Ann. Funct. Anal. 7 (2016), no. 3, 496–507
http://dx.doi.org/10.1215/20088752-3624940
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

NONLINEAR MAPS PRESERVING THE JORDAN TRIPLE $*$-PRODUCT ON VON NEUMANN ALGEBRAS

CHANGJING LI,[1*] FANGYAN LU,[2] and TING WANG[3]

Communicated by S.-H. Kye

Abstract. This article investigates a bijective map Φ between two von Neumann algebras, one of which has no central abelian projections, satisfying $\Phi([[A,B]_*,C]_*) = [[\Phi(A),\Phi(B)]_*,\Phi(C)]_*$ for all A,B,C in the domain, where $[A,B]_* = AB - BA^*$ is the skew Lie product of A and B. We show that the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, where $\Phi(I)$ is a self-adjoint central element in the range with $\Phi(I)^2 = I$.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra and let η be a nonzero scalar. For $A,B \in \mathcal{A}$, define the Jordan η-$*$-product of A and B by $A\diamondsuit_\eta B = AB + \eta BA^*$. The Jordan (-1)-$*$-product, which is customarily called the *skew Lie product*, was extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [11], [12], [10]) and in the problem of characterizing ideals (see, for example, [2], [9]). A map Φ between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan η-$*$-product if $\Phi(A\diamondsuit_\eta B) = \Phi(A)\diamondsuit_\eta\Phi(B)$ for all $A,B \in \mathcal{A}$. Recently, many authors have paid more attention to the maps preserving the Jordan η-$*$-product between $*$-algebra (see, for example, [1], [3], [7]). In [4], Dai and Lu proved that if Φ is a bijective map preserving the Jordan η-$*$-product between two von Neumann algebras, one of which has no central abelian projections, then Φ is a linear $*$-isomorphism if η is not real, and Φ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real.

Received Oct. 30, 2015; Accepted Feb. 11, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47B48; Secondary 46L10.
Keywords. Jordan triple $*$-product, isomorphism, von Neumann algebras.

Recently, Huo et al. in [5] studied a more general problem. They considered the Jordan triple η-$*$-product of three elements A, B, and C in a $*$-algebra $\mathcal{A}$ defined by $A\diamondsuit_\eta B\diamondsuit_\eta C = (A\diamondsuit_\eta B)\diamondsuit_\eta C$ (we should be aware that $\diamondsuit_\eta$ is not necessarily associative). A map Φ between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan triple η-$*$-product if $\Phi(A\diamondsuit_\eta B\diamondsuit_\eta C) = \Phi(A)\diamondsuit_\eta\Phi(B)\diamondsuit_\eta\Phi(C)$ for all $A, B, C \in \mathcal{A}$. Clearly, a map between $*$-algebras preserving the Jordan η-$*$-product also preserves the Jordan triple η-$*$-product. However, the map $\Phi : \mathbb{C} \to \mathbb{C}, \Phi(\alpha + \beta i) = -4(\alpha^3 + \beta^3 i)$ is a bijection which preserves the Jordan triple (-1)-$*$-product and the Jordan triple 1-$*$-product, but it does not preserve the Jordan (-1)-$*$-product or Jordan 1-$*$-product. So, the class of those maps preserving the Jordan triple η-$*$-product is, in principle, wider than the class of maps preserving the Jordan η-$*$-product. In [5], let $\eta \neq -1$ be a nonzero complex number, and let Φ be a bijection between two von Neumann algebras, one of which has no central abelian projections, satisfying $\Phi(I) = I$ and preserving the Jordan triple η-$*$-product. In [5] it was shown that Φ is a linear $*$-isomorphism if η is not real and that Φ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real.

On the one hand, [5] did not include the case $\eta = -1$. Obviously, the Jordan (triple) (-1)-$*$-product is very important and meaningful. On the other hand, it is easy to see that a map Φ preserving the Jordan triple η-$*$-product does not need to satisfy $\Phi(I) = I$. Indeed, let $\Phi(A) = -A$ for all $A \in \mathcal{A}$. Then Φ preserves the Jordan triple η-$*$-product but $\Phi(I) = -I$. Because of the two reasons above, in the current article we will discuss maps preserving the Jordan triple (-1)-$*$-product without the assumption $\Phi(I) = I$. We will prove that, if Φ is a bijective map preserving the Jordan triple (-1)-$*$-product between two von Neumann algebras, one of which has no central abelian projections, then the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, where $\Phi(I)$ is a self-adjoint central element in the range with $\Phi(I)^2 = I$. We mention that the methods in [5] do not fit for solving our problem since their proofs heavily depend on the assumption $\Phi(I) = I$.

Before stating the main results, we need some notation and preliminaries. Throughout this paper, we often write the Jordan (-1)-$*$-product by $[A, B]_*$, that is, $[A, B]_* = AB - BA^*$. All algebras and spaces are over the complex number field $\mathbb{C}$. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. The set $\mathcal{Z}(\mathcal{A}) = \{S \in \mathcal{A} : ST = TS \text{ for all } T \in \mathcal{A}\}$ is called the *center* of $\mathcal{A}$. A projection P is called a *central abelian projection* if $P \in \mathcal{Z}(\mathcal{A})$ and $P\mathcal{A}P$ is abelian. Recall that the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P satisfying $PA = A$. It is not difficult to see that the central carrier of A is the projection onto the closed subspace spanned by $\{BA(x) : B \in \mathcal{A}, x \in H\}$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \leq A\}$. If P is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be *core-free* if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$.

Lemma 1.1 ([8, Lemma 4])**.** *Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections. Then there exists a projection $P \in \mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = I$.*

Lemma 1.2. *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space H. Let A be an operator in $\mathcal{A}$ and let $P \in \mathcal{A}$ be a projection with $\overline{P} = I$. If $ABP = 0$ for all $B \in \mathcal{A}$, then $A = 0$.*

Proof. This is easy to see from the fact that $\{BP(x) : B \in \mathcal{A}, x \in H\}$ is dense in H. □

Lemma 1.3. *Let $\mathcal{A}$ be an arbitrary von Neumann algebra. Then $AB = BA^*$ for all $B \in \mathcal{A}$ implies that $A \in \mathcal{Z}(\mathcal{A})$ and that $A = A^*$.*

Proof. In fact, take $B = I$, then $A = A^*$. So $AB = BA$ for all $B \in \mathcal{A}$, which implies that $A \in \mathcal{Z}(\mathcal{A})$. □

2. Additivity

The main result in this section is the following.

Theorem 2.1. *Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections and let $\mathcal{B}$ be a $*$-algebra. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([[A, B]_*, C]_*) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$. Then Φ is additive.*

Before the proof, we note that the hypothesis "$\mathcal{A}$ containing no central abelian projections" is needed in the above theorem. For example, for $\alpha, \beta \in \mathbb{R}$, define $\Phi(\alpha+\beta i) = 4(\alpha^3+\beta^3 i)$. Then Φ is a bijection from $\mathbb{C}$ onto itself. It is not difficult to verify that Φ preserves the skew Lie triple product. However, it is obviously not additive.

Proof. First, we give a key technique. Suppose that $A_1, A_2, \ldots, A_n$ and T are in $\mathcal{A}$ such that $\Phi(T) = \sum_{i=1}^n \Phi(A_i)$. Then for all $S_1, S_2 \in \mathcal{A}$, we have

$$\Phi\big(\big[[S_1, S_2]_*, T\big]_*\big) = \big[\big[\Phi(S_1), \Phi(S_2)\big]_*, \Phi(T)\big]_* = \sum_{i=1}^n \Phi\big(\big[[S_1, S_2]_*, A_i\big]_*\big), \quad (2.1)$$

$$\Phi\big(\big[[S_1, T]_*, S_2\big]_*\big) = \big[\big[\Phi(S_1), \Phi(T)\big]_*, \Phi(S_2)\big]_* = \sum_{i=1}^n \Phi\big(\big[[S_1, A_i]_*, S_2\big]_*\big), \quad (2.2)$$

and

$$\Phi\big(\big[[T, S_1]_*, S_2\big]_*\big) = \big[\big[\Phi(T), \Phi(S_1)\big]_*, \Phi(S_2)\big]_* = \sum_{i=1}^n \Phi\big(\big[[A_i, S_1]_*, S_2\big]_*\big). \quad (2.3)$$

Claim 1. *We have $\Phi(0) = 0$.*

For every $A \in \mathcal{A}$, we have

$$\Phi(0) = \Phi\big(\big[[0, A]_*, A\big]_*\big) = \big[\big[\Phi(0), \Phi(A)\big]_*, \Phi(A)\big]_*.$$

Since Φ is surjective, there exists $A \in \mathcal{A}$ such that $\Phi(A) = 0$. So $\Phi(0) = 0$.

By Lemma 1.1, there exists a projection P such that $\underline{P} = 0$ and $\overline{P} = I$. Let $P_1 = P$ and $P_2 = I - P$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. In the remainder of this article, when we write A_{ij} it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

Claim 2. *For every $A_{12} \in \mathcal{A}_{12}, B_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21}).$$

Choose $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{12}) + \Phi(B_{21}).$$

Since

$$\big[\big[i(P_2 - P_1), I\big]_*, A_{12}\big]_* = \big[\big[i(P_2 - P_1), I\big]_*, B_{21}\big]_* = 0,$$

it follows from (2.1) and Claim 1 that

$$\Phi\big(\big[\big[i(P_2 - P_1), I\big]_*, T\big]_*\big) = 0.$$

From this, we get $[[i(P_2 - P_1), I]_*, T]_* = 0$. So $T_{11} = T_{22} = 0$.

Since $[[A_{12}, P_1]_*, I]_* = 0$, it follows from (2.3) and Claim 1 that

$$\Phi\big(\big[\big[T, P_1\big]_*, I\big]_*\big) = \Phi\big(\big[\big[B_{21}, P_1\big]_*, I\big]_*\big).$$

By the injectivity of Φ, we obtain

$$2(TP_1 - P_1T^*) = \big[\big[T, P_1\big]_*, I\big]_* = \big[\big[B_{21}, P_1\big]_*, I\big]_* = 2(B_{21} - B_{21}^*).$$

Hence $T_{21} = B_{21}$. Similarly, $T_{12} = A_{12}$, proving the claim.

Claim 3. *For every $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$, we have*

$$\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})$$

and

$$\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Let $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}).$$

It follows from (2.1) and Claim 2 that

$$\begin{aligned}
&\Phi\big(2i(P_2T + TP_2)\big)\\
&\quad = \Phi\big(\big[\big[iP_2, I\big]_*, T\big]_*\big)\\
&\quad = \Phi\big(\big[\big[iP_2, I\big]_*, A_{11}\big]_*\big) + \Phi\big(\big[\big[iP_2, I\big]_*, B_{12}\big]_*\big) + \Phi\big(\big[\big[iP_2, I\big]_*, C_{21}\big]_*\big)\\
&\quad = \Phi(2iB_{12}) + \Phi(2iC_{21})\\
&\quad = \Phi\big(2i(B_{12} + C_{21})\big).
\end{aligned}$$

Thus $P_2T + TP_2 = B_{12} + C_{21}$, which implies that $T_{22} = 0$, $T_{12} = B_{12}$, and $T_{21} = C_{21}$. Now we get $T = T_{11} + B_{12} + C_{21}$.

Since

$$\big[\big[i(P_2 - P_1), I\big]_*, B_{12}\big]_* = \big[\big[i(P_2 - P_1), I\big]_*, C_{21}\big]_* = 0,$$

it follows from (2.1) that

$$\Phi\big(\big[\big[i(P_2-P_1),I\big]_*,T\big]_*\big)=\Phi\big(\big[\big[i(P_2-P_1),I\big]_*,A_{11}\big]_*\big),$$

from which we get $T_{11}=A_{11}$. Consequently, $\Phi(A_{11}+B_{12}+C_{21})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})$.

Similarly, we can get that $\Phi(B_{12}+C_{21}+D_{22})=\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22})$.

Claim 4. *For every $A_{11}\in\mathcal{A}_{11}$, $B_{12}\in\mathcal{A}_{12}$, $C_{21}\in\mathcal{A}_{21}$, and $D_{22}\in\mathcal{A}_{22}$, we have*

$$\Phi(A_{11}+B_{12}+C_{21}+D_{22})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

Let $T=\sum_{i,j=1}^{2}T_{ij}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

It follows from (2.1) and Claim 3 that

$$\begin{aligned}\Phi(2iP_1T+2iTP_1)&=\Phi\big(\big[\big[iP_1,I\big]_*,T\big]_*\big)\\&=\Phi\big(\big[\big[iP_1,I\big]_*,A_{11}\big]_*\big)+\Phi\big(\big[\big[iP_1,I\big]_*,B_{12}\big]_*\big)\\&\quad+\Phi\big(\big[\big[iP_1,I\big]_*,C_{21}\big]_*\big)+\Phi\big(\big[\big[iP_1,I\big]_*,D_{22}\big]_*\big)\\&=\Phi(4iA_{11})+\Phi(2iB_{12})+\Phi(2iC_{21})\\&=\Phi(4iA_{11}+2iB_{12}+2iC_{21}).\end{aligned}$$

Thus

$$P_1T+TP_1=2A_{11}+B_{12}+C_{21},$$

and then $T_{11}=A_{11}$, $T_{12}=B_{12}$, and $T_{21}=C_{21}$.

Similarly, we can get

$$\Phi(2iP_2T+2iTP_2)=\Phi(4iD_{22}+2iB_{12}+2iC_{21}).$$

From this, we get $T_{22}=D_{22}$, proving the claim.

Claim 5. *For every $C_{jk},D_{jk}\in\mathcal{A}_{jk}$, $1\le j\ne k\le 2$, we have*

$$\Phi(C_{jk}+D_{jk})=\Phi(C_{jk})+\Phi(D_{jk}).$$

For every $A_{jk},B_{jk}\in\mathcal{A}_{jk}$, since

$$\Big[\Big[\frac{i}{2}I,P_j+A_{jk}\Big]_*,P_k+B_{jk}\Big]_*=i(A_{jk}+B_{jk})+i(A_{jk}^*)+i(B_{jk}A_{jk}^*),$$

we get from Claim 4 that

$$\begin{aligned}&\Phi\big(i(A_{jk}+B_{jk})\big)+\Phi(iA_{jk}^*)+\Phi\big(i(B_{jk}A_{jk}^*)\big)\\&\quad=\Phi\Big(\Big[\Big[\frac{i}{2}I,P_j+A_{jk}\Big]_*,P_k+B_{jk}\Big]_*\Big)\\&\quad=\Big[\Big[\Phi\Big(\frac{i}{2}I\Big),\Phi(P_j+A_{jk})\Big]_*,\Phi(P_k+B_{jk})\Big]_*\\&\quad=\Big[\Big[\Phi\Big(\frac{i}{2}I\Big),\Phi(P_j)+\Phi(A_{jk})\Big]_*,\Phi(P_k)+\Phi(B_{jk})\Big]_*\\&\quad=\Big[\Big[\Phi\Big(\frac{i}{2}I\Big),\Phi(P_j)\Big]_*,\Phi(P_k)\Big]_*+\Big[\Big[\Phi\Big(\frac{i}{2}I\Big),\Phi(P_j)\Big]_*,\Phi(B_{jk})\Big]_*\end{aligned}$$

$$
\begin{aligned}
&+ \Big[\Big[\Phi\Big(\frac{i}{2}I\Big), \Phi(A_{jk})\Big]_*, \Phi(P_k)\Big]_* \\
&+ \Big[\Big[\Phi\Big(\frac{i}{2}I\Big), \Phi(A_{jk})\Big]_*, \Phi(B_{jk})\Big]_* \\
&= \Phi(iB_{jk}) + \Phi\big(i(A_{jk} + A_{jk}^*)\big) + \Phi\big(i(B_{jk}A_{jk}^*)\big) \\
&= \Phi(iB_{jk}) + \Phi(iA_{jk}) + \Phi(iA_{jk}^*) + \Phi\big(i(B_{jk}A_{jk}^*)\big),
\end{aligned}
$$

which implies that $\Phi(i(A_{jk} + B_{jk})) = \Phi(iB_{jk}) + \Phi(iA_{jk})$. Let $A_{jk} = -iC_{jk}$ and $B_{jk} = -iD_{jk}$. Then

$$\Phi(C_{jk} + D_{jk}) = \Phi\big(i(A_{jk} + B_{jk})\big) = \Phi(iB_{jk}) + \Phi(iA_{jk}) = \Phi(C_{jk}) + \Phi(D_{jk}).$$

Claim 6. *For every $A_{jj}, B_{jj} \in \mathcal{A}_{jj}, 1 \le j \le 2$, we have*

$$\Phi(A_{jj} + B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

Let $T = \sum_{i,j=1}^2 T_{ij} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

For $1 \le j \neq k \le 2$, it follows from (2.1) that

$$\Phi\big(\big[[iP_k, I]_*, T\big]_*\big) = \Phi\big(\big[[iP_k, I]_*, A_{jj}\big]_*\big) + \Phi\big(\big[[iP_k, I]_*, B_{jj}\big]_*\big) = 0.$$

Hence $P_kT + TP_k = 0$, implying that $T_{jk} = T_{kj} = T_{kk} = 0$. Now we get $T = T_{jj}$.

For every $C_{jk} \in \mathcal{A}_{jk}, j \neq k$, it follows from (2.2) and Claim 5 that

$$
\begin{aligned}
\Phi(2iT_{jj}C_{jk}) &= \Phi\big(\big[[iP_j, T_{jj}]_*, C_{jk}\big]_*\big) \\
&= \Phi\big(\big[[iP_j, A_{jj}]_*, C_{jk}\big]_*\big) + \Phi\big(\big[[iP_j, B_{jj}]_*, C_{jk}\big]_*\big) \\
&= \Phi(2iA_{jj}C_{jk}) + \Phi(2iB_{jj}C_{jk}) \\
&= \Phi\big(2i(A_{jj}C_{jk} + B_{ii}C_{jk})\big).
\end{aligned}
$$

Hence

$$(T_{jj} - A_{jj} - B_{jj})C_{jk} = 0$$

for all $C_{jk} \in \mathcal{A}_{jk}$; that is, $(T_{jj} - A_{jj} - B_{jj})CP_j = 0$ for all $C \in \mathcal{A}$. It follows from Lemma 1.2 that $T_{jj} = A_{jj} + B_{jj}$, proving the claim.

Claim 7. *We have Φ is additive.*

Let $A = \sum_{i,j=1}^2 A_{ij}, B = \sum_{i,j=1}^2 B_{ij} \in \mathcal{A}$. By Claims 4, 5, and 6, we have

$$
\begin{aligned}
\Phi(A + B) &= \Phi\Big(\sum_{i,j=1}^2 A_{ij} + \sum_{i,j=1}^2 B_{ij}\Big) = \Phi\Big(\sum_{i,j=1}^2 (A_{ij} + B_{ij})\Big) \\
&= \sum_{i,j=1}^2 \Phi(A_{ij} + B_{ij}) = \sum_{i,j=1}^2 \Phi(A_{ij}) + \sum_{i,j=1}^2 \Phi(B_{ij}) \\
&= \Phi\Big(\sum_{i,j=1}^2 A_{ij}\Big) + \Phi\Big(\sum_{i,j=1}^2 B_{ij}\Big) = \Phi(A) + \Phi(B).
\end{aligned}
$$

□

3. Main result

Our main result in this paper reads as follows.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras, one of which has no central abelian projections. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([[A,B]_*,C]_*) = [[\Phi(A),\Phi(B)]_*,\Phi(C)]_*$ for all $A,B,C \in \mathcal{A}$. Then the following statements hold:*

1. *$\Phi(I)$ is a self-adjoint central element in $\mathcal{B}$ with $\Phi(I)^2 = I$;*
2. *let $\phi(A) = \Phi(I)\Phi(A)$ for all $A \in \mathcal{A}$; then there exists a central projection $E \in \mathcal{A}$ such that the restriction of ϕ to $\mathcal{A}E$ is a linear $*$-isomorphism and the restriction of ϕ to $\mathcal{A}(I-E)$ is a conjugate linear $*$-isomorphism.*

The proof of Theorem 3.1 will be organized into several claims. First, note that Φ is additive. Indeed, if $\mathcal{A}$ has no central abelian projections, then Theorem 2.1 assures that Φ is additive. If $\mathcal{B}$ has no central abelian projections, note that $\Phi^{-1} : \mathcal{B} \to \mathcal{A}$ is a bijection and preserves the skew Lie triple product. Applying Theorem 2.1 to Φ^{-1}, we know that Φ^{-1} (and hence Φ) is additive. In what follows, without loss of generality, we assume that $\mathcal{B}$ has no central abelian projections.

Claim 1. *We have the following:*

1. *$\Phi(I)^2 = I$ and $\Phi(I)$ is a self-adjoint central element in $\mathcal{B}$;*
2. *$\Phi(iI)^2 = -I$ and $\Phi(iI)$ is a conjugate self-adjoint central element in $\mathcal{B}$;*
3. *$\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$;*
4. *$\Phi(\mathcal{Z}(\mathcal{A})) = \mathcal{Z}(\mathcal{B})$.*

Proof. Let $A \in \mathcal{A}$ be arbitrary. Since Φ is surjective, there exists $B \in \mathcal{A}$ such that $\Phi(B) = I$. Then

$$\begin{aligned} 0 &= \Phi\big(\big[[I,A]_*,B\big]_*\big) \\ &= \big[\big[\Phi(I),\Phi(A)\big]_*,I\big]_* \\ &= \Phi(I)\Phi(A) - \Phi(A)\Phi(I)^* - \Phi(A)^*\Phi(I)^* + \Phi(I)\Phi(A)^* \end{aligned}$$

holds true for all $A \in \mathcal{A}$. That is,

$$\Phi(I)\big(\Phi(A)+\Phi(A)^*\big) = \big(\Phi(A)+\Phi(A)^*\big)\Phi(I)^*$$

holds true for all $A \in \mathcal{A}$. So $\Phi(I)B = B\Phi(I)^*$ holds true for all $B = B^* \in \mathcal{B}$. Since for every $B \in \mathcal{B}$, $B = B_1 + iB_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2i}$, it follows that $\Phi(I)B = B\Phi(I)^*$ holds true for all $B \in \mathcal{B}$. It follows from Lemma 1.3 that $\Phi(I)^* = \Phi(I) \in \mathcal{Z}(\mathcal{B})$.

For all $A \in \mathcal{A}$, since $\Phi(I)$ is a self-adjoint central element, then

$$\begin{aligned} 2\Phi(A-A^*) &= \Phi\big(\big[[A,I]_*,I\big]_*\big) \\ &= \big[\big[\Phi(A),\Phi(I)\big]_*,\Phi(I)\big]_* \\ &= 2\Phi(I)^2\big(\Phi(A)-\Phi(A)^*\big). \end{aligned} \tag{3.1}$$

Consequently, for every $A = -A^* \in \mathcal{A}$,

$$\Phi(A) = \Phi(I)^2\Big(\Phi\Big(\frac{A}{2}\Big) - \Phi\Big(\frac{A}{2}\Big)^*\Big) \tag{3.2}$$

which ensures that $\Phi(A) = -\Phi(A)^*$. Note that Φ^{-1} has the same properties as Φ, and note that we have Φ preserving the conjugate self-adjoint elements in both directions (i.e., $A = -A^*$ if and only if $\Phi(A) = -\Phi(A)^*$). It follows from the additivity of Φ and (3.2) that $\Phi(A) = \Phi(I)^2\Phi(A)$ for all $A = -A^* \in \mathcal{A}$, and then $B = \Phi(I)^2 B$ for all $B = -B^* \in \mathcal{B}$. For every $B \in \mathcal{B}$, we have $B = B_1 + iB_2$, where $B_1 = \frac{B-B^*}{2}$ and $B_2 = \frac{B+B^*}{2i}$ are conjugate self-adjoint elements. Then $B = \Phi(I)^2 B$ for all $B \in \mathcal{B}$. So $\Phi(I)^2 = I$, which, together with (3.1) and the additivity of Φ, implies that $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$.

Let $Z \in \mathcal{Z}(\mathcal{A})$ be arbitrary. For every $A = A^* \in \mathcal{A}$,

$$0 = \Phi\big(\big[[A, Z]_*, C\big]_*\big) = \big[[\Phi(A), \Phi(Z)]_*, \Phi(C)\big]_*$$

holds true for all $C \in \mathcal{A}$. It follows from Lemma 1.3 that

$$\big[\Phi(A), \Phi(Z)\big]_* \in \mathcal{Z}(\mathcal{B}).$$

Note that we have shown that Φ preserves star operator. Hence

$$\big[\Phi(A), \Phi(Z)\big] = \Phi(A)\Phi(Z) - \Phi(Z)\Phi(A) \in \mathcal{Z}(\mathcal{B}).$$

By the Kleinecke–Shirokov theorem (see [6]), $[\Phi(A), \Phi(Z)]$ is quasinilpotent, and therefore, being central, is zero. Then $B\Phi(Z) = \Phi(Z)B$ holds true for all $B = B^* \in \mathcal{B}$. Thus $\Phi(Z)B = B\Phi(Z)$ holds true for all $B \in \mathcal{B}$. Hence $\Phi(Z) \in \mathcal{Z}(\mathcal{B})$. Applying the similar process to Φ^{-1}, we get $\Phi(\mathcal{Z}(\mathcal{A})) = \mathcal{Z}(\mathcal{B})$.

Note that $\Phi(iI) \in \mathcal{Z}(\mathcal{B})$ and that $\Phi(iI)^* = -\Phi(iI)$. Then

$$-4\Phi(I) = \Phi\big(\big[[iI, I]_*, iI\big]_*\big) = \big[[\Phi(iI), \Phi(I)]_*, \Phi(iI)\big]_* = 4\Phi(I)\Phi(iI)^2.$$

So $\Phi(iI)^2 = -I$. □

Now, defining a map $\phi : \mathcal{A} \to \mathcal{B}$ by $\phi(A) = \Phi(I)\Phi(A)$ for all $A \in \mathcal{A}$, it is easy to see that ϕ has the following properties.

Claim 2. *The following hold:*

(1) *ϕ is an additive bijection and satisfies*

$$\phi\big(\big[[A, B]_*, C\big]_*\big) = \big[[\phi(A), \phi(B)]_*, \phi(C)\big]_*$$

for all $A, B, C \in \mathcal{A}$;

(2) *$\phi(I) = I, \phi(iI)^2 = -I$ and $\phi(iI)$ is conjugate self-adjoint central element in $\mathcal{B}$;*

(3) *$\phi(A^*) = \phi(A)^*$ for all $A \in \mathcal{A}$;*

(4) *$\phi(\mathcal{Z}(\mathcal{A})) = \mathcal{Z}(\mathcal{B})$.*

Claim 3. *We have that P is a projection in $\mathcal{A}$ if and only if $\phi(P)$ is a projection in $\mathcal{B}$.*

Proof. To prove the necessity, we suppose that P is a projection in $\mathcal{A}$. On the one hand, we have

$$\phi(iP) = \frac{1}{4}\phi\big(\big[[iI, I]_*, P\big]_*\big) = \frac{1}{4}\big[[\phi(iI), I]_*, \phi(P)\big]_* = \phi(iI)\phi(P).$$

On the other hand, we also have

$$\phi(iP) = \frac{1}{4}\phi\big([[iP, I]_*, P]_*\big) = \frac{1}{4}\big[[\phi(iP), I]_*, \phi(P)\big]_* = \phi(iI)\phi(P)^2.$$

Hence $\phi(iI)(\phi(P) - \phi(P)^2) = 0$. Since $\phi(iI)^2 = -I$, we get $\phi(P) = \phi(P)^2$. Together with part (3) of Claim 2, this implies that $\phi(P)$ is a projection in $\mathcal{B}$.

So far we have established the necessity. Note that the preceding proof does not use the condition that $\mathcal{B}$ has no central abelian projections. Therefore, the previous result can apply to ϕ^{-1}. Now, if $\phi(P)$ is a projection in $\mathcal{B}$, then $P = \phi^{-1}(\phi(P))$ is a projection in $\mathcal{A}$, proving the sufficiency. □

Since $\mathcal{B}$ has no central abelian projections, by Lemma 1.1 there exists a projection Q_1 in $\mathcal{B}$ such that $\underline{Q_1} = 0$ and $\overline{Q_1} = I$. Then by Claim 3, $P_1 = \phi^{-1}(Q_1)$ is a projection in $\mathcal{A}$. Set $P_2 = I - P_1$ and set $Q_2 = I - Q_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ and $\mathcal{B}_{ij} = Q_i\mathcal{B}Q_j$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$ and $\mathcal{B} = \sum_{i,j=1}^{2} \mathcal{B}_{ij}$.

Claim 4. *We have $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}, \phi(\mathcal{A}_{ii}) \subseteq \mathcal{B}_{ii}, 1 \leq i \neq j \leq 2$.*

Proof. Let A_{12} be an arbitrary element in $\mathcal{A}_{12}$. Then from

$$\begin{aligned}
-2\phi(A_{12}) &= \phi\big(\big[[iI, P_1]_*, iA_{12}\big]_*\big) \\
&= \big[[\phi(iI), Q_1]_*, \phi(iA_{12})\big]_* \\
&= 2\phi(iI)\big(Q_1\phi(iA_{12}) + \phi(iA_{12})Q_1\big),
\end{aligned}$$

we get that $Q_2\phi(A_{12})Q_2 = 0$; and from

$$\begin{aligned}
-2\phi(A_{12}) &= \phi\big(\big[[iI, P_2]_*, iA_{12}\big]_*\big) \\
&= \big[[\phi(iI), Q_2]_*, \phi(iA_{12})\big]_* \\
&= 2\phi(iI)\big(Q_2\phi(iA_{12}) + \phi(iA_{12})Q_2\big),
\end{aligned}$$

we get $Q_1\phi(A_{12})Q_1 = 0$. Hence $\phi(A_{12}) = B_{12} + B_{21}$ for some $B_{12} \in \mathcal{B}_{12}$ and $B_{21} \in \mathcal{B}_{21}$.

Now to show that $\phi(A_{12}) \subseteq \mathcal{B}_{12}$, we have to show that $B_{21} = 0$. This can be seen from

$$\begin{aligned}
0 &= \phi\big(\big[[iI, A_{12}]_*, P_1\big]_*\big) \\
&= \big[[\phi(iI), \phi(A_{12})]_*, Q_1\big]_* \\
&= 2\phi(iI)(B_{21} + B_{21}^*).
\end{aligned}$$

So $\Phi(A_{12}) \subseteq \mathcal{B}_{12}$. Hence by considering ϕ^{-1}, we have $\Phi(\mathcal{A}_{12}) = \mathcal{B}_{12}$. Similarly, we have $\Phi(\mathcal{A}_{21}) = \mathcal{B}_{21}$.

Let A_{ii} be an arbitrary element in $\mathcal{A}_{ii}$. Then for $j \neq i$, we have

$$0 = \phi\big(\big[[iI, P_j]_*, A_{ii}\big]_*\big) = \big[[\phi(iI), Q_j]_*, \phi(A_{ii})\big]_* = 2\phi(iI)\big(Q_j\phi(A_{ii}) + \phi(A_{ii})Q_j\big),$$

which implies that $Q_i\phi(A_{ii})Q_j = Q_j\phi(A_{ii})Q_i = Q_j\phi(A_{ii})Q_j = 0$. So $\phi(A_{ii}) = Q_i\phi(A_{ii})Q_i \subseteq \mathcal{B}_{ii}$. □

Claim 5. *We have that ϕ is multiplicative.*

Proof. Let A and B be in $\mathcal{A}$. Write $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. To show $\phi(AB) = \phi(A)\phi(B)$, by the additivity of ϕ, it suffices to show that $\phi(A_{ij}B_{kl}) = \phi(A_{ij})\phi(B_{kj})$ for all $i, j, k, l \in \{1, 2\}$. Since if $j \neq k$, then $\phi(A_{ij}B_{kl}) = \phi(A_{ij})\phi(B_{kj}) = 0$ by Claim 4, we only need to consider the cases with $j = k$.

First of all, $\phi(B_{12})\phi(A_{11})^* = 0$, which implies that

$$\begin{aligned}\phi(A_{11}B_{12}) - \phi(B_{12}^*A_{11}^*) &= \phi\big(\big[\big[A_{11}, B_{12}\big]_*, I\big]_*\big)\\ &= \big[\big[\phi(A_{11}), \phi(B_{12})\big]_*, I\big]_*\\ &= \phi(A_{11})\phi(B_{12}) - \phi(B_{12})^*\phi(A_{11})^*.\end{aligned}$$

Thus $\phi(A_{11}B_{12}) = \phi(A_{11})\phi(B_{12})$ by Claim 4. Similarly, we can prove that $\phi(A_{22}B_{21}) = \phi(A_{22})\phi(B_{21})$.

For $D_{12} \in \mathcal{B}_{12}$, we have $C_{12} = \phi^{-1}(D_{12}) \in \mathcal{A}_{12}$ by Claim 4. Thus

$$\phi(A_{11}B_{11})D_{12} = \phi(A_{11}B_{11}C_{12}) = \phi(A_{11})\phi(B_{11}C_{12}) = \phi(A_{11})\phi(B_{11})D_{12}$$

for all $D_{12} \in \mathcal{B}_{12}$. Since $\overline{Q_2} = I$, by Lemma 1.2 and Claim 4, $\phi(A_{11}B_{11}) = \phi(A_{11})\phi(B_{11})$. Similarly, we can prove that $\phi(A_{22}B_{22}) = \phi(A_{22})\phi(B_{22})$.

By Claim 4, we have

$$\begin{aligned}\phi(A_{12}B_{21}) - \phi(B_{21}A_{12}) &= \phi\big(\big[\big[A_{12}, I\big]_*, B_{21}\big]_*\big)\\ &= \big[\big[\phi(A_{12}), I\big]_*, \phi(B_{21})\big]_*\\ &= \phi(A_{12})\phi(B_{21}) - \phi(B_{21})\phi(A_{12}).\end{aligned}$$

Thus $\phi(A_{12}B_{21}) = \phi(A_{12})\phi(B_{21})$ and $\phi(A_{21}B_{12}) = \phi(A_{21})\phi(B_{12})$.

For $D_{21} \in \mathcal{B}_{21}$, we have $C_{21} = \phi^{-1}(D_{21}) \in \mathcal{A}_{12}$. Thus

$$\phi(A_{12}B_{22})D_{21} = \phi(A_{12}B_{22}C_{21}) = \phi(A_{12})\phi(B_{22}C_{21}) = \phi(A_{12})\phi(B_{21})D_{21}$$

for all $D_{21} \in \mathcal{B}_{21}$. Since we have $\overline{Q_1} = I$, by Lemma 1.2 and Claim 4, $\phi(A_{12}B_{22}) = \phi(A_{12})\phi(B_{22})$. Similarly, we can prove that $\phi(A_{21}B_{11}) = \phi(A_{21})\phi(B_{11})$. □

Claim 6. *There is a central projection $E \in \mathcal{A}$ such that the restriction of ϕ to $\mathcal{A}E$ is a linear $*$-isomorphism and the restriction of ϕ to $\mathcal{A}(I - E)$ is a conjugate linear $*$-isomorphism.*

Proof. For every rational number q, we have $\phi(qI) = qI$. Indeed, since q is rational number, there exist two integers r and s such that $q = \frac{r}{s}$. Since $\phi(I) = I$ and ϕ is additive, we get

$$\phi(qI) = \phi\Big(\frac{r}{s}I\Big) = r\phi\Big(\frac{1}{s}I\Big) = \frac{r}{s}\phi(I) = qI.$$

Now we show that ϕ is real linear. Let A be a positive element in $\mathcal{A}$. Then $A = B^2$ for some self-adjoint element $B \in \mathcal{A}$. It follows from Claim 5 that $\phi(A) = \phi(B)^2$. By Claim 2(3), we get that $\phi(B)$ is self-adjoint. So $\phi(A)$ is positive. This shows that ϕ preserves positive elements. Let $\lambda \in \mathbb{R}$. Choose sequence $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. It follows from

$$a_nI \leq \lambda I \leq b_nI$$

that

$$a_n I \leq \phi(\lambda I) \leq b_n I.$$

Taking the limit, we get $\phi(\lambda I) = \lambda I$. Hence for all $A \in \mathcal{A}$,

$$\phi(\lambda A) = \phi\big((\lambda I)A\big) = \phi(\lambda I)\phi(A) = \lambda\phi(A).$$

Hence ϕ is real linear.

Let $F = \frac{I - i\phi(iI)}{2}$. Then it is easy to verify that F is a central projection in $\mathcal{B}$ by Claim 2(2). Since $\phi(iI) = i(2F - I)$, we have

$$F\phi(iI) = iF \text{and} (I - F)\phi(iI) = i(F - I).$$

Let $E = \phi^{-1}(F)$. Then by Claim 2(4) and Claim 3, E is a central projection in $\mathcal{A}$. Moreover, for $A \in \mathcal{A}$, the following hold:

$$\phi(iAE) = \phi(A)\phi(E)\phi(iI) = i\phi(A)F = i\phi(AE),$$

and

$$\phi\big(iA(I - E)\big) = \phi(A)\phi(I - E)\phi(iI) = -i\phi(A)(I - F) = -i\phi\big(A(I - E)\big).$$

That is, the restriction of ϕ to $\mathcal{A}E$ is linear and the restriction of ϕ to $\mathcal{A}(I - E)$ is conjugate linear. This, together with Claim 2 and Claim 5, shows Claim 6. $\square$

Acknowledgments. The authors would like to thank the referees for the very thorough reading of the paper and many helpful comments. The first author was supported by the National Natural Science Foundation of China grant No. 11526123 and the Natural Science Foundation of Shandong Province grant No. ZR2015PA010. The second author was supported by National Natural Science Foundation of China grant No. 11571247.

References

1. Z. Bai and S. Du, *Maps preserving products $XY - YX^*$ on von Neumann algebras*, J. Math. Anal. Appl. **386** (2012), no. 1, 103–109. Zbl 1232.47032. MR2834869. DOI 10.1016/j.jmaa.2011.07.052. 496
2. M. Brešar and M. Fošner, *On rings with involution equipped with some new product*, Publ. Math. Debrecen **57** (2000), no. 1–2, 121–134. Zbl 0969.16013. MR1771679. 496
3. J. Cui and C.-K. Li, *Maps preserving product $XY - YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **431** (2009), no. 5–7, 833–842. Zbl 1183.47031. MR2535555. DOI 10.1016/j.laa.2009.03.036. 496
4. L. Dai and F. Lu, *Nonlinear maps preserving Jordan $*$-products*, J. Math. Anal. Appl. **409** (2014), no. 1, 180–188. Zbl pre06409237. MR3095028. DOI 10.1016/j.jmaa.2013.07.019. 496
5. D. Huo, B. Zheng, and H. Liu, *Nonlinear maps preserving Jordan triple η-$*$-products*, J. Math. Anal. Appl. **430** (2015), no. 2, 830–844. Zbl 1331.47054. MR3351983. DOI 10.1016/j.jmaa.2015.05.021. 497
6. D. C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc. **8** (1957), 535–536. Zbl 0079.12904. MR0087914. 503
7. C. Li, F. Lu, and X. Fang, *Nonlinear mappings preserving new product $XY + YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **438** (2013), no. 5, 2339–2345. Zbl 1276.47047. MR3005295. DOI 10.1016/j.laa.2012.10.015. 496
8. C. R. Miers, *Lie homomorphisms of operator algebras*, Pacific J. Math. **38** (1971) 717–735. Zbl 0204.14803. MR0308804. 498

9. L. Molnár, *A condition for a subspace of $\mathcal{B}(H)$ to be an ideal*, Linear Algebra Appl. **235** (1996), 229–234. Zbl 0852.46021. MR1374262. DOI 10.1016/0024-3795(94)00143-X. 496
10. P. Šemrl, *On Jordan $*$-derivations and an application*, Colloq. Math. **59** (1990), no. 2, 241–251. Zbl 0723.46044. MR1090656. 496
11. P. Šemrl, *Quadratic functionals and Jordan $*$-derivations*, Studia Math. **97** (1991), no. 3, 157–165. Zbl 0761.46047. MR1100685. 496
12. P. Šemrl, *Quadratic and quasi-quadratic functionals*, Proc. Amer. Math. Soc. **119** (1993), no. 4, 1105–1113. Zbl 0803.15024. MR1158008. DOI 10.2307/2159972. 496

[1]School of Mathematical Sciences, Shandong Normal University, Jinan 250014, P. R. China.

E-mail address: lcjbxh@163.com

[2]Department of Mathematics, Soochow University, Suzhou 215006, P. R. China.

E-mail address: fylu@suda.edu.cn

[3]Department of Mathematics and Statistics, Nanyang Normal University, Nanyang 47306, P. R. China.

E-mail address: tingwang526@126.com