Ann. Funct. Anal. 10 (2019), no. 4, 483–495
https://doi.org/10.1215/20088752-2019-0004
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

APPROXIMATE ρ_λ^v-ORTHOGONALITY AND ITS PRESERVATION

MOHAMMED Y. ABED, MAHDI DEHGHANI,* and RUHOLLAH JAHANIPUR

Communicated by S. Barza

ABSTRACT. We introduce and study a generalized approximate orthogonality relation in real normed linear spaces, namely, approximate ρ_λ^v-orthogonality. We investigate the relation between this generalized approximate orthogonality and approximate Birkhoff–James orthogonality. In particular, we show that every approximately ρ_λ^v-orthogonality-preserving linear mapping is necessarily a scalar multiple of an almost isometry.

1. Introduction and preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. It is known that there is one orthogonality relation derived from an inner product in X. In fact, the vectors $x, y \in X$ are orthogonal (written as $x \perp y$) if and only if $\langle x, y \rangle = 0$. The situation is completely different in general normed linear spaces. However, there is no unique way of defining the notion of orthogonality in general normed linear spaces. Many mathematicians have introduced different types of orthogonality relations in normed linear spaces, all of which are generalizations of orthogonality in an inner product space. Birkhoff–James orthogonality is one of the most important orthogonality types that was introduced by Birkhoff in [4] and then later developed by James in [14]. Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least 2. A vector $x \in X$ is said to be *orthogonal* to a vector $y \in X$ in the sense

Received Oct. 25, 2018; Accepted Jan. 11, 2019.
First published online Oct. 23, 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 46C50, 47B99.
Keywords. approximate orthogonality, approximate orthogonality-preserving property, norm derivatives, ρ_λ^v-orthogonality, smooth normed linear spaces.

of Birkhoff–James, written as $x \perp_B y$, if

$$\|x\| \leq \|x+ty\| \quad (\forall t \in \mathbb{R}).$$

One of the possible notions of orthogonality is connected with so-called *norm derivatives*, which are defined by Amir in [3] as follows:

$$\rho_{\pm}(x,y) := \lim_{t\to 0^{\pm}} \frac{\|x+ty\|^2-\|x\|^2}{2t} = \|x\| \lim_{t\to 0^{\pm}} \frac{\|x+ty\|-\|x\|}{t}.$$

A vector $x \in X$ is said to be $\rho_{\pm}$-*orthogonal* to a vector $y \in X$ if $\rho_{\pm}(x,y) = 0$ (see [2]). Also, x is called ρ-*orthogonal* to y if $\rho(x,y) = 0$, where the functional ρ is defined in [16] by $\rho := \frac{\rho_- + \rho_+}{2}$. (For more information about norm derivatives, $\rho_{\pm}$-orthogonality, ρ-orthogonality, and Birkhoff–James orthogonality, we refer the reader to [2], [9], [10], [13], [14], and [19] and the references therein.)

Recently, some generalizations of norm derivatives have been introduced to extend orthogonality relations related to norm derivatives. In [22], $\rho_{\pm}$- and ρ-orthogonality have been extended to ρ_λ-orthogonality for $\lambda \in [0,1]$, by introducing the functional $\rho_\lambda : X \times X \to \mathbb{R}$ as follows:

$$\rho_\lambda(x,y) = \lambda\rho_-(x,y) + (1-\lambda)\rho_+(x,y) \quad (x,y \in X).$$

More generally, in [11] we defined and studied a generalized orthogonality relation, namely, ρ_λ^v-orthogonality. For each $x,y \in X$, $\lambda \in [0,1]$, and $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$, the functional $\rho_\lambda^v : X \times X \to \mathbb{R}$ is defined by

$$\rho_\lambda^v(x,y) = \lambda\rho_-^v(x,y)\rho_+^{1-v}(x,y) + (1-\lambda)\rho_+^v(x,y)\rho_-^{1-v}(x,y).$$

We say that x is ρ_λ^v-orthogonal to y, denoted by $x \perp_{\rho_\lambda^v} y$, if $\rho_\lambda^v(x,y) = 0$.

We remark here that other orthogonality relations which are taken from norm derivatives are special cases of ρ_λ^v-orthogonality for different mods of λ and v. In fact, let $x,y \in X$. Then

$$\rho_\lambda^1(x,y) = \rho_\lambda(x,y), \qquad \rho_{\frac{1}{2}}^1(x,y) = \rho(x,y),$$
$$\rho_0^1(x,y) = \rho_+(x,y) \qquad \text{and} \qquad \rho_1^1(x,y) = \rho_-(x,y).$$

Therefore, ρ_λ^1-orthogonality coincides with ρ_λ-orthogonality for all $\lambda \in [0,1]$, and $\rho_{\frac{1}{2}}^1$-orthogonality is equivalent to ρ-orthogonality. Also, ρ_0^1-orthogonality is equivalent to ρ_+-orthogonality and ρ_1^1-orthogonality is equivalent to ρ_--orthogonality.

Alsina, Sikorska, and Tomás in [2] offer thorough explanations of $\rho_{\pm}$- and ρ-orthogonality relations. Additionally, Chmieliński and Wójcik (see [10], [20]) studied $\rho_{\pm}$- and ρ-orthogonality and their approximate counterparts. The notion of ρ_*-orthogonality is given by

$$x \perp_{\rho_*} y \qquad \text{if and only if} \qquad \rho_*(x,y) := \rho_-(x,y)\rho_+(x,y) = 0.$$

This concept was introduced in [6], and more of its properties have been investigated in [18]. Note that since

$$\rho_0^v(x,y) = \rho_+^v(x,y)\rho_-^{1-v}(x,y) \qquad \text{and} \qquad \rho_1^v(x,y) = \rho_-^v(x,y)\rho_+^{1-v}(x,y)$$

for all $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$, it is easy to see that $\perp_{\rho_0^v} = \perp_{\rho_*}$ and $\perp_{\rho_1^v} = \perp_{\rho_*}$ for all $v \neq 1$.

Let $\varepsilon \in [0, 1)$. For an inner product space $(X, \langle \cdot, \cdot \rangle)$, Chmieliński [7] studied an approximate orthogonality (ε-orthogonality) of vectors $x, y \in X$ naturally defined by

$$x \perp^\varepsilon y \qquad \text{if and only if} \qquad \big|\langle x, y\rangle\big| \leq \varepsilon \|x\| \|y\|.$$

In addition, the notions of approximate $\rho_\pm$- and ρ-orthogonality have been studied extensively in [9] and [10]. In a similar way, we can define approximate ρ_λ^v-orthogonality in real normed linear spaces as follows.

Definition 1.1. Let $(X, \|\cdot\|)$ be a real normed linear space, and let $x, y \in X$. If $\lambda \in [0, 1]$ and $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$, then

$$x \perp_{\rho_\lambda^v}^\varepsilon y \Leftrightarrow \big|\rho_\lambda^v(x, y)\big| \leq \varepsilon \|x\| \|y\|.$$

In particular, for $v = 1$, we get

$$x \perp_{\rho_\lambda}^\varepsilon y \Leftrightarrow \big|\rho_\lambda(x, y)\big| \leq \varepsilon \|x\| \|y\|.$$

We recall that the notion of approximate ρ_*-orthogonality is defined and studied in [12]:

$$x \perp_{\rho_*}^\varepsilon y \Leftrightarrow \big|\rho_*(x, y)\big| \leq \varepsilon^2 \|x\|^2 \|y\|^2.$$

For an approximate Birkhoff–James orthogonality, we will follow the definition from [8]:

$$x \perp_B^\varepsilon y \Leftrightarrow \|x + ty\|^2 \geq \|x\|^2 - 2\varepsilon \|x\| \|ty\| \quad (\forall t \in \mathbb{R}).$$

According to [8], if the norm comes from an inner product, then

$$\perp_{\rho_\lambda^v}^\varepsilon = \perp_B^\varepsilon = \perp_{\rho_*}^\varepsilon = \perp^\varepsilon$$

and for $\varepsilon = 0$ all the previous approximated orthogonality relations are equivalent to the related exact orthogonality.

The main aim of the present note is to study the approximate ρ_λ^v-orthogonality and its preservation. First, some illustrated examples are presented to show that the relations $\perp_\diamondsuit^\varepsilon$ for $\diamondsuit \in \{\rho_\pm, \rho, \rho_*, \rho_\lambda^v\}$ are incomparable in real normed linear spaces. Next, we show that approximate Birkhoff–James orthogonality and approximate ρ_λ^v-orthogonality are equivalent in a real normed linear space X if and only if X is smooth. In particular, although $\perp_\diamondsuit^\varepsilon$ for $\diamondsuit \in \{\rho_\pm, \rho, \rho_*\}$, need not be equivalent to $\perp_{\rho_\lambda^v}^\varepsilon$ $(v \neq 1)$, unless we assume the smoothness of the norm, we will prove that an approximate ρ_λ^v-orthogonality-preserving linear mapping is necessarily a scalar multiple of an almost isometry.

2. Approximate ρ_λ^v-orthogonality

We start this section with some examples to show that the relations $\perp_{\rho_-}^\varepsilon$, $\perp_{\rho_+}^\varepsilon$, $\perp_\rho^\varepsilon$, $\perp_{\rho_*}^\varepsilon$, and $\perp_{\rho_\lambda^v}^\varepsilon$ are incomparable in general normed linear spaces.

Example 2.1. Consider the real normed linear space $X = \mathbb{R}^3$ with the norm $\|x\| = \sum_{i=1}^{3} |x_i|$, where $x = (x_1, x_2, x_3)$. If $x = (1, 0, 0)$ and $y = (1, 1, \varepsilon)$, then $\|x\| = 1$, $\|y\| = 2 + \varepsilon$, and some easy computations show that $\rho_+(x, y) = 2 + \varepsilon$ and $\rho_-(x, y) = -\varepsilon$. Now, let $v = \frac{1}{5}$, $\varepsilon = \frac{1}{2}$, and $\lambda = \frac{9}{10}$. Then

$$\begin{aligned}\rho_\lambda^v(x, y) &= \lambda \rho_-^v(x, y) \rho_+^{1-v}(x, y) + (1 - \lambda) \rho_+^v(x, y) \rho_-^{1-v}(x, y) \\ &= \lambda(-\varepsilon)^v (2 + \varepsilon)^{1-v} + (1 - \lambda)(2 + \varepsilon)^v (-\varepsilon)^{1-v} \\ &= \Big(\frac{9}{10}\Big)\Big(-\frac{1}{2}\Big)^{\frac{1}{5}}\Big(\frac{5}{2}\Big)^{\frac{4}{5}} + \Big(\frac{1}{10}\Big)\Big(\frac{5}{2}\Big)^{\frac{1}{5}}\Big(-\frac{1}{2}\Big)^{\frac{4}{5}} \\ &= \frac{1}{10}\Big(\frac{5}{2}\Big)^{\frac{1}{5}}\Big(\frac{1}{2}\Big)^{\frac{4}{5}}\big[1 - 9(5)^{\frac{3}{5}}\big].\end{aligned}$$

It is easy to check that $|\rho_\lambda^v(x, y)| > \varepsilon \|x\| \|y\|$. Hence, $x \not\perp_{\rho_\lambda^v}^\varepsilon y$. On the other hand, it is clear that $x \perp_{\rho_-}^\varepsilon y$, for all $\varepsilon \in (0, 1)$ and

$$x \perp_\rho^\varepsilon y \qquad \text{and} \qquad x \perp_{\rho_*}^\varepsilon y$$

for all $\varepsilon \geq \sqrt{2} - 1$. Then, in particular, for $\varepsilon = \frac{1}{2}$, we have

$$\perp_{\rho_-}^\varepsilon \not\subseteq \perp_{\rho_\lambda^v}^\varepsilon, \qquad \perp_\rho^\varepsilon \not\subseteq \perp_{\rho_\lambda^v}^\varepsilon, \qquad \text{and} \qquad \perp_{\rho_*}^\varepsilon \not\subseteq \perp_{\rho_\lambda^v}^\varepsilon .$$

Now, let $z = (-1, \frac{1}{2}, 0)$. Then $\rho_-(x, z) = -\frac{3}{2}$ and $\rho_+(x, z) = -\frac{1}{2}$. Similarly, for $\lambda = \frac{9}{10}$ and $\varepsilon = v = \frac{1}{3}$ we have $x \not\perp_{\rho_\lambda^v}^\varepsilon z$, while $x \perp_{\rho_+}^\varepsilon z$. Therefore, $\perp_{\rho_+}^\varepsilon \not\subseteq \perp_{\rho_\lambda^v}^\varepsilon$.

Example 2.2. Consider the real normed linear space $X = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. Let $\varepsilon \in (0, 1)$, let $x = (1, 1)$, let $y = (1, -1)$, and let $z = (0, 2(1 + \varepsilon))$. Then $\|x\| = \|y\| = 1$, $\|z\| = 2(1 + \varepsilon)$ and so

$$\rho_-(x, y) = -1, \quad \rho_+(x, y) = 1, \quad \rho_-(x, z) = 0 \quad \text{and} \quad \rho_+(x, z) = 2(1 + \varepsilon).$$

It is clear that $x \not\perp_{\rho_+}^\varepsilon y$, $x \not\perp_{\rho_-}^\varepsilon y$, and $x \not\perp_{\rho_*}^\varepsilon y$ for all $\varepsilon \in (0, 1)$. On the other hand, for each $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$ we have

$$\begin{aligned}\rho_\lambda^v(x, y) &= \lambda \rho_-^v(x, y) \rho_+^{1-v}(x, y) + (1 - \lambda) \rho_+^v(x, y) \rho_-^{1-v}(x, y) \\ &= \lambda(-1)^v (1)^{1-v} + (1 - \lambda)(1)^v (-1)^{1-v} \\ &= 1 - 2\lambda.\end{aligned}$$

Hence, if $|1 - 2\lambda| \leq \varepsilon$, then $x \perp_{\rho_\lambda^v}^\varepsilon y$. Therefore,

$$\perp_{\rho_\lambda^v}^\varepsilon \not\subseteq \perp_{\rho_-}^\varepsilon, \qquad \perp_{\rho_\lambda^v}^\varepsilon \not\subseteq \perp_{\rho_+}^\varepsilon, \qquad \text{and} \qquad \perp_{\rho_\lambda^v}^\varepsilon \not\subseteq \perp_{\rho_*}^\varepsilon .$$

Also, we have $\rho_\lambda^v(x, z) = 0$, which follows that $x \perp_{\rho_\lambda^v}^\varepsilon z$, while

$$x \perp_\rho^\varepsilon z \Leftrightarrow \varepsilon \geq \frac{1}{2}.$$

Now, if $\varepsilon < \frac{1}{2}$, then $x \not\perp_\rho^\varepsilon z$. Therefore,

$$\perp_{\rho_\lambda^v}^\varepsilon \not\subseteq \perp_\rho^\varepsilon .$$

We recall that norm derivatives characterize approximate Birkhoff–James orthogonality in the following sense.

Lemma 2.3 ([10, Theorem 3.1])**.** *Let $(X, \|\cdot\|)$ be a real normed linear space, and let $\varepsilon \in [0, 1)$. Then for arbitrary $x, y \in X$*

$$x \perp_B^\varepsilon y \Leftrightarrow \rho_-(x, y) - \varepsilon\|x\|\|y\| \le 0 \le \rho_+(x, y) + \varepsilon\|x\|\|y\|.$$

In particular,

$$x \perp_B y \Leftrightarrow \rho_-(x, y) \le 0 \le \rho_+(x, y).$$

The following result is an approximate version of Proposition 2.9 of [11].

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a real normed linear space, and let $\varepsilon \in [0, 1)$. Then $\perp_{\rho_\lambda^v}^\varepsilon \subseteq \perp_B^\varepsilon$ for all $\lambda \in [0, 1]$ and all $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$.*

Proof. Let $x, y \in X$ be such that $x \perp_{\rho_\lambda^v}^\varepsilon y$. If $\rho_*(x, y) = 0$, then $x \perp_{\rho_*} y$ and so $x \perp_{\rho_*}^\varepsilon y$. It follows from [12, Theorem 2.3] that $x \perp_B^\varepsilon y$. Now, assume that $\rho_*(x, y) \neq 0$. Then there are two cases.

Case 1. If $\rho_*(x, y) > 0$, then $0 < \rho_-(x, y) \le \rho_+(x, y)$ or $\rho_-(x, y) \le \rho_+(x, y) < 0$. Since $x \perp_{\rho_\lambda^v}^\varepsilon y$, we conclude that

$$-\varepsilon\|x\|\|y\| \le \lambda\rho_-^v(x, y)\rho_+^{1-v}(x, y) + (1-\lambda)\rho_+^v(x, y)\rho_-^{1-v}(x, y) \le \varepsilon\|x\|\|y\|. \quad (2.1)$$

If $0 < \rho_-(x, y) \le \rho_+(x, y)$ (since the functions $f(x) = x^v$ and $g(x) = x^{1-v}$ $(x > 0)$ are monotone increasing for all $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$), then from (2.1) we get

$$\begin{aligned}\rho_-(x, y) &= \lambda\rho_-^v(x, y)\rho_-^{1-v}(x, y) + (1-\lambda)\rho_-^v(x, y)\rho_-^{1-v}(x, y)\\ &\le \lambda\rho_-^v(x, y)\rho_+^{1-v}(x, y) + (1-\lambda)\rho_+^v(x, y)\rho_-^{1-v}(x, y)\\ &\le \varepsilon\|x\|\|y\|,\end{aligned}$$

which implies that

$$\rho_-(x, y) - \varepsilon\|x\|\|y\| \le 0. \quad (2.2)$$

Also, we have

$$\rho_+(x, y) > 0 > -\varepsilon\|x\|\|y\|. \quad (2.3)$$

It follows from (2.2) and (2.3) that

$$\rho_-(x, y) - \varepsilon\|x\|\|y\| \le 0 \le \rho_+(x, y) + \varepsilon\|x\|\|y\|.$$

Therefore, Lemma 2.3 implies that $x \perp_B^\varepsilon y$.

If $\rho_-(x, y) \le \rho_+(x, y) < 0$, then, using a similar argument from (2.1), since the function $f(x) = x^v$ is monotone increasing and the function $g(x) = x^{1-v}$ $(x < 0)$ is monotone decreasing for all $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$ and $\rho_+^{1-v}(x, y) > 0$ and $\rho_+^v(x, y) < 0$, we conclude that

$$\begin{aligned}\rho_+(x, y) &= \lambda\rho_+^v(x, y)\rho_+^{1-v}(x, y) + (1-\lambda)\rho_+^v(x, y)\rho_+^{1-v}(x, y)\\ &\ge \lambda\rho_-^v(x, y)\rho_+^{1-v}(x, y) + (1-\lambda)\rho_+^v(x, y)\rho_-^{1-v}(x, y)\\ &\ge -\varepsilon\|x\|\|y\|.\end{aligned}$$

Also, obviously we have $\rho_-(x, y) < 0 \le \varepsilon\|x\|\|y\|$. Hence,

$$\rho_-(x, y) - \varepsilon\|x\|\|y\| \le 0 \le \rho_+(x, y) + \varepsilon\|x\|\|y\|.$$

Therefore, Lemma 2.3 implies that $x \perp_B^{\varepsilon} y$.

Case 2. If $\rho_*(x,y) < 0$, then $\rho_-(x,y) < 0 < \rho_+(x,y)$. Hence,

$$\rho_-(x,y) - \varepsilon\|x\|\|y\| \le \rho_-(x,y) < 0 < \rho_+(x,y) \le \rho_+(x,y) + \varepsilon\|x\|\|y\|$$

and so $x \perp_B^{\varepsilon} y$. □

As a direct consequence of Theorem 2.4 for the special case $v = 1$, we have the following result.

Corollary 2.5. *Let $(X, \|\cdot\|)$ be a real normed linear space, and let $\varepsilon \in [0,1)$. Then $\perp_{\rho_\lambda}^{\varepsilon} \subseteq \perp_B^{\varepsilon}$ for all $\lambda \in [0,1]$.*

Remark 2.6. Note that if we put $\lambda = \frac{1}{2}$ in Corollary 2.5, then we conclude that $\perp_{\rho}^{\varepsilon} \subseteq \perp_B^{\varepsilon}$. Also, if $\lambda = 1$, then we get $\perp_{\rho_-}^{\varepsilon} \subseteq \perp_B^{\varepsilon}$, and if $\lambda = 0$, then $\perp_{\rho_+}^{\varepsilon} \subseteq \perp_B^{\varepsilon}$.

It should be noted that there are nonsmooth normed linear spaces for which $\perp_B^{\varepsilon} \nsubseteq \perp_{\rho_\lambda^v}^{\varepsilon}$. In fact, if we consider Example 2.1, then

$$\begin{aligned}\|x+ty\|^2 - \|x\|^2 + 2\varepsilon\|x\|\|ty\| &= \big(|1+t| + |t| + |t|\varepsilon\big)^2 - 1 + 2\varepsilon(2+\varepsilon)|t| \\ &\ge \big(1 + |t|\varepsilon\big)^2 - 1 + 2\varepsilon(2+\varepsilon)|t| \\ &= 2\varepsilon(3+\varepsilon)|t| + |t|^2\varepsilon^2 \ge 0\end{aligned}$$

for all $t \in \mathbb{R}$ and all $\varepsilon \in [0,1)$. Hence, $x \perp_B^{\varepsilon} y$. On the other hand, for $\varepsilon = \frac{4}{10}$, $v = \frac{1}{5}$, and $\lambda = \frac{9}{10}$, we have

$$\begin{aligned}\rho_\lambda^v(x,y) &= \lambda\rho_-^v(x,y)\rho_+^{1-v}(x,y) + (1-\lambda)\rho_+^v(x,y)\rho_-^{1-v}(x,y) \\ &= \lambda(-\varepsilon)^v(2+\varepsilon)^{1-v} + (1-\lambda)(2+\varepsilon)^v(-\varepsilon)^{1-v} \\ &= \Big(\frac{9}{10}\Big)\Big(-\frac{4}{10}\Big)^{\frac{1}{5}}\Big(\frac{24}{10}\Big)^{\frac{4}{5}} + \Big(\frac{1}{10}\Big)\Big(\frac{24}{10}\Big)^{\frac{1}{5}}\Big(-\frac{4}{10}\Big)^{\frac{4}{5}} \\ &= \frac{1}{25}(6)^{\frac{1}{5}}\big[1 - 9(6)^{\frac{3}{5}}\big].\end{aligned}$$

It is easy to check that $|\rho_\lambda^v(x,y)| > \varepsilon\|x\|\|y\|$ and so $x \not\perp_{\rho_\lambda^v}^{\varepsilon} y$. Therefore, $\perp_B^{\varepsilon} \nsubseteq \perp_{\rho_\lambda^v}^{\varepsilon}$.

Recall that a real normed linear space $(X, \|\cdot\|)$ is said to be *smooth* at the point $x \in X \setminus \{0\}$ if there is a unique $x^* \in X^*$ such that $x^*(x) = \|x\|$ and $\|x^*\| = 1$. Also, X is smooth if X is smooth at each point $x \ne 0$ in X (see, e.g., [13]). It is known that X is smooth at the point $x \in X \setminus \{0\}$ if and only if $\rho_-(x,y) = \rho_+(x,y)$ for all $y \in X$ (see [2], [13]).

It was proved in [10, Theorem 3.4] that the smoothness of a normed linear space X results from $\perp_{\rho_\pm}^{\varepsilon} = \perp_B^{\varepsilon}$ and $\perp_{\rho}^{\varepsilon} = \perp_B^{\varepsilon}$ for some $\varepsilon \in (0,1)$. Analogously, in the following theorem we prove that approximate Birkhoff–James orthogonality and approximate ρ_λ^v-orthogonality in a real normed linear space X are equivalent if and only if X is smooth.

Theorem 2.7. *Let $(X, \|\cdot\|)$ be a normed linear space, let $\varepsilon \in [0,1)$, let $\lambda \in (0,1]$, and let $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. If $\perp_B^{\varepsilon} = \perp_{\rho_\lambda^v}^{\varepsilon}$, then X is smooth.*

Proof. By the assumption, $\perp_{\rho_+}^{\varepsilon} \subset \perp_{\rho_\lambda^v}^{\varepsilon}$. It follows from the proof of Theorem 3.3 of [10] that, for a fixed unit vector $x \in X$, there exists a unit vector $y \in X$ such that $\rho_+(x,y) = -\varepsilon$. Then $x \perp_{\rho_+}^{\varepsilon} y$ and so $x \perp_{\rho_\lambda^v}^{\varepsilon} y$. Thus

$$\begin{aligned} -\varepsilon \leq \rho_\lambda^v(x,y) &= \lambda \rho_-^v(x,y)\rho_+^{1-v}(x,y) + (1-\lambda)\rho_+^v(x,y)\rho_-^{1-v}(x,y) \\ &\leq \lambda \rho_+^v(x,y)\rho_+^{1-v}(x,y) + (1-\lambda)\rho_+^v(x,y)\rho_+^{1-v}(x,y) \\ &= \lambda\rho_+(x,y) + (1-\lambda)\rho_+(x,y) = -\varepsilon. \end{aligned}$$

Hence, $\rho_\lambda^v(x,y) = -\varepsilon$. This implies that

$$\lambda \rho_-^v(x,y)\varepsilon^{1-v} - (1-\lambda)\rho_-^{1-v}(x,y)\varepsilon^v = -\varepsilon,$$

which is equivalent to

$$(1-\lambda)\rho_-^{1-v}(x,y) - \lambda\varepsilon^{1-2v}\rho_-^v(x,y) - \varepsilon^{1-v} = 0. \tag{2.4}$$

Now, let $\rho_-^v(x,y) = t$. Then $\rho_-^{1-v}(x,y) = t^{\frac{1}{v}-1} = t^{2(k-1)}$ $(k \in \mathbb{N})$ and so (2.4) is equivalent to

$$(1-\lambda)t^{2(k-1)} - \lambda\varepsilon^{1-2v}t - \varepsilon^{1-v} = 0. \tag{2.5}$$

Define $f(t) = (1-\lambda)t^{2(k-1)} - \lambda\varepsilon^{1-2v}t - \varepsilon^{1-v}$. Then $f'(t) < 0$ for all $t < 0$. It follows that f is strictly monotone decreasing on $(-\infty, 0)$. Since $f(t) \to +\infty$ as $t \to -\infty$ and $f(0) < 0$, we can infer that the equation $f(t)$ has exactly one negative root, namely, $t = -\varepsilon^v$. Since $\rho_-(x,y) \leq \rho_+(x,y) = -\varepsilon < 0$, we conclude that $\rho_-(x,y) = t^v = -\varepsilon$. Thus, $\rho_-(x,y) = \rho_+(x,y) = -\varepsilon$. Again similar to the proof of Theorem 3.3 of [10], we conclude that X is smooth. ☐

3. Linear mappings approximately preserving ρ_λ^v-orthogonality

The problem of determining the structure of linear mappings between normed linear spaces, which leave certain properties invariant, has been considered in several papers. The study of linear orthogonality-preserving mappings can be considered as a part of the theory of linear preservers. The orthogonality-preserving property has recently been the focus of intensive study in connection with functional analysis and operator theory (see [5], [6], [10], [15], [17]). Similar investigations have been carried out in the framework of vector lattices in [1]. In addition, the approximate orthogonality-preserving property has been considered also in the setting of normed spaces with respect to various definitions of orthogonality in general normed spaces. Chmieliński [7] studied the linear mappings that preserve the approximate orthogonality in the setting of inner product spaces. Mojškerc and Turnšek [17] considered the class of linear mappings approximately preserving the Birkhoff–James orthogonality. More precisely, they proved the following theorem.

Theorem 3.1 ([17, Theorem 3.5, Remark 3.1])**.** *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed linear spaces, let $\varepsilon \in [0, \frac{1}{8})$, and let $T : X \longrightarrow Y$ be a linear mapping* *satisfying*

$$x \perp_B y \Rightarrow Tx \perp_B^{\varepsilon} Ty \quad (x, y \in X).$$

Then

$$(1 - 8\varepsilon)\|T\|\|x\| \le \|Tx\| \le \|T\|\|x\| \quad (x \in X).$$

Suppose that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are real normed linear spaces. We say that a linear mapping $T : X \to Y$ preserves the approximate $\diamondsuit$-orthogonality if

$$x \perp_{\diamondsuit} y \Rightarrow Tx \perp_{\diamondsuit}^{\varepsilon} Ty \quad (\forall x, y \in X),$$

where $\diamondsuit \in \{B, \rho_*, \rho_\lambda^v\}$. In particular, we say that T is $\diamondsuit$-orthogonality-preserving if

$$x \perp_{\diamondsuit} y \Rightarrow Tx \perp_{\diamondsuit} Ty \quad (x, y \in X).$$

Chmieliński and Wójcik in [10] and [21] investigated approximate $\rho_\pm$- and ρ-orthogonality-preserving mappings. In particular, they showed that every mapping that preserves approximate $\diamondsuit$-orthogonality with $\diamondsuit \in \{\rho_\pm, \rho\}$ is necessarily a scalar multiple of an almost isometry. The same result was proved in [12, Corollary 2.10] for approximate ρ_*-orthogonality-preserving mappings. In conclusion, combining these results and Theorem 3.1, we have the following characterizations of linear mappings approximately preserving the orthogonality relations.

Theorem 3.2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces, and let $\varepsilon \in [0, 1)$. If $T : X \to Y$ is a nonzero linear mapping, then the following conditions are equivalent.*

(1) *T approximately preserves ρ_--orthogonality.*
(2) *T approximately preserves ρ_+-orthogonality.*
(3) *T approximately preserves ρ-orthogonality.*
(4) *T approximately preserves ρ_*-orthogonality.*
(5) *T approximately preserves B-orthogonality.*

Moreover, if $\varepsilon \in [0, \frac{1}{8})$, then each of the above conditions implies that

$$(1 - 8\varepsilon)\|T\|\|x\| \le \|Tx\| \le \|T\|\|x\| \quad (x \in X).$$

In this section we will prove that an approximate ρ_λ^v-orthogonality-preserving linear mapping is necessarily a scalar multiple of an almost isometry.

Recall that a set $S \subset X$ is said to be *star-shaped* if for all $\alpha \in \mathbb{R}$ and $x \in S$, then $\alpha x \in S$. Also, for fixed $x^* \in X^*$ and $\alpha \in \mathbb{R}$, the set $M := \{x \in X : x^*(x) = \alpha\}$ will be called a *hyperplane*. Also, we need the following Lemma from [20].

Lemma 3.3 ([20, Lemma 4.1])**.** *Let $S \subset X$ be a dense, star-shaped subset, and let M be a hyperplane such that $0 \notin M$. Then $\overline{M \cap S} = M$.*

Theorem 3.4. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed linear spaces, let $\varepsilon \in [0, 1)$, let $\lambda \in [0, 1]$, and let $v = \frac{1}{2k+1}$ $(k \in \mathbb{N})$. If $T : X \to Y$ is a nonzero linear mapping, then the following conditions are equivalent:*

(1) *For each $x, y \in X$, $x \perp_{\rho_\lambda^v} y \Rightarrow Tx \perp_{\rho_\lambda^v}^{\varepsilon} Ty$.*
(2) *For each $x, y \in X$, $x \perp_B y \Rightarrow Tx \perp_B^{\varepsilon} Ty$.*

Proof. To prove that (1) ⇒ (2), let $x, y \in X$ be such that $x \perp_B y$. It follows from Lemma 2.3 that

$$\rho_-(x,y) \le 0 \le \rho_+(x,y). \tag{3.1}$$

Assume that $x \neq 0$. Since $x \perp_{\rho_\pm} \frac{-\rho_\pm(x,y)}{\|x\|^2}x + y$ and $\perp_{\rho_\pm} \subseteq \perp_{\rho_\lambda^v}$, we get

$$x \perp_{\rho_\lambda^v} \frac{-\rho_+(x,y)}{\|x\|^2}x + y$$

and

$$x \perp_{\rho_\lambda^v} \frac{-\rho_-(x,y)}{\|x\|^2}x + y.$$

It follows from (1) that

$$Tx \perp^\varepsilon_{\rho_\lambda^v} \frac{-\rho_+(x,y)}{\|x\|^2}Tx + Ty$$

and

$$Tx \perp^\varepsilon_{\rho_\lambda^v} \frac{-\rho_-(x,y)}{\|x\|^2}Tx + Ty.$$

So from Theorem 2.4, we conclude that

$$Tx \perp^\varepsilon_B \frac{-\rho_+(x,y)}{\|x\|^2}Tx + Ty \tag{3.2}$$

and

$$Tx \perp^\varepsilon_B \frac{-\rho_-(x,y)}{\|x\|^2}Tx + Ty. \tag{3.3}$$

Therefore, (3.2), (3.3), and Lemma 2.3 imply that

$$0 \le \rho_+\Big(Tx, \frac{-\rho_+(x,y)}{\|x\|^2}Tx + Ty\Big) + \varepsilon\|Tx\|\Big\|\frac{-\rho_+(x,y)}{\|x\|^2}Tx + Ty\Big\| \tag{3.4}$$

and

$$\rho_-\Big(Tx, \frac{-\rho_-(x,y)}{\|x\|^2}Tx + Ty\Big) - \varepsilon\|Tx\|\Big\|\frac{-\rho_-(x,y)}{\|x\|^2}Tx + Ty\Big\| \le 0. \tag{3.5}$$

Since $\varepsilon \in [0,1)$, by (3.1) and (3.4) we obtain

$$\begin{aligned}
0 &\le \frac{-\rho_+(x,y)}{\|x\|^2}\|Tx\|^2 + \rho_+(Tx,Ty) + \varepsilon\|Tx\|\Big\|\frac{-\rho_+(x,y)}{\|x\|^2}Tx + Ty\Big\| \\
&\le \frac{-\rho_+(x,y)}{\|x\|^2}\|Tx\|^2 + \rho_+(Tx,Ty) + \varepsilon\|Tx\|^2\frac{\rho_+(x,y)}{\|x\|^2} + \varepsilon\|Tx\|\|Ty\| \\
&= (\varepsilon - 1)\|Tx\|^2\frac{\rho_+(x,y)}{\|x\|^2} + \rho_+(Tx,Ty) + \varepsilon\|Tx\|\|Ty\| \\
&\le \rho_+(Tx,Ty) + \varepsilon\|Tx\|\|Ty\|.
\end{aligned}$$

Thus,

$$0 \le \rho_+(Tx,Ty) + \varepsilon\|Tx\|\|Ty\|. \tag{3.6}$$

Similarly, from (3.1) and (3.5), we get

$$\begin{aligned}
&\rho_-(Tx,Ty)-\varepsilon\|Tx\|\|Ty\|\\
&\quad\le(\varepsilon-1)\|Tx\|^2\frac{\rho_-(x,y)}{\|x\|^2}+\rho_-(Tx,Ty)-\varepsilon\|Tx\|\|Ty\|\\
&\quad=\frac{-\rho_-(x,y)}{\|x\|^2}\|Tx\|^2+\rho_-(Tx,Ty)-\varepsilon\|Tx\|^2\frac{-\rho_-(x,y)}{\|x\|^2}-\varepsilon\|Tx\|\|Ty\|\\
&\quad\le\rho_-\Big(Tx,\frac{-\rho_-(x,y)}{\|x\|^2}Tx+Ty\Big)-\varepsilon\|Tx\|\Big\|\frac{-\rho_-(x,y)}{\|x\|^2}Tx+Ty\Big\|\\
&\quad\le 0.
\end{aligned}$$

Hence,

$$\rho_-(Tx,Ty)-\varepsilon\|Tx\|\|Ty\|\le 0. \tag{3.7}$$

Therefore, (3.6) and (3.7) imply that

$$\begin{aligned}
&\rho_-(Tx,Ty)-\varepsilon\|Tx\|\|Ty\|\\
&\qquad\le 0\le\rho_+(Tx,Ty)+\varepsilon\|Tx\|\|Ty\|.
\end{aligned}$$

Consequently, by Lemma 2.3, we get $Tx\perp_B^{\varepsilon}Ty$.

Now, we prove the implication (2) ⇒ (1). Suppose that (2) holds. It has been proved in [21, Theorem 5.5] that if a linear mapping approximately preserves B-orthogonality, then it approximately preserves the ρ-orthogonality (see Theorem 3.2). Then from (2), we conclude that $x\perp_\rho y\Rightarrow Tx\perp_\rho^{\varepsilon}Ty$ $(x,y\in X)$. From the proof of Theorem 5.3 of [21], we obtain that T is injective and that $S=D_{\mathrm{sm}}(X)\cap T^{-1}(D_{\mathrm{sm}}(Y))$ is dense and star-shaped, where

$$D_{\mathrm{sm}}(X)=\{x\in X: X \text{ is smooth at } x\}\cup\{0\}.$$

We consider two cases.

Case 1. Suppose that $\dim(X)=\dim(Y)=2$, $y\in X$, and $x\in S\setminus\{0\}$. Since $x\in S$, we get $Tx\in D_{\mathrm{sm}}(Y)$, and so

$$\rho_-(x,\cdot)=\rho_+(x,\cdot)=\rho_\lambda^v(x,\cdot)$$

and

$$\rho_-(Tx,\cdot)=\rho_+(Tx,\cdot)=\rho_\lambda^v(Tx,\cdot).$$

It is clear that

$$x\perp_{\rho_\lambda^v}\frac{-\rho_+(x,y)}{\|x\|^2}x+y. \tag{3.8}$$

But, by [11, Proposition 2.9], $\perp_{\rho_\lambda^v}\subseteq\perp_B$. Then $x\perp_B\frac{-\rho_+(x,y)}{\|x\|^2}x+y$ and so by (2) we conclude that $Tx\perp_B^{\varepsilon}\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty$. It follows from Lemma 2.3 that

$$\begin{aligned}
&\rho_-\Big(Tx,\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty\Big)-\varepsilon\|Tx\|\Big\|\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty\Big\|\\
&\qquad\le 0\le\rho_+\Big(Tx,\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty\Big)+\varepsilon\|Tx\|\Big\|\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty\Big\|,
\end{aligned}$$

which implies that

$$\begin{aligned}\frac{-\rho_+(x,y)}{\|x\|^2}\|Tx\|^2+\rho_-(Tx,Ty)-\varepsilon\|Tx\|\Big\|\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty)\Big\|\\ \le 0\\ \le\frac{-\rho_+(x,y)}{\|x\|^2}\|Tx\|^2+\rho_+(Tx,Ty)+\varepsilon\|Tx\|\Big\|\frac{-\rho_+(x,y)}{\|x\|^2}Tx+Ty)\Big\|. \quad (3.9)\end{aligned}$$

Therefore, we can substitute ρ_- and ρ_+ by ρ_λ^v in (3.9) to obtain

$$\begin{aligned}\frac{-\rho_\lambda^v(x,y)}{\|x\|^2}\|Tx\|^2+\rho_\lambda^v(Tx,Ty)-\varepsilon\|Tx\|\Big\|\frac{-\rho_\lambda^v(x,y)}{\|x\|^2}Tx+Ty\Big\|\\ \le 0\le\frac{-\rho_\lambda^v(x,y)}{\|x\|^2}\|Tx\|^2+\rho_\lambda^v(Tx,Ty)+\varepsilon\|Tx\|\Big\|\frac{-\rho_\lambda^v(x,y)}{\|x\|^2}Tx+Ty\Big\|.\end{aligned}$$

That is,

$$\Big|\frac{-\rho_\lambda^v(x,y)}{\|x\|^2}\|Tx\|^2+\rho_\lambda^v(Tx,Ty)\Big|\le\varepsilon\|Tx\|\Big\|\frac{-\rho_\lambda^v(x,y)}{\|x\|^2}Tx+Ty\Big\|. \quad (3.10)$$

Now, suppose that $a,b\in X$ are linearly independent. Define $x^*\in X^*$ by $x^*(\alpha a+\beta b):=\alpha$. Then $M:=\{x\in X : x^*(x)=1\}=\{a+tb : t\in\mathbb{R}\}$ is a hyperplane and $0\notin M$. According to Lemma 3.3, we have $(\overline{M\cap S})=M$. Hence, there is a sequence $\{a+t_nb\}$ in S such that $\lim_{n\to\infty}(a+t_nb)=a$ and $\lim_{n\to\infty}t_n=0$. Let $x=a+t_nb$ and $y=b$ in (3.10); thus, we get

$$\begin{aligned}\Big|\frac{-\rho_\lambda^v(a+t_nb,b)}{\|a+t_nb\|^2}\big\|T(a+t_nb)\big\|^2+\rho_\lambda^v\big(T(a+t_nb),Tb\big)\Big|\\ \le\varepsilon\big\|T(a+t_nb)\big\|\Big\|\frac{-\rho_\lambda^v(a+t_nb,b)}{\|a+t_nb\|^2}T(a+t_nb)+Tb)\Big\|. \quad (3.11)\end{aligned}$$

Since the mappings $t\mapsto\rho_\pm(x+ty,y)$ are continuous at zero (see [2, Corollary 2.1.2]), we conclude that $\rho_\lambda^v(x,y)=\lim_{t\to0}\rho_\lambda^v(x+ty,y)$ for all $x,y\in X$. Then (3.11) becomes

$$\Big|\frac{-\rho_\lambda^v(a,b)}{\|a\|^2}\|Ta\|^2+\rho_\lambda^v(Ta,Tb)\Big|\le\varepsilon\|Ta\|\Big\|\frac{-\rho_\lambda^v(a,b)}{\|a\|^2}Ta+Tb\Big\| \quad (3.12)$$

for all linearly independent $a,b\in X$. Now, suppose that $x,y\in X\setminus\{0\}$ such that $x\perp_{\rho_\lambda^v}y$. Then it is clear that x and y are linearly independent. It follows from (3.12) that $|\rho_\lambda^v(Tx,Ty)|\le\varepsilon\|Tx\|\|Ty\|$. Therefore, $x\perp^\varepsilon_{\rho_\lambda^v}y$.

Case 2. Assume that $\dim(X)\ge2$ and $\dim(Y)\ge2$. If $x,y\in X\setminus\{0\}$ and $x\perp_{\rho_\lambda^v}y$, then x and y are linearly independent. Define $\tilde{T}:\operatorname{span}\{x,y\}\longrightarrow W$ such that $W\subseteq Y$ is a 2-dimensional subspace such that $\tilde{T}(\operatorname{span}\{x,y\})\subseteq W$ and $\tilde{T}=T|_{\operatorname{span}\{x,y\}}$. By case 1, $\tilde{T}$ approximately preserves ρ_λ^v-orthogonality. So we get $\tilde{T}x\perp^\varepsilon_{\rho_\lambda^v}\tilde{T}y$ and therefore $Tx\perp^\varepsilon_{\rho_\lambda^v}Ty$. □

As a special case, if we take $\varepsilon=0$ in the foregoing theorem, then since every B-orthogonality-preserving linear mapping is a similarity (see [5], [15]), we conclude that every ρ_λ^v-orthogonality-preserving linear mapping is a similarity.

Remark 3.5. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed linear spaces, let $\varepsilon \in [0,1)$, and let $\lambda \in (0,1)$ be such that $\lambda \neq \frac{1}{2}$. If we substitute ρ_λ instead of ρ in the proof of Theorem 5.3 of [21], then we conclude that every approximately ρ_λ-orthogonality-preserving linear mapping $T : X \to Y$ is an approximately ρ_+-preserving mapping and so is an approximately B-orthogonality-preserving mapping, by Theorem 3.2. Also, since $x \perp_{\rho_\lambda} \frac{-\rho_\lambda(x,y)}{\|x\|^2}x + y$ for all nonzero $x \in X$ and all $y \in X$, if we replace $\rho_+(x,y)$ with $\rho_\lambda(x,y)$ in (3.8), then it is easy to see that the implication (2) $\Rightarrow$ (1) of Theorem 3.4 is valid for $v = 1$. Thus, every approximately B-orthogonality-preserving linear mapping is an approximately ρ_λ-orthogonality-preserving mapping. Consequently, we have the next result.

Corollary 3.6. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed linear spaces, let $\varepsilon \in [0,1)$, and let $\lambda \in (0,1)$ be such that $\lambda \neq \frac{1}{2}$. If $T : X \to Y$ is a nonzero linear mapping, then the following conditions are equivalent:*

(1) *For each $x, y \in X$, $x \perp_{\rho_\lambda} y \Rightarrow Tx \perp^{\varepsilon}_{\rho_\lambda} Ty$.*
(2) *For each $x, y \in X$, $x \perp_B y \Rightarrow Tx \perp^{\varepsilon}_B Ty$.*

In particular, if we take $\varepsilon = 0$ in Corollary 3.6, then we obtain that every ρ_λ-orthogonality-preserving mapping is a similarity. We should note that this result was already shown in [22, Theorem 4.4] with a different approach.

Finally, combining Theorems 3.2 and 3.4, Remark 3.5, and Corollary 3.6, we get the following result.

Corollary 3.7. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed linear spaces, let $\varepsilon \in [0,1)$, let $\lambda \in [0,1]$, and let $v = \frac{1}{2k+1}$ $(k \in \mathbb{N})$. A linear mapping $T : X \to Y$ approximately preserves ρ^v_λ-orthogonality if and only if T approximately preserves $\diamondsuit$-orthogonality for which $\diamondsuit \in \{\rho_\pm, \rho, \rho_*, \rho_\lambda, B\}$. Moreover, if $\varepsilon \in [0, \frac{1}{8})$ and T approximately preserves ρ^v_λ-orthogonality, then*

$$(1 - 8\varepsilon)\|T\|\|x\| \leq \|Tx\| \leq \|T\|\|x\| \qquad (x \in X).$$

Acknowledgments. The authors would like to express their sincere gratitude to the anonymous referees for helpful comments that improved the presentation of this article.

References

1. N. Abasov and M. Pliev, *Disjointness-preserving orthogonally additive operators in vector lattices*, Banach J. Math. Anal. **12** (2018), no. 3, 730–750. Zbl 06946079. MR3824749. DOI 10.1215/17358787-2018-0001. 489
2. C. Alsina, J. Sikorska, and M. S. Tomás, *Norm Derivatives and Characterizations of Inner Product Spaces*, World Scientific, Hackensack, NJ, 2010. Zbl 1196.46001. MR2590240. 484, 488, 493
3. D. Amir, *Characterizations of Inner Product Spaces*, Oper. Theory Adv. Appl. **20**, Birkhäuser, Basel, 1986. Zbl 0617.46030. MR0897527. DOI 10.1007/978-3-0348-5487-0. 484
4. G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), no. 2, 169–172. Zbl 0012.30604. MR1545873. DOI 10.1215/S0012-7094-35-00115-6. 483
5. A. Blanco and A. Turnšek, *On maps that preserve orthogonality in normed spaces*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), no. 4, 709–716. Zbl 1115.46016. MR2250441. DOI 10.1017/S0308210500004674. 489, 493

6. C. Chen and F. Lu, *Linear maps preserving orthogonality*, Ann. Funct. Anal. **6** (2015), no. 4, 70–76. Zbl 1330.47044. MR3365982. DOI 10.15352/afa/06-4-70. 484, 489
7. J. Chmieliński, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **304** (2005), no. 1, 158–169. Zbl 1090.46017. MR2124655. DOI 10.1016/j.jmaa.2004.09.011. 485, 489
8. J. Chmieliński, *On an ε-Birkhoff orthogonality*, JIPAM J. Inequal. Pure Appl. Math. **6** (2005), no. 3, art. ID 79. Zbl 1095.46011. MR2164320. 485
9. J. Chmieliński and P. Wójcik, *On a ρ-orthogonality*, Aequationes Math. **80** (2010), no. 1–2, 45–55. Zbl 1208.46015. MR2736939. DOI 10.1007/s00010-010-0042-1. 484, 485
10. J. Chmieliński and P. Wójcik, "On ρ-orthogonality and its preservation revisited" in *Recent Developments in Functional Equations and Inequalities*, Banach Center Publ. **99**, Polish Acad. Sci. Inst. Math., Warsaw, 2013, 17–30. Zbl 1292.46007. MR3204073. DOI 10.4064/bc99-0-2. 484, 485, 487, 488, 489, 490
11. M. Dehghani, M. Abed, and R. Jahanipur, *A generalized orthogonality relation via norm derivatives in real normed linear spaces*, Aequationes Math. **93** (2019), no. 4, 651–667. Zbl 07086047. MR3984319. DOI 10.1007/s00010-018-0598-8. 484, 487, 492
12. M. Dehghani and A. Zamani, *Linear mappings approximately preserving ρ_*-orthogonality*, Indag. Math. (N.S.) **28** (2017), no. 5, 992–1001. Zbl 1383.46014. MR3697043. DOI 10.1016/j.indag.2017.07.001. 485, 487, 490
13. S. S. Dragomir, *Semi-Inner Products and Applications*, Nova Sci., Hauppauge, NY, 2004. Zbl 1060.46001. MR2047793. 484, 488
14. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947). 265–292. Zbl 0037.08001. MR0021241. DOI 10.2307/1990220. 483, 484
15. A. Koldobsky, *Operators preserving orthogonality are isometries*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 5, 835–837. Zbl 0806.46013. MR1249689. DOI 10.1017/S0308210500029528. 489, 493
16. P. M. Miličić, *Sur la g-orthogonalité dans des espaces normés*, Mat. Vesnik **39** (1987), no. 3, 325–334. Zbl 0652.46011. MR0935690. 484
17. B. Mojškerc and A. Turnšek, *Mappings approximately preserving orthogonality in normed spaces*, Nonlinear Anal. **73** (2010), no. 12, 3821–3831. Zbl 1208.46016. MR2728557. DOI 10.1016/j.na.2010.08.007. 489
18. M. S. Moslehian, A. Zamani, and M. Dehghani, *Characterizations of smooth spaces by ρ_*-orthogonality*, Houston J. Math. **43** (2017), no. 4, 1187–1208. Zbl 1395.46012. MR3766364. 484
19. K. Paul, D. Sain, A. Mal, and K. Mandal, *Orthogonality of bounded linear operators on complex Banach spaces*, Adv. Oper. Theory **3** (2018), no. 3, 699–709. Zbl 1404.46015. MR3795110. DOI 10.15352/aot.1712-1268. 484
20. P. Wójcik, *Linear mappings preserving ρ-orthogonality*, J. Math. Anal. Appl. **386** (2012), no. 1, 171–176. Zbl 1238.46014. MR2834875. DOI 10.1016/j.jmaa.2011.07.062. 484, 490
21. P. Wójcik, *Linear mappings approximately preserving orthogonality in real normed spaces*, Banach J. Math. Anal. **9** (2015), no. 2, 134–141. Zbl 1334.46015. MR3296111. DOI 10.15352/bjma/09-2-11. 490, 492, 494
22. A. Zamani and M. S. Moslehian, *An extension of orthogonality relations based on norm derivatives*, Q. J. Math. **70** (2019), no. 2, 379–393. Zbl 07086668. MR3975515. DOI 10.1093/qmath/hay048. 484, 494

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, P. O. Box 87317-53153, Kashan, Iran.

E-mail address: alhussanawe12@gmail.com; mohammed.y@uokerbala.edu.iq; m.dehghani@kashanu.ac.ir; e.g.mahdi@gmail.com; jahanipu@kashanu.ac.ir