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APPROXIMATE DUALS AND MORPHISMS OF HILBERT C^*-MODULES

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ABSTRACT. In the following we show that, under some conditions, φ-morphisms preserve duals and approximate duals of frames in Hilbert C^*-modules. Moreover, using φ-morphisms and some concepts related to frame theory such as modular Riesz bases, canonical duals, tensor products, and Bessel multipliers, we construct new approximate duals, considering in particular the approximate duals constructed by compact operators and morphisms.

1. Introduction and preliminaries

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Now suppose that $\mathfrak{A}$ is a unital C^*-algebra and that E is a left $\mathfrak{A}$-module such that the linear structures of $\mathfrak{A}$ and E are compatible. E is a pre-Hilbert $\mathfrak{A}$-module if E is equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathfrak{A}$ such that

(i) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
(ii) $\langle ax, y\rangle = a\langle x, y\rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$, for each $x, y \in E$;
(iv) $\langle x, x\rangle \geq 0$, for each $x \in E$ and if $\langle x, x\rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ and $|x| = \langle x, x\rangle^{\frac{1}{2}}$. If E is complete with $\|\cdot\|$, it is called a *Hilbert $\mathfrak{A}$-module* or a *Hilbert C^*-module* over $\mathfrak{A}$.

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A Hilbert $\mathfrak{A}$-module E is *finitely generated* if there exists some set $\{x_1, \ldots, x_n\}$ in E such that every element $x \in E$ can be expressed as an $\mathfrak{A}$-linear combination $x = \sum_{i=1}^{n} a_i x_i$, $a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$-module E is *countably generated* if there exists a countable set $\{x_i\}_{i\in I} \subseteq E$ such that E equals the norm-closure of the $\mathfrak{A}$-linear hull of $\{x_i\}_{i\in I}$. In this note, all Hilbert C^*-modules are assumed to be finitely or countably generated.

Let E and F be Hilbert C^*-modules. An operator $T : E \longrightarrow F$ is called *adjointable* if there exists an operator $T^* : F \longrightarrow E$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for each $x \in E$ and $y \in F$. We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E, F)$. Note that $\mathfrak{L}(E, E)$ is a C^*-algebra and it is denoted by $\mathfrak{L}(E)$. (For more details about Hilbert C^*-modules, see [9]; Frank and Larson presented a general approach to frame theory for Hilbert C^*-modules in [4].)

Definition 1.1. Let E be a Hilbert $\mathfrak{A}$-module. A family $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E$ is a *frame* for E if there exist real constants $0 < A_{\mathcal{F}} \le B_{\mathcal{F}} < \infty$ such that for each $x \in E$,

$$A_{\mathcal{F}}\langle x, x\rangle \le \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \le B_{\mathcal{F}}\langle x, x\rangle,$$

that is, there exist real constants $0 < A_{\mathcal{F}} \le B_{\mathcal{F}} < \infty$ such that $\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ converges in the ultraweak operator topology to some element in the universal enveloping von Neumann algebra of $\mathfrak{A}$ such that the inequality holds for each $x \in E$.

The numbers $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ are called the *lower* and *upper bound* of the frame, respectively. In this case, we call it an $(A_{\mathcal{F}}, B_{\mathcal{F}})$ *frame*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum converges in norm, the frame is called *standard*. (For more study about standard frames, see [1].)

Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be standard Bessel sequences in E. Then we say that $\mathcal{G}$ (resp., $\mathcal{F}$) is an *alternate dual* or a *dual* of $\mathcal{F}$ (resp., $\mathcal{G}$) if $x = \sum_{i\in I}\langle x, f_i\rangle g_i$ or, equivalently, $x = \sum_{i\in I}\langle x, g_i\rangle f_i$, for each $x \in E$ (see [5, Proposition 3.8]).

Let $\mathcal{F}$ and $\mathcal{G}$ be standard Bessel sequences in E. Then it is easy to see that the operator $S_{\mathcal{F},\mathcal{G}} : E \longrightarrow E$ defined by $S_{\mathcal{F},\mathcal{G}}(x) = \sum_{i\in I}\langle x, g_i\rangle f_i$ is adjointable with $S^*_{\mathcal{F},\mathcal{G}} = S_{\mathcal{G},\mathcal{F}}$. We call $S_{\mathcal{F},\mathcal{G}}$ the operator constructed by $\mathcal{F}$ and $\mathcal{G}$. It is clear that $\mathcal{G}$ is a dual for $\mathcal{F}$ if and only if $S_{\mathcal{F},\mathcal{G}} = \mathrm{Id}_E$.

2. Stability of approximate duals under φ-morphisms

Two standard Bessel sequences in a Hilbert C^*-module are approximately duals if the distance (with respect to the norm) between the identity operator on the Hilbert C^*-module and the operator constructed by the Bessel sequences is strictly less than 1.

Approximate duality of frames in Hilbert spaces was recently investigated in [3], and some interesting applications of approximate duals were obtained. For example, it was shown how approximate duals can be obtained via perturbation theory; additionally, some applications of approximate duals to Gabor frames—especially Gabor frames generated by the Gaussian—were presented. Moreover, in Section 6 in [3], the authors described various numerical approaches to the construction of approximate duals. Also, it was shown in [7] that approximate duals are stable under small perturbations and that they are useful for erasures. We mention that approximate duals of frames have been generalized to Hilbert C^*-modules in [10].

Definition 2.1. Two standard Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ are *approximately dual* frames if $\|\mathrm{Id}_E - S_{\mathcal{F},\mathcal{G}}\| < 1$ (equivalently, $\|\mathrm{Id}_E - S_{\mathcal{G},\mathcal{F}}\| < 1$), where $S_{\mathcal{F},\mathcal{G}}$ is the operator constructed by the Bessel sequences. In this case, we say that $\mathcal{G}$ (resp., $\mathcal{F}$) is an *approximate dual* of $\mathcal{F}$ (resp., $\mathcal{G}$).

It follows from Corollary 3.3 in [10] that if $\mathcal{F}$ and $\mathcal{G}$ are *approximately duals*, then they are standard frames.

Now we state the definition of φ-morphisms and refer to [2] for more details about the properties of these operators.

Definition 2.2. Let E and F be Hilbert C^*-modules over C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a morphism of C^*-algebras. A map $\Phi : E \longrightarrow F$ is said to be a φ-morphism of Hilbert C^*-modules if

$$\big\langle \Phi(x), \Phi(y) \big\rangle = \varphi\big(\langle x, y \rangle\big),$$

for each $x, y \in E$.

It is easy to see that each φ-morphism is a linear operator and $\Phi(ax) = \varphi(a)\Phi(x)$, for each $a \in \mathfrak{A}$ and also

$$\big\|\Phi(x)\big\| = \big\|\big\langle \Phi(x), \Phi(x) \big\rangle\big\|^{\frac{1}{2}} = \big\|\varphi\big(\langle x, x \rangle\big)\big\|^{\frac{1}{2}} \leq \|x\|,$$

so $\|\Phi\| \leq 1$.

Proposition 2.3. *Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be an isomorphism of C^*-algebras. If $\Phi : E \longrightarrow F$ is a φ-morphism and $\mathcal{F} = \{f_i\}_{i\in I}$, $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E$ are standard Bessel sequences such that $\Phi\mathcal{G} = \{\Phi(g_i)\}_{i\in I}$ is an approximate dual of $\Phi\mathcal{F} = \{\Phi(f_i)\}_{i\in I}$, then Φ is surjective.*

Proof. Since $\Phi\mathcal{G} \subseteq F$ and $\mathcal{F} \subseteq E$ are standard Bessel sequences, it is easy to see that $\sum_{i\in I} \varphi^{-1}(\langle y, \Phi(g_i)\rangle) f_i$ converges in E, for each $y \in F$ and the operator $R : F \longrightarrow E$ defined by $Ry = \sum_{i\in I} \varphi^{-1}(\langle y, \Phi(g_i)\rangle) f_i$ is bounded. Thus,

$$S_{\Phi\mathcal{F},\Phi\mathcal{G}}\, y = \sum_{i\in I} \big\langle y, \Phi(g_i) \big\rangle \Phi(f_i) = \Phi\Big(\sum_{i\in I} \varphi^{-1}\big(\big\langle y, \Phi(g_i) \big\rangle\big) f_i\Big) = \Phi R(y).$$

Because $\Phi\mathcal{G}$ is an approximate dual of $\Phi\mathcal{F}$, that is, $\|\mathrm{Id}_F - S_{\Phi\mathcal{F},\Phi\mathcal{G}}\| < 1$, by Neumann algorithm $S_{\Phi\mathcal{F},\Phi\mathcal{G}}$ is invertible and consequently Φ is surjective. □

Example 2.4. Let X be a compact Hausdorff space, and let Y be a closed, nonempty subset of X with dense complement. Let $\mathfrak{A} = C(X) = F$, and let $E = \{f \in \mathfrak{A} : f(Y) = 0\}$. If $\Phi : E \longrightarrow F$ is the inclusion map, then it is easy to see that Φ is an $\mathrm{Id}_{\mathfrak{A}}$-morphism but Φ is not adjointable, that is, $\Phi \notin \mathfrak{L}(E, F)$. Of course, $\mathrm{Id}_{\mathfrak{A}}$ is an isomorphism and Φ is not surjective, so by Proposition 2.3, Φ does not preserve any duals and approximate duals.

The following example shows that Proposition 2.3 is not necessarily true for the case that φ is not an isomorphism.

Example 2.5. Let $\mathfrak{A}$ be a proper C^*-subalgebra of $\mathfrak{B}$ with $1_{\mathfrak{A}} = 1_{\mathfrak{B}} = 1$. Consider $\mathfrak{A}$ and $\mathfrak{B}$ as Hilbert C^*-modules over themselves. If $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is the inclusion map and $\varphi = \Phi$ is assumed as a morphism of C^*-algebras, then Φ is a φ-morphism and $\{\Phi(1)\}$ is a dual for itself, but clearly Φ is not surjective.

Let $T \in \mathfrak{L}(E, F)$, and let $\mathcal{G}$ be a dual for $\mathcal{F}$ in E. Then it is easy to see that $T\mathcal{G} = \{Tg_i\}_{i\in I}$ is a dual for $T\mathcal{F} = \{Tf_i\}_{i\in I}$ if and only if T is co-isometric. Therefore, if $\mathcal{G}$ is a dual for $\mathcal{F}$ and $T \in \mathfrak{L}(E, F)$ is invertible, then $T\mathcal{G}$ is not necessarily a dual for $T\mathcal{F}$. The next result shows that surjective φ-morphisms have this property.

Proposition 2.6. *Let Φ be a surjective φ-morphism.*

(i) *If $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E$ is a standard Bessel sequence, then $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$ is a standard Bessel sequence for F.*
(ii) *If $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E$ is a dual for $\mathcal{F} = \{f_i\}_{i\in I}$, then $\Phi\mathcal{G} = \{\Phi g_i\}_{i\in I}$ is a dual for $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$.*
(iii) *If $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E$ is a standard frame, then $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$ is a standard frame for F.*

Proof. (i) Let B be an upper bound for $\mathcal{F}$, and let $y \in F$. Since Φ is surjective, there exists some $x \in E$ such that $\Phi(x) = y$. Now we have

$$\begin{aligned}\sum_{i\in I}\big\langle y, \Phi(f_i)\big\rangle\big\langle \Phi(f_i), y\big\rangle &= \sum_{i\in I}\varphi\big(\langle x, f_i\rangle\big)\varphi\big(\langle f_i, x\rangle\big)\\ &= \varphi\Big(\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle\Big) \le B\varphi\big(\langle x, x\rangle\big)\\ &= B\big\langle \Phi(x), \Phi(x)\big\rangle = B\langle y, y\rangle,\end{aligned}$$

so $\{\Phi f_i\}_{i\in I}$ is a standard Bessel sequence for F.

(ii) Let $y \in F$ and $x \in E$ with $\Phi(x) = y$. Then

$$\begin{aligned}\sum_{i\in I}\big\langle y, \Phi(f_i)\big\rangle\Phi(g_i) &= \sum_{i\in I}\varphi\big(\langle x, f_i\rangle\big)\Phi(g_i)\\ &= \Phi\Big(\sum_{i\in I}\langle x, f_i\rangle g_i\Big) = \Phi(x) = y.\end{aligned}$$

This yields that $\Phi\mathcal{G}$ is a dual for $\Phi\mathcal{F}$.

(iii) Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a standard frame for E. Then it is well known that $\mathcal{F}$ admits at least one dual, so by (i) and (ii), $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$ is a standard Bessel sequence which has one dual; consequently, $\Phi\mathcal{F}$ is a standard frame. □

Corollary 2.7. *Let Φ be a surjective φ-morphism. If $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E$ is an approximate dual for $\mathcal{F} = \{f_i\}_{i\in I}$, then $\{(\Phi \circ S_{\mathcal{G},\mathcal{F}}^{-1})g_i\}_{i\in I}$ is a dual for $\{\Phi(f_i)\}_{i\in I}$ and for each $y \in F$, the following reconstruction formula holds:*

$$y = \sum_{i\in I} \big\langle y, \Phi(f_i)\big\rangle (\Phi \circ S_{\mathcal{G},\mathcal{F}}^{-1}) g_i = \sum_{i\in I} \sum_{n=0}^{\infty} \big\langle y, \Phi(f_i)\big\rangle \Phi(\mathrm{Id}_E - S_{\mathcal{G},\mathcal{F}})^n (g_i).$$

Proof. Since $\mathcal{G}$ is an approximate dual of $\mathcal{F}$, Neumann algorithm implies that $S_{\mathcal{G},\mathcal{F}}$ is invertible with $S_{\mathcal{G},\mathcal{F}}^{-1} = \sum_{n=0}^{\infty} (\mathrm{Id}_E - S_{\mathcal{G},\mathcal{F}})^n$ and it is clear that $\{S_{\mathcal{G},\mathcal{F}}^{-1} g_i\}_{i\in I}$ is a dual for $\mathcal{F}$. Now the result follows from Proposition 2.6. □

If T is an invertible operator in $\mathfrak{L}(E, F)$ and $T\mathcal{G}$ is a dual of $T\mathcal{F}$, then it is not necessarily true that $\mathcal{G}$ is a dual for $\mathcal{F}$. For example, let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let $E = F = H$, $\mathcal{F} = \mathcal{G} = \{\frac{1}{2} e_n\}_{n=1}^{\infty}$, and $T = 2 \cdot \mathrm{Id}_E$. The following theorem shows that injective φ-morphisms have this property. (Recall that a Hilbert C^*-module E is called *full* if the linear span $\{\langle x, y\rangle : x, y \in E\}$ is dense in $\mathfrak{A}$.)

Theorem 2.8. *Let $\mathcal{G} = \{g_i\}_{i\in I}$ and $\mathcal{F} = \{f_i\}_{i\in I}$ be standard Bessel sequences in E. Then we have the following.*

- (i) *If φ or Φ is injective and $\Phi\mathcal{G} = \{\Phi g_i\}_{i\in I}$ is a dual of $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$, then $\mathcal{G}$ is a dual for $\mathcal{F}$.*
- (ii) *Assume that φ is injective or Φ is an isometry. If Φ is surjective and $\mathcal{G}$ is an approximate dual for $\mathcal{F}$, then $\Phi\mathcal{G} = \{\Phi g_i\}_{i\in I}$ is an approximate dual for $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$. Also, if $\Phi\mathcal{G}$ is an approximate dual for $\Phi\mathcal{F}$, then $\mathcal{G}$ is an approximate dual for $\mathcal{F}$.*
- (iii) *Let Φ be injective on a full Hilbert C^*-module E. If Φ is surjective and $\mathcal{G}$ is an approximate dual for $\mathcal{F}$, then $\Phi\mathcal{G}$ is an approximate dual for $\Phi\mathcal{F}$. Also, if $\Phi\mathcal{G}$ is an approximate dual for $\Phi\mathcal{F}$, then $\mathcal{G}$ is an approximate dual for $\mathcal{F}$.*

Proof. First, we note that Theorem 2.3 in [2] says that if φ is injective, then Φ is isometric and so it is injective.

(i) For each $x \in E$, we have

$$\Phi(x) = \sum_{i\in I} \big\langle \Phi(x), \Phi(g_i)\big\rangle \Phi(f_i) = \Phi\Big(\sum_{i\in I} \langle x, g_i\rangle f_i\Big),$$

and since Φ is injective, we have

$$\sum_{i\in I} \langle x, g_i\rangle f_i = x, \quad \forall x \in E.$$

(ii) Let $\mathcal{G} = \{g_i\}_{i\in I}$ be an approximate dual of $\mathcal{F}$. Then there exists $K < 1$ such that

$$\Big\| x - \sum_{i\in I} \langle x, g_i\rangle f_i \Big\| \le K\|x\|, \quad \forall x \in E.$$

Now let $y \in F$ and $x \in E$ with $\Phi(x) = y$. Therefore,

$$\Big\|y - \sum_{i\in I}\big\langle y, \Phi(g_i)\big\rangle\Phi(f_i)\Big\| = \Big\|\Phi\Big(x - \sum_{i\in I}\langle x, g_i\rangle f_i\Big)\Big\|$$
$$= \Big\|x - \sum_{i\in I}\langle x, g_i\rangle f_i\Big\| \leq K\|x\| = K\|y\|,$$

so $\|\mathrm{Id}_F - S_{\Phi\mathcal{F},\Phi\mathcal{G}}\| \leq K < 1$ and this means that $\Phi\mathcal{G}$ is an approximate dual of $\Phi\mathcal{F}$.

For the opposite implication, let $K < 1$ such that

$$\Big\|\sum_{i\in I}\big\langle y, \Phi(g_i)\big\rangle\Phi(f_i) - y\Big\| \leq K\|y\|, \quad \forall y \in F.$$

Now for each $x \in E$ by putting $y = \Phi(x)$ in the above inequality, we get

$$\Big\|\Phi\Big(\sum_{i\in I}\langle x, g_i\rangle f_i - x\Big)\Big\| \leq K\big\|\Phi(x)\big\|, \quad \forall x \in E,$$

and since Φ is an isometry, we have

$$\Big\|\sum_{i\in I}\langle x, g_i\rangle f_i - x\Big\| \leq K\|x\|, \quad \forall x \in E,$$

and the result follows.

(iii) By the assumption and Theorem 2.3 in [2], we conclude that φ is an injection and Φ is isometric. Now the result is obtained using (ii). □

We recall the following definition from [2, Definition 2.8].

Definition 2.9. A map $\Phi : E \longrightarrow F$ is called a *unitary* if there exists an injective morphism of C^*-algebras $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that Φ is a surjective φ-morphism.

Note that if Φ is a unitary φ-morphism, then it is surjective, and since φ is an injection, then Φ is isometric and so it is invertible.

If $\mathcal{G}$ is an approximate dual for $\mathcal{F}$ and T is an invertible operator in $\mathfrak{L}(E, F)$, then it is not necessarily true that $T\mathcal{G}$ is an approximate dual of $T\mathcal{F}$. For example, let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let $E = F = H$, $\mathcal{F} = \mathcal{G} = \{e_n\}_{n=1}^{\infty}$, and $T = \alpha\cdot\mathrm{Id}_E$ with $\alpha > 2$. Moreover, if $T\mathcal{G}$ is an approximate dual of $T\mathcal{F}$ and T is invertible, then it does not always imply that $\mathcal{G}$ is an approximate dual of $\mathcal{F}$. As an example, consider $\mathcal{F} = \{e_n\}_{n=1}^{\infty}$ as an orthonormal basis for a Hilbert space H, and let $\mathcal{G} = \{2e_n\}_{n=1}^{\infty}$ and $T = \frac{1}{4}\mathrm{Id}_H$. Now we get the following result using Proposition 2.6 and Theorem 2.8 for unitary φ-morphisms.

Corollary 2.10. *Suppose that Φ is a unitary φ-morphism, and suppose that $\mathcal{F} = \{f_i\}_{i\in I}$, $\mathcal{G} = \{g_i\}_{i\in I}$ are standard Bessel sequences in E. Then $\mathcal{G}$ is a dual (resp., an approximate dual) of $\mathcal{F}$ if and only if $\{\Phi g_i\}_{i\in I}$ is a dual (resp., an approximate dual) of $\{\Phi f_i\}_{i\in I}$.*

Example 2.11. If E is a Hilbert C^*-module over $\mathfrak{A}$ and $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is an isomorphism of C^*-algebras, then it is easy to see that E can be regarded as a Hilbert $\mathfrak{B}$-module such that the identity map on E is a unitary φ-morphism. Hence, by Corollary 2.10, if $\mathcal{G}$ is an approximate dual (resp., a dual) of $\mathcal{F}$ in E over $\mathfrak{A}$, then $\mathcal{G}$ is also an approximate dual (resp., a dual) of $\mathcal{F}$ in E over $\mathfrak{B}$.

Proposition 2.12. *Let E and F be Hilbert C^*-modules over C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Suppose that $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is an isomorphism, and suppose that $\Phi : E \longrightarrow F$ is a unitary φ-morphism. Then for two standard Bessel sequences $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ in F, we have that $\mathcal{G}$ is a dual (resp., an approximate dual) of $\mathcal{F}$ if and only if $\Phi^{-1}\mathcal{G} = \{\Phi^{-1}g_i\}_{i\in I}$ is a dual (resp., an approximate dual) of $\Phi^{-1}\mathcal{F} = \{\Phi^{-1}f_i\}_{i\in I}$.*

Proof. Let $y_1, y_2 \in F$ and $x_1, x_2 \in E$ with $x_1 = \Phi^{-1}(y_1)$ and $x_2 = \Phi^{-1}(y_2)$. Then $\varphi(\langle x_1, x_2\rangle) = \langle \Phi(x_1), \Phi(x_2)\rangle$, so $\varphi(\langle \Phi^{-1}(y_1), \Phi^{-1}(y_2)\rangle) = \langle y_1, y_2\rangle$, and consequently $\varphi^{-1}(\langle y_1, y_2\rangle) = \langle \Phi^{-1}(y_1), \Phi^{-1}(y_2)\rangle$. Thus, Φ^{-1} is a unitary φ^{-1}-morphism. Now the result follows from Corollary 2.10. □

Remark 2.13. It follows from Proposition 2.5 in [2] that if $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a morphism of C^*-algebras and $\Phi : E \longrightarrow F$ is a φ-morphism, then $\operatorname{Im}\Phi$ is a Hilbert C^*-module over the C^*-algebra $\operatorname{Im}\varphi \subseteq \mathfrak{B}$. Hence, if Φ is not surjective, then using Proposition 2.6 and Theorem 2.8, we can consider duality and approximate duality of standard Bessel sequences in that Hilbert C^*-module $\operatorname{Im}\Phi$.

Example 2.14. Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be three C^*-algebras, and let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ and $\psi : \mathfrak{B} \longrightarrow \mathfrak{C}$ be morphisms of C^*-algebras. If $\Phi : E_1 \longrightarrow E_2$ is a φ-morphism and $\Psi : E_2 \longrightarrow E_3$ is a ψ-morphism, then $\Psi\Phi : E_1 \longrightarrow E_3$ is a $\psi\varphi$-morphism. Since the composition of two injective (resp., surjective, invertible) φ, ψ-morphisms is injective (resp., surjective, invertible), using the results obtained in this section, new duals and approximate duals can be constructed.

Remark 2.15. It follows from Example 2.2 and Lemma 2.4 in [2] that if $\Phi : E \longrightarrow F$ is a φ-morphism, then $\frac{E}{\operatorname{Ker}\Phi}$ can be considered as a Hilbert C^*-module over $\frac{\mathfrak{A}}{\operatorname{Ker}\varphi}$ and $q : E \longrightarrow \frac{E}{\operatorname{Ker}\Phi}$ is a π-morphism, where q and $\pi : \mathfrak{A} \longrightarrow \frac{\mathfrak{A}}{\operatorname{Ker}\varphi}$ are the quotient maps. Now since q is surjective, the stability of standard Bessel sequences, frames, duals, and approximate duals under q can be studied. Moreover, the stability of standard Bessel sequences, frames, duals, and approximate duals under $\hat{\Phi}$ can be considered, where $\hat{\Phi} : \frac{E}{\operatorname{Ker}\Phi} \longrightarrow F$ is defined by $\hat{\Phi}(q(x)) = \Phi(x)$ and it is a $\hat{\varphi}$-morphism with $\hat{\varphi} : \frac{\mathfrak{A}}{\operatorname{Ker}\varphi} \longrightarrow \mathfrak{B}$, $\hat{\varphi}(\pi(a)) = \varphi(a)$.

3. Construction of new approximate duals using φ-morphisms

In this section, we construct new approximate duals using φ-morphisms and some concepts related to frame theory such as modular Riesz bases, canonical duals, tensor products, and Bessel multipliers.

For a unital C^*-algebra $\mathfrak{A}$, $\ell^2(I, \mathfrak{A})$, defined by

$$\ell^2(I, \mathfrak{A}) = \Big\{\{a_i\}_{i\in I} \subseteq \mathfrak{A} : \sum_{i\in I} a_i a_i^* \text{ converges in } \|\cdot\|\Big\},$$

is a Hilbert $\mathfrak{A}$-module with inner product $\langle \{a_i\}_{i\in I}, \{b_i\}_{i\in I}\rangle = \sum_{i\in I} a_i b_i^*$.

For a standard Bessel sequence $\mathcal{F} = \{f_i\}_{i\in I}$ with an upper bound $B_{\mathcal{F}}$, the operator $T_{\mathcal{F}} : \ell^2(I, \mathfrak{A}) \longrightarrow E$ which is defined by $T_{\mathcal{F}}(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i$ is called the *synthesis operator* of $\mathcal{F}$. It is adjointable with $T^*_{\mathcal{F}}(x) = \{\langle x, f_i\rangle\}_{i\in I}$, and $\|T_{\mathcal{F}}\| \leq \sqrt{B_{\mathcal{F}}}$. $T^*_{\mathcal{F}}$ is the *analysis operator* of $\mathcal{F}$. Now we define the operator

$S_{\mathcal{F}} : E \longrightarrow E$ by $S_{\mathcal{F}}(x) = T_{\mathcal{F}}T_{\mathcal{F}}^*(x) = \sum_{i\in I}\langle x, f_i\rangle f_i$. If $\mathcal{F}$ is a standard $(A_{\mathcal{F}}, B_{\mathcal{F}})$ frame, then $A_{\mathcal{F}}.\mathrm{Id}_E \leq S_{\mathcal{F}} \leq B_{\mathcal{F}}.\mathrm{Id}_E$. The operator $S_{\mathcal{F}}$ is called the *frame operator* of $\mathcal{F}$.

It is easy to see that if $\mathcal{F}$ is an $(A_{\mathcal{F}}, B_{\mathcal{F}})$ standard frame, then $\widetilde{\mathcal{F}} = \{S_{\mathcal{F}}^{-1}f_i\}_{i\in I}$ is a $(\frac{1}{B_{\mathcal{F}}}, \frac{1}{A_{\mathcal{F}}})$ standard frame with $x = \sum_{i\in I}\langle x, S_{\mathcal{F}}^{-1}f_i\rangle f_i = \sum_{i\in I}\langle x, f_i\rangle S_{\mathcal{F}}^{-1}f_i$, for each $x \in E$. Hence, $\widetilde{\mathcal{F}} = \{S_{\mathcal{F}}^{-1}f_i\}_{i\in I}$ is a dual of $\mathcal{F}$, called the *canonical dual* of $\mathcal{F}$.

A standard frame $\{f_i\}_{i\in I}$ for E is a *modular Riesz basis* if it has the following property: if an $\mathfrak{A}$-linear combination $\sum_{i\in\Omega} a_if_i$ with coefficients $\{a_i : i \in \Omega\} \subseteq \mathfrak{A}$ and $\Omega \subseteq I$ is equal to zero, then $a_i = 0$, for each $i \in \Omega$ (see [6]).

Theorem 3.1. *Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a surjective morphism of C^*-algebras, let $\Phi : E \longrightarrow F$ be a bijective φ-morphism, and let $\mathcal{F} = \{f_i\}_{i\in I}$ be a modular Riesz basis in E. Then $\Phi\mathcal{F} = \{\Phi(f_i)\}_{i\in I}$ is a modular Riesz basis and if $\mathcal{G} = \{g_i\}_{i\in I}$ is an approximate dual of $\mathcal{F}$, then $\widetilde{\Phi\mathcal{F}} = \{\Phi\widetilde{f_i}\}_{i\in I} = \{\sum_{n=0}^{\infty}\Phi(\mathrm{Id}_E - S_{\mathcal{G},\mathcal{F}})^n g_i\}_{i\in I}$.*

Proof. It follows from Proposition 2.6 that $\Phi\mathcal{F}$ is a standard frame. Now let $\Omega \subseteq I$ and $b_i \in \mathfrak{B}$ be such that $\sum_{i\in\Omega} b_i\Phi(f_i) = 0$. If $a_i \in \mathfrak{A}$ with $\varphi(a_i) = b_i$, then the injectivity of Φ and the fact that $\mathcal{F}$ is a modular Riesz basis imply that $a_i = 0$ and consequently $b_i = 0$, for each $i \in I$. Therefore, $\Phi\mathcal{F}$ is a modular Riesz basis. Also, by Proposition 2.6, $\{\Phi\widetilde{f_i}\}_{i\in I}$ is a dual for $\Phi\mathcal{F}$ and since $\Phi\mathcal{F}$ is a modular Riesz basis, by Proposition 3.1 in [6], its canonical dual is the unique dual of $\Phi\mathcal{F}$, so $\widetilde{\Phi\mathcal{F}} = \{\Phi\widetilde{f_i}\}_{i\in I}$. Now the equality $\{\Phi\widetilde{f_i}\}_{i\in I} = \{\sum_{n=0}^{\infty}\Phi(\mathrm{Id}_E - S_{\mathcal{G},\mathcal{F}})^n g_i\}_{i\in I}$ follows from Corollary 2.7 and the fact that $S_{\mathcal{G},\mathcal{F}}^{-1} = \sum_{n=0}^{\infty}(\mathrm{Id}_E - S_{\mathcal{G},\mathcal{F}})^n$. □

Proposition 3.2. *Let $\Phi : E \longrightarrow F$ be a unitary φ-morphism, and let $0 \leq \lambda_1, \lambda_2 < 1$, $A, B, \varepsilon > 0$, and $K = \lambda_1 + \frac{\varepsilon}{\sqrt{A}} + \frac{\lambda_2[(1+\lambda_1)\sqrt{A}+\varepsilon]}{\sqrt{A}(1-\lambda_2)}$. If $\mathcal{F} = \{f_i\}_{i\in I}$ is an (A, B) frame and $\mathcal{G} = \{g_i\}_{i\in I}$ is a sequence satisfying*

$$\Big\|\sum_{i\in\Omega} a_if_i - \sum_{i\in\Omega} a_ig_i\Big\| \leq \lambda_1\Big\|\sum_{i\in\Omega} a_if_i\Big\| + \lambda_2\Big\|\sum_{i\in\Omega} a_ig_i\Big\| + \varepsilon\Big\|\sum_{i\in\Omega} |a_i|^2\Big\|^{\frac{1}{2}},$$

for each subset Ω in I, $\{a_i\}_{i\in\Omega} \subseteq \mathfrak{A}$ with $K < 1$, then $\widetilde{\Phi\mathcal{F}}$ is an approximate dual of $\Phi\mathcal{G} = \{\Phi g_i\}_{i\in I}$.

Proof. It follows from Proposition 3.8 in [10] that $\widetilde{\mathcal{F}} = \{\widetilde{f_i}\}_{i\in I}$ is an approximate dual of $\mathcal{G}$. Hence, by Corollary 2.10, $\{\Phi\widetilde{f_i}\}_{i\in I}$ is an approximate dual of $\Phi\mathcal{G}$. It just remains to show that $\widetilde{\Phi\mathcal{F}} = \{\Phi\widetilde{f_i}\}_{i\in I}$. According to the definition of a unitary φ-morphism, φ is injective and Φ is surjective, so by Theorem 2.3 in [2], Φ is also injective and so it is invertible. By Proposition 2.6, $\Phi\mathcal{F} = \{\Phi f_i\}_{i\in I}$ is a frame and we have $\widetilde{\Phi(f_i)} = S_{\Phi\mathcal{F}}^{-1}(\Phi(f_i))$. Now one can easily obtain that $S_{\Phi\mathcal{F}}^{-1} = \Phi S_{\mathcal{F}}^{-1}\Phi^{-1}$ and consequently

$$S_{\Phi\mathcal{F}}^{-1}\big(\Phi(f_i)\big) = \Phi S_{\mathcal{F}}^{-1}f_i = \Phi(\widetilde{f_i}).$$

This means that $\widetilde{\Phi\mathcal{F}} = \{\Phi(\widetilde{f_i})\}_{i\in I}$. □

Now we recall tensor products of C^*-algebras and Hilbert C^*-modules from [12] and [9], respectively.

Suppose that $\mathfrak{A}$ and $\mathfrak{A}'$ are two C^*-algebras. Then $\mathfrak{A}\otimes\mathfrak{A}'$ is a C^*-algebra with the spatial norm and for each $a\in\mathfrak{A}$ and $a'\in\mathfrak{A}'$, we have $\|a\otimes a'\|=\|a\|\|a'\|$. The multiplication and involution on simple tensors are defined by $(a\otimes a')(b\otimes b')=ab\otimes a'b'$ and $(a\otimes a')^*=a^*\otimes a'^*$, respectively.

Now let E be a Hilbert $\mathfrak{A}$-module, and let E' be a Hilbert $\mathfrak{A}'$-module. Then the tensor product $E\otimes E'$ is a Hilbert $(\mathfrak{A}\otimes\mathfrak{A}')$-module. The module action and inner product for simple tensors are defined by $(a\otimes a')(x\otimes x')=(ax)\otimes(a'x')$ and $\langle x\otimes x', y\otimes y'\rangle=\langle x,y\rangle\otimes\langle x',y'\rangle$, respectively. Let U and U' be adjointable operators on E and E', respectively. Then the tensor product $U\otimes U'$ is an adjointable operator on $E\otimes E'$. Also $(U\otimes U')^*=U^*\otimes U'^*$ and $\|U\otimes U'\|=\|U\|\|U'\|$. For more results about tensor products of C^*-algebras and Hilbert C^*-modules, see [12] and [9].

Assume that $\varphi:\mathfrak{A}\longrightarrow\mathfrak{B}$ and $\psi:\mathfrak{A}'\longrightarrow\mathfrak{B}'$ are morphisms of C^*-algebras. Then it is easy to see that $\varphi\otimes\psi:\mathfrak{A}\otimes\mathfrak{A}'\longrightarrow\mathfrak{B}\otimes\mathfrak{B}'$ is also a morphism. Now suppose that $\Phi:E\longrightarrow F$ and $\Psi:E'\longrightarrow F'$ are adjointable φ- and ψ-morphisms, respectively. Then using the relation

$$\begin{aligned}\big\langle(\Phi\otimes\Psi)(x\otimes x'),(\Phi\otimes\Psi)(y\otimes y')\big\rangle&=\big\langle\Phi(x),\Phi(y)\big\rangle\otimes\big\langle\Psi(x'),\Psi(y')\big\rangle\\&=(\varphi\otimes\psi)\big(\big\langle(x\otimes x'),(y\otimes y')\big\rangle\big),\end{aligned}$$

for each $x,y\in E$, $x',y'\in E'$ and continuity of $\Phi\otimes\Psi$ and $\varphi\otimes\psi$, we can get that $\Phi\otimes\Psi$ is a $(\varphi\otimes\psi)$-morphism.

Here, using the above explanation, Proposition 3.6 in [10], and Theorem 2.8, we present the following example to construct new approximate duals for the tensor product of Hilbert C^*-modules.

Example 3.3. Assume that Φ and Ψ are the morphisms as stated above, and assume further that they are isometric and surjective. Then it is easy to see that $\Phi\otimes\Psi$ is surjective and isometric. If $\mathcal{G}=\{g_i\}_{i\in I}\subseteq E$ is an approximate dual of $\mathcal{F}=\{f_i\}_{i\in I}$ and $\mathcal{G}'=\{g'_j\}_{j\in J}\subseteq E'$ is a dual of $\mathcal{F}'=\{f'_j\}_{j\in J}$, then $\{(\Phi\otimes\Psi)(g_i\otimes g'_j)\}_{(i,j)\in I\times J}=\{\Phi(g_i)\otimes\Psi(g'_j)\}_{(i,j)\in I\times J}$ is an approximate dual of $\{(\Phi\otimes\Psi)(f_i\otimes f'_j)\}_{(i,j)\in I\times J}$.

Recall that $\ell^\infty(I,\mathfrak{A})$ is $\{\{a_i\}_{i\in I}\subseteq\mathfrak{A}:\|\{a_i\}_{i\in I}\|_\infty=\sup\{\|a_i\|:i\in I\}<\infty\}$, and $\mathcal{Z}(\mathfrak{A})=\{a\in\mathfrak{A}:ab=ba,\forall b\in\mathfrak{A}\}$ is called the *center* of $\mathfrak{A}$. Note that if $a\in\mathcal{Z}(\mathfrak{A})$, then $a^*\in\mathcal{Z}(\mathfrak{A})$, and if $a\in\mathcal{Z}(\mathfrak{A})$ and $T\in\mathfrak{L}(E)$, then the operator $aT: E\longrightarrow E$ which is defined by $(aT)(x)=aT(x)$ is adjointable with $(aT)^*=a^*T^*$. In the rest of this note, m is a sequence $\{m_i\}_{i\in I}\in\ell^\infty(I,\mathfrak{A})$ with $m_i\in\mathcal{Z}(\mathfrak{A})$, for each $i\in I$. Each sequence with these properties is called a *symbol.*

Let E_1 and E_2 be Hilbert $\mathfrak{A}$-modules, and let $\mathcal{F}=\{f_i\}_{i\in I}\subseteq E_1$ and $\mathcal{G}=\{g_i\}_{i\in I}\subseteq E_2$ be standard Bessel sequences. It was proved in [8] that the operator $S_{m,\mathcal{G},\mathcal{F}}:E_1\longrightarrow E_2$ which is defined by $S_{m,\mathcal{G},\mathcal{F}}(x)=\sum_{i\in I}m_i\langle x,f_i\rangle g_i$, is adjointable.

Definition 3.4. $S_{m,\mathcal{G},\mathcal{F}}$ is called the *Bessel multiplier* for the Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ with symbol m. If $m_i=1_{\mathfrak{A}}$, for each $i\in I$, then $S_{m,\mathcal{G},\mathcal{F}}$ is denoted by $S_{\mathcal{G},\mathcal{F}}$.

We recall the following definition from [11, Definition 3.4].

Definition 3.5. Let m be a symbol, and let $a \in \mathcal{Z}(\mathfrak{A})$. Let $\mathcal{F}$ and $\mathcal{G}$ be standard Bessel sequences in E. Then we say that $\mathcal{G}$ is an (a,m)-*dual* (resp., (a,m)-*approximate dual*) of $\mathcal{F}$ if $\mathrm{Id}_E = aS_{m,\mathcal{G},\mathcal{F}}$ (resp., $\|\mathrm{Id}_E - aS_{m,\mathcal{G},\mathcal{F}}\| < 1$).

Proposition 3.6. *Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a surjective morphism of C^*-algebras, and let $\Phi : E \longrightarrow F$ be a surjective φ-morphism. If $\mathcal{G}$ is an (a,m)-dual of $\mathcal{F}$ in E, then $\Phi\mathcal{G}$ is a $(\varphi(a), \{\varphi(m_i)\}_{i\in I})$-dual of $\Phi\mathcal{F}$.*

Proof. The surjectivity of φ yields that $\varphi(a)$ and the $\varphi(m_i)$'s belong to $\mathcal{Z}(\mathfrak{B})$, and clearly $\{\varphi(m_i)\}_{i\in I} \in \ell^\infty(I, \mathfrak{B})$. On the other hand, (a,m)-duality of $\mathcal{G}$ for $\mathcal{F}$ is equivalent to saying that $\{am_ig_i\}_{i\in I}$ is a dual for $\mathcal{F}$. Now by Proposition 2.6,

$$\big\{\Phi(am_ig_i)\big\}_{i\in I} = \big\{\varphi(a)\varphi(m_i)\Phi(g_i)\big\}_{i\in I}$$

is a dual for $\Phi\mathcal{F}$; equivalently, $\Phi\mathcal{G}$ is a $(\varphi(a), \{\varphi(m_i)\}_{i\in I})$-dual of $\Phi\mathcal{F}$. □

Similar to the proof of the above proposition and using Theorem 2.8, we obtain the following result for approximate duals.

Proposition 3.7. *Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be an isomorphism of C^*-algebras, and let $\Phi : E \longrightarrow F$ be a surjective φ-morphism. If $\mathcal{G}$ is an (a,m)-approximate dual of $\mathcal{F}$, then $\Phi\mathcal{G}$ is a $(\varphi(a), \{\varphi(m_i)\}_{i\in I})$-approximate dual of $\Phi\mathcal{F}$.*

Let E and F be Hilbert C^*-modules. For each $x \in E$, $y \in F$, the operator $\theta_{x,y} : F \longrightarrow E$ is defined by $\theta_{x,y}(z) = \langle z, y\rangle x$. It is easy to check that $\theta_{x,y} \in \mathfrak{L}(F,E)$, with $(\theta_{x,y})^* = \theta_{y,x}$. We say that an operator $T \in \mathfrak{L}(F,E)$ is compact if it is in the closed linear subspace of $\mathfrak{L}(F,E)$ spanned by $\{\theta_{x,y} : x \in E, y \in F\}$.

Theorem 3.8. *Let E be a finitely generated Hilbert C^*-module, and let $\mathcal{F} = \{f_i\}_{i=1}^m \subseteq E$. Then for each $x_j, y_j \in E$ $(1 \le j \le n)$, the following are equivalent:*

(i) *$\{(\sum_{j=1}^n \theta_{x_j,y_j})f_i\}_{i=1}^m$ is a frame for E.*
(ii) *There exists a frame $\mathcal{G} = \{g_i\}_{i=1}^m \subseteq E$ such that $\sum_{j=1}^n \langle f, x_j\rangle S_{\mathcal{G},\mathcal{F}} y_j = f$, for each $f \in E$.*
(iii) *There exist a frame $\mathcal{G} = \{g_i\}_{i=1}^m \subseteq E$ and a positive number $K < 1$ such that $\|\sum_{j=1}^n \langle f, x_j\rangle S_{\mathcal{G},\mathcal{F}} y_j - f\| < K\|f\|$, for each $f \in E$.*

Proof. (i) $\Longrightarrow$ (ii) Since $\{(\sum_{j=1}^n \theta_{x_j,y_j})f_i\}_{i=1}^m$ is a frame, it admits a frame dual called $\mathcal{G} = \{g_i\}_{i=1}^m$. Now for each $f \in E$, we have

$$\sum_{i=1}^m \sum_{j=1}^n \langle f, x_j\rangle\langle y_j, f_i\rangle g_i = \sum_{i=1}^m \Big\langle f, \sum_{j=1}^n \langle f_i, y_j\rangle x_j\Big\rangle g_i = f;$$

equivalently,

$$\sum_{j=1}^n \langle f, x_j\rangle S_{\mathcal{G},\mathcal{F}} y_j = f.$$

The implication (ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i) The relation $\|\sum_{j=1}^n \langle f, x_j\rangle S_{\mathcal{G},\mathcal{F}} y_j - f\| < K\|f\|$ yields that

$$\Big\|\sum_{i=1}^m \Big\langle f, \sum_{j=1}^n \theta_{x_j,y_j}(f_i)\Big\rangle g_i - f\Big\| < K\|f\|,$$

for each $f \in E$. Hence, $\mathcal{G}$ is an approximate dual of $\{(\sum_{j=1}^{n} \theta_{x_j,y_j}) f_i\}_{i=1}^{m}$ and because $\{(\sum_{j=1}^{n} \theta_{x_j,y_j}) f_i\}_{i=1}^{m}$ admits an approximate dual, by [10, Corollary 3.3], it is a standard frame. □

Proposition 3.9. *Let E and F be two finitely generated Hilbert C^*-modules over C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Suppose that $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is an injective morphism, that $\Phi : E \longrightarrow F$ is a unitary φ-morphism, and that $\Phi^+ : \mathfrak{L}(E) \longrightarrow \mathfrak{L}(F)$ is defined by $\Phi^+(T) = \Phi T \Phi^{-1}$. Then for a sequence $\mathcal{F} = \{f_i\}_{i=1}^{m}$ in F and $x_j, y_j \in E$ $(1 \leq j \leq n)$, the following are equivalent:*

(i) *$\{\Phi^+(\sum_{j=1}^{n} \theta_{x_j,y_j}) f_i\}_{i=1}^{m}$ is a frame for F;*
(ii) *there exists a frame $\mathcal{G} = \{g_i\}_{i=1}^{m} \subseteq F$ with $S_{\mathcal{G},\mathcal{F}} \Phi(\sum_{j=1}^{n} \langle \Phi^{-1}(f), x_j \rangle y_j) = f$ for each $f \in F$;*
(iii) *there exist a frame $\mathcal{G} = \{g_i\}_{i=1}^{m} \subseteq F$ and a positive number $K < 1$ such that $\|S_{\mathcal{G},\mathcal{F}} \Phi(\sum_{j=1}^{n} \langle \Phi^{-1}(f), x_j \rangle y_j) - f\| < K\|f\|$ for each $f \in F$.*

Proof. Using Proposition 2.11 in [2], we get $\Phi^+(\sum_{j=1}^{n} \theta_{x_j,y_j}) = \sum_{j=1}^{n} \theta_{\Phi(x_j),\Phi(y_j)}$. Now the result is obtained by Theorem 3.8 and using the relation

$$\begin{aligned}\sum_{j=1}^{n} \big\langle f, \Phi(x_j) \big\rangle S_{\mathcal{G},\mathcal{F}} \Phi(y_j) &= \sum_{j=1}^{n} \varphi\big(\big\langle \Phi^{-1}(f), x_j \big\rangle\big) S_{\mathcal{G},\mathcal{F}} \Phi(y_j) \\ &= S_{\mathcal{G},\mathcal{F}} \Phi\Big(\sum_{j=1}^{n} \big\langle \Phi^{-1}(f), x_j \big\rangle y_j\Big),\end{aligned}$$

for each $f \in F$. □

Proposition 3.10. *Let E be a finitely generated Hilbert C^*-module, and let $\mathcal{F} = \{f_i\}_{i=1}^{m} \subseteq E$ be a standard frame. Then for each $x_j, y_j \in E$ $(1 \leq j \leq n)$, the following are equivalent.*

(i) *Every dual of $\{f_i\}_{i=1}^{m}$ is an approximate dual of $\{(\sum_{j=1}^{n} \theta_{x_j,y_j}) f_i\}_{i=1}^{m}$.*
(ii) *The frame $\{f_i\}_{i=1}^{m}$ admits a dual which is an approximate dual of $\{(\sum_{j=1}^{n} \theta_{x_j,y_j}) f_i\}_{i=1}^{m}$.*
(iii) *There exists $0 < K < 1$ such that $\| \sum_{j=1}^{n} \langle f, x_j \rangle y_j - f\| < K\|f\|$, for each $f \in E$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is clear, and the implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) can be concluded from the proof of Theorem 3.8 using the fact that if $\mathcal{G} = \{g_i\}_{i=1}^{m}$ is a dual for $\mathcal{F} = \{f_i\}_{i=1}^{m}$, then $S_{\mathcal{G},\mathcal{F}} = \mathrm{Id}_E$ and it is an approximate dual for $\{(\sum_{j=1}^{n} \theta_{x_j,y_j}) f_i\}_{i=1}^{m}$ if and only if there exists $0 < K < 1$ such that

$$\Big\| \sum_{j=1}^{n} \langle f, x_j \rangle y_j - f \Big\| = \Big\| \sum_{j=1}^{n} \langle f, x_j \rangle S_{\mathcal{G},\mathcal{F}} y_j - f \Big\| < K\|f\|,$$

for each $f \in E$. □

Corollary 3.11. *Let E and F be two finitely generated Hilbert C^*-modules over C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Suppose that $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is an injective morphism, that $\Phi : E \longrightarrow F$ is a unitary φ-morphism, and that $\Phi^+ : \mathfrak{L}(E) \longrightarrow$*

$\mathfrak{L}(F)$ is defined by $\Phi^+(T) = \Phi T \Phi^{-1}$. Then for a frame $\{f_i\}_{i=1}^m$ in F and $x_j, y_j \in E$ ($1 \leq j \leq n$), the following are equivalent:

(i) *Every dual of $\{f_i\}_{i=1}^m$ is an approximate dual of $\{\Phi^+(\sum_{j=1}^n \theta_{x_j,y_j})f_i\}_{i=1}^m$.*

(ii) *The frame $\{f_i\}_{i=1}^m$ has a dual which is an approximate dual of $\{\Phi^+(\sum_{j=1}^n \theta_{x_j,y_j})f_i\}_{i=1}^m$.*

(iii) *There exists $0 < K < 1$ such that $\|(\sum_{j=1}^n \langle \Phi^{-1}f, x_j\rangle y_j) - \Phi^{-1}f\| < K\|f\|$, for each $f \in F$.*

Proof. The result is obtained using Propositions 3.9 and 3.10 and the fact that Φ is an isometry because φ is injective. □

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