Ann. Funct. Anal. 10 (2019), no. 4, 515–524
https://doi.org/10.1215/20088752-2019-0010
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

WIGNER'S THEOREM ON THE TSIRELSON SPACE T

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Communicated by J. Chmieliński

Abstract. We say that a map $f : X \to Y$ between two real normed spaces is a *phase-isometry* if $\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\}$ holds for all $x, y \in X$. Two maps $f, g : X \to Y$ are called *phase-equivalent* if there is a phase function $\varepsilon : X \to \{-1, 1\}$ such that $\varepsilon f = g$. By studying the properties of surjective phase-isometries on the Tsirelson space T, we show that such maps are phase-equivalent to linear isometries. This gives a real version of Wigner's theorem for the Tsirelson space.

1. Introduction and preliminaries

Let X and Y be real normed spaces. A map $f : X \to Y$ is called an *isometry* if it satisfies

$$\big\|f(x) - f(y)\big\| = \|x - y\|$$

for all $x, y \in X$. The well-known Mazur–Ulam theorem (see [10]) states that surjective isometries between X and Y are affine (for another proof of the theorem, see [1, Theorem 14.1]). A map f from X to Y is called a *phase-isometry* if

$$\big\{\big\|f(x) + f(y)\big\|, \big\|f(x) - f(y)\big\|\big\} = \big\{\|x + y\|, \|x - y\|\big\}, \tag{1.1}$$

for all $x, y \in X$. Two maps $f, g : X \to Y$ are said to be *phase-equivalent* if there is a phase function $\varepsilon : X \to \{-1, 1\}$ such that $\varepsilon f = g$. Via this approach, we get the notion of maps which are phase-equivalent to linear isometries. The famous Wigner's theorem gives a characterization of mappings that are phase-equivalent

Received Sep. 16, 2018; Accepted Feb. 4, 2019.
First published online Oct. 29, 2019.
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2010 *Mathematics Subject Classification.* Primary 46B04; Secondary 46B20.
Keywords. Wigner's theorem, T-space, phase-isometry, phase-equivalent.

to linear isometries in Hilbert spaces. That is, if X and Y are real Hilbert spaces, then the phase-isometries $f : X \to Y$ are precisely the solution of the functional equation

$$\big|\big\langle f(x), f(y)\big\rangle\big| = \big|\langle x, y\rangle\big|, \quad (x, y \in X).$$

One can easily see that, when X and Y are real normed spaces, all maps $f : X \to Y$ that are phase-equivalent to linear isometries are obviously phase-isometries. In [9], Maksa and Páles proved that the converse also holds provided that X and Y are real inner product spaces; at the end of their article, they posed the following question: Is it still true that every phase-isometry from X to Y is phase-equivalent to a linear isometry when X and Y are real normed but not necessarily inner product space? Huang and the second author [6] gave a positive answer to the above question for atomic L_p spaces with $0 < p < \infty$, while in [7], Jia and the second author also answered the question affirmatively for $L^\infty(\Gamma)$-type spaces. Wigner's theorem plays a fundamental role in quantum mechanics and has several equivalent formulations and extensions (see, e.g., [4], [5], [11]–[15], [18]).

The aim of this note is to study the same result in the case of surjective phase-isometries on the Tsirelson space (see [17]). The main result (see Theorem 2.8) demonstrates that every surjective phase-isometry on the Tsirelson space is phase-equivalent to a linear isometry. The Tsirelson space is the first example of a reflexive Banach space in which neither an ℓ_p space nor a c_0 space can be embedded (see [8], [17]). Figiel and Johnson gave in [3] the implicit expression of the norm of T which is displayed in (1.2) below. It is the dual of the original that came to be known as *Tsirelson space*, and it is denoted in the literature by T. It is our belief that one of the goals of functional analysis ought to be the study of individual spaces. Our hope is that the result of the present article can give some impetus to the study of T and its generalized spaces (see [2]).

For the reader's convenience, we present the description by Figiel and Johnson from [3] (see [1] and [2] for more information about the space T). Let c_{00} be the space of all finitely supported real sequences whose unit vector basis will be denoted by $\{e_i\}$. For any finite subset E of $\mathbb{N}$ and $x \in c_{00}$, Ex is the sequence which agrees with x at coordinates in E and is zero in other coordinates; that is, if $x = \sum a_i e_i$, then $Ex = \sum_{i\in E} a_i e_i$. We will consider 1-unconditional norms on c_{00}. Thus, $\|Ex\| \le \|x\|$.

Given two finite subsets E, F of $\mathbb{N}$, we write $E < F$ if $\max E < \min F$. If $E = \{i\}$ is a singleton, then we simply write i instead of $\{i\}$. A collection $\{E_j\}_{j=1}^k$ of finite subsets of $\mathbb{N}$ is called *admissible* if $k \le E_1 < \cdots < E_k$.

The sup norm on c_{00} is denoted by $\|\cdot\|_{c_0}$. Define a sequence of norms on c_{00} as follows. Let $\|x\|_0 = \|x\|_{c_0}$, and for all $m \ge 0$, let

$$\|x\|_{m+1} = \max\Big(\|x\|_{c_0}, \frac{1}{2}\sup\Big\{\sum_{j=1}^{k} \|E_j x\|_m\Big\}\Big),$$

where the sup is taken over all admissible collections $\{E_j\}_{j=1}^k$. It is easy to check that $\|x\| = \lim_{m\to\infty} \|x\|_m$ exists. The Tsirelson space T is the completion of $(c_{00}, \|\cdot\|)$. The norm $\|\cdot\|$ satisfies the implicit equation

$$\|x\| = \max\Big(\|x\|_{c_0}, \frac{1}{2}\sup\Big\{\sum_{j=1}^{k}\|E_j x\|\Big\}\Big), \tag{1.2}$$

where the sup is taken over all admissible collections $\{E_j\}_{j=1}^{k}$.

Throughout this paper, we use S_T to denote the unit sphere of T. All the vector spaces we consider here are assumed to be real. For any $a, b \in \mathbb{R}$, we will write $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$. For every $x \in T$,

$$\operatorname{supp} x := \big\{i \in \mathbb{N}, x(i) \neq 0\big\}.$$

2. Main results

For our main results, we need a basic lemma from [16].

Lemma 2.1 ([16, Lemma 2.1])**.** *Let x, y be in S_T. Then we have the following:*

(1) $\|x + z\| \wedge \|x - z\| \leq 1$ *for all* $z \in S_T$ *if and only if* $x \in \{\pm e_1, \pm e_2\}$*;*
(2) *if y is supported in $[3, \infty)$ and satisfies $\|y+z\| \wedge \|y-z\| \leq \frac{3}{2}$ for all $z \in S_T$, then either $|y(3)| = 1$ with $|y(i)| \leq \frac{1}{2}$ for all $i \neq 3$ or $y = \theta e_m + a e_3$ for some $m \geq 4$, some $\theta \in \{\pm 1\}$, and some $|a| \leq \frac{1}{2}$.*

We give a lemma demonstrating that every phase-isometry on any normed space X enjoys some interesting properties. One may note that every phase-isometry f from a normed-space X to another normed space Y preserves the norm; that is, $\|f(x)\| = \|x\|$. Indeed, it is easy to see from (1.1) that

$$\big\{2\|f(x)\|, 0\big\} = \big\{\|f(x) + f(x)\|, \|f(x) - f(x)\|\big\} = \big\{2\|x\|, 0\big\} \quad (\forall x \in X).$$

Lemma 2.2. *Let X and Y be two normed-spaces, and let $f : X \to Y$ be a surjective phase-isometry. Then f is odd and injective.*

Proof. Note first that $f(0) = 0$ and $f(x) = 0$ if and only if $x = 0$. To see that f is odd, given $0 \neq x \in X$, suppose that $f(y) = -f(x)$ for some $y \in X$. Since

$$\big\{\|y + x\|, \|y - x\|\big\} = \big\{\|f(y) + f(x)\|, \|f(y) - f(x)\|\big\} = \big\{0, 2\|f(x)\|\big\},$$

it follows that $y = -x$ since $y = x$ is impossible. Thus, f is odd. Now assume that $f(x_1) = f(x_2)$ with $x_1, x_2 \in X$. Then the equations

$$\|x_1 - x_2\| \wedge \|x_1 + x_2\| = \|f(x_1) - f(x_2)\| \wedge \|f(x_1) + f(x_2)\| = 0$$

prove that $x_1 = x_2$ or $x_1 = -x_2$. We induce from the fact that f is odd that $x_1 = x_2$. So f is injective. The proof is complete. □

Lemma 2.2 reveals that the inverse of every surjective phase-isometry is also a surjective phase-isometry. By this and Lemma 2.1, we will give a characterization of every surjective phase-isometry on T.

Lemma 2.3. *Let $f : T \to T$ be a surjective phase-isometry. Then f has one of the following forms:*

(1) $f(e_j) = \pm e_j$ *for all* $j \geq 1$,
(2) $f(e_1) = \pm e_2$, $f(e_2) = \pm e_1$, $f(e_j) = \pm e_j$ *for all* $j \geq 3$.

Proof. By the hypothesis, for every $y \in S_T$ there is an $x \in S_T$ satisfying $f(x) = y$. Since f satisfies (1.1), it follows that

$$\big\|f(e_1) - y\big\| \wedge \big\|f(e_1) + y\big\| = \|e_1 - x\| \wedge \|e_1 + x\| \leq 1.$$

Lemma 2.1(1) allows us to conclude that $f(e_1) = \pm e_1$ or $f(e_1) = \pm e_2$. Similar arguments to those above show that $f(e_2) = \pm e_1$ or $f(e_2) = \pm e_2$. Notice that

$$\big\|f(e_1) \pm f(e_2)\big\| = 1.$$

It must actually be

$$f(e_1) = \pm e_1, \qquad f(e_2) = \pm e_2,$$

or

$$f(e_1) = \pm e_2, \qquad f(e_2) = \pm e_1.$$

Now we deal with the case where $j \geq 3$. Since f is a phase-isometry, for $i = 1, 2$, we have

$$\big\{\big\|f(e_j) + f(e_i)\big\|, \big\|f(e_j) - f(e_i)\big\|\big\} = \big\{\|e_j + e_i\|, \|e_j - e_i\|\big\} = \{1\}.$$

It follows that $\operatorname{supp} f(e_j) \subset [3, \infty)$. Note that for every $y \in S_T$ with $f(x) = y$, we have

$$\big\|f(e_j) - y\big\| \wedge \big\|f(e_j) + y\big\| = \|e_j - x\| \wedge \|e_j + x\| \leq 3/2.$$

Thus, by Lemma 2.1(2) there is an $i_j \in \mathbb{R}$ such that $|f(e_j)(i_j)| = 1$. For every $k \neq j$, note that

$$\begin{aligned}\big(1 \pm f(e_k)(i_j)\big) \vee \big(1 \pm f(e_j)(i_k)\big) &\leq \big\|f(e_k) + f(e_j)\big\| \vee \big\|f(e_k) - f(e_j)\big\| \\ &= \|e_k + e_j\| \vee \|e_k - e_j\| = 1.\end{aligned}$$

Therefore, $f(e_j)(i_k) = f(e_k)(i_j) = 0$. Additionally, the equations $\|f(e_j) \pm f(e_k)\| = 1$ ensure that

$$f(e_j) = \pm e_{i_j}.$$

To see our conclusion, it is enough to prove that $i_j = j$. By Lemma 2.2, the inverse $f^{-1} : T \to T$ is a surjective phase-isometry. Thus,

$$\bigcup_{j \geq 1} \operatorname{supp} f(e_j) = \mathbb{N}.$$

Consider the unit vector $x_j = \sum_{k=j}^{2j-1} \frac{2}{j} e_k$. Observe from the identity (1.2) that, for every $3 \leq j \leq k \leq 2j - 1$,

$$\big\|f(e_k) + f(x_j)\big\| \wedge \big\|f(e_k) - f(x_j)\big\| = \|e_k + x_j\| \wedge \|e_k - x_j\| = \frac{3}{2} - \frac{2}{j}.$$

Thus the support of $f(e_k)$, which is i_k, satisfies $i_k \in \operatorname{supp} f(x_j)$ and

$$\big|f(x_j)(i_k)\big| = \frac{2}{j}. \tag{2.1}$$

On the other hand, for every $k \geq 2j$ and $k < j$, we have

$$\big\|f(e_k) + f(x_j)\big\| = \big\|f(e_k) - f(x_j)\big\|.$$

It follows that

$$\operatorname{supp} f(x_j) \cap \operatorname{supp} f(e_k) = \emptyset,$$

for all $k \geq 2j$ and $k < j$. As a consequence,

$$\operatorname{supp} f(x_j) = \{i_j, \dots, i_{2j-1}\}.$$

This together with (2.1) proves that $i_j \geq j$. Since Lemma 2.2 yields that f^{-1} is a surjective phase-isometry, we apply the same arguments as above to f^{-1} to give $i_j = j$. The proof is complete. □

Using the previous characterization, we will establish that every phase-isometry on T has a property for the unit vector basis $\{e_j\}$ in terms of homogeneity.

Lemma 2.4. *Let $f : T \to T$ be a surjective phase-isometry. Then $f(\lambda e_j) = \pm\lambda f(e_j)$ for every $j \in \mathbb{N}$ and $0 \leq \lambda \in \mathbb{R}$.*

Proof. We may assume that f has form (1) of Lemma 2.3 since a minor modification of the proof also applies to the case that f is of form (2). Then it follows from this that, for every $j \geq 1$ and every $0 \leq \lambda \in \mathbb{R}$,

$$\begin{aligned}\big\{\big\|f(\lambda e_j) + e_j\big\|, \big\|f(\lambda e_j) - e_j\big\|\big\} &= \big\{\big\|f(\lambda e_j) + f(e_j)\big\|, \big\|f(\lambda e_j) - f(e_j)\big\|\big\} \\ &= \big\{\lambda + 1, |\lambda - 1|\big\}.\end{aligned}$$

As a consequence, $f(\lambda e_j)(j) = \pm\lambda$. On the other hand, for every $i \neq j$,

$$\big\|f(\lambda e_j) + f(\lambda e_i)\big\| = \big\|f(\lambda e_j) - f(\lambda e_i)\big\| = \lambda.$$

This implies that $f(\lambda e_j)(i) = 0$. The proof is complete. □

We are ready for a proposition that will be used repeatedly in what follows. It establishes that every surjective phase-isometry on T preserves the absolute value of vectors' coordinates.

Proposition 2.5. *Let $f : T \to T$ be a surjective phase-isometry. Then for every $x = (\xi_j) \in T$, we have $f(x) = (\eta_j) \in T$ satisfying either $|\eta_j| = |\xi_j|$ for all $j \in \mathbb{N}$ or $|\eta_2| = |\xi_1|$, $|\eta_1| = |\xi_2|$, $|\eta_j| = |\xi_j|$ for all $j \geq 3$.*

Proof. To prove our proposition, we first give an observation that, for every $j \in \mathbb{N}$, the equality

$$\|\theta M e_j + x\| = M + |\xi_j| \tag{2.2}$$

holds for all $x = (\xi_i) \in T$ with $\theta = \operatorname{sign} \xi_j$ and $M \geq 2\|x\|$. It is clear that

$$\big\|E(\theta M e_j + x)\big\| = \|Ex\| \leq \|x\|$$

for any finite subset E of $\mathbb{N}$ which does not contain j. By the definition of the T norm (using the identity (1.2)), it suffices to consider those admissible collections $\{E_i\}_{i=1}^k$ for which j is included in some E_{i_0}, where $1 \leq i_0 \leq k$. Then $E_i(\theta M e_j + x) = E_i x$ for all $i \neq i_0$, and thus

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{k}\big\|E_i(\theta Me_j+x)\big\| &= \frac{1}{2}\Big(\sum_{i=1,i\neq i_0}^{k}\|E_ix\|+\big\|E_{i_0}(\theta Me_j+x)\big\|\Big)\\ &\le \frac{1}{2}\Big(\sum_{i=1,i\neq i_0}^{k}\|E_ix\|+\|E_{i_0}x\|+\big\|E_{i_0}(\theta Me_j)\big\|\Big)\\ &\le \|x\|+\frac{M}{2}\le M.\end{aligned}$$

This and the identity (1.2) guarantee that (2.2) is true. For every $x=(\xi_j)\in T$, there is $y=(\eta_j)\in T$ such that $f(x)=y$. By Lemmas 2.3 and 2.4 and (2.2), we see that for every $j\ge 3$,

$$\begin{aligned}2\big\|f(x)\big\|+|\eta_j| &= \|2\|x\|f(e_j)+f(x)\|\vee\|2\|x\|f(e_j)-f(x)\|\\ &= \big\|f\big(2\|x\|e_j\big)+f(x)\big\|\vee\big\|f\big(2\|x\|e_j\big)-f(x)\big\|\\ &= \|2\|x\|e_j+x\|\vee\|2\|x\|e_j-x\| = 2\|x\|+|\xi_j|.\end{aligned}$$

It follows that $|\eta_j|=|\xi_j|$. A minor modification of the previous proof also applies to the case of $j=1,2$. This finishes the proof. $\square$

Given $x\in T$ and $i,j\in\mathbb{N}$, let $\Theta_{ij}(x)=(\theta_n)$, where $\theta_n=\operatorname{sign}(x(n))$ if $n=i,j$ and $\theta_n=0$ if $n\neq i$ or j. It is easily seen that the element $\Theta_{ij}(x)$ relies on the signs of $x(i)$ and $x(j)$. As Proposition 2.5 indicates that the absolute values of vectors' coordinates are invariant under any surjective phase-isometry of T, the maps $\Theta_{ij}(\cdot)$ will play a role in recognizing the signs of vectors' coordinates under the surjective phase-isometry.

Lemma 2.6. *Let $f: T\to T$ be a surjective phase-isometry. Then for every $x\in T$ and all $i,j\in\mathbb{N}$, we have*

$$f\big(\lambda\Theta_{ij}(x)\big)=\pm\lambda f\big(\Theta_{ij}(x)\big)\qquad\textit{and}\qquad \Theta_{ij}\big(f(x)\big)=\pm f\big(\Theta_{ij}(x)\big)$$

for all $i,j\in\mathbb{N}$ and $\lambda>0$.

Proof. To simplify the notation, we may assume that f has form (1) of Lemma 2.3. We first show that for every $\lambda>0$,

$$f\big(\lambda\Theta_{ij}(x)\big)=\pm\lambda f\big(\Theta_{ij}(x)\big) \tag{2.3}$$

for all i, j and $0\neq x\in T$. By Lemma 2.4 and Proposition 2.5, we may set $f(\Theta_{ij}(x))=(\theta_n)$ and $f(\lambda\Theta_{ij}(x))=(\lambda\theta'_n)$, where $|\theta'_i|=|\theta'_j|=|\theta_i|=|\theta_j|=1$ and $\theta'_n=\theta_n=0$ for $n\notin\{i,j\}$. Since f is a phase-isometry, we have

$$\begin{aligned}&\big\{|\lambda\theta'_i-\theta_i|\vee|\lambda\theta'_j-\theta_j|,|\lambda\theta'_i+\theta_i|\vee|\lambda\theta'_j+\theta_j|\big\}\\ &\quad=\big\{\big\|f\big(\lambda\Theta_{ij}(x)\big)-f\big(\Theta_{ij}(x)\big)\big\|,\big\|f\big(\lambda\Theta_{ij}(x)\big)+f\big(\Theta_{ij}(x)\big)\big\|\big\}\\ &\quad=\big\{\lambda+1,|\lambda-1|\big\}.\end{aligned}$$

As a result, $\theta'_i=\theta_i$, $\theta'_j=\theta_j$ or $\theta'_i=-\theta_i$, $\theta'_j=-\theta_j$. Hence the desired identity (2.3) is proved. To establish that $\Theta_{ij}(f(x))=\pm f(\Theta_{ij}(x))$, by Proposition 2.5 it is enough to consider two coordinates i and j since other coordinates are all zero. By (2.3) and (1.1), we have

$$\begin{aligned}
&\big\|3\|x\|f\big(\Theta_{ij}(x)\big)+f(x)\big\|\wedge\big\|3\|x\|f\big(\Theta_{ij}(x)\big)-f(x)\big\|\\
&\quad=\big\|f\big(3\|x\|\Theta_{ij}(x)\big)+f(x)\big\|\wedge\big\|f\big(3\|x\|\Theta_{ij}(x)\big)-f(x)\big\|\\
&\quad=\big\|3\|x\|\theta_{ij}(x)+x\big\|\wedge\big\|3\|x\|\theta_{ij}(x)-x\big\|. \qquad (2.4)
\end{aligned}$$

Similar arguments to those in the proof of (2.2) in Proposition 2.5, using the definition of the T norm in (1.2), enable us to conclude that

$$\begin{aligned}
&\big\|3\|x\|\theta_{ij}(x)+x\big\|\wedge\big\|3\|x\|\theta_{ij}(x)-x\big\|=\big\|3\|x\|\theta_{ij}(x)-x\big\|\\
&\quad=\max\{3\|x\|\theta_{ij}(x)-x\big\|_{c_0},3\|x\|\theta_{ij}(x)-x\big\|_T\}\\
&\quad=\max\{3\|x\|-|x(i)|\wedge|x(j)|,3\|x\|\theta_{ij}(x)-x\big\|_T\}, \qquad (2.5)
\end{aligned}$$

where $\|\cdot\|_T$ stands for the T norm.

We deduce from (2.5), (2.4), Proposition 2.5, and the identity (1.2) that

$$\Theta_{ij}\big(f(x)\big)=\pm f\big(\Theta_{ij}(x)\big)$$

as required. The proof is finished. □

We will give a lemma showing that every surjective phase-isometry has a property close to linearity. For all $x,y\in T$, to simplify the notation we write $x\perp y$ if $\operatorname{supp}x\cap\operatorname{supp}y=\emptyset$.

Lemma 2.7. *Let $f:T\to T$ be a surjective phase-isometry. Then we have the following:*

(1) *$f(\lambda x)=\pm\lambda f(x)$ for every $x\in X$, $\lambda\in\mathbb{R}$;*
(2) *given $x,y\in T$ with $x\perp y$, there exist* (unique) *$\alpha(x,y)$ and $\beta(x,y)$ with absolute value 1 such that*

$$f(x+y)=\alpha(x,y)f(x)+\beta(x,y)f(y). \qquad (2.6)$$

Proof. (1) We may assume that $\lambda>0$. Fix $x\in T$. By Lemma 2.6, we conclude that for all $i,j\in\mathbb{N}$,

$$\Theta_{ij}\big(f(\lambda x)\big)=\pm f\big(\Theta_{ij}(\lambda x)\big)=\pm f\big(\Theta_{ij}(x)\big)=\pm\Theta_{ij}\big(f(x)\big).$$

This combined with Proposition 2.5 yields the desired conclusion.

(2) We induce from Proposition 2.5 that $f(x)\perp f(y)$. Moreover, for all $i,j\in\operatorname{supp}x$, it follows from Lemma 2.6 that

$$\Theta_{ij}\big(f(x+y)\big)=\pm f\big(\Theta_{ij}(x+y)\big)=\pm\Theta_{ij}\big(f(x)\big).$$

Similar equalities also hold by considering "y" instead of "x." Thus, by Proposition 2.5 we get (2.6). □

Now, we will present our main result.

Theorem 2.8. *Let $f:T\to T$ be a surjective phase-isometry. Then f is phase-equivalent to a linear isometry.*

Proof. Let L be a subset of the unit sphere of T such that, for every $0 \neq y \in T$, either $y/\|y\|$ or $-y/\|y\|$ belongs to L. Then for every $x \in T$, there exists exactly one $y \in L$ of norm 1 such that $x = \lambda y$ for some $\lambda \in \mathbb{R}$. We define $f' : T \to T$ by $f'(0) = 0$ and

$$f'(x) = f'(\lambda y) = \lambda f(y), \quad \forall x = \lambda y \in T.$$

Then f' is well defined, homogeneous, and phase-equivalent to f by Lemma 2.7. Therefore, we may assume that f is homogeneous. Fix $i_0 \in \mathbb{N}$, and let

$$Z := \{x \in T : x \perp e_{i_0}\}.$$

By Lemma 2.7(2), for every $z \in Z$ we can write

$$f(z + e_{i_0}) = \alpha(z) f(z) + \beta(z) f(e_{i_0}), \quad \big|\alpha(z)\big| = \big|\beta(z)\big| = 1$$

for all $z \in Z$. Since f is homogeneous, for every $\lambda \neq 0$, we have

$$f(z + \lambda e_{i_0}) = \alpha\Big(\frac{z}{\lambda}\Big) f(z) + \beta\Big(\frac{z}{\lambda}\Big) \lambda f(e_{i_0}).$$

Define a mapping $g : T \to T$ as

$$g(z) = \alpha(z)\beta(z) f(z) \qquad \text{and} \qquad g(z + \lambda e_{i_0}) = \alpha\Big(\frac{z}{\lambda}\Big)\beta\Big(\frac{z}{\lambda}\Big) f(z) + \lambda f(e_{i_0})$$

for all $z \in Z$ and $0 \neq \lambda \in \mathbb{R}$. It is clear that g is phase-equivalent to f. Hence, g is also a phase-isometry. We will show that for every $z \in Z$ and $0 \neq \lambda \in \mathbb{R}$, we have

$$\alpha(\lambda z)\beta(\lambda z) = \alpha(z)\beta(z). \tag{2.7}$$

It suffices to show that (2.7) holds for every z in the unit sphere of Z. Note first that

$$\begin{aligned}&\big\{\big\|g(\lambda z + e_{i_0}) + g(\lambda z - e_{i_0})\big\|, \big\|g(\lambda z + e_{i_0}) - g(\lambda z - e_{i_0})\big\|\big\}\\ &\qquad = \big\{2|\lambda|, 2\big\}.\end{aligned}$$

It follows that

$$\alpha(\lambda z)\beta(\lambda z) = \alpha(-\lambda z)\beta(-\lambda z).$$

Moreover, for every $\lambda > 0$, we have

$$\begin{aligned}&\big\{\big\|(1+\lambda)z + 2e_{i_0}\big\|, |1-\lambda|\big\}\\ &\quad = \big\{\big\|g(\lambda z + e_{i_0}) + g(z + e_{i_0})\big\|, \big\|g(\lambda z + e_{i_0}) - g(z + e_{i_0})\big\|\big\}\\ &\quad = \big\{\big\|\big(\lambda\alpha(\lambda z)\beta(\lambda z) + \alpha(z)\beta(z)\big) f(z) + 2f(e_{i_0})\big\|,\\ &\qquad \big\|\big(\lambda\alpha(\lambda z)\beta(\lambda z) - \alpha(z)\beta(z)\big) f(z)\big\|\big\}.\end{aligned}$$

We deduce from this and $\|f(z)\| = \|z\| = 1$ that $\alpha(\lambda z)\beta(\lambda z) = \alpha(z)\beta(z)$. The proof of (2.7) is finished. So $g(Z) = Z$. We will use (2.7) to show that g is an isometry on Z, and thus g is linear on T by the Mazur–Ulam theorem and its definition. For all z_1, z_2 in Z, choose a positive large enough number λ. It is easy to check that

$$\begin{aligned}\{\|g(z_1)-g(z_2)\|, \|g(z_1)+g(z_2)+2\lambda g(e_{i_0})\|\} \\ = \{\|g(z_1+\lambda e_{i_0})-g(z_2+\lambda e_{i_0})\|, \|g(z_1+\lambda e_{i_0})+g(z_2+\lambda e_{i_0})\|\} \\ = \{\|z_1-z_2\|, \|z_1+z_2+2\lambda e_{i_0}\|\}.\end{aligned}$$

Since λ is a positive large number, we conclude that $\|g(z_1)-g(z_2)\| = \|z_1-z_2\|$ for all $z_1, z_2 \in Z$. This completes the proof. □

Acknowledgments. The authors would like to thank the referees for their constructive comments and suggestions. This paper was written during the second author's visit to the University of Illinois at Urbana-Champaign, and she thanks that institution for its hospitality.

Tan's work was partially supported by National Natural Science Foundation of China grants 11371201, 11201337, and 11201338.

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