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A NOTE ON THE PAPER "MATRIX INEQUALITIES FOR THE DIFFERENCE BETWEEN ARITHMETIC MEAN AND HARMONIC MEAN"

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Abstract. In this short article, we mainly give some examples to prove that the main results of Section 4 of the paper [W. S. Liao and J. Wu, Ann. Func. Anal. 6 (2015), no. 3, 191–202] are not true. Then we give the corrected matrix inequalities.

1. Introduction

Let M_n be the algebra of $n \times n$ complex matrices, let M_n^+ be the cone of positive semidefinite matrices in M_n, and let M_n^{++} be the cone of positive invertible matrices in M_n. For two Hermitian matrices A and B in M_n, we consider that $A \geq B (A > B)$ when $A - B \in M_n^+$ $(A - B \in M_n^{++})$. For $A, B \in M_n^{++}$, the operator arithmetic, geometric, and harmonic means are defined, respectively, by

$$A \nabla_v B = vA + (1-v)B, A \sharp_v B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^v A^{\frac{1}{2}}, A !_v B = \big(vA^{-1} + (1-v)B^{-1}\big)^{-1},$$

where $0 \leq v \leq 1$. When $v = \frac{1}{2}$, we drop the v from the above notation.

The Hilbert–Schmidt norm of $A = [a_{ij}] \in M_n$ is defined by $\|A\|_2 = (\sum_{i,j=1}^n |a_{ij}|^2)^{\frac{1}{2}}$. It is well known that the Hilbert–Schmidt norm is unitarily

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invariant in the sense that $\|UAV\|_2 = \|A\|_2$ for all unitary matrices $U, V \in M_n$. As is also well known, the famous Young's inequality for real numbers is that

$$a^v b^{1-v} \le va + (1-v)b, \quad a, b > 0 \text{ and } 0 \le v \le 1, \tag{1.1}$$

which is also called the *$v-$weighted arithmetic-geometric mean inequality*. The first refinement of (1.1) is the squared version proved in [2],

$$(a^v b^{1-v})^2 + \min\{v, 1-v\}^2 (a-b)^2 \le \big(va + (1-v)b\big)^2, \tag{1.2}$$

where $a, b > 0$ and $0 \le v \le 1$. Later, similar to (1.2), Kittaneh and Manasrah obtained the following interesting refinement in [4]:

$$a^v b^{1-v} + \min\{v, 1-v\}(\sqrt{a} - \sqrt{b})^2 \le va + (1-v)b$$

for $a, b > 0$ and $0 \le v \le 1$. Of the many results about Young inequalities presented in recent years, we refer the reader to [3] and [7] for some insightful results.

Related to arithmetic-geometric-harmonic mean inequality, Alzer [1] proved that

$$\left(\frac{\nu}{\tau}\right)^\lambda \le \frac{(a\nabla_\nu b)^\lambda - (a\sharp_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (a\sharp_\tau b)^\lambda} \le \left(\frac{1-\nu}{1-\tau}\right)^\lambda \tag{1.3}$$

for $0 < \nu \le \tau < 1$ and $\lambda \ge 1$. By a similar technique, Liao [5] showed that

$$\left(\frac{\nu}{\tau}\right)^\lambda \le \frac{(a\nabla_\nu b)^\lambda - (a!_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (a!_\tau b)^\lambda} \le \left(\frac{1-\nu}{1-\tau}\right)^\lambda \tag{1.4}$$

for $0 < \nu \le \tau < 1$ and $\lambda \ge 1$. Sababheh [6] generalized (1.3) and (1.4) by convexity of f

$$\left(\frac{\nu}{\tau}\right)^\lambda \le \frac{((1-\nu)f(0) + \nu f(1))^\lambda - f^\lambda(\nu)}{((1-\tau)f(0) + \tau f(1))^\lambda - f^\lambda(\tau)} \le \left(\frac{1-\nu}{1-\tau}\right)^\lambda,$$

where $0 < \nu \le \tau < 1$ and $\lambda \ge 1$.

In [5], Liao and Wu proved similar inequalities for matrices. By giving some examples, we mainly prove that Theorem 4.1, Corollary 4.2, and Corollary 4.3 of Section 4 in [5] are not true. Moreover, we give their correct Hilbert–Schmidt norm inequalities.

2. Hilbert–Schmidt norm inequalities for arithmetic-harmonic mean

In [5], Liao and Wu obtained the following results.

Theorem 2.1 ([5, Theorem 4.1])**.** *Let $X \in M_n$ and $A, B \in M_n^{++}$ for $0 < \nu \le \tau < 1$. Then*

$$\frac{\nu^2}{\tau^2} \le \frac{\|\nu AX + (1-\nu)XB\|_2^2 - \|[\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}]^{-1}\|_2^2}{\|\tau AX + (1-\tau)XB\|_2^2 - \|[\tau X^{-1}A^{-1} + (1-\tau)B^{-1}X^{-1}]^{-1}\|_2^2} \le \frac{(1-\nu)^2}{(1-\tau)^2}.$$

Corollary 2.2 ([5, Corollary 4.2])**.** *Let $X \in M_n$ and $A, B \in M_n^{++}$ for $0 \le \nu \le 1$. Then*

$$\big\|\nu AX + (1-\nu)XB\big\|_2^2 \ge \|A^\nu X B^{1-\nu}\|_2^2 \ge \big\|\big[\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}\big]^{-1}\big\|_2^2.$$

Corollary 2.3 ([5, Corollay 4.3])**.** *Let $X \in M_n$ and $A, B \in M_n^{++}$ for $0 \le \nu \le \frac{1}{2}$. Then*

$$
\begin{aligned}
&4\nu^2\Big[\Big\|\frac{AX+XB}{2}\Big\|_2^2 - \Big\|\Big(\frac{X^{-1}A^{-1}+B^{-1}X^{-1}}{2}\Big)^{-1}\Big\|_2^2\Big] \\
&\quad \le \big\|\nu AX + (1-\nu)XB\big\|_2^2 - \big\|\big[\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}\big]^{-1}\big\|_2^2 \\
&\quad \le 4(1-\nu)^2\Big[\Big\|\frac{AX+XB}{2}\Big\|_2^2 - \Big\|\Big(\frac{X^{-1}A^{-1}+B^{-1}X^{-1}}{2}\Big)^{-1}\Big\|_2^2\Big].
\end{aligned}
$$

In the proofs of Theorem 2.1, Corollary 2.2 (the second inequality), and Corollary 2.3, by using the following equality (2.1), the authors transformed scalar inequalities into matrix inequalities

$$
\big[\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}\big]^{-1} = U\big[\big(\nu\mu_i^{-1} + (1-\nu)\nu_j^{-1}\big)^{-1} y_{ij}\big]V^*, \quad (2.1)
$$

where U, V are unitary matrices such that $A = U\Lambda_1 U^*, B = V\Lambda_2 V^*$ for $\Lambda_1 = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, $\Lambda_2 = \operatorname{diag}(\nu_1, \nu_2, \ldots, \nu_n)$, and $Y = V^*XV = [y_{il}]$.

But, unfortunately, (2.1) is not true in general. Next, we give some examples to show these results are wrong except the first inequality in Corollary 2.2.

Example 2.4. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, and $X = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ for $\nu = \frac{1}{2}$ and $\tau = \frac{2}{3}$. By a careful calculation, we can show that

$$
\big\|\nu AX + (1-\nu)XB\big\|_2^2 = \left\|\begin{pmatrix} 1 & \frac{1}{2} + \frac{v}{2} \\ 0 & 1 - \frac{2}{3}v \end{pmatrix}\right\|_2^2 = \frac{81 - 30\nu + 25\nu^2}{36}, \quad (2.2)
$$

and

$$
\begin{aligned}
&\big\|\big(\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}\big)^{-1}\big\|_2^2 \\
&\quad = \left\|\begin{pmatrix} 1 & -\frac{1}{2} - \frac{5v}{2} \\ 0 & 1 + 2v \end{pmatrix}^{-1}\right\|_2^2 = \frac{9 + 26\nu + 41\nu^2}{4 + 16\nu + 16\nu^2}. \quad (2.3)
\end{aligned}
$$

Hence by (2.2) and (2.3) we have

$$
\begin{aligned}
&\big\|\nu AX + (1-\nu)XB\big\|_2^2 - \big\|\big(\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}\big)^{-1}\big\|_2^2 \\
&\quad = \frac{15\nu - 35\nu^2 - 5\nu^3 + 25\nu^4}{9 + 36\nu + 36\nu^2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\big\|\nu AX + (1-\nu)XB\big\|_2^2 - \big\|\big[\nu X^{-1}A^{-1} + (1-\nu)B^{-1}X^{-1}\big]^{-1}\big\|_2^2 \\
&\quad = -\frac{5}{576} < 0, \quad (2.4)
\end{aligned}
$$

and that

$$
\begin{aligned}
&\big\|\tau AX + (1-\tau)XB\big\|_2^2 - \big\|\big[\tau X^{-1}A^{-1} + (1-\tau)B^{-1}X^{-1}\big]^{-1}\big\|_2^2 \\
&\quad = -\frac{170}{3969}. \quad (2.5)
\end{aligned}
$$

Now, (2.4) and (2.5) imply that

$$\frac{\|\nu AX+(1-\nu)XB\|_2^2-\|[\nu X^{-1}A^{-1}+(1-\nu)B^{-1}X^{-1}]^{-1}\|_2^2}{\|\tau AX+(1-\tau)XB\|_2^2-\|[\tau X^{-1}A^{-1}+(1-\tau)B^{-1}X^{-1}]^{-1}\|_2^2} = \frac{441}{2176} < \frac{9}{16} = \frac{\nu^2}{\tau^2},$$

which shows that the first inequality of Theorem 2.1 is clearly not true.

Example 2.5. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{7} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, and $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ for $\nu = \frac{1}{2}$ and $\tau = \frac{2}{3}$, we can obtain that

$$\left\|\nu AX+(1-\nu)XB\right\|_2^2 = \left\|\begin{pmatrix} 2 & \frac{1}{4}+\frac{3v}{4} \\ 1-\frac{6v}{7} & \frac{1}{4}-\frac{3v}{28} \end{pmatrix}\right\|_2^2 = \frac{2009-546\nu+513\nu^2}{392}$$

and that

$$\left\|\left(\nu X^{-1}A^{-1}+(1-\nu)B^{-1}X^{-1}\right)^{-1}\right\|_2^2 = \left\|\begin{pmatrix} 1 & -1-6v \\ 3v-4 & 8+6v \end{pmatrix}^{-1}\right\|_2^2 = \frac{82+84\nu+81\nu^2}{(4-15\nu+18\nu^2)^2}.$$

So then we have

$$\frac{\|\nu AX+(1-\nu)XB\|_2^2-\|[\nu X^{-1}A^{-1}+(1-\nu)B^{-1}X^{-1}]^{-1}\|_2^2}{(1-\nu)^2} = -557.977\cdots$$

and

$$\frac{\|\tau AX+(1-\tau)XB\|_2^2-\|[\tau X^{-1}A^{-1}+(1-\tau)B^{-1}X^{-1}]^{-1}\|_2^2}{(1-\tau)^2} = -348.497\cdots,$$

which is not true clearly for the second inequality of Theorem 2.1.

Now we begin to prove that the second inequalities in Corollary 2.2 and the two inequalities in Corollary 2.3 are not true. First, by taking A, B, X and v in Example 2.4, we have that the inequalities in Corollary 2.2 are reduced to $2.0069\cdots \geq \frac{11}{6} \geq 2.0156\cdots$, which is not possible. Next, by taking A, B and X in Example 2.4, then the inequalities in Corollary 2.3 are reduced to

$$-\frac{5}{144}v^2 \leq \frac{15\nu-35\nu^2-5\nu^3+25\nu^4}{9+36\nu+36\nu^2} \leq -\frac{5}{144}(1-v)^2,$$

but the second inequality is not true for $v = \frac{1}{3}$. Last, if by taking A, B and X in Example 2.5, then the first inequality in Corollary 2.3 is reduced to

$$4v^2\Big(4+\frac{25}{64}+\frac{16}{49}+\frac{121}{3136}-121-16-\frac{25}{4}-1\Big) \leq \frac{2009-546\nu+513\nu^2}{392} - \frac{82+84\nu+81\nu^2}{(4-15\nu+18\nu^2)^2},$$

which is not true for $v = \frac{1}{3}$.

Now we give the following correct form of Theorem 2.1.

Theorem 2.6. *Let $X \in M_n$ and $B \in M_n^{++}$ for $0 < \nu \leq \tau < 1$. Then we have*

$$\begin{aligned}\frac{\nu^2}{\tau^2} &\leq \frac{\|\nu X + (1-\nu)XB\|_2^2 - \|[\nu X^{-1} + (1-\nu)B^{-1}X^{-1}]^{-1}\|_2^2}{\|\tau X + (1-\tau)XB\|_2^2 - \|[\tau X^{-1} + (1-\tau)B^{-1}X^{-1}]^{-1}\|_2^2} \\ &\leq \frac{(1-\nu)^2}{(1-\tau)^2}. \end{aligned} \tag{2.6}$$

Proof. Since B is a positive definite, it follows by spectral theorem that there exist unitary matrices $V \in M_n$, such that

$$B = V\Lambda V^*,$$

where $\Lambda = \operatorname{diag}(\nu_1, \nu_2, \dots, \nu_n)$ and ν_i are eigenvalues of B, so $\nu_l > 0, l = 1, 2, \dots, n$.

Let $Y = V^*XV = [y_{il}]$. Then

$$\begin{aligned} &vX + (1-v)XB \\ &\quad = V\big[vY + (1-v)Y\Lambda\big]V^* \\ &\quad = V\big[\big(v + (1-v)\nu_l\big)y_{il}\big]V^*, \end{aligned}$$

and

$$\begin{aligned} &\big[vX^{-1} + (1-v)B^{-1}X^{-1}\big]^{-1} \\ &\quad = V\big[vY^{-1} + (1-v)\Lambda_2^{-1}Y^{-1}\big]^{-1}V^* \\ &\quad = V\big[\big(v + (1-v)\nu_l^{-1}\big)^{-1}y_{il}\big]V^*. \end{aligned}$$

Now, by (1.4) and the unitary invariant of the Hilbert–Schmidt norm, we have

$$\begin{aligned} &\big\|vX + (1-v)XB\big\|_2^2 - \big\|\big[vX^{-1} + (1-v)B^{-1}X^{-1}\big]^{-1}\big\|_2^2 \\ &\quad = \sum_{i,l=1}^{n}\big(v + (1-v)\nu_l\big)^2|y_{il}|^2 - \sum_{i,l=1}^{n}\big(v + (1-v)\nu_l^{-1}\big)^{-2}|y_{il}|^2 \\ &\quad = \sum_{i,l=1}^{n}\big[\big(v + (1-v)\nu_l\big)^2 - \big(v + (1-v)\nu_l^{-1}\big)^{-2}\big]|y_{il}|^2 \\ &\quad \leq \frac{(1-v)^2}{(1-\tau)^2}\sum_{i,l=1}^{n}\big[\big(\tau + (1-\tau)\nu_l\big)^2 - \big(\tau + (1-\tau)\nu_l^{-1}\big)^{-2}\big]|y_{il}|^2 \\ &\quad = \frac{(1-v)^2}{(1-\tau)^2}\Big[\sum_{i,l=1}^{n}\big(\tau + (1-\tau)\nu_l\big)^2|y_{il}|^2 - \sum_{i,l=1}^{n}\big(\tau + (1-\tau)\nu_l^{-1}\big)^{-2}|y_{il}|^2\Big] \\ &\quad = \frac{(1-v)^2}{(1-\tau)^2}\big[\big\|\tau X + (1-\tau)XB\big\|_2^2 - \big\|\big[\tau X^{-1} + (1-\tau)B^{-1}X^{-1}\big]^{-1}\big\|_2^2\big]. \end{aligned}$$

Here we complete the proof the second inequality of (2.6). Using the same method, we can prove the first inequality of (2.6) easily. So we omit it. ☐

To keep concision, here we only give correct inequalities of Corollary 2.2 and Corollary 2.3 in [5] as follows.

Corollary 2.7. *Let $X \in M_n$ and $B \in M_n^{++}$ for $0 < \nu \le \tau < 1$. Then we have*

$$\left\|\nu X + (1-\nu)XB\right\|_2^2 \ge \|XB^{1-\nu}\|_2^2 \ge \left\|\left[\nu X^{-1} + (1-\nu)B^{-1}X^{-1}\right]^{-1}\right\|_2^2$$

and

$$\begin{aligned} &4\nu^2\left[\left\|\frac{X+XB}{2}\right\|_2^2 - \left\|\left(\frac{X^{-1}+B^{-1}X^{-1}}{2}\right)^{-1}\right\|_2^2\right] \\ &\le \left\|\nu X + (1-\nu)XB\right\|_2^2 - \left\|\left[\nu X^{-1} + (1-\nu)B^{-1}X^{-1}\right]^{-1}\right\|_2^2 \\ &\le 4(1-\nu)^2\left[\left\|\frac{X+XB}{2}\right\|_2^2 - \left\|\left(\frac{X^{-1}+B^{-1}X^{-1}}{2}\right)^{-1}\right\|_2^2\right]. \end{aligned}$$

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