsilon(t-),y)(\epsilon N^{\epsilon^{-1}}(dt,dy) - \nu(dy)\,dt) \\ X^\epsilon(0) = x_0 \in \overline{D(A)}, \quad \epsilon \in (0,1]. \end{cases} \tag{6}$$

As we have proved in [13], under ($\mathbf{H}1$)–($\mathbf{H}4$), (6) has a unique solution. Denote the solution by $(X^\epsilon(\cdot,x_0), K^\epsilon(\cdot,x_0))$. Moreover, it follows from the classical Yamada-Watanabe theorem that there exists a measurable function $\mathcal{G}^\epsilon$ such that

$$X^\epsilon(\cdot,x_0) = \mathcal{G}^\epsilon(x_0, \sqrt{\epsilon}W, \epsilon N^{\epsilon^{-1}}). \tag{7}$$

Our main result is the following theorem.

THEOREM 3.1
Under ($\mathbf{H}1$)–($\mathbf{H}4$), *the solution* $\{X^\epsilon(t,x_0), \epsilon \in (0,1], t \in [0,T]\}$ *to (6) satisfies the uniform large deviation principle with rate function* I_{x_0} *defined by*

$$I_{x_0}(f) := \inf_{\{q=(g,h)\in S: f = X^q\}} \big\{L_{1,T}(g) + L_{2,T}(h)\big\}, \tag{8}$$

where (X^q, K^q) *solves*

$$\begin{cases} dX^q(t) \in -A(X^q(t))\,dt + b(X^q(t))\,dt + \sigma(X^q(t))g(t)\,dt \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X^q(t),y)(h(t,y) - 1)\nu(dy)\,dt \\ X^q(0) = x_0. \end{cases} \tag{9}$$

NOTE
Throughout this section, $\mathbb{D} = \mathcal{D}([0,T] \times \overline{D(A)}; \overline{D(A)})$, $\mathbb{D}_0 = \overline{D(A)}$.

By Theorem 2.15, to prove the theorem, we need to verify ($\mathbf{C}1$) and ($\mathbf{C}2$), which is approached through a few propositions and lemmas.

For $u_\epsilon = (\psi_\epsilon, \varphi_\epsilon) \in \mathcal{U}_M$, consider the following controlled equation:

$$
(10)\qquad \begin{cases} dX^{\epsilon,u_\epsilon}(t) \in -A(X^{\epsilon,u_\epsilon}(t))\,dt + b(X^{\epsilon,u_\epsilon}(t))\,dt + \sigma(X^{\epsilon,u_\epsilon}(t))\psi_\epsilon(t)\,dt \\ \qquad\qquad + \sqrt{\epsilon}\sigma(X^{\epsilon,u_\epsilon}(t))\,dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X^{\epsilon,u_\epsilon}(t-),y)(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(dt,dy) - \nu(dy)\,dt) \\ X^{\epsilon,u_\epsilon}(0) = x_0 \in \overline{D(A)}, \quad \epsilon \in (0,1]. \end{cases}
$$

By Girsanov's theorem (see Lemma 2.11), it admits a unique solution. We denote the solution as $(X^{\epsilon,u_\epsilon}(\cdot,x_0), K^{\epsilon,u_\epsilon}(\cdot,x_0))$.

PROPOSITION 3.2

Let $(X^{\epsilon,u_\epsilon}(\cdot,x_1), K^{\epsilon,u_\epsilon}(\cdot,x_1))$, and let $(X^{\epsilon,u_\epsilon}(\cdot,x_2), K^{\epsilon,u_\epsilon}(\cdot,x_2))$ be the solutions of (10) with initial values x_1 and x_2, respectively. Then for any $\epsilon \in (0,1)$ and $p \geq 1$, there exists a constant C such that

$$
\bar{E}\sup_{t\leq T} |X^{\epsilon,u_\epsilon}(t,x_1) - X^{\epsilon,u_\epsilon}(t,x_2)|^{2p} \leq C|x_1 - x_2|^{2p}.
$$

Proof

Denote

$$
\begin{aligned}
Z^{\epsilon,u_\epsilon}(t) &:= X^{\epsilon,u_\epsilon}(t,x_1) - X^{\epsilon,u_\epsilon}(t,x_2) \\
\Lambda(t) &:= \sigma\big(X^{\epsilon,u_\epsilon}(t,x_1)\big) - \sigma\big(X^{\epsilon,u_\epsilon}(t,x_2)\big) \\
R(t) &:= \gamma\big(X^{\epsilon,u_\epsilon}(t,x_1),y\big) - \gamma\big(X^{\epsilon,u_\epsilon}(t,x_2),y\big).
\end{aligned}
$$

By Itō's formula, (**H**2), (**H**4), and Lemma 2.4, for $p \geq 2$,

$$
\begin{aligned}
&|Z^{\epsilon,u_\epsilon}(t)|^p \\
&\quad = |x_1 - x_2|^p \\
&\qquad + p\int_0^t |Z^{\epsilon,u_\epsilon}(s)|^{p-2}\big\langle Z^{\epsilon,u_\epsilon}(s), b\big(X^{\epsilon,u_\epsilon}(s,x_1)\big) - b\big(X^{\epsilon,u_\epsilon}(s,x_2)\big)\big\rangle\,ds \\
&\qquad + p\int_0^t |Z^{\epsilon,u_\epsilon}(s)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s), \Lambda(s)\psi_\epsilon(s)\rangle\,ds \\
&\qquad + p\sqrt{\epsilon}\int_0^t |Z^{\epsilon,u_\epsilon}(s)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s), \Lambda(s)\,dW(s)\rangle \\
&\qquad + p\int_0^t\int_{\mathbb{Y}} |Z^{\epsilon,u_\epsilon}(s)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s), R(s)\rangle\big(\varphi_\epsilon(s,y) - 1\big)\nu(dy)\,ds \\
&\qquad - p\int_0^t |Z^{\epsilon,u_\epsilon}(s)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s), dK^{\epsilon,u_\epsilon}(s,x_1) - dK^{\epsilon,u_\epsilon}(s,x_2)\rangle \\
&\qquad + p\epsilon\int_0^t |Z^{\epsilon,u_\epsilon}(s)|^{p-2}\big(\|\Lambda(s)\|^2 + (p-1)\langle Z^{\epsilon,u_\epsilon}(s), \Lambda(s)\Lambda^*(s)Z^{\epsilon,u_\epsilon}(s)\rangle \\
&\qquad /|Z^{\epsilon,u_\epsilon}(s)|^2\big)\,ds \\
&\qquad + \int_0^t\int_{\mathbb{Y}}\big(|Z^{\epsilon,u_\epsilon}(s-) + \epsilon R(s-)|^p - |Z^{\epsilon,u_\epsilon}(s-)|^p\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times\big(N^{\epsilon^{-1}\varphi_\epsilon}(ds,dy)-\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds\big)\\
&\quad +\int_0^t\int_{\mathbb{Y}}\big(|Z^{\epsilon,u_\epsilon}(s-)+\epsilon R(s-)|^p-|Z^{\epsilon,u_\epsilon}(s-)|^p\\
&\quad -p|Z^{\epsilon,u_\epsilon}(s-)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s-),\epsilon R(s-)\rangle\big)\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds\\
&\le |x_1-x_2|^p+C\int_0^t|Z^{\epsilon,u_\epsilon}(s)|^p\,ds+p\int_0^t|Z^{\epsilon,u_\epsilon}(s)|^p\|\psi_\epsilon(s)\|_{l^2}\,ds\\
&\quad +p\int_0^t\int_\Gamma|Z^{\epsilon,u_\epsilon}(s)|^p\varphi_\epsilon(s,y)\nu(dy)\,ds+|I_1(t)|+|I_2(t)|+|V(t)|,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(t)&:=p\sqrt{\epsilon}\int_0^t|Z^{\epsilon,u_\epsilon}(s)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s),\Lambda(s)\,dW(s)\rangle\\
I_2(t)&:=\int_0^t\int_\Gamma\big(|Z^{\epsilon,u_\epsilon}(s-)+\epsilon R(s-)|^p-|Z^{\epsilon,u_\epsilon}(s-)|^p\big)\\
&\qquad\times\big(N^{\epsilon^{-1}\varphi_\epsilon}(ds,dy)-\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds\big)\\
V(t)&:=\int_0^t\int_\Gamma\big(|Z^{\epsilon,u_\epsilon}(s-)+\epsilon R(s-)|^p-|Z^{\epsilon,u_\epsilon}(s-)|^p\\
&\qquad -p|Z^{\epsilon,u_\epsilon}(s-)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s-),\epsilon R(s-)\rangle\big)\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds.
\end{aligned}
$$

Using Taylor's expansion and ($\mathbf{H}2$), we obtain

$$
\begin{aligned}
V(t)&=\int_0^t\int_\Gamma\big(|Z^{\epsilon,u_\epsilon}(s-)+\epsilon R(s-)|^p-|Z^{\epsilon,u_\epsilon}(s-)|^p\\
&\qquad -p|Z^{\epsilon,u_\epsilon}(s-)|^{p-2}\langle Z^{\epsilon,u_\epsilon}(s-),\epsilon R(s-)\rangle\big)\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds\\
&\le C_p\int_0^t\int_\Gamma|\epsilon R(s-)|^2\big(|Z^{\epsilon,u_\epsilon}(s-)|^{p-2}+|\epsilon R(s-)|^{p-2}\big)\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds\\
&\le C_p\epsilon\int_0^t\int_\Gamma|Z^{\epsilon,u_\epsilon}(s-)|^p\varphi_\epsilon(s,y)\nu(dy)\,ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
|Z^{\epsilon,u_\epsilon}(t)|^p&\le|x_1-x_2|^p+C\int_0^t|Z^{\epsilon,u_\epsilon}(s)|^p\,ds+p\int_0^t|Z^{\epsilon,u_\epsilon}(s)|^p\|\psi_\epsilon(s)\|_{l^2}\,ds\\
&\qquad +C\int_0^t\int_\Gamma|Z^{\epsilon,u_\epsilon}(s)|^p\varphi_\epsilon(s,y)\nu(dy)\,ds+|I_1(t)|+|I_2(t)|.
\end{aligned}\tag{11}
$$

Since u_ϵ is S_M-valued,

$$
\int_0^T\|\psi_\epsilon(s)\|_{l^2}^2\,ds\le 2M,\qquad\int_0^T\int_\Gamma\varphi_\epsilon(s,y)\nu(dy)\,ds\le M,\quad\text{almost surely.}\tag{12}
$$

Applying Gronwall's lemma to (11) yields

$$
\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^p\le e^{CM}\Big(|x_1-x_2|^p+\sup_{s\le t}|I_1(s)|+\sup_{s\le t}|I_2(s)|\Big),
$$

which furthermore leads to

$$\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^{2p} \le C\Big(|x_1-x_2|^{2p} + \sup_{s\le t}|I_1(s)|^2 + \sup_{s\le t}|I_2(s)|^2\Big). \tag{13}$$

By Doob's maximal inequality, Jensen's inequality, (**H**2), and (12),

$$\begin{aligned}\bar{E}\sup_{s\le t}|I_1(s)|^2 &\le 4\bar{E}|I_1(t)|^2\\ &\le C_p\epsilon\bar{E}\int_0^t \sup_{r\le s}|Z^{\epsilon,u_\epsilon}(r)|^{2p}\,ds \le C_p\epsilon\int_0^t \bar{E}\sup_{r\le s}|Z^{\epsilon,u_\epsilon}(r)|^{2p}\,ds.\end{aligned}$$

Similarly by Taylor's expansion and Doob's maximal inequality,

$$\begin{aligned}&\bar{E}\sup_{s\le t}|I_2(s)|^2\\ &\le C\bar{E}\int_0^t\int_\Gamma \big(|Z^{\epsilon,u_\epsilon}(s-)+\epsilon R(s-)|^p - |Z^{\epsilon,u_\epsilon}(s-)|^p\big)^2\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\\ &\le C\bar{E}\int_0^t\int_\Gamma |\epsilon R(s-)|^2\big(|Z^{\epsilon,u_\epsilon}(s-)|^{p-1}+|\epsilon R(s-)|^{p-1}\big)^2\epsilon^{-1}\varphi_\epsilon(s,y)\nu(dy)\,ds\\ &\le C\epsilon\bar{E}\Big(\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^{2p}\int_0^t\int_\Gamma \varphi_\epsilon(s,y)\nu(dy)\,ds\Big)\\ &\le CM\epsilon\bar{E}\Big(\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^{2p}\Big).\end{aligned}$$

Substituting the two estimates in (13), we get

$$\begin{aligned}&\bar{E}\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^{2p}\\ &\le C\Big(|x_1-x_2|^{2p} + \bar{E}\sup_{s\le t}|I_1(s)|^2 + \bar{E}\sup_{s\le t}|I_2(s)|^2\Big)\\ &\le C\Big(|x_1-x_2|^{2p} + \epsilon\int_0^t \bar{E}\sup_{r\le s}|Z^{\epsilon,u_\epsilon}(r)|^{2p}\,ds\Big) + CM\epsilon\bar{E}\Big(\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^{2p}\Big).\end{aligned}$$

Hence for $\epsilon \le \frac{1}{2CM}$,

$$\bar{E}\sup_{s\le t}|Z^{\epsilon,u_\epsilon}(s)|^{2p} \le C\Big(|x_1-x_2|^{2p} + \epsilon\int_0^t \bar{E}\sup_{r\le s}|Z^{\epsilon,u_\epsilon}(r)|^{2p}\,ds\Big).$$

Using Gronwall's lemma again we obtain

$$\bar{E}\sup_{t\le T}|Z^{\epsilon,u_\epsilon}(t)|^{2p} \le C|x_1-x_2|^{2p}, \quad p\ge 2.$$

Now the proof is completed using Hölder's inequality for the case $1\le p<2$. □

PROPOSITION 3.3

For $x_0\in\overline{D(A)}$, denote by $(X^{\epsilon,u_\epsilon}(t,x_0), K^{\epsilon,u_\epsilon}(t,x_0))$ the solution (10) with the initial value x_0. Then under the assumptions, there exists a constant $C=C(T,$

$M, x_0)$ such that for any $p \geq 1$,

$$\bar{E}\sup_{t\leq T}|X^{\epsilon,u_\epsilon}(t,x_0)|^{2p} + \bar{E}|K^{\epsilon,u_\epsilon}(\cdot,x_0)|_T^0 \leq C.$$

Proof

In the proof of Proposition 3.2 we have actually proved

$$\bar{E}\sup_{t\leq T}|X^{\epsilon,u_\epsilon}(t,x_0)|^{2p} \leq C(p,x_0,M,T). \tag{14}$$

Let a be the same as in Lemma 2.5. We get by Itō's formula and **(H2)**,

$$\begin{aligned}
&|X^{\epsilon,u_\epsilon}(t,x_0)-a|^2\\
&\quad= |x_0-a|^2 + 2\int_0^t \big\langle X^{\epsilon,u_\epsilon}(s,x_0)-a, b\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\big\rangle\, ds\\
&\qquad+ 2\int_0^t \big\langle X^{\epsilon,u_\epsilon}(s,x_0)-a, \sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\psi_\epsilon(s)\big\rangle\, ds\\
&\qquad+ 2\sqrt{\epsilon}\int_0^t \big\langle X^{\epsilon,u_\epsilon}(s,x_0)-a, \sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\, dW(s)\big\rangle\\
&\qquad+ \int_0^t\int_{\mathbb{Y}} \big\langle X^{\epsilon,u_\epsilon}(s,x_0)-a, \gamma\big(X^{\epsilon,u_\epsilon}(s,x_0),y\big)\big(\varphi_\epsilon(s,y)-1\big)\nu(dy)\, ds\big\rangle\\
&\qquad+ \int_0^t\int_{\mathbb{Y}} \big(\epsilon|\gamma(X^{\epsilon,u_\epsilon}(s-,x_0),y)|^2 + 2\langle X^{\epsilon,u_\epsilon}(s-,x_0)-a, \gamma(X^{\epsilon,u_\epsilon}(s-,x_0),y)\rangle\big)\\
&\qquad\times \big(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(ds,dy) - \varphi_\epsilon(s,y)\nu(dy)\, ds\big)\\
&\qquad- 2\int_0^t \langle X^{\epsilon,u_\epsilon}(s,x_0)-a, dK^{\epsilon,u_\epsilon}(s,x_0)\rangle\\
&\qquad+ \epsilon\int_0^t \big\|\sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\big\|_{l^2}^2\, ds + \epsilon\int_0^t\int_{\mathbb{Y}} \big|\gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big|^2 \varphi_\epsilon(s,y)\nu(dy)\, ds\\
&\quad\leq |x_0-a|^2 + C\Big(1+\int_0^t |X^{\epsilon,u_\epsilon}(s,x_0)-a|^2\Big)\, ds\\
&\qquad+ \sqrt{\epsilon}\int_0^t \big\langle X^{\epsilon,u_\epsilon}(s,x_0)-a, \sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\, dW(s)\big\rangle\\
&\qquad+ \int_0^t\int_{\mathbb{Y}} C\big(\epsilon + 2|X^{\epsilon,u_\epsilon}(s-,x_0)-a|\big)\big(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(ds,dy) - \varphi_\epsilon(s,y)\nu(dy)\, ds\big)\\
&\qquad- 2\int_0^t \langle X^{\epsilon,u_\epsilon}(s,x_0)-a, dK^{\epsilon,u_\epsilon}(s,x_0)\rangle.
\end{aligned}$$

Note that by Lemma 2.5,

$$\begin{aligned}
&2\int_0^T \langle X^{\epsilon,u_\epsilon}(s,x_0)-a, dK^{\epsilon,u_\epsilon}(s,x_0)\rangle\\
&\qquad\geq 2\gamma|K^{\epsilon,u_\epsilon}|_T^0 - \mu\int_0^T |X^{\epsilon,u_\epsilon}(s,x_0)-a|\, ds - \gamma\mu T,
\end{aligned}$$

which together with (14) gives the desired result. □

Now consider the deterministic equation

$$(15)\quad \begin{aligned} X^u(t,x_0) &= x_0 + \int_0^t b\big(X^u(s,x_0)\big)\,ds + \int_0^t \sigma\big(X^u(s,x_0)\big)\psi(s)\,ds \\ &\quad + \int_0^t \int_{\mathbb{Y}} \gamma\big(X^u(s,x_0),y\big)\big(\varphi(s,y)-1\big)\nu(dy)\,ds - K^u(t,x_0). \end{aligned}$$

Note that it can be solved uniquely (see [2]). Suppose that $(X^u(t,x_0), K^u(t,x_0))$ is the solution.

LEMMA 3.4

Let $u_\epsilon = (\psi_\epsilon, \varphi_\epsilon), u = (\psi, \varphi) \in \mathcal{U}_M$ *such that as* $\epsilon \to 0$, $u_\epsilon \xrightarrow{\text{a.s.}} u$ *as random variables in* S_M. *Denote*

$$v_{1,\epsilon}(t) := \int_0^t \sigma\big(X^u(s,x_0)\big)\big(\psi_\epsilon(s)-\psi(s)\big)\,ds$$

$$v_{2,\epsilon}(t) := \int_0^t \int_{\mathbb{Y}} \gamma\big(X^u(s,x_0),y\big)\big(\varphi_\epsilon(s,y)-\varphi(s,y)\big)\nu(dy)\,ds.$$

Then there exists a subsequence (still denoted by $\{\epsilon\}$*) such that* $\sup_{t\le T}|v_{1,\epsilon}(t)| \xrightarrow{\text{a.s.}} 0$ *and* $\sup_{t\le T}|v_{2,\epsilon}(t)| \xrightarrow{\text{a.s.}} 0$ *as* $\epsilon \to 0$.

Proof

(1) $\sup_{t\le T}|v_{1,\epsilon}(t)| \xrightarrow{\text{a.s.}} 0$ obviously follows from Ascoli-Arzela's theorem (see also [4, Lemma 3.2]).

(2) To prove $\sup_{t\le T}|v_{2,\epsilon}(t)| \xrightarrow{\text{a.s.}} 0$, we first prove that for any $t \in [0,T]$,

$$(16)\quad \int_0^t \int_{\Gamma} \gamma\big(X^u(s,x_0),y\big)\big(\varphi_\epsilon(s,y)-\varphi(s,y)\big)\nu(dy)\,ds \xrightarrow{\text{a.s.}} 0.$$

By the topology endowed to $S_{2,M}$, $\varphi_\epsilon \to \varphi$ if and only if for any compact subset $K \subset \mathbb{Y}$ and any bounded continuous function f, as $\epsilon \to 0$,

$$(17)\quad \int_0^t \int_K f\nu_T^{\varphi_\epsilon}(ds,dy) - \int_0^t \int_K f\nu_T^{\varphi}(ds,dy) = \int_0^t \int_K f(\varphi_\epsilon - \varphi)\nu(dy)\,ds \to 0.$$

We divide the proof of (16) into two cases.

(i) If $\int_0^T \int_\Gamma \varphi(s,y)\nu(dy)\,ds = 0$, then $\varphi = 0$, $\nu(dy) \times ds$ almost surely. In this case, (16) can be written as

$$(18)\quad \int_0^t \int_{\Gamma} \gamma\big(X^u(s,x_0),y\big)\varphi_\epsilon(s,y)\nu(dy)\,ds \to 0.$$

Since γ is measurable and bounded, denote the bound by C_0. By (17), we have as $\epsilon \to 0$,

$$\Big|\int_0^t \int_{\Gamma} \gamma\big(X^u(s,x_0),y\big)\varphi_\epsilon(s,y)\nu(dy)\,ds\Big| \le C_0 \int_0^t \int_{\Gamma} \varphi_\epsilon(s,y)\nu(dy)\,ds \to 0,$$

and (18) follows.

(ii) If $\int_0^T\int_\Gamma \varphi(s,y)\nu(dy)\,ds>0$, by taking $f\equiv 1$ in (17), we have as $\epsilon\to 0$,

$$\nu_T^{\varphi_\epsilon}([0,T]\times\Gamma)-\nu_T^{\varphi}([0,T]\times\Gamma)=\int_0^T\int_\Gamma\big(\varphi_\epsilon(s,y)-\varphi(s,y)\big)\nu(dy)\,ds\to 0. \tag{19}$$

Thus for ϵ small enough,

$$0<\frac{1}{2}\nu_T^{\varphi}([0,T]\times\Gamma)<\nu_T^{\varphi_\epsilon}([0,T]\times\Gamma)\le M. \tag{20}$$

Set

$$d\mu_\epsilon:=\frac{\varphi_\epsilon\nu(dy)\,ds}{\nu_T^{\varphi_\epsilon}([0,T]\times\Gamma)},\qquad d\mu:=\frac{\varphi\nu(dy)\,ds}{\nu_T^{\varphi}([0,T]\times\Gamma)},\qquad d\theta:=\frac{\nu(dy)\,ds}{\nu_T([0,T]\times\Gamma)}.$$

Then μ_ϵ, μ, and θ are all probability measures on $[0,T]\times\Gamma$. Moreover, μ_ϵ converges to μ weakly by (17) and (19)–(20). On the other hand, since $\varphi_\epsilon\in S_{2,M}$,

$$\begin{aligned}R(\mu_\epsilon\|\theta)&=\int\log\Big(\frac{d\mu_\epsilon}{d\theta}\Big)d\mu_\epsilon\\&\le\frac{1}{\nu_T^{\varphi_\epsilon}([0,T]\times\Gamma)}\\&\quad\times\int\varphi_\epsilon\big(\log\varphi_\epsilon+\log\nu_T([0,T]\times\Gamma)-\log\nu_T^{\varphi_\epsilon}([0,T]\times\Gamma)\big)\nu(dy)\,ds\\&\le C(M)<\infty,\end{aligned}$$

where C does not depend on ϵ. That is to say, $\sup_\epsilon R(\mu_\epsilon\|\theta)<\infty$. Therefore by Proposition 2.8,

$$\int\gamma\,d\mu_\epsilon\to\int\gamma\,d\mu,$$

that is,

$$\frac{1}{\nu_T^{\varphi_\epsilon}([0,T]\times\Gamma)}\int\gamma\varphi_\epsilon\nu(dy)\,ds\to\frac{1}{\nu_T^{\varphi}([0,T]\times\Gamma)}\int\gamma\varphi\nu(dy)\,ds.$$

Using (19) and (20) again, we get

$$\int\gamma\varphi_\epsilon\nu(dy)\,ds\to\int\gamma\varphi\nu(dy)\,ds.$$

Hence we get (16).

Now we turn to prove that $\sup_{t\le T}|v_{2,\epsilon}(t)|\to 0$. Recall that $\theta(ds,dy)=\frac{\nu(dy)\,ds}{\nu_T([0,T]\times\Gamma)}$ is a probability measure on $[0,T]\times\Gamma$ and that for $g(x):=x\log x-x+1$,

$$\lim_{x\to\infty}\frac{g(x)}{x}=\infty,\qquad\sup_\epsilon E^\theta\big(g(\varphi_\epsilon(s,y))\big)\le\frac{M}{\nu_T([0,T]\times\Gamma)}<\infty.$$

Thus $\{\varphi_\epsilon\}$ is uniformly integrable with respect to θ, leading to the fact that the indefinite integration of $\{\varphi_\epsilon\}$ with respect to $\nu(dy)\,ds$ is uniformly bounded and uniformly absolutely continuous. Note that γ is bounded. Thus $\{\gamma(X^u(s,x_0),y)(\varphi_\epsilon(s,y)-\varphi(s,y))\}$ is uniformly integrable with respect to $\nu(dy)\,ds$ as well;

whence we can deduce that $\{v_{2,\epsilon}\}$ is uniformly bounded and equicontinuous on $[0,T]$. Therefore $\sup_{t\le T}|v_{2,\epsilon}(t)|\to 0$ follows from Ascoli-Arzela's theorem and the above arguments. $\square$

PROPOSITION 3.5

Let $x_\epsilon, x_0\in\overline{D(A)}$ such that $x_\epsilon\to x_0$, $u_\epsilon:=(\psi^\epsilon,\varphi^\epsilon)\to u:=(\psi,\varphi)$ almost surely as random variables in S_M. Let $(X^{\epsilon,u_\epsilon}(\cdot,x_\epsilon),K^{\epsilon,u_\epsilon}(\cdot,x_\epsilon))$ solve (10) with initial value x_ϵ, and let $(X^u(x_0),K^u(x_0))$ solve (15) with x_0 being the initial value. Then $X^{\epsilon,u_\epsilon}(x_\epsilon)\xrightarrow{\bar P}X^u(x_0)$.

Proof

(1) First, we prove $X^{\epsilon,u_\epsilon}(t,x_0)\xrightarrow{\bar P}X^u(t,x_0)$.

Denote $\omega_\epsilon(s,x_0):=X^{\epsilon,u_\epsilon}(s,x_0)-X^u(s,x_0)$. By Itō's formula, (**H**2), and (**H**4), we have

$$
\begin{aligned}
&|X^{\epsilon,u_\epsilon}(t,x_0)-X^u(t,x_0)|^2\\
&\quad=2\int_0^t\langle\omega_\epsilon(s,x_0),b\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)-b\big(X^u(s,x_0)\big)\rangle\,ds\\
&\qquad+2\int_0^t\big\langle\omega_\epsilon(s,x_0),\big(\sigma(X^{\epsilon,u_\epsilon}(s,x_0))-\sigma(X^u(s,x_0))\big)\psi_\epsilon(s)\big\rangle\,ds\\
&\qquad+2\int_0^t\big\langle\omega_\epsilon(s,x_0),\sigma\big(X^u(s,x_0)\big)\big(\psi_\epsilon(s)-\psi(s)\big)\big\rangle\,ds\\
&\qquad+2\sqrt{\epsilon}\int_0^t\big\langle\omega_\epsilon(s,x_0),\sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\,dW(s)\big\rangle\\
&\qquad+2\int_0^t\Big\langle\omega_\epsilon(s,x_0),\int_{\mathbb{Y}}\big(\gamma(X^{\epsilon,u_\epsilon}(s,x_0),y)-\gamma(X^u(s,x_0),y)\big)\Big\rangle\\
&\qquad\times\big(\varphi_\epsilon(s,y)-1\big)\nu(dy)\,ds\\
&\qquad+2\int_0^t\Big\langle\omega_\epsilon(s,x_0),\int_{\mathbb{Y}}\gamma\big(X^u(s,x_0),y\big)\Big\rangle\big(\varphi_\epsilon(s,y)-\varphi(s,y)\big)\nu(dy)\,ds\\
&\qquad+\int_0^t\int_{\mathbb{Y}}\big[\epsilon|\gamma(X^{\epsilon,u_\epsilon}(s-,x_0),y)|^2+2\big\langle\omega_\epsilon(s-,x_0),\gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big\rangle\big]\\
&\qquad\times\big(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}-\varphi_\epsilon(s,y)\nu(dy)\,ds\big)-2\int_0^t\langle\omega_\epsilon(s,x_0),dK^{\epsilon,u_\epsilon}(s,x_0)-dK^u(s,x_0)\rangle\\
&\qquad+\epsilon\int_0^t\big\|\sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\big\|_{l^2}^2\,ds\\
&\qquad+\epsilon\int_0^t\int_{\mathbb{Y}}\big|\gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big|^2\varphi_\epsilon(s,y)\nu(dy)\,ds\\
&\quad\le C\int_0^t|\omega_\epsilon(s,x_0)|^2\,ds+C\int_0^t|\omega_\epsilon(s,x_0)|^2\|\psi_\epsilon(s)\|_{l^2}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C\int_0^t |\omega_\epsilon(s,x_0)|^2 \int_\Gamma \big(\varphi_\epsilon(s,y)-1\big)\nu(dy)\,ds \\
&\quad + |J_1(t)| + |J_2(t)| + |J_3(t)| + |J_4(t)| \\
&\quad + \epsilon\int_0^t \big\|\sigma(X^{\epsilon,u_\epsilon}(s,x_0))\big\|_{l^2}^2\,ds \\
&\quad + \epsilon\int_0^t\int_\Gamma \big|\gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big|^2 \varphi_\epsilon(s,y)\nu(dy)\,ds \\
&\le C\epsilon + C\int_0^t |\omega_\epsilon(s,x_0)|^2\,ds + C\int_0^t |\omega_\epsilon(s,x_0)|^2\|\psi_\epsilon(s)\|_{l^2}\,ds \\
&\quad + C\int_0^t |\omega_\epsilon(s,x_0)|^2\int_\Gamma \varphi_\epsilon(s,z)\nu(dy)\,ds + |J_1(t)| + |J_2(t)| + |J_3(t)| + |J_4(t)|,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1(t) &:= 2\int_0^t \langle \omega_\epsilon(s,x_0), dv_{1,\epsilon}(s)\rangle \\
J_2(t) &:= 2\sqrt{\epsilon}\int_0^t \big\langle \omega_\epsilon(s,x_0), \sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\,dW(s)\big\rangle \\
J_3(t) &:= 2\int_0^t \langle \omega_\epsilon(s,x_0), dv_{2,\epsilon}(s)\rangle \\
J_4(t) &:= \int_0^t\int_\Gamma \big[\epsilon\big|\gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big|^2 \\
&\qquad + 2\big\langle \omega_\epsilon(s-,x_0), \gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big\rangle\big]\big(\epsilon N^{\epsilon^{-1}\varphi_\epsilon} - \varphi_\epsilon(s,y)\nu(dy)\,ds\big).
\end{aligned}
$$

Notice that $\int_0^T \|\psi_\epsilon(s)\|_{l^2}^2\,ds \le 2M$, $\int_0^T\int_{\mathbb{Y}} \varphi_\epsilon(s,y)\nu(dy)\,ds \le M$ almost surely. By Gronwall's lemma, we get

$$
\begin{aligned}
&\sup_{s\le t} |X^{\epsilon,u_\epsilon}(s,x_0) - X^u(s,x_0)|^2 \\
&\quad \le e^{CM}\Big(C\epsilon + \sup_{s\le t}|J_1(t)| + \sup_{s\le t}|J_2(t)| + \sup_{s\le t}|J_3(t)| + \sup_{s\le t}|J_4(t)|\Big).
\end{aligned}
$$

By (**H**2), BDG's inequality, and Young's inequality,

$$
\begin{aligned}
\bar{E}\Big(\sup_{s\le t}|J_2(s)|\Big) &\le C\sqrt{\epsilon}\bar{E}\Big(\int_0^t |\omega_\epsilon(s,x_0)|^2\big\|\sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\big\|_{l^2}^2\,ds\Big)^{1/2} \\
&\le C\sqrt{\epsilon}\bar{E}\Big(\int_0^t |\omega_\epsilon(s,x_0)|^2\,ds\Big)^{1/2} \le C\epsilon + \frac{1}{4}\bar{E}\Big(\sup_{s\le t}|\omega_\epsilon(s,x_0)|^2\Big).
\end{aligned}
$$

Similarly by Burkholder's inequality, we get

$$
\begin{aligned}
\bar{E}\sup_{s\le t}|J_4(s)| &\le CM\epsilon^{3/2} + C\sqrt{\epsilon}\bar{E}\Big(\int_0^t |\omega_\epsilon(s-,x_0)|^2\int_\Gamma \varphi_\epsilon(s,y)\nu(dy)\,ds\Big)^{1/2} \\
&\le CM\epsilon^{3/2} + CM\epsilon + \frac{1}{4}\bar{E}\sup_{s\le t}|\omega_\epsilon(s,x_0)|^2.
\end{aligned}
$$

Thus

$$\bar{E}\sup_{s\le t}|\omega_\epsilon(s,x_0)|^2 \le C\Big(\epsilon+\bar{E}\sup_{s\le t}|J_1(s)|+\bar{E}\sup_{s\le t}|J_3(s)|\Big). \tag{21}$$

By Itō's formula,

$$\begin{aligned}
&\langle\omega_\epsilon(t,x_0),v_{1,\epsilon}(t)\rangle\\
&=\int_0^t v_{1,\epsilon}(s)\,d\omega_\epsilon(s,x_0)-\int_0^t \omega_\epsilon(s,x_0)\,dv_{1,\epsilon}(s)\\
&=\int_0^t v_{1,\epsilon}(s)\big(b(X^{\epsilon,u_\epsilon}(s,x_0))-b(X^u(s,x_0))\big)\,ds-\int_0^t \omega_\epsilon(s,x_0)\,dv_{1,\epsilon}(s)\\
&\quad+\int_0^t v_{1,\epsilon}(s)\big(\sigma(X^{\epsilon,u_\epsilon}(s,x_0))\psi_\epsilon(s)-\sigma(X^u(s,x_0))\psi(s)\big)\,ds\\
&\quad+\sqrt{\epsilon}\int_0^t v_{1,\epsilon}(s)\sigma\big(X^{\epsilon,u_\epsilon}(s,x_0)\big)\,dW(s)\\
&\quad-\int_0^t v_{1,\epsilon}(s)\big(dK^{\epsilon,u_\epsilon}(s,x_0)-dK^u(s,x_0)\big)\\
&\quad+\int_0^t\int_{\mathbb{Y}} v_{1,\epsilon}(s)\big[\gamma\big(X^{\epsilon,u_\epsilon}(s,x_0),y\big)\big(\varphi_\epsilon(s,y)-1\big)\\
&\quad-\gamma\big(X^u(s,x_0),y\big)\big(\varphi(s,y)-1\big)\big]\nu(dy)\,ds\\
&\quad+\int_0^t\int_{\mathbb{Y}} v_{1,\epsilon}(s-)\gamma\big(X^{\epsilon,u_\epsilon}(s-,x_0),y\big)\big(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}-\varphi_\epsilon(s,y)\nu(dy)\,ds\big)\\
&:=\sum_{i=1}^6 L_i(t)-\frac{1}{2}J_1(t).
\end{aligned}$$

By (14), for any $\delta>0$,

$$\begin{aligned}
&\bar{P}\Big(\sup_{t\le T}|\langle\omega_\epsilon(t,x_0),v_{1,\epsilon}(t)\rangle|>\delta\Big)\\
&\quad=\bar{P}\Big(\sup_{t\le T}|\langle\omega_\epsilon(t,x_0),v_{1,\epsilon}(t)\rangle|>\delta,\sup_\epsilon\sup_{t\le T}|\omega_\epsilon(t,x_0)|<N\Big)\\
&\qquad+\bar{P}\Big(\sup_{t\le T}|\langle\omega_\epsilon(t,x_0),v_{1,\epsilon}(t)\rangle|>\delta,\sup_\epsilon\sup_{t\le T}|\omega_\epsilon(t,x_0)|\ge N\Big)\\
&\quad\le\bar{P}\Big(\sup_{t\le T}|v_{1,\epsilon}(t)|\ge\frac{\delta}{N}\Big)+\bar{P}\Big(\sup_\epsilon\sup_{t\le T}|\omega_\epsilon(t,x_0)|\ge N\Big).
\end{aligned}$$

Let $\epsilon\to 0$, and then let $N\to\infty$. By Proposition 3.3 and Lemma 3.4,

$$\bar{P}\Big(\sup_{t\le T}|\langle\omega_\epsilon(t,x_0),v_{1,\epsilon}(t)\rangle|>\delta\Big)\to 0.$$

Analogously by (**H**2),

$$\sup_{t\le T}|L_i(t)|\to 0,\quad \text{in probability},\ i=1,2.$$

By Proposition 3.3 and Lemma 3.4,

$$\sup_{t\le T} L_4(t) \le \sup_{t\le T}|v_{1,\epsilon}(t)|(|K^{\epsilon,u_\epsilon}|_T^0 + |K^u|_T^0) \to 0.$$

Moreover, using Doob's martingale inequality and Lemma 3.4 again, we get

$$\bar{E}\Big(\sup_{t\le T}|L_3(t)|\Big) \le C\epsilon, \quad \bar{E}\Big(\sup_{t\le T}|L_6(t)|\Big) \le C\epsilon.$$

As to L_5,

$$\begin{aligned}
L_5(t) = \int_0^t \int_{\mathbb{Y}} & v_{1,\epsilon}(s)\big[\gamma\big(X^{\epsilon,u_\epsilon}(s,x_0),y\big)\big(\varphi_\epsilon(s,y)-1\big) \\
& - \gamma\big(X^u(s,x_0),y\big)\big(\varphi(s,y)-1\big)\big]\nu(dy)\,ds \\
= & \int_0^t \int_{\mathbb{Y}} \big\langle v_{1,\epsilon}(s), \gamma\big(X^{\epsilon,u_\epsilon}(s,x_0),y\big) - \gamma\big(X^u(s,x_0),y\big)\big\rangle\big(\varphi_\epsilon(s,y)-1\big)\nu(dy)\,ds \\
& + \int_0^t \int_{\mathbb{Y}} v_{1,\epsilon}(s)\gamma\big(X^u(s,x_0),y\big)\big(\varphi_\epsilon(s,y)-\varphi(s,y)\big)\nu(dy)\,ds.
\end{aligned}$$

Thus for any $\delta > 0$,

$$\begin{aligned}
&\bar{P}\Big(\sup_{t\le T}|L_5(t)|>\delta\Big) \\
&\quad \le \bar{P}\Big(\int_0^T \int_\Gamma |v_{1,\epsilon}(s)||\omega_\epsilon(s,x_0)|\big(\varphi_\epsilon(s,y)+1\big)\nu(dy)\,ds > \frac{\delta}{2C_1}\Big) \\
&\qquad + \bar{P}\Big(\int_0^T \int_\Gamma |v_{1,\epsilon}(s)|\big(\varphi_\epsilon(s,y)+\varphi(s,y)\big)\nu(dy)\,ds > \frac{\delta}{2C}\Big) \\
&\quad = \bar{P}\Big(\int_0^T \int_\Gamma |v_{1,\epsilon}(s)||\omega_\epsilon(s,x_0)|\big(\varphi_\epsilon(s,y)+1\big)\nu(dy)\,ds > \frac{\delta}{2C_1}, \sup_{t\le T}|\omega_\epsilon(t,x_0)|<N\Big) \\
&\qquad + \bar{P}\Big(\int_0^T \int_\Gamma |v_{1,\epsilon}(s)||\omega_\epsilon(s,x_0)|\big(\varphi_\epsilon(s,y)+1\big)\nu(dy)\,ds \\
&\quad > \frac{\delta}{2C_1}, \sup_{t\le T}|\omega_\epsilon(t,x_0)|\ge N\Big) \\
&\qquad + \bar{P}\Big(\int_0^T \int_\Gamma |v_{1,\epsilon}(s)|\big(\varphi_\epsilon(s,y)+\varphi(s,y)\big)\nu(dy)\,ds > \frac{\delta}{2C}\Big) \\
&\quad \le \bar{P}\Big(\int_0^T \int_\Gamma |v_{1,\epsilon}(s)|\big(\varphi_\epsilon(s,y)+1\big)\nu(dy)\,ds > \frac{\delta}{2C_1N}\Big) \\
&\qquad + \bar{P}\Big(\sup_{t\le T}|\omega_\epsilon(t,x_0)|\ge N\Big) + \bar{P}\Big(\sup_{t\le T}|v_{1,\epsilon}(t)| > \frac{\delta}{4CM}\Big).
\end{aligned}$$

With Proposition 3.3 and Lemma 3.4, we have

$$\lim_{\epsilon\to 0}\bar{P}\Big(\sup_{t\le T}|L_5(t)|>\delta\Big) = 0.$$

Summing up all these inequalities gives

$$\sup_{t\le T}|J_1(t)| = 2\sup_{t\le T}\big|\langle\omega_\epsilon(t,x_0), v_{1,\epsilon}(t)\rangle\big| + \sup_{t\le T}\sum_{i=1}^{6}|L_i(t)| \xrightarrow{\bar{P}} 0. \tag{22}$$

In the same way, we can prove that

$$\sup_{t\le T}|J_3(t)| \to 0 \quad \text{in probability.} \tag{23}$$

Note that by Proposition 3.3,

$$\sup_\epsilon \bar{E}\Big(\sup_{t\le T}|J_1(t)|^2\Big) < \infty, \qquad \sup_\epsilon \bar{E}\Big(\sup_{t\le T}|J_3(t)|^2\Big) < \infty. \tag{24}$$

Therefore combining (22)–(24) gives

$$\bar{E}\Big(\sup_{t\le T}\big(|J_1(t)| + |J_3(t)|\big)\Big) \to 0.$$

Substituting this estimate in (21), we get

$$\bar{E}\Big(\sup_{t\le T}|X^{\epsilon,u_\epsilon}(t,x_0) - X^u(t,x_0)|^2\Big) \le C\epsilon.$$

(2) By Chebyshev's inequality and Proposition 3.2, for any $\delta > 0$,

$$\begin{aligned}
&\bar{P}\Big(\sup_{t\le T}|X^{\epsilon,u_\epsilon}(t,x_\epsilon) - X^u(t,x_0)| > \delta\Big)\\
&\quad\le \frac{1}{\delta^2}\bar{E}\Big(\sup_{t\le T}|X^{\epsilon,u_\epsilon}(t,x_\epsilon) - X^u(t,x_0)|^2\Big)\\
&\quad\le \frac{1}{\delta^2}\bar{E}\Big(\sup_{t\le T}\big(|X^{\epsilon,u_\epsilon}(t,x_\epsilon) - X^{\epsilon,u_\epsilon}(t,x_0)|^2 + |X^{\epsilon,u_\epsilon}(t,x_0) - X^u(t,x_0)|^2\big)\Big)\\
&\quad\le \frac{1}{\delta^2}C(|x_\epsilon - x_0|^2 + \epsilon) \to 0, \quad \text{as } \epsilon \to 0,
\end{aligned}$$

and we complete the proof. □

Proof of Theorem 3.1.
Define $\mathcal{G}^0 : \mathbb{D}_0 \times \mathbb{V} \to \mathbb{D}$ as

$$\mathcal{G}^0(x,w,m) := \begin{cases} X^q(\cdot,x) & \text{if } (w,m) = (\int_0^\cdot g(s)\,ds, \nu_T^h) \in \mathbb{V} \\ & \quad \text{and } q = (g,h) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $u_\epsilon, u \in \mathcal{U}_M$ and $u_\epsilon \to u$ in distribution. For the solution X^{ϵ,u_ϵ} to (10) with the initial value x_ϵ, like (7), it has a representation

$$X^{\epsilon,u_\epsilon} = \mathcal{G}^\epsilon\Big(x_\epsilon, \sqrt{\epsilon}W(\cdot) + \int_0^\cdot \psi_\epsilon(s)\,ds, \epsilon N^{\epsilon^{-1}\varphi_\epsilon}\Big).$$

Since S_M is compact, the law of (u_ϵ, W, N) is tight. By Prohorov's theorem, there exists a subsequence the law of which converges weakly to, say μ. By Skorohod's representation theorem, there exists another probability space $(\Omega', \mathcal{F}', P')$ and $(u'_\epsilon, W'_\epsilon, N'_\epsilon)$ and (u', W', N') on it satisfying the following:

(1) $(u'_\epsilon, W'_\epsilon, N'_\epsilon)$ has the same law as (u_ϵ, W, N);
(2) the law of (u', W', N') is μ;
(3) $(u'_\epsilon, W'_\epsilon, N'_\epsilon)$ converges to (u', W', N') almost surely.

Then by Proposition 3.5,

$$\mathcal{G}^\epsilon\Big(x_\epsilon, \sqrt{\epsilon}W(\cdot) + \int_0^\cdot \psi_\epsilon(s)\,ds, \epsilon N^{\epsilon^{-1}\varphi_\epsilon}\Big) \Rightarrow \mathcal{G}^0\Big(x_0, \int_0^\cdot \psi(s)\,ds, \nu_T^\varphi\Big),$$

and ($\mathbf{C}1$) thus follows. Assertion ($\mathbf{C}2$) can be proved in a similar way to Proposition 3.5. Now it remains to verify that $x \to I_x$ is lower semicontinuous. By (8),

$$I_x(f) = \inf_{\{q=(g,h)\in S: f = X^q\}} \big\{L_{1,T}(g) + L_{2,T}(h)\big\}, \quad x \in \overline{D(A)},$$

where X^q solves (15). If $\overline{D(A)} \ni x_n \to x \in \overline{D(A)}$, then it follows from an argument similar to the proof of Proposition 3.2 that

$$f_n(t) := X^q(t, x_n) \xrightarrow{\text{a.s.}} X^q(t,x) =: f(t), \quad \forall t \in [0,T].$$

Thus $\liminf_n I(f_n) \geq I(f)$ follows from Remark 2.13. Summing up these arguments gives

$$\liminf_n I_{x_n} = \liminf_n I(f_n) \geq I(f) = I_x,$$

and the proof is now complete. $\square$

NOTE

It is worth mentioning that the large deviation principle can be established for MSDEs in Banach spaces similarly. Only the calculations may be more complicated.

4. Examples

In this section, we apply the result to SDEs and reflected SDEs driven by Brownian motions plus a Poisson random measure.

4.1. SDEs with Brownian motions and Poisson random measure

If $A = 0$, then our result covers the example given in [6] as an application; that is, the large deviation principle is established under ($\mathbf{H}2$)–($\mathbf{H}4$) for solutions to the following SDE:

$$(25) \qquad \begin{cases} dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t) + \int_{\mathbb{Y}} \gamma(X(t-), y)\tilde{N}(dt, dy), \\ X(0) = x_0 \in \mathbb{R}^d. \end{cases}$$

4.2. SDEs with reflecting boundaries

Let D be a convex closed region with nonempty interior, and let $I_D(x)$ be the indicator function of D. We have

$$I_D(x) = \begin{cases} 0 & x \in D, \\ \infty & \text{otherwise}. \end{cases}$$

Take A to be the subdifferential operator ∂I_D defined by

$$\partial I_D(x) = \begin{cases} \{0\} & x \in \mathrm{Int}(D), \\ \Pi_x & x \in \partial D, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then (1) becomes the following SDE with reflecting boundary (see [12]):

$$\begin{cases} X^\epsilon(t) = \int_0^t b(X^\epsilon(s))\,ds + \int_0^t \sqrt{\epsilon}\sigma(X^\epsilon(s))\,dW(s) \\ \qquad\qquad + \int_0^t \int_{\mathbb{Y}} \gamma(X^\epsilon(s-), y)(\epsilon N^{\epsilon^{-1}}(ds, dy) - \nu(dy)\,ds) - K^\epsilon(t) \\ X^\epsilon(0) = x_0 \in \bar{D}, \quad \epsilon \in (0, 1], \end{cases}$$

and our result shows that a uniform large deviation principle holds for the solutions to such reflected SDEs under (**H**2)–(**H**4).

Acknowledgments. The author is very grateful to Professor Jiagang Ren for proposing this subject and for valuable discussions, and to the referees for helpful comments and suggestions.

References

[1] D. Applebaum, *Levy Processes and Stochastic Calculus*, Cambridge Univ. Press, Cambridge, 2004.

[2] H. Brézis, *Operateurs Maximaux Monotones*, North-Holland, Amsterdam, 1973.

[3] A. Budhiraja and P. Dupuis, *A variational representation for certain functionals of Brownian motion*, Ann. Probab. **26** (1998), 1641–1659.

[4] ———, *A variational representation for positive functionals of infinite-dimensional Brownian motion*, Probab. Math. Statist. **20** (1998), 39–61.

[5] A. Budhiraja, P. Dupuis, and V. Maroulas, *Large deviations for infinite-dimensional stochastic dynamical systems*, Ann. Probab. **36** (2008), 1390–1420.

[6] ———, *Variational representations for continuous time processes*, preprint.

[7] E. Cépa, *Problème de Skorohod multivoque*, Ann. Probab. **26** (1998), 500–532.

[8] J. Doob, *Measure Theory*, Springer, New York, 1993.

[9] P. Dupuis and R. S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley, New York, 1997.

[10] M. Loève, *Probability Theory I*, 4th ed., Springer, Berlin, 1977.

[11] V. Maroulas, *Uniform large deviations for ∞-dimensional stochastic systems with jumps*, preprint.

[12] J. L. Menaldi and M. Robin, *Reflected diffusion processes with jumps*, Ann. Probab. **13** (1985), 319–341.

[13] J. Ren and J. Wu, Multivalued stochastic differential equations driven by Poisson point processes, to appear in *Workshop on Stochastic Analysis and Finance (City Univ. Hong Kong, July, 2009)*, Birkhäuser.

[14] J. Ren, J. Wu, and X. Zhang, *Exponential ergodicity of non-Lipschitz multivalued stochastic differential equations*, Bull. Sci. Math. **134** (2010), 391–404.

[15] J. Ren, S. Xu, and X. Zhang, *Large deviations for multivalued stochastic differential equations*, to appear in J. Theoret. Probab.

[16] J. Ren and X. Zhang, *Freidlin-Wenzell's large deviations for homeomorphism flows of non-Lipschitz SDEs*, Bull. Sci. Math. **129/8** (2010), 643–655.

[17] J. Wu, *Wiener-Poisson typed multivalued stochastic evolution equations in Banach spaces*, to appear in Stoch. Dyn.

[18] X. Zhang, *A variational representation for random functionals on abstract Wiener spaces*, J. Math. Kyoto Univ. **49** (2009), 475–490.

[19] ———, *Clark-Ocone formula and variational representation for Poisson functionals*, Ann. Probab. **37** (2009), 506–529.

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