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ON THE CLASSIFICATION AND GEOMETRY FOR COWEN–DOUGLAS OPERATORS OF HIGH-RANK CASE

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Abstract. In this paper we introduce a subclass of Cowen–Douglas operators of high-rank case denoted by $\mathcal{FB}_{n,1}(\Omega)$. Each operator $T \in \mathcal{FB}_{n,1}(\Omega)$ is induced by one Cowen–Douglas operator with rank n, another Cowen–Douglas operator with rank 1, and an intertwining operator between them. By using this definition, we can construct plenty of Cowen–Douglas operators with high rank. By discussing the curvature of line bundle and second fundamental form of some rank 2 bundle and its subbundle, we give the unitary classification of operators in $\mathcal{FB}_{n,1}(\Omega)$ and we reduce the number of unitary invariants of this kind of operators from $(n+1)^2$ to two.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. Let $\mathrm{Gr}(n,\mathcal{H})$ denote an n-dimensional Grassmann manifold, the set of all n-dimensional subspaces of $\mathcal{H}$. If $\dim \mathcal{H} < +\infty$, then $\mathrm{Gr}(n,\mathcal{H})$ is a complex manifold. Let Ω be an open connected subset of $\mathbb{C}$. In [2], Cowen and Douglas introduced a class of operators denoted by $\mathcal{B}_n(\Omega)$ which contains a bounded, open set Ω as eigenvalues of constant multiplicity n. The class of Cowen–Douglas operator with rank n: $\mathcal{B}_n(\Omega)$ is defined as follows (see [2]):

$$\mathcal{B}_n(\Omega) := \Big\{ T \in \mathcal{L}(\mathcal{H}) :$$

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$$\left.\begin{array}{l}(1)\ \Omega \subset \sigma(T) := \{w \in \mathbb{C} : T - wI \text{ is not invertible}\},\\ (2)\ \bigvee_{w\in\Omega} \operatorname{Ker}(T-w) = \mathcal{H},\\ (3)\ \operatorname{Ran}(T-w) = \mathcal{H},\\ (4)\ \dim \operatorname{Ker}(T-w) = n, \forall w \in \Omega.\end{array}\right\}$$

It follows that $\pi : E_T \to \Omega$, where

$$E_T = \big\{\operatorname{Ker}(T-w) : w \in \Omega, \pi\big(\operatorname{Ker}(T-w)\big) = w\big\}$$

defines a Hermitian holomorphic vector bundle on Ω. In the present article, they make a rather detailed study of certain aspects of complex geometry as well as introduce the following concepts: let E be a Hermitian holomorphic vector bundle; then by following [2], a curvature function for E can be defined as

$$K(w) = -\frac{\partial}{\partial \overline{w}}\Big(h^{-1}\frac{\partial h}{\partial w}\Big), \quad \text{for all } w \in \Omega,$$

where the metric h is defined as

$$h(w) = \big(\big(\langle \gamma_j(w), \gamma_i(w)\rangle\big)\big)_{n\times n}, \quad \forall w \in \Omega,$$

where $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a frame of E over Ω. Covariant partial derivatives of curvature are defined as follows: let E be a Hermitian holomorphic vector bundle. For any C^∞-bundle map ϕ on E, and given frame σ of E, we have

(1) $\phi_{\overline{w}}(\sigma) = \frac{\partial}{\partial \overline{w}}(\phi(\sigma))$,
(2) $\phi_w(\sigma) = \frac{\partial}{\partial w}(\phi(\sigma)) + [h^{-1}\frac{\partial}{\partial w}h, \phi(\sigma)]$.

Since curvature can also be regarded as a bundle map, we can get covariant partial derivatives of curvature, denoted by $\mathcal{K}_{w^i\overline{w}^j}$, $i, j \in \mathbb{N} \cup \{0\}$, by using the inductive formulas above. The curvature $\mathcal{K}$ and its covariant partial derivatives $\mathcal{K}_{w^i\overline{w}^j}$ are the complete unitary invariants of Hermitian holomorphic vector bundle E (see [2]).

Theorem 1.1 (see [2, Proposition 2.18])**.** *Let T and S be two Cowen–Douglas operators and let E_T, E_S be two Hermitian holomorphic vector bundles induced by T and S. Then $E_T \sim_u E_S$ if and only if there exists an isometry $V : E_T \to E_S$ such that*

$$V\mathcal{K}_{T,w^i\overline{w}^j} = \mathcal{K}_{S,w^i\overline{w}^j}V, \quad \forall i, j = 0, 1, \ldots, n-1.$$

For any Cowen–Douglas operator T with rank larger than 1, the curvature $\mathcal{K}_T$ and the covariant partial derivatives of curvature $\mathcal{K}_{T,w^i\overline{w}^j}, 0 \le i, j \le n$, are not easy to compute. In the other words, it is hard to determine whether two Cowen–Douglas operators with higher rank are unitarily equivalent by calculating their curvatures and the covariant partial derivatives of curvatures. So it is natural to isolate a subset of Cowen–Douglas operators for which a complete set of tractable unitary invariants is relatively easy to identify. In [20], Misra and Roy gave some examples to show that some covariant partial derivatives of the curvature are not needed to determine when two Cowen–Douglas operators with high rank are unitarily equivalent. In that article, Misra and Roy also showed that all the covariant partial derivatives of curvature are not needed to give the complete unitary invariants of Cowen–Douglas operators with higher rank. For example, they constructed a certain class of homogeneous operators in the Cowen–Douglas class; the curvature $\mathcal{K}_T$ and its covariant partial derivatives $\mathcal{K}_{T,\overline{w}}$ provide complete unitary invariants. So it is natural and also necessary to reduce the number of the unitary invariants for the Cowen–Douglas operators of high-rank case.

In [7] and [8], Jiang, Ji, and Misra introduced a class of Cowen–Douglas operators with rank 2, which was denoted by $\mathcal{FB}_2(\Omega)$. The numbers of unitary invariants of this class Cowen–Douglas operators are reduced from four to two. In this note, we introduce a Cowen–Douglas operator class with rank $n+1$ ($n \in \mathbb{N}$), denoted by $\mathcal{FB}_{n,1}(\Omega)$ which are induced by a Cowen–Douglas operator with rank n and a Cowen–Douglas operator with rank 1. Note that the operator class $\mathcal{FB}_{n,1}(\Omega)$ is equal to $\mathcal{FB}_2(\Omega)$ when $n = 1$. In the main theorem, we reduced the numbers of unitary invariants of operators in $\mathcal{F}B_{n,1}(\Omega)$ from $(n+1)^2$ to 2.

2. The operator class $\mathcal{FB}_{n,1}(\Omega)$

Definition 2.1. Let Ω be a connected open set of the complex plane and let n be a positive integer. We let $\mathcal{FB}_{n,1}(\Omega)$ denote the set of all operators T defined on a Hilbert space $\mathcal{H}$ which admits a decomposition of the form

$$T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1,$$

where $T_0 \in \mathcal{B}_n(\Omega)$, $T_1 \in \mathcal{B}_1(\Omega)$, and S is a nonzero intertwining operator between T_0 and T_1 with dense range (i.e., $T_0S = ST_1$, $\overline{\operatorname{Ran}}(S) = \mathcal{H}_0$).

First of all, we can see that any operator $T \in \mathcal{FB}_{n,1}(\Omega)$ is a Cowen–Douglas operator with rank $n+1$. In fact, from Fredholm operator index theory, it follows that the operator T is Fredholm and $\operatorname{ind}(T) = \operatorname{ind}(T_0) + \operatorname{ind}(T_1)$ (cf. [1, page 360]). Therefore, we only need to prove that the vectors in the kernel $\ker(T - w)$, $w \in \Omega$, span the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Now, let $\{\gamma_1, \ldots, \gamma_{n-1}\}$ and t_1 be holomorphic frame for the bundles E_{T_0} and E_{T_1} corresponding to the operators T_0 and T_1, respectively. For each $w \in \Omega$, the operator $T_0 - w$ is surjective. Therefore, we can find a vector $\alpha(w)$ in $\mathcal{H}_0$ such that $(T_0 - w)\alpha(w) = -S(t_1(w))$, $w \in \Omega$. Setting $\gamma_n(w) = \alpha(w) + t_1(w)$, we see that

$$(T - w)\gamma_n(w) = 0 = (T - w)\gamma_1(w) = \cdots = (T - w)\gamma_{n-1}(w).$$

Thus $\{\gamma_1(w), \ldots, \gamma_n(w)\} \subseteq \ker(T - w)$ for w in Ω. If x is any vector orthogonal to $\ker(T - w)$, $w \in \Omega$, then in particular it is orthogonal to the vectors $\{\gamma_1(w), \ldots, \gamma_{n-1}(w)\}$ and $\gamma_n(w)$, $w \in \Omega$, forcing it to be the zero vector.

An operator $T \in \mathcal{FB}_{n,1}(\Omega)$ can be realized as the adjoint of the multiplication operator on a reproducing kernel Hilbert space of holomorphic $\mathbb{C}^2$-valued functions on Ω^*. Let t_1 be a non-zero holomorphic section of E_{T_1} and $S(t_1)$ be a section of E_{T_0} such that

$$\overline{\operatorname{Span}\{S(t_1(w)) : w \in \Omega\}} = \mathcal{H}_0.$$

Set $\gamma_0(w) := S(t_1(w)), \gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$. Clearly, $(T-w)(\gamma_0(w)) = (T-w)(\gamma_1(w)) = 0$.

Define the map $\Gamma_\gamma : \mathcal{H} \to \mathcal{O}(\Omega^*, \mathbb{C}^2)$ as

$$\Gamma_\gamma(x)(z) = \begin{pmatrix} \langle x, \gamma_0(\bar z)\rangle \\ \langle x, \gamma_1(\bar z)\rangle \end{pmatrix}, \quad z \in \Omega^*, x \in \mathcal{H},$$

where $\mathcal{O}(\Omega^*, \mathbb{C}^2)$ is the space of holomorphic $\mathbb{C}^2$-valued functions defined on Ω^*. It is easy to see that the map Γ_γ is an injective map. We transfer the inner product from $\mathcal{H}$ on the range of Γ_γ. Thus, the map Γ_γ is a unitary map from $\mathcal{H}$ onto $\mathcal{H}_\gamma := \operatorname{ran}\Gamma_\gamma$. Define K_γ to be the function on $\Omega^* \times \Omega^*$ taking values in the 2×2 matrices $\mathcal{M}_2(\mathbb{C})$:

$$\begin{aligned} K_\gamma(z,w) &= \big(\big(\big\langle \gamma_j(\bar w), \gamma_i(\bar z)\big\rangle\big)\big)_{i,j=0}^{1} \\ &= \begin{pmatrix} \langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle & \frac{\partial}{\partial \bar w}\langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle \\ \frac{\partial}{\partial z}\langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle & \frac{\partial^2}{\partial z\,\partial \bar w}\langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle + \langle t_1(\bar w), t_1(\bar z)\rangle \end{pmatrix} \\ &= \begin{pmatrix} K_0(z,w) & \frac{\partial}{\partial \bar w}K_0(z,w) \\ \frac{\partial}{\partial z}K_0(z,w) & \frac{\partial^2}{\partial z\,\partial \bar w}K_0(z,w) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K_1(z,w) \end{pmatrix}, \end{aligned}$$

where $K_0(z,w) = \langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle$ and $K_1(z,w) = \langle t_1(\bar w), t_1(\bar z)\rangle$ for $z, w \in \Omega^*$. Set $(K_\gamma)_w(\cdot) = K_\gamma(\cdot, w)$; here K has the reproducing property, namely,

$$\begin{aligned} \big\langle \Gamma_\gamma(x)(\cdot), (K_\gamma)_w(\cdot)\eta\big\rangle_{\operatorname{ran}\Gamma_\gamma} &= \Big\langle \Gamma_\gamma(x)(\cdot), \sum_{i=0}^{1} \Gamma_\gamma\big(\gamma_i(\bar w)\big)(\cdot)\eta_i\Big\rangle_{\operatorname{ran}\Gamma_\gamma} \\ &= \sum_{i=0}^{1} \bar\eta_i \big\langle \Gamma_\gamma(x)(\cdot), \Gamma_\gamma\big(\gamma_i(\bar w)\big)(\cdot)\big\rangle_{\operatorname{ran}\Gamma_\gamma} \\ &= \sum_{i=0}^{1} \big\langle x, \gamma_i(\bar w)\big\rangle_{\mathcal{H}} \bar\eta_i \\ &= \big\langle \Gamma_\gamma(x)(w), \eta\big\rangle_{\mathbb{C}^2}, \quad x \in \mathcal{H}, \eta \in \mathbb{C}^2, w \in \Omega^*. \end{aligned}$$

Consider,

$$\begin{aligned} \Gamma_\gamma(T^*x)(w) &= \begin{pmatrix} \langle T^*x, \gamma_0(\bar w)\rangle \\ \langle T^*x, \gamma_1(\bar w)\end{pmatrix} \\ &= \begin{pmatrix} \langle x, T\gamma_0(\bar w)\rangle \\ \langle x, T\gamma_1(\bar w)\rangle \end{pmatrix} \\ &= w \begin{pmatrix} \langle x, \gamma_0(\bar w)\rangle \\ \langle x, \gamma_1(\bar w)\rangle \end{pmatrix} \\ &= w\Gamma_\gamma(x)(w) \\ &= \big(M_z\big(\Gamma_\gamma(x)\big)\big)(w). \end{aligned}$$

Hence,

$$\Gamma_\gamma T^* = M_z \Gamma_\gamma.$$

2.1. Spanning holomorphic cross-sections/second fundamental forms. In [22], Zhu defined the spanning holomorphic cross-section for a Hermitian holomorphic vector bundle corresponding to the Cowen–Douglas operator. Let $T \in \mathcal{B}_n(\Omega)$, and let E_T be a Hermitian holomorphic vector over Ω. A holomorphic section of vector bundle E_T is a holomorphic function $\gamma : \Omega \to \mathcal{H}$ such that, for each $w \in \Omega$, the vector $\gamma(w)$ belongs to the fiber of E_T over w. Here γ is called a *spanning holomorphic section* for E_T if $\overline{\text{Span}}\{\gamma(w) : w \in \Omega\} = \mathcal{H}$. In [22], it was proved that, for any Cowen–Douglas operator $T \in \mathcal{B}_n(\Omega)$, the Hermitian holomorphic vector bundle E_T possesses a spanning holomorphic cross-section. Suppose that T and $\widetilde{T}$ belongs to $\mathcal{B}_n(\Omega)$, then T and $\widetilde{T}$ are unitarily equivalent if and only if there exist spanning holomorphic cross-sections γ_T and $\gamma_{\widetilde{T}}$ for E_T and E_S, respectively, such that $\gamma_T \sim_u \gamma_{\widetilde{T}}$.

Let T_0 and T_1 be bounded linear operators defines on a Hilbert space $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively. Let $T_0 \in \mathcal{B}_n(\Omega), T_1 \in \mathcal{B}_1(\Omega)$ and $S : \mathcal{H}_1 \to \mathcal{H}_0$ be a non-zero bounded linear operator such that $T_0 S = S T_1$. If t_1 be a frame of the vector bundle E_{T_1}, then S has dense range if and only if $S(t_1)$ is a spanning section of E_{T_0}. Thus, two natural Hermitian holomorphic bundles induced by T_1 and S is defined as follows.

Definition 2.2. Suppose that $T = \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right) \in \mathcal{FB}_{n,1}(\Omega)$ and that $t_1(w) \in \text{Ker}(T_1 - w)$. Then the line bundle induced by S, denoted as E_0^S, is defined as

$$E_0^S(\omega) = \bigvee \big\{ S(t_1)(w) \big\}, \quad \forall \omega \in \Omega,$$

and the Hermitian holomorphic bundle with rank 2 induced by S, denoted as E^S, is defined as

$$E^S(\omega) = \bigvee \big\{ S(t_1)(w), \big(S(t_1)(w)\big)' - t_1(w) \big\}, \quad \forall \omega \in \Omega.$$

If we choose another frame $\widetilde{t}_1$ of E_{T_1} with $\widetilde{t}_1(w) = \phi(w) t_1(w)$ for some holomorphic function ϕ on Ω, then we obtain

$$\big(S(\widetilde{t}_1)(w), \big(S(\widetilde{t}_1)(w)\big)' - \widetilde{t}_1(w)\big) = \big(S(t_1)(w), \big(S(t_1)(w)\big)' - t_1(w)\big) \begin{pmatrix} \phi(w) & \phi'(w) \\ 0 & \phi(w) \end{pmatrix}.$$

In the following, we introduce the concept of second fundamental form for the Hermitian holomorphic vector bundle with rank 2. Let $\{\gamma_0, \gamma_1\}$ be the frames of a rank 2 bundle E, and let E_0 be the subbundle of E with the frame $\{\gamma_0\}$. Now set $h(w) = \langle \gamma_0(w), \gamma_0(w) \rangle$. We obtain an orthogonal frame, say $\{e_0(w), e_1(w)\}$, from the holomorphic frame by the usual Gram–Schmidt process:

$$e_0 = h^{-1/2} \gamma_0,$$

and

$$\begin{aligned} e_1 &= \frac{\gamma_1 - \frac{\gamma_0 \langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2}}{\|\gamma_1 - \frac{\gamma_0 \langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2}\|} \\ &= \frac{\gamma_1 - \frac{\gamma_0 \langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2}}{(\|\gamma_1\|^2 - \frac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2})^{1/2}}. \end{aligned}$$

Let D denote the canonical Hermitian connection for E_T which preserves both the Hermitian and holomorphic structures. Following the method given in [3, page 2244], the second fundamental form θ of E_0 in E is defined as

$$\theta := \langle De_1, e_0 \rangle.$$

Applying this definition to our case, we obtain the second fundamental form of $E_0^S \subset E^S$ denoted by θ_S:

$$\theta_S = \frac{\mathcal{K}_{E_0^S}}{(-\mathcal{K}_{E_0^S} + \frac{\|t_1\|^2}{\|S(t_1)\|^2})^{1/2}}.$$

By a direct computation, we can see that the second fundamental form does not depend on the choice of t_1, which is the frame of E_{T_1}.

Before we give the proof of our main theorem, we need the following concepts and lemmas: let $T, S \in \mathcal{L}(\mathcal{H})$. The Rosenblum operator introduced in [21] is defined as $\sigma_{T,S} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ as $\sigma_{T,S}(X) = TX - XS, \forall X \in \mathcal{L}(\mathcal{H})$. Furthermore, when $T = S$, $\tau_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is defined as $\tau_T(X) = TX - XT, \forall X \in \mathcal{L}(\mathcal{H})$.

Recall that operator T defined on a Hilbert space $\mathcal{H}$ is considered quasinilpotent if the spectrum of T is just $\{0\}$ (i.e., $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$).

Lemma 2.3 ([6, page 232])**.** *Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that $A \in \operatorname{Ran} \sigma_T$ and that $AT = TA$. Then A is a quasinilpotent.*

Lemma 2.4. *Let $T, \tilde{T} \in \mathcal{B}_n(\Omega)$ and let X be a bounded operator such that $XT = \tilde{T}X$. If X is a injective map, then the range of X is dense.*

Proof. Since X is injective, it maps a frame $\{e_i : 1 \leq n\}$ of E_T to a frame $\{X(e_i) : 1 \leq n\}$ of $E_{\tilde{T}}$, and hence spans closer to $\{X(e_1), \ldots, X(e_n)\}$ and is whole Hilbert space $\widetilde{\mathcal{H}}$. It follows that X has a dense range. □

By a similar proof, we also have the following lemma.

Lemma 2.5. *Let $T \in \mathcal{B}_n(\Omega)$, let $\tilde{T} \in \mathcal{B}_1(\Omega)$, and let Y be a operator such that $YT = \tilde{T}Y$. If Y is nonzero, then the range of Y is dense.*

2.2. Main theorem.

Theorem 2.6. *Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ be two Cowen–Douglas operators which belong to $\mathcal{FB}_{n,1}(\Omega)$. Then the operators T and $\tilde{T}$ are unitarily equivalent if and only if*

$$\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}, \qquad \theta_S = \theta_{\widetilde{S}},$$

where θ_S and $\theta_{\widetilde{S}}$ are the second fundamental forms of $E_0^S (\subset E_S)$ and $E_0^{\widetilde{S}} (\subset E_{\widetilde{S}})$, respectively.

Proof. Suppose that T and $\tilde{T}$ are unitarily equivalent. Let $U = ((U_{i,j}))_{1 \leq i,j \leq 2} : \mathcal{H} \to \widetilde{\mathcal{H}}$ be the unitary map such that $UT = \tilde{T}U$. From the relation $UT = \tilde{T}U$, $TU^* = U^*\tilde{T}$, we get the following equations:

$$U_{21}S + U_{22}T_1 = \tilde{T}_1 U_{22}, \tag{2.1}$$
$$U_{21}T_0 = \tilde{T}_1 U_{21}, \tag{2.2}$$
$$T_1 U_{12}^* = U_{12}^* \tilde{T}_0. \tag{2.3}$$

By (2.1), (2.2), and (2.3), we obtain

$$\begin{aligned} U_{21}SU_{12}^*\tilde{S} &= \tilde{T}_1 U_{22}U_{12}^*\tilde{S} - U_{22}U_{12}^*\tilde{S}\tilde{T}_1 \\ &= \tau_{\tilde{T}_1}(U_{22}U_{12}^*\tilde{S}). \end{aligned}$$

By (2.2) and (2.3), we have that $U_{21}SU_{12}^*\tilde{S}$ belongs to the commutant of $\widetilde{T}_1$. Thus $U_{21}SU_{12}^*\tilde{S} \in \operatorname{Ran}\tau_{\tilde{T}_1} \cap \ker \tau_{\tilde{T}_1}$. So by Proposition 2.3, $U_{21}SU_{12}^*\tilde{S}$ is a quasinilpotent operator. Since $U_{21}SU_{12}^*\tilde{S} \in \ker \tau_{\tilde{T}_1}$ and $\tilde{T}_1 \in B_1(\Omega)$, we then obtain

$$U_{21}SU_{12}^*\tilde{S} = 0.$$

Since $\tilde{S}$ has dense range, we have $U_{21}SU_{12}^* = 0$. Either $U_{12}^* = 0$ or $U_{12}^* \neq 0$. If $U_{12}^* \neq 0$, then from equation (2.3) it holds that U_{12}^* will have dense range. Since S also has dense range, we have $U_{21} = 0$. Since U is unitary and $UT = \tilde{T}U$, this implies that $U_{12} = 0$.

Suppose that $U_{12} = 0$. Since U is unitary and $TU^* = U^*\tilde{T}$, it follows that $U_{21} = 0$.

Thus, U takes the form $\left(\begin{smallmatrix} U_0 & 0 \\ 0 & U_1 \end{smallmatrix}\right)$ for some pair of unitary operators U_0 and U_1. Hence, we have $U_1 t_1 = \phi \tilde{t}_1$ for some holomorphic function ϕ. The intertwining relation $U_0 S = \tilde{S}U_1$ implies that $U_0(S(t_1)) = \phi(\tilde{S}(\tilde{t}_1))$. Thus $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$ and

$$\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|U_0(S(t_1))\|^2}{\|U_1(t_1)\|^2} = \frac{\|\phi\tilde{S}(\tilde{t}_1)\|^2}{\|\phi\tilde{t}_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}.$$

Conversely, if we assume that invariants are the same, then there exist a non-zero holomorphic function ϕ defined on Ω such that

$$\big\|t_1(w)\big\| = \big|\phi(w)\big|\big\|\tilde{t}_1(w)\big\| \quad \text{and} \quad \big\|S\big(t_1(w)\big)\big\| = \big|\phi(w)\big|\big\|\tilde{S}\big(\tilde{t}_1(w)\big)\big\|.$$

Set $t_0(w) := S(t_1(w))$ and $\tilde{t}_0(w) := \tilde{S}(\tilde{t}_1(w))$. For $0 \leq i \leq 1$, define an operator $U_i : \mathcal{H}_i \to \widetilde{\mathcal{H}}_i$ as follows:

$$U_i\big(t_i(w)\big) = \phi(w)\tilde{t}_i(w), \quad w \in \Omega.$$

For $0 \leq i \leq 1$,

$$\begin{aligned} \big\|U_i\big(t_i(w)\big)\big\| &= \big\|\phi(w)\tilde{t}_i(w)\big\| \\ &= \big|\phi(w)\big|\big\|\tilde{t}_i(w)\big\| \\ &= \big\|t_i(w)\big\|. \end{aligned}$$

Thus U_i extends to an isometry from $\mathcal{H}_i$ to $\widetilde{\mathcal{H}}_i$ and $U_iT_i = \tilde{T}_iU_i$. Since U_i is isometric and $U_iT_i = \tilde{T}_iU_i$, it follows, using Lemma 2.4, that U_i is a unitary operator. It also follows that $U_0S = \tilde{S}U_1$. Hence setting $U = \left(\begin{smallmatrix} U_0 & 0 \\ 0 & U_1 \end{smallmatrix}\right)$, we see that U is unitary and $UT = \tilde{T}U$ which completes the proof. □

Corollary 2.7. *Let* $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ *and* $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S} \\ 0 & T_1 \end{pmatrix}$ *be two Cowen–Douglas operators which belong to* $\mathcal{FB}_{n,1}(\Omega)$. *Suppose* T_0 *is irreducible operator. Then we have the following statements:*

(1) T *is unitarily equivalent to* $\tilde{T}$ *if and only if* $\tilde{S} = e^{i\theta}S$, *where* θ *is a real number.*

(2) *Let* μ *be a positive real number. Set,* $T_\mu = \begin{pmatrix} T_0 & \mu S \\ 0 & T_1 \end{pmatrix} \in \mathcal{FB}_{n,1}(\Omega)$. T_μ *is unitarily equivalent to* $T_{\tilde{\mu}}$ *if and only if* $\mu = \tilde{\mu}$.

2.3. Irreducibility and strongly irreducibility. In 1968, Paul Richard Halmos [5] introduced the concept of irreducible operator, as in the following.

Definition 2.8. An operator $T \in \mathcal{L}(\mathcal{H})$ is called an *irreducible operator* if there exists no nontrivial projection (self-adjoint) in its commutant.

In 1972, Gilfeather [4], followed by Jiang [14] in 1978, introduced the concept of *strongly irreducible operators* independently.

Definition 2.9. An operator $T \in \mathcal{L}(\mathcal{H})$ is considered a strongly irreducible operator if there exists no nontrivial idempotent (nonself-adjoint) in its commutant.

In the finite-dimensional Hilbert space, strongly irreducible operators are similar to Jordan blocks. In the infinite-dimensional, separable Hilbert space, strongly irreducible operators could be regarded as the generalization of Jordan blocks. A Cowen–Douglas operator with rank 1 is strongly irreducible.

Remark 2.10. [12] A strongly irreducible operator $T \in \mathcal{L}(\mathcal{H})$ has the following properties.

(1) The spectrum of T is connected.
(2) For any complex number w, $T - w$ is not finite rank.
(3) There is no any non-zero polynomial p such that $p(T) = 0$.
(4) There is no any singular points in the semi-Fredholm field of T.

In the general case, it is not easy to determine the case when a Cowen–Douglas operator T with high rank is irreducible or strongly irreducible. However, for operators in $\mathcal{F}B_{n,1}(\Omega)$, we have the following proposition about the irreducibility and strongly Irreducibility.

Proposition 2.11. *Suppose that* $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} \in \mathcal{FB}_{n,1}(\Omega)$. *Then we have the following statements.*

(1) *If* T_0 *is irreducible, then* T *is irreducible.*
(2) *If* T_0 *is strongly irreducible and* S *is invertible, then* T *is strongly irreducible.*
(3) *If* T_0 *is strongly irreducible, then* T *is strongly irreducible if and only if* $S \notin \operatorname{Ran} \tau_{T_0,T_1}$.

Proof. For the statement (1), let $P = (P_{ij})_{2\times 2}$ be a projection with $PT = TP$. Then it follows that $P_{11}T_0 = T_0P_{11} + SP_{21}$, $P_{11}S + P_{12}T_1 = T_0P_{12} + SP_{22}$, $P_{21}T_0 = T_1P_{21}$ and $P_{21}S + P_{22}T_1 = T_1P_{22}$. Since $T_0S = ST_1$, we obtain $P_{21}S \in \ker \tau_{T_1}$. Also note that, $P_{21}S = T_1P_{22} - P_{22}T_1$. Hence $P_{21}S \in \operatorname{Ran} \sigma_{T_1} \cap \ker \sigma_{T_1}$. Thus by Lemma 2.3, $P_{21}S$ is quasinilpotent. Since $P_{21}S$ is in commutant of T_1 and $T_1 \in B_1(\Omega)$, so $P_{21}S = 0$. As operator S has dense range, so it will follows that $P_{21} = 0$. Since P is self-adjoint, we have $P_{12} = 0$. Thus $P_{11}T_0 = T_0P_{11}$, $P_{22}T_1 = T_1P_{22}$ and $P_{11}S = SP_{22}$. It follows that $P_{11} = P_{22} = I$ or $P_{11} = P_{22} = 0$. This proves that T is irreducible.

For the statement (2), set $X = \left(\begin{smallmatrix} I & 0 \\ 0 & S \end{smallmatrix}\right)$. As X is invertible, so $XTX^{-1} = \left(\begin{smallmatrix} T_0 & I \\ 0 & T_0 \end{smallmatrix}\right)$. Since we get $\left(\begin{smallmatrix} T_0 & I \\ 0 & T_0 \end{smallmatrix}\right)$ is strongly irreducible. Thus, T is also strongly irreducible. For the last statement, let $P \in \{T\}'$ be an idempotent operator, by a similar proof of the main theorem, we know that $P_{21} = 0$. Commuting relation $PT = TP$ gives us $P_{11}T_0 = T_0P_{11}$, $P_{22}T_1 = T_1P_{22}$ and

$$P_{11}S - SP_{22} = T_0P_{12} - P_{12}T_1. \tag{2.4}$$

Since $P_{i+1i+1} \in \{T_i\}'$, for $0 \le i \le 1$, so P_{ii} can be either I or 0. If either $P_{11} = I$, $P_{22} = 0$ or $P_{11} = 0$, $P_{22} = I$, then $S \in \operatorname{Ran}\tau_{T_0,T_1}$ which is a contradiction to our assumption that $S \notin \operatorname{Ran}\tau_{T_0,T_1}$. Thus, we have $P_{11} = P_{22} = 0$ or $P_{11} = P_{22} = I$. Then it follows that $P_{12} = 0$. Hence T is strongly irreducible operator.

Conversely suppose that T is strongly irreducible and that $S \in \operatorname{Ran}\tau_{T_0,T_1}$ so that we can find an operator $\widetilde{P}$ such that $S = T_0\widetilde{P} - \widetilde{P}T_1$. Then the operator $P = \left(\begin{smallmatrix} I & \tilde{P} \\ 0 & 0 \end{smallmatrix}\right)$ is a nontrivial idempotent in $\{T\}'$. This means T is not strongly irreducible. □

In [10], C. Jiang proved that for any strongly irreducible Cowen–Douglas operators T and S, T and S are similarly denoted by $T \sim S$ if and the only if the order-K_0 group of $T \oplus S$ is isomorphic to $(\mathbb{Z}, \mathbb{N}, 1)$ (for more details, see [10], [11]). So for the similarity of operators in $\mathcal{F}B_{n,1}(\Omega)$, we have the following.

Corollary 2.12. *Let* $T = \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right)$ *and* $\tilde{T} = \left(\begin{smallmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{smallmatrix}\right)$ *be two Cowen–Douglas operators which belong to* $\mathcal{FB}_{n,1}(\Omega)$ *and satisfy the condition (2) or (3) in Proposition 2.11. Then the following statements are equivalent:*

(1) $T \sim \tilde{T}$,
(2) $(K_0(\{T \oplus \tilde{T}\}'), \bigvee(\{T \oplus \tilde{T}\}'), 1_{(\{T\oplus\tilde{T}\}')}) \cong (\mathbb{Z}, \mathbb{N}, 1)$.

Similar to the proof of the main theorem, we can show that any invertible operator which intertwines T and $\widetilde{T}$ would be upper-triangular under the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Then we also have the following corollary.

Corollary 2.13. *Let* $T = \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right)$ *and* $\tilde{T} = \left(\begin{smallmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{smallmatrix}\right)$ *be two Cowen–Douglas operators which belong to* $\mathcal{FB}_{n,1}(\Omega)$. *If* $T \sim \tilde{T}$, *then we have*

$$\Big(K_0(\{T_i \oplus \tilde{T}_i\}'), \bigvee(\{T_i \oplus \tilde{T}_i\}'), 1_{(\{T_i\oplus\tilde{T}_i\}')}\Big) \cong (\mathbb{Z}, \mathbb{N}, 1), \quad i = 0, 1.$$

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