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KY FAN MINIMAX INEQUALITIES FOR SET-VALUED MAPPINGS ON DENSE SETS

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ABSTRACT. We give a natural Ky Fan minimax inequality version of set-valued maps, and we deal with a type of vector equilibrium problem for set-valued mappings on a special dense set not on the whole domain. We use these results as applications to study the solutions of a generalized set-valued vector variational inequality.

1. Introduction and preliminaries

There are several inequalities due to Ky Fan in various fields, such as the Ky Fan matrix inequality, mean inequality, minimax inequality, sharpening of the von Neumann inequality, generalization of Szász's inequality, and so on (see, e.g., [15]). It is well known that Ky Fan minimax inequalities (see [8]) play an important role in many fields, such as game theory, variational inequalities, mathematical economics, control theory, and fixed point theory. Because of their wide application, Ky Fan minimax inequalities for real-valued functions have been studied extensively by many scholars (see, e.g., [2], [9], [10], [4], [14], [6]).

To the author's best knowledge, only a small number of articles have investigated Ky Fan minimax inequalities for set-valued mappings. Zhang and Li [17] obtained two types of Ky Fan minimax inequalities for set-valued mappings by the Kakutani–Fan–Glicksberg fixed point theorem, and in [18] they studied some generalized Ky Fan inequalities for scalar set-valued mappings with nonconvex

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domains. Zhang, Li, and Li [19] investigated Ky Fan minimax inequalities for set-valued mappings by the Ky Fan lemma and the Kakutani–Fan–Glicksberg fixed point theorem. The concept of a self-segment-dense set was recently introduced by László and Viorel in [12] and has been used in set-valued equilibrium problems. Alleche and Rădulescu [1] obtained solutions for set-valued equilibrium problems applied to Browder variational inclusions and fixed point theorems.

Motivated by [17]–[19], [12], [1], [20], [13], [3], and [11], we propose new types of Ky Fan minimax inequalities for set-valued mappings involving conditions that are not imposed on the whole domain, but on self-segment-dense sets of set-valued mappings, which is different from that described by Zhang, S. Li, M. Li, and Zhu (see [17]–[20]). Moreover, we obtain results for set-valued vector equilibrium problems. We apply these results to prove the existence of solutions for a variational inequality.

Throughout this paper, let X be a Hausdorff locally convex topological vector space, and let V be a Banach space. Let $A \subset X$ be a subset. We denote its closure by $\operatorname{cl} A$ and its interior by $\operatorname{int} A$. Let C be a pointed closed convex cone in V. We define the partial order as

$$x \leq y \Leftrightarrow x \in y - C, \quad \forall x, y \in V.$$

Here, $y - C$ denotes the Minkowski addition of two sets $\{y\}$ and $-C$, that is, $y - C = \{y\} + (-C) = \{y + c : c \in -C\}$.

Definition 1.1 ([1, p. 254], [2, Chapter 3], [12, p. 54]). Let $F : X \to 2^V$ be a set-valued mapping.

(i) F is called *upper semicontinuous* at $x_0 \in X$ if for any neighborhood N of $F(x_0)$ there exists a neighborhood U of x_0 such that

$$F(x) \subset N, \quad \forall x \in U.$$

(ii) F is called *lower semicontinuous* at $x_0 \in X$ if for any open neighborhood N in V satisfying $F(x_0) \cap N \neq \emptyset$ there exists a neighborhood U of x_0 such that

$$F(x) \cap N \neq \emptyset, \quad \forall x \in U.$$

(iii) F is called *continuous* at $x_0 \in X$ if F is both upper and lower semicontinuous at x_0.

Remark 1.2. A set-valued map $F : X \to 2^V$ is called lower semicontinuous at $x_0 \in X$ if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_0 \in F(x_0)$ there exists $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y_0$.

Proposition 1.3 ([1, Proposition 2.4])**.** *Let $F : X \to 2^V$ be a set-valued mapping, and let S be a subset of X. Then the following statements are equivalent:*

(i) *F is lower semicontinuous on S;*

(ii) *for any open subset N of V, one has*

$$\big\{x \in X : F(x) \cap N\big\} \cap S = \operatorname{int}\big(\big\{x \in X : F(x) \cap N\big\}\big) \cap S;$$

(iii) *for any closed subset B of V, one has*

$$\{x \in X : F(x) \subset B\} \cap S = \mathrm{cl}\big(\{x \in X : F(x) \subset B\}\big) \cap S.$$

Proposition 1.4 ([1, Proposition 2.4])**.** *Let $F : X \to 2^V$ be a set-valued mapping, and let S be a subset of X. Then the following statements are equivalent:*

(i) *F is upper semicontinuous on S;*

(ii) *for any open subset N of V, one has*

$$\{x \in X : F(x) \subset N\} \cap S = \mathrm{int}\big(\{x \in X : F(x) \subset N\}\big) \cap S;$$

(iii) *for any closed subset B of V, one has*

$$\{x \in X : F(x) \cap B\} \cap S = \mathrm{cl}\big(\{x \in X : F(x) \cap B\}\big) \cap S.$$

From [17], [12], [1], [13], we describe the concept in the following.

Definition 1.5. Let X_0 be a nonempty subset of X, and let $F : X \to 2^V$ be a set-valued mapping.

(i) F is called *properly cone-quasiconcave* on X_0 if for any $x_1, x_2 \in X_0$ and $t \in [0,1]$ such that $tx_1 + (1-t)x_2 \in D$ one has either

$$F(x_1) \subset F\big(tx_1 + (1-t)x_2\big) - C \qquad \text{or} \qquad F(x_2) \subset F\big(tx_1 + (1-t)x_2\big) - C.$$

(ii) F is called *properly cone-quasiconvex* on X_0 if for any $x_1, x_2 \in X_0$ and $t \in [0,1]$ such that $tx_1 + (1-t)x_2 \in D$ one has either

$$F(x_1) \subset F\big(tx_1 + (1-t)x_2\big) + C \qquad \text{or} \qquad F(x_2) \subset F\big(tx_1 + (1-t)x_2\big) + C.$$

(iii) F is called *cone-convex* on X_0 if for any $x_1, x_2 \in X_0$ and $t \in [0,1]$ such that $tx_1 + (1-t)x_2 \in D$, then

$$tF(x_1) + (1-t)F(x_2) \subset F\big(tx_1 + (1-t)x_2\big) + C.$$

(iv) F is called *cone-concave* on X_0 if for any $x_1, x_2 \in X_0$ and $t \in [0,1]$ such that $tx_1 + (1-t)x_2 \in D$, then

$$tF(x_1) + (1-t)F(x_2) \subset F\big(tx_1 + (1-t)x_2\big) - C.$$

(v) F is called *convex* on X_0 if for any $x_1, x_2 \in X_0$ and $t \in [0,1]$ such that $tx_1 + (1-t)x_2 \in D$, then

$$tF(x_1) + (1-t)F(x_2) \subset F\big(tx_1 + (1-t)x_2\big).$$

(vi) F is called *concave* on X_0 if for any $x_1, x_2 \in X_0$ and $t \in [0,1]$ such that $tx_1 + (1-t)x_2 \in D$, then

$$F\big(tx_1 + (1-t)x_2\big) \subset tF(x_1) + (1-t)F(x_2).$$

Note that these notions do not ensure the convexity of X_0. However, it is easy to see that the convexity of F implies cone-convexity of F or cone-concavity of F. If X_0 is convex, then Definition 1.5(i)(ii) coincides with [17, Definition 2.3(i)], and Definition 1.5(iii)(iv) coincides with [13, Definition 2.2(i)(iii)].

Example 1.6. Let $X = V = \mathbb{R}$, $C = \{x \in \mathbb{R} : x \geq 0\}$, and $X_0 = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Let $F : X_0 \to 2^V$ be a set-valued mapping defined by

$$F(x) = \{y \in \mathbb{R} : -\cos x \leq y \leq 1 + x^2\}, \quad x \in X_0.$$

Then it is easy to check that F is cone-convex on X_0, but F is not convex on X_0.

Example 1.7. Let $X = V = \mathbb{R}$, $C = \{x \in \mathbb{R} : x \geq 0\}$, and $X_0 = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Let $F : X_0 \to 2^V$ be a set-valued mapping defined by

$$F(x) = \{y \in \mathbb{R} : \cos x \leq y \leq 3 - x^2\}, \quad x \in X_0.$$

Then it is easy to check that F is cone-concave on X_0, but F is not convex on X_0.

Definition 1.8 ([16, Definition 2.2]). Let D be a nonempty subset of V.

(i) A point $z \in V$ is said to be the *supremum* of D, denoted by $\sup D$, if it satisfies the following:
(a) $x \leq z$, $\forall x \in D$; (b) $x \leq z_1$, $\forall x \in D \Rightarrow z \leq z_1$.
In an analogous way, we define the infimum of D by $\inf D$.

(ii) C is said to be a *strongly minihedral cone* if for any nonempty subset of D which is upper (resp., lower) bounded, one has the supremum (resp., infimum).

It is easy to see that $\{x \in \mathbb{R} : x \geq 0\}$ is a strongly minihedral cone.

Remark 1.9. The nonempty subset $D \subset V$ is said to be *upper bounded* if there exists a point $b \in V$ such that $x \leq b$ for all $x \in D$. Similarly, we define *lower bounded* of D. A nonempty subset $D \subset V$ is said to be *bounded* if D is both upper and lower bounded.

Proposition 1.10. *Let X_0 be a nonempty subset of X, and let $F : X \to 2^V$ be a set-valued mapping.*

(i) *If F is properly cone-quasiconvex and lower bounded on X_0, then $x \to \inf F(x)$ is properly cone-quasiconvex on X_0.*

(ii) *If F is properly cone-quasiconcave and upper bounded on X_0, then $x \to \sup F(x)$ is properly cone-quasiconcave on X_0.*

Proof. (i) Let $\forall y_1, y_2 \in X_0$, $t \in [0,1]$ be such that $ty_1 + (1-t)y_2 \in X_0$. By the proper cone-quasiconvexity of F, one has either

$$F(y_1) \subset F\big(ty_1 + (1-t)y_2\big) + C \qquad \text{or} \qquad F(y_2) \subset F\big(ty_1 + (1-t)y_2\big) + C.$$

Hence, we have either

$$\inf F\big(ty_1 + (1-t)y_2\big) \leq \inf F(y_1) \qquad \text{or} \qquad \inf F\big(ty_1 + (1-t)y_2\big) \leq \inf F(y_2).$$

Thus, $\inf F(x)$ is properly cone-quasiconvex on X_0.

(ii) Similar to the proof of (i), we show that (ii) holds. □

Definition 1.11 ([13, Definition 2.3], [17, Definition 2.1], [18, Definition 2.1], [19, Definition 2.1], [20, Definition 2.1]). Let A be a nonempty subset in V.

(i) A point $x \in A$ is called a *minimal point* of A if $A \cap (x - C) = \{x\}$; $\operatorname{Min} A$ denotes the set of all minimal points of A.

(ii) A point $x \in A$ is called a *maximal point* of A if $A \cap (x + C) = \{x\}$; $\operatorname{Max} A$ denotes the set of all maximal points of A.

Definition 1.12 ([5, Definition 2.4]). Let $f : X \to V$ be a vector-valued function.

(i) The function f is called *cone-lower semicontinuous* on X if for any $r \in V$ the set $\{x \in X : f(x) \leq r\}$ is closed in X.
(ii) The function f is called *cone-upper semicontinuous* on X if for any $r \in V$ the set $\{x \in X : f(x) \geq r\}$ is closed in X.

Remark 1.13. In [5], if X is a compact set and $f : X \to V$ is cone-upper semicontinuous, then $\operatorname{Max} f(X) = \operatorname{Max} \bigcup_{x \in X} f(x) \neq \emptyset$. If X is a compact set and $f : X \to V$ is cone-lower semicontinuous, then $\operatorname{Min} f(X) = \operatorname{Min} \bigcup_{x \in X} f(x) \neq \emptyset$.

Lemma 1.14. *Let X_0 and Y_0 be two nonempty subsets of X, let C be a strongly minihedral cone in V, and let $F : X_0 \times Y_0 \to 2^V$ be a set-valued mapping. Suppose that $F(x, y)$ is bounded on $X_0 \times Y_0$. Then we have the following conclusions:*

(a) *If $F(x, y)$ is lower semicontinuous x on X_0 for each fixed $y \in Y_0$, then the vector-valued function $\phi : X_0 \to V$ defined by $\phi(x) = \inf_{y \in Y_0} F(x, y)$ is cone-upper semicontinuous X_0.*
(b) *If $F(x, y)$ is lower semicontinuous y on Y_0 for each fixed $x \in X_0$. then the vector-valued function $\varphi : Y_0 \to V$ defined by $\varphi(y) = \sup_{x \in X_0} F(x, y)$ is cone-lower semicontinuous Y_0.*

Proof. By the assumptions and Definition 1.8, we know that the vector-valued functions ϕ and φ are well defined. Now we show that $\phi : X_0 \to V$ is cone-upper semicontinuous. Let $r \in V$. We take any net $\{x_\alpha\} \subset \{x \in X : \phi(x) \geq r\}$ such that

$$x_\alpha \to x_0.$$

Obviously, we get $\bigcup_{y \in Y_0} F(x_\alpha, y) \subset r + C$; then we obtain $F(x_\alpha, y) \subset r + C$ for each $y \in Y_0$. By the lower semicontinuity of $F(\cdot, y)$, for any $z_0 \in F(x_0, y)$, there exists $z_\alpha \in F(x_\alpha, y)$ such that $z_\alpha \to z_0$. It follows from the closeness of C and $z_\alpha \in r + C$ that $z_0 \in r + C$. By the arbitrary of z_0, we get

$$\phi(x_0) \geq r.$$

Thus, $\phi(x)$ is cone-upper semicontinuous on X_0 and so conclusion (a) is true. Similarly, we can show that conclusion (b) holds. □

Remark 1.15. Obviously, for each $y \in Y_0$, we have that $\inf F(\cdot, y)$ is cone-upper semicontinuous on X_0. For each $x \in X_0$, we get that $\sup F(x, \cdot)$ is cone-lower semicontinuous on Y_0.

Definition 1.16. Let M be a nonempty subset of X. A set-valued map $F : M \to V$ is called a KKM (Knaster–Kuratowski–Mazurkiewicz) map if for every finite subset $\{x_1, x_2, \ldots, x_n\} \subset M$ one has

$$\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

Lemma 1.17 ([7, Lemma 1]). *Let X_0 be a nonempty subset of X. Let $K : X_0 \to 2^X$ be a KKM mapping with closed values. If there exists $x_0 \in X_0$ such that $K(x_0)$ is compact, then $\bigcap_{x \in X_0} K(x) \neq \emptyset$.*

Definition 1.18 ([12, Definition 2.2]). Let A be a nonempty convex subset of X, and let $D \subset A$ be a subset. We say that D is *self-segment-dense* in A if $A \subset \operatorname{cl} D$ and $\forall x, y \in D$, $[x, y] \subset \operatorname{cl}([x, y] \cap A)$, where $[x, y] = \{z \in X : z = x + t(y - x), t \in [0, 1]\}$.

Remark 1.19. László and Viorel in [12] introduced self-segment-denseness to study the existence of solutions for equilibrium problems. Clearly, a self-segment-dense set is equivalent to the notion of a dense set in one dimension. However, a dense set does not imply the concept of self-segment-dense (see [12, Examples 2.1 and 2.2]).

Lemma 1.20 ([12, Lemma 3.1]). *Let A be a nonempty convex subset in X, and let $D \subset A$ be a self-segment-dense set in A. If for every finite subset $\{x_1, x_2, \ldots, x_n\} \subset D$, then*

$$\operatorname{cl}\big(\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \cap D\big) = \operatorname{conv}\{x_1, x_2, \ldots, x_n\}.$$

2. Ky Fan minimax inequalities for set-valued mappings

Theorem 2.1. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment dense set, and let C be a strongly minihedral cone in V. Let $F : X_0 \times X_0 \to 2^V$ be a set-valued mapping satisfying the following conditions.*

(i) *For each $x \in D$, $y \to F(x, y)$ is lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x, y)$ is properly cone-quasiconcave on D.*
(iii) *For each $y \in X_0$, $x \to F(x, y)$ is lower semicontinuous on $X_0 \setminus D$.*
(iv) *$F(x, y)$ is upper bounded for every $(x, y) \in X_0 \times X_0$.*

Then there exists a point $y_0 \in X_0$ such that

$$\sup_{x \in X_0} F(x, y_0) \leq \sup_{x \in X_0} F(x, x).$$

Proof. By assumption (iv) and the fact that C is a strongly minihedral cone, we have

$$\sup_{x \in X_0} F(x, y) \neq \emptyset, \quad \forall y \in X_0; \qquad \sup_{x \in X_0} F(x, x) \neq \emptyset.$$

Let $\beta = \sup_{x \in X_0} F(x, x)$. Define a set-valued map $G : D \to 2^{X_0}$ by

$$G(x) = \big\{y \in X_0 : F(x, y) \subset \beta - C\big\},$$

for all $x \in D$.

First, $G(x) \neq \emptyset$, since $x \in G(x)$. Second, we show that $G(x)$ is closed for each $x \in D$. Indeed, fix $x \in D$, and let any net $\{y_\alpha\} \subset G(x)$ be such that $y_\alpha \to \widehat{y}$. By the lower semicontinuity of $F(x, \cdot)$, for any $z_0 \in F(x, \widehat{y})$, there exists $z_\alpha \in F(x, y_\alpha)$ such that $z_\alpha \to z_0$. By the closeness of C, we have $z_0 \in \beta - C$. It follows from the arbitrary of z_0 that $\widehat{y} \in G(x)$. Hence, $G(x) \subset X_0$ is a closed for every $x \in D$. By the compactness of X_0, we have that $G(x)$ is a compact subset for every $x \in D$.

Now, we prove that $G(x)$ is a KKM map on D. Indeed, for every finite subset $\{x_1, x_2, \dots, x_n\} \subset D$, one has

$$\operatorname{conv}\{x_1, x_2, \dots, x_n\} \cap D \subset \bigcup_{i=1}^{n} G(x_i).$$

Suppose to the contrary that there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of D such that $\operatorname{conv}\{x_1, x_2, \dots, x_n\} \cap D \not\subset \bigcup_{i=1}^{n} G(x_i)$. Then there exist $\alpha_i \geq 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^{n} \alpha_i = 1$ such that $\overline{y} = \sum_{i=1}^{n} \alpha_i x_i \in D$ and

$$F(x_i, \overline{y}) \not\subset \beta - C, \quad i = 1, 2, \dots, n.$$

By assumption (ii), we have $F(x_{i_0}, \overline{y}) \subset F(\overline{y}, \overline{y}) - C$ for some $i_0 \in \{1, 2, \dots, n\}$. Thus, $F(\overline{y}, \overline{y}) \not\subset \beta - C$, which is a contradiction. Hence,

$$\operatorname{conv}\{x_1, x_2, \dots, x_n\} \cap D \subset \bigcup_{i=1}^{n} G(x_i),$$

and so

$$\operatorname{cl}\big(\operatorname{conv}\{x_1, x_2, \dots, x_n\}\big) \cap D \subset \operatorname{cl}\Big(\bigcup_{i=1}^{n} G(x_i)\Big).$$

By Lemma 1.20 and the fact that $G(x_i)$ is closed, we get

$$\operatorname{conv}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i).$$

So, G is a KKM map. By applying Lemma 1.17, we get

$$\bigcap_{x \in D} G(x) \neq \emptyset.$$

It follows from $y_0 \in \bigcap_{x \in D} G(x)$ that $D \subset \{x \in X_0 : F(x, y_0) \subset \beta - C\}$. At this point, let $x \in X_0 \setminus D$; since D is dense in X_0, we have

$$X_0 \subset \operatorname{cl} D \subset \operatorname{cl}\big\{x \in X_0 : F(x, y_0) \subset \beta - C\big\}.$$

Then $x \in \operatorname{cl}\{x \in X_0 : F(x, y_0) \subset \beta - C\} \cap (X_0 \setminus D)$. By assumption (iii) and Proposition 1.3, we obtain $x \in \{x \in X_0 : F(x, y_0) \subset \beta - C\} \cap (X_0 \setminus D)$ and so $\sup_{x \in X_0} F(x, y_0) \leq \sup_{x \in X_0} F(x, x)$. □

Remark 2.2. (a) The conditions of Theorem 2.1 are imposed on a self-segment-dense set, but not on the whole domain of mapping. Meanwhile, Theorem 2.1 does not assume that X_0 is a convex set, and our results are also different from those in [17]–[19], and [20].

(b) If F is a real-valued function, then Theorem 2.1 reduces to the classical Ky Fan minimax inequality.

When F is a vector-valued mapping, we get the following result.

Corollary 2.3. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let C be a strongly minihedral cone in V. Let $F : X_0 \times X_0 \to V$ be a vector-valued function satisfying the following conditions:*

(i) *For each $x \in D$, $y \to F(x,y)$ is cone-lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x,y)$ is properly cone-quasiconcave on D.*
(iii) *For each $y \in X_0$, $x \to F(x,y)$ is cone-lower semicontinuous on $X_0 \setminus D$.*
(iv) *$F(x,y)$ is upper bounded for every $(x,y) \in X_0 \times X_0$.*

Then there exists a point $y_0 \in X_0$ such that

$$\sup_{x\in X_0} F(x,y_0) \leq \sup_{x\in X_0} F(x,x).$$

Theorem 2.4. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let C be a strongly minihedral cone in V. Let $F : X_0 \times X_0 \to 2^V$ be a set-valued mapping satisfying the following conditions:*

(i) *For each $x \in D$, $y \to F(x,y)$ is lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x,y)$ is properly cone-quasiconcave on D.*
(iii) *$F(x,y)$ is upper bounded for every $(x,y) \in X_0 \times X_0$.*

Then

$$\operatorname{Min} \bigcup_{y\in X_0} \sup_{x\in D} F(x,y) \subset \sup_{x\in X_0} F(x,x) + V\backslash\big(C\backslash\{0\}\big).$$

Proof. By Lemma 1.14 and Remark 1.15, we have

$$\operatorname{Min} \bigcup_{y\in X_0} \sup_{x\in D} F(x,y) \neq \emptyset.$$

Similar to the proof of Theorem 2.1, there exists a point $y_0 \in X_0$ such that

$$\sup_{x\in D} F(x,y_0) \leq \sup_{x\in X_0} F(x,x).$$

Thus, we have

$$z \notin \sup_{x\in X_0} F(x,x) + C\backslash\{0\}, \quad \forall z \in \operatorname{Min} \bigcup_{y\in X_0} \sup_{x\in D} F(x,y).$$

□

When F is a vector-valued mapping, we get the following result.

Corollary 2.5. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let C be a strongly minihedral cone in V. Suppose that $F : X_0 \times X_0 \to V$ is a vector-valued function satisfying the following conditions:*

(i) *For each $x \in D$, $y \to F(x,y)$ is a cone-lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x,y)$ is properly cone-quasiconcave on D.*
(iii) *$F(x,y)$ is upper bounded for every $(x,y) \in X_0 \times X_0$.*

Then

$$\operatorname{Min} \bigcup_{y\in X_0} \sup_{x\in D} F(x,y) \subset \sup_{x\in X_0} F(x,x) + V\backslash\big(C\backslash\{0\}\big).$$

In what follows, we obtain the solution of a set-valued equilibrium problem.

Theorem 2.6. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let $C \subset V$ be a pointed closed convex cone with* $\operatorname{int} C \neq \emptyset$. *Let $F : X_0 \times X_0 \to 2^V$ be a set-valued mapping satisfying the following conditions:*

(i) *For each $x \in D$, $y \to F(x, y)$ is lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x, y)$ is properly cone-quasiconcave on D.*
(iii) *We have $F(x, x) \subset V \setminus \operatorname{int} C$, $\forall x \in D$.*
(iv) *For each $y \in X_0$, $x \to F(x, y)$ is lower semicontinuous on $X_0 \setminus D$.*

Then there exists a point $y_0 \in X_0$ such that

$$F(x, y_0) \subset V \setminus \operatorname{int} C, \quad \forall x \in X_0.$$

Proof. Define a set-valued map $T : D \to 2^{X_0}$ by

$$T(x) = \big\{y \in X_0 : F(x, y) \subset V \setminus \operatorname{int} C\big\},$$

for all $x \in D$.

First, by condition (iii), we get $T(x) \neq \emptyset$. Second, we show that $T(x)$ is closed for each $x \in D$. Indeed, for each $x \in D$, let any net $\{y_\alpha\} \subset T(x)$ be such that $y_\alpha \to \widehat{y}$. Since $F(x, \cdot)$ is lower semicontinuous, for any $z_0 \in F(x, \widehat{y})$, there exists $z_\alpha \in F(x, y_\alpha)$ such that $z_\alpha \to z_0$. It follows from $z_\alpha \in V \setminus \operatorname{int} C$ and the closeness of $V \setminus \operatorname{int} C$ that $z_0 \in V \setminus \operatorname{int} C$. By the arbitrary of z_0, we get $\widehat{y} \in T(x)$. Thus, $T(x)$ is a closed set for every $x \in D$. By the compactness of X_0, we obtain that $T(x) \subset X_0$ is a compact subset for all $x \in D$.

Now, we prove that $T(x)$ is a KKM map. Indeed, for every finite subset $\{x_1, x_2, \ldots, x_n\} \subset D$, one has

$$\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \cap D \subset \bigcup_{i=1}^{n} T(x_i).$$

Suppose to the contrary that there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of D such that $\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \cap D \not\subset \bigcup_{i=1}^{n} T(x_i)$. Then there exist $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$ such that $\overline{y} = \sum_{i=1}^{n} \alpha_i x_i \in D$ and

$$F(x_i, \overline{y}) \cap \operatorname{int} C \neq \emptyset, \quad i = 1, 2, \ldots, n.$$

By assumption (ii), we have $F(x_{i_0}, \overline{y}) \subset F(\overline{y}, \overline{y}) - C$ for some $i_0 \in \{1, 2, \ldots, n\}$. Thus, there exist $\eta_{i_0} \in F(x_{i_0}, \overline{y}) \cap \operatorname{int} C$ and $\eta \in F(\overline{y}, \overline{y})$ such that

$$\eta_{i_0} \in \eta - C.$$

So $\eta \in \eta_{i_0} + C \subset \operatorname{int} C$, which contradicts condition (iii). Hence,

$$\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \cap D \subset \bigcup_{i=1}^{n} T(x_i).$$

We have

$$\operatorname{cl}\big(\operatorname{conv}\{x_1, x_2, \ldots, x_n\}\big) \cap D \subset \operatorname{cl}\Big(\bigcup_{i=1}^{n} T(x_i)\Big).$$

By Lemma 1.20 and the fact that $T(x_i)$ is closed, we get

$$\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i).$$

Thus, T is a KKM map. It follows from Lemma 1.17 that $\bigcap_{x\in D} T(x) \neq \emptyset$. Let $y_0 \in \bigcap_{x\in D} T(x)$. Then $D \subset \{x \in X_0 : F(x, y_0) \subset V \setminus \operatorname{int} C\}$. At this point, let $x \in X_0 \setminus D$. Since D is dense in X_0, we have

$$X_0 \subset \operatorname{cl} D \subset \operatorname{cl}\big\{x \in X_0 : F(x, y_0) \subset V \setminus \operatorname{int} C\big\}.$$

Then $x \in \operatorname{cl}\{x \in X_0 : F(x, y_0) \subset V \setminus \operatorname{int} C\} \cap (X_0 \setminus D)$. By assumption (iv) and Proposition 1.3, we then obtain $x \in \{x \in X_0 : F(x, y_0) \subset V \setminus \operatorname{int} C\} \cap (X_0 \setminus D)$, so $F(x, y_0) \subset V \setminus \operatorname{int} C$ for all $x \in X_0$. □

Remark 2.7. (a) The conclusion of Theorem 2.6 is true if condition (ii) is replaced by "for each $y \in D$, $x \to F(x, y)$ is cone-quasiconcave on D." In fact, similar to the proof of Theorem 2.6, we only prove that for every finite subset $\{x_1, x_2, \ldots, x_n\} \subset D$, one has $\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \cap D \subset \bigcup_{i=1}^{n} T(x_i)$. Suppose to the contrary that there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of D such that $\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \cap D \not\subset \bigcup_{i=1}^{n} T(x_i)$. Then there exist $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$ such that $\overline{y} = \sum_{i=1}^{n} \alpha_i x_i \in D$ and $F(x_i, \overline{y}) \cap \operatorname{int} C \neq \emptyset$, $i = 1, 2, \ldots, n$. By the above assumption, we have $\sum_{i=1}^{n} \alpha_i F(x_i, \overline{y}) \subset F(\overline{y}, \overline{y}) - C$. Thus, there exist $\eta_i \in F(x_i, \overline{y}) \cap \operatorname{int} C$ for all $i \in \{1, 2, 3 \ldots, n\}$ and $\eta \in F(\overline{y}, \overline{y})$ such that

$$\sum_{i=1}^{n} \alpha_i \eta_i \in \eta - C.$$

So $\eta \in \sum_{i=1}^{n} \alpha_i \eta_i + C \subset \operatorname{int} C$, which contradicts condition (iii).

(b) We replace condition (ii) by "$x \to F(x, y)$ is convex on D." By Definition 1.5 and Remark 2.7(a), Theorem 3.3 is also true. If F is a real-valued mapping, then Theorem 2.6 reduces to [12, Theorem 3.2].

By arguments similar to those of Theorem 2.6 and Remark 2.7, we have the followings results.

Corollary 2.8. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let $C \subset V$ be a pointed closed convex cone with $\operatorname{int} C \neq \emptyset$. Let $F : X_0 \times X_0 \to 2^V$ be a set-valued mapping satisfying the following conditions.*

(i) *For each $x \in D$, $y \to F(x, y)$ is lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x, y)$ is convex on D.*
(iii) *We have $F(x, x) \subset V \setminus -\operatorname{int} C$, $\forall x \in D$.*
(iv) *For each $y \in X_0$, $x \to F(x, y)$ is lower semicontinuous on $X_0 \setminus D$.*

Then there exists a point $y_0 \in X_0$ such that

$$F(x, y_0) \subset V \setminus -\operatorname{int} C, \quad \forall x \in X_0.$$

Remark 2.9. Theorem 3.1 of [12] is a special case of Corollary 2.8.

Theorem 2.10. *Let X_0 be a nonempty closed convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let $C \subset V$ be a pointed closed convex cone with* $\operatorname{int} C \neq \emptyset$. *Let $F : X_0 \times X_0 \to 2^V$ be a set-valued mapping satisfying the following conditions.*

(i) *For each $x \in D$, $y \to F(x, y)$ is lower semicontinuous on X_0.*
(ii) *For each $y \in D$, $x \to F(x, y)$ is properly cone-quasiconcave on D.*
(iii) *We have $F(x, x) \subset V \setminus -\operatorname{int} C$, $\forall x \in D$.*
(iv) *For each $y \in X_0$, $x \to F(x, y)$ is lower semicontinuous on $X_0 \setminus D$.*
(v) *There exist a compact subset $K \subset X$ and $x_0 \in D$ such that $F(x_0, y) \subset -\operatorname{int} C$, for all $y \in X_0 \setminus K$.*

Then there exists a point $y_0 \in X_0$ such that

$$F(x, y_0) \subset V \setminus -\operatorname{int} C, \quad \forall x \in X_0.$$

Proof. Define a set-valued map $T : D \to 2^{X_0}$ by

$$T(x) = \{y \in X_0 : F(x, y) \subset V \setminus -\operatorname{int} C\},$$

for all $x \in D$. Similar to the proof of Theorem 2.6, we have that $T(x)$ is a closed-valued KKM map on D. Now, we show that $T(x_0)$ is compact. It is sufficient to prove that $T(x_0) \subset K$. Assume that $T(x_0) \not\subset K$. There exists a point $\overline{y} \in T(x_0)$, $\overline{y} \in X_0 \setminus K$ such that $F(x_0, \overline{y}) \subset V \setminus -\operatorname{int} C$, which contradicts condition (v). So, $T(x_0)$ is a compact subset of K. Again, by an argument similar to that of Theorem 2.6, the rest of the proof is completed. □

Remark 2.11. Theorem 2.10 does not assume the compactness of the domain X_0. If F is a vector-valued mapping, then Theorem 2.10 reduces to [11, Theorem 3.1].

3. Applications

In this section, we use the results in Section 2 to obtain the solutions of a generalized set-valued vector variational inequality. Let $L(X, V)$ denote the set of all continuous linear operators from X into V. For $x \in X$ and a subset E of $L(X, V)$, we put $\langle E, x\rangle = \{\langle f, x\rangle : f \in E\}$.

Theorem 3.1. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let C be a strongly minihedral cone in V. Suppose that a set-valued mapping $T : X_0 \times X_0 \to 2^{L(X,V)}$ satisfies the following conditions.*

(i) *For each $y \in D$, $x \to \langle Tx, x - y\rangle$ is properly cone-quasiconcave on D.*
(ii) *For each $y \in X_0$, $x \to \langle Tx, x - y\rangle$ is lower semicontinuous on $X_0 \setminus D$.*
(iii) *$\langle Tx, x - y\rangle$ is upper bounded on $X_0 \times X_0$.*

Then there exists a point $y_0 \in X_0$ such that

$$\langle Tx, x - y_0\rangle \subset -C, \quad \forall x \in X_0.$$

Proof. For each $(x, y) \in X_0 \times X_0$, let $F(x, y) = \langle Tx, x - y\rangle$. It is easy to see that for $x \in D$, $y \to F(x, y)$ is lower semicontinuous on X_0. Obviously, we get $F(x, x) = \{0\}$. All conditions of Theorem 2.1 are satisfied. Then there exists a point $y_0 \in X_0$ such that

$$\sup_{x\in X_0} F(x, y_0) \leq \sup_{x\in X_0} F(x, x) = 0.$$

Thus, $\langle Tx, x - y_0\rangle \subset -C$, $\forall x \in X_0$. □

By an argument similar to that of Theorem 3.1, we get the following result.

Corollary 3.2. *Let X_0 be a nonempty compact convex subset of X, let $D \subset X_0$ be a self-segment-dense set, and let C be a strongly minihedral cone in V. Suppose that a vector-valued function $T : X_0 \times X_0 \to L(X, V)$ satisfies the following conditions.*

(i) *For each $y \in D$, $x \to \langle Tx, x - y\rangle$ is properly cone-quasiconcave on D.*
(ii) *For each $y \in X_0$, $x \to \langle Tx, x - y\rangle$ is lower semicontinuous on $X_0 \setminus D$.*
(iii) *$\langle Tx, x - y\rangle$ is upper bounded on $X_0 \times X_0$.*

Then there exists a point $y_0 \in X_0$ such that

$$\langle Tx, x - y_0\rangle \leq 0, \quad \forall x \in X_0.$$

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References

1. B. Alleche and V. D. Rădulescu, *Set-valued equilibrium problems with applications to Browder variational inclusions and to fixed point theory*, Nonlinear Anal. Real World Appl. **28** (2016), 251–268. Zbl 1327.49029. MR3422823. DOI 10.1016/j.nonrwa.2015.10.002. 497, 498
2. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York, 1984. Zbl 0641.47066. MR0749753. 496, 497
3. G. Y. Chen and S. J. Li, *Existence of solutions for a generalized vector quasivariational inequality*, J. Optim. Theory Appl. **90** (1996), no. 2, 321–334. Zbl 0896.49005. MR1399437. DOI 10.1007/BF02190001. 497
4. L. J. Chu, *On Fan's minimax inequality*, J. Math. Anal. Appl. **201** (1996), no. 1, 103–113. Zbl 0849.49008. MR1397889. DOI 10.1006/jmaa.1996.0244. 496
5. H. W. Corley, *An existence result for maximizations with respect to cones*, J. Optim. Theory Appl. **31** (1980), no. 2, 277–281. Zbl 0416.90068. MR0600387. DOI 10.1007/BF00934115. 500
6. M. R. Delavar, S. A Mohammadzadeh, and M. Roohi, *Fan's minimax inequality in MG-convex spaces*, Nonlinear Funct. Anal. Appl. **16** (2011), no. 2, 201–210. Zbl 1258.49003. 496
7. K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1960/61), 305–310. Zbl 0093.36701. MR0131268. DOI 10.1007/BF01353421. 501
8. K. Fan, "A minimax inequality and applications" in *Inequalities III (Los Angeles, 1969)*, Academic Press, New York, 1972, 103–113. Zbl 0302.49019. MR0341029. 496
9. F. Ferro, *Minimax type theorem for n-valued functions*, Ann. Mat. Pura Appl. (4) **132** (1982), no. 1, 113–130. Zbl 0508.49010. MR0696040. DOI 10.1007/BF01760978. 496
10. F. Ferro, *A minimax theorem for vector-valued functions*, J. Optim. Theory Appl. **60** (1989), no. 1, 19–31. Zbl 0631.90077. MR0981942. DOI 10.1007/BF00938796. 496
11. S. László, *Vector equilibrium problems on dense sets*, J. Optim. Theory Appl. **170** (2016), no. 2, 437–457. Zbl 1351.47042. MR3527704. DOI 10.1007/s10957-016-0915-0. 497, 506

12. S. László and A. Viorel, *Densely defined equilibrium problems*, J. Optim. Theory Appl. **166** (2015), no. 1, 52–75. Zbl 1321.47132. MR3366104. DOI 10.1007/s10957-014-0702-8. 497, 498, 501, 505
13. S. J. Li, G. Y. Chen, and G. M. Lee, *Minimax theorems for set-valued mappings*, J. Optim. Theory Appl. **106** (2000), no. 1, 183–199. Zbl 1054.90086. MR1780114. DOI 10.1023/A:1004667309814. 497, 498, 499
14. Z. F. Li and S. Y. Wang, *A type of minimax inequality for vector-valued mappings*, J. Math. Anal. Appl. **227** (1998), no. 1, 68–80. Zbl 0913.90275. MR1652910. DOI 10.1006/jmaa.1998.6076. 496
15. M. S. Moslehian, *Ky Fan inequalities*, Linear Multilinear Algebra **60** (2012), no. 11–12, 1313–1325. Zbl 1252.26001. MR2989765. DOI 10.1080/03081087.2011.641545. 496
16. C. L. Zhang, *The minimax problem of two vector-valued functions*, Nonlinear Funct. Anal. Appl. **23** (2018), no. 1, 21–30. Zbl 1393.49003. 499
17. Y. Zhang and S. J. Li, *Ky Fan minimax inequalities for set-valued mappings*, Fixed Point Theory Appl. **2012**, no. 64. Zbl 1273.49012. MR2927695. DOI 10.1186/1687-1812-2012-64. 496, 497, 498, 499, 502
18. Y. Zhang and S. J. Li, *Minimax theorems for scalar set-valued mappings with nonconvex domains and applications*, J. Global Optim. **57** (2013), no. 4, 1359–1373. Zbl 1281.49023. MR3121799. DOI 10.1007/s10898-012-9992-2. 496, 497, 499, 502
19. Y. Zhang, S. J. Li, and M. H. Li, *Minimax inequalities for set-valued mappings*, Positivity **16** (2012), no. 4, 751–770. Zbl 1255.49014. MR2994082. DOI 10.1007/s11117-011-0144-6. 497, 499, 502
20. Y. Zhang, S. J. Li, and S. K. Zhu, *Minimax problems for set-valued mappings*, Numer. Funct. Anal. Optim. **33** (2012), no. 2, 239–253. Zbl 1237.49010. MR2876780. DOI 10.1080/01630563.2011.610915. 497, 499, 502

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