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THE REDUCIBILITY OF COMPRESSED SHIFTS ON A CLASS OF QUOTIENT MODULES OVER THE BIDISK

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Abstract. In this paper, we show that, for the rational inner function $\theta(z,w)=\frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$, S_z is reducible on the quotient module $\mathcal{K}_\theta=H^2\ominus\theta H^2$ over the bidisk if and only if θ is the product of two one-variable inner functions.

1. Introduction

Let $\mathbb{D}^2$ denote the open-unit bidisk in $\mathbb{C}^2$, and let $\mathbb{T}^2$ denote the distinguished boundary of $\mathbb{D}^2$. The Hardy space $H^2=H^2(\mathbb{D}^2)$ is the closure of polynomials in the usual square integrable space $L^2(\mathbb{T}^2)$. On $H^2(\mathbb{D}^2)$, the Toeplitz operators T_z and T_w are unilateral shifts of infinity multiplicity. For the two variable inner functions $\theta(z,w)$ on $\mathbb{D}^2$ (namely, a function holomorphic on $\mathbb{D}^2$ with boundary values of modulus 1 almost everywhere on $\mathbb{T}^2$), the associated model space is defined by

$$\mathcal{K}_\theta=H^2(\mathbb{D}^2)\ominus\theta H^2(\mathbb{D}^2),$$

and the compressed shifts are defined by

$$S_z=P_\theta T_z|_{\mathcal{K}_\theta},\qquad S_w=P_\theta T_w|_{\mathcal{K}_\theta},$$

where P_θ is the orthogonal projection from $H^2(\mathbb{D}^2)$ onto $\mathcal{K}_\theta$. For a bounded analytic function φ on $\mathbb{D}^2$, we also denote $S_\varphi=P_\theta T_\varphi|_{\mathcal{K}_\theta}$. Important recent work has been done on the operator theory and function theory on Hardy space over the polydisk (see, e.g., [3], [4], [9], [10], [12] and the references therein).

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In 1990, Agler [1] showed that every analytic contraction θ admits the following decomposition:

$$\frac{1-\overline{\theta(\lambda_1,\lambda_2)}\theta(z,w)}{(1-\overline{\lambda}_1 z)(1-\overline{\lambda}_2 w)} = \frac{K_1(z,w,\lambda_1,\lambda_2)}{1-\overline{\lambda}_1 z} + \frac{K_2(z,w,\lambda_1,\lambda_2)}{1-\overline{\lambda}_2 w},$$

where $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ are two positive kernels. Therefore, for an inner function θ, we see that $\mathcal{K}_\theta$ can be decomposed (see [2]) as

$$\mathcal{K}_\theta = \mathcal{S}_1 \oplus \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{H}(\frac{K_1}{1-\overline{\lambda}_1 z})$ and $\mathcal{S}_2 = \mathcal{H}(\frac{K_2}{1-\overline{\lambda}_2 w})$ are z-invariant and w-invariant, respectively, and $\mathcal{H}(K)$ denotes the reproducing kernel Hilbert space with reproducing kernel K. These types of subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are called *Agler subspaces* of θ.

Let T be a bounded linear operator on a Hilbert space H. A closed subspace M of H is called a *reducing subspace* of T if $TM \subset M$ and $T^*M \subset M$. It is obvious that $\{0\}$ and H are trivial reducing subspaces for T. If T has a nontrivial reducing subspace, we say that T is reducible, otherwise, we say that T is irreducible. The classification of reducing subspaces of various operators on function spaces has proved to be one very rewarding research problem in analysis: insights on the reducing subspaces of multiplication operators on the Bergman space can be found in [6], [8], [13], and insights on the reducing subspace of truncated Toeplitz operators can be found in [5] and [11]. In [4], it was proved that, for a rational inner function θ on $\mathbb{D}^2$, there is a pair of Agler subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ which are reducing subspaces for S_z if and only if θ is the product of two one-variable inner functions. The main result that we focus on in this present paper is the reducibility of S_z. For the rational inner function $\theta(z,w) = \frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$ of degree $(1,1)$, we extend the result in [4] as follows.

Theorem 1.1 (Main Theorem). *Let $\theta(z,w) = \frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$ be a rational inner function in $H^2(\mathbb{D}^2)$. Then S_z is reducible on $\mathcal{K}_\theta$ if and only if θ is the product of two one-variable inner functions.*

2. Proof of the main theorem

In this section, we will prove the main Theorem 1.1, and the proof consists of several steps. First, we need some notation.

Let $\theta = \frac{q}{p}$ be a rational inner function on $\mathbb{D}^2$ such that p and q are polynomials with no common factors. The degree of θ, denoted by $\deg\theta = (m,n)$, is defined by the following: m is the highest degree of z and n the highest degree of w appearing in either p or q. Moreover, if θ is rational with $\deg\theta = (m,n)$, there is an almost unique polynomial p with no zeros in $\mathbb{D}^2$ such that $\theta = \frac{\widetilde{p}}{p}$, where $\widetilde{p} = z^m w^n \overline{p(\frac{1}{\bar{z}}, \frac{1}{\bar{w}})}$ (see [12, Theorem 5.2.5]). Then we see that the rational function θ with degree $(1,1)$ is of the form $\theta = \frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$.

In the following, denote $p(z,w) = 1+bz+cw+dzw$ if $\theta = \frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$, and let $Z(p)$ denote the zero set of p. For $\lambda \in \mathbb{D}$, let $\varphi_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$ be the Blaschke factor.

The following lemma comes from [9], which affords the crucial simplification for our proof.

Lemma 2.1 ([9, Proposition 3.7, Theorem 3.2])**.** *Let* $\theta = \frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$ *with* $d \neq bc$. *Then we have the following.*

(1) *If* $Z(p) \cap \mathbb{T}^2 = \emptyset$, *then* S_z *on* $\mathcal{K}_\theta = H^2 \ominus \theta H^2$ *is unitarily equivalent to* $S_{\varphi_{\lambda_1}(z)}$ *or* $S_{\varphi_{\lambda_2}(w)}$ *on* $H^2 \ominus \frac{zw-t}{1-tzw} H^2$ *for some* t *with* $0 < t < 1$, *and* $\lambda_1, \lambda_2 \in \mathbb{D}$.
(2) *If* $Z(p) \cap \mathbb{T}^2 \neq \emptyset$, *then* S_z *on* $\mathcal{K}_\theta = H^2 \ominus \theta H^2$ *is unitarily equivalent to* $S_{\varphi_{\lambda_1}(z)}$ *or* $S_{\varphi_{\lambda_2}(w)}$ *on* $H^2 \ominus \frac{zw-tz-(1-t)w}{1-tw-(1-t)z} H^2$ *for some* t *with* $0 < t < 1$, *nd* $\lambda_1, \lambda_2 \in \mathbb{D}$.

Since p has no more than 1 zero on $\mathbb{T}^2$ (see [10]), we divide into two cases $Z(p) \cap \mathbb{T}^2 = \emptyset$ and $Z(p) \cap \mathbb{T}^2 \neq \emptyset$. We first consider the case for $\theta = \frac{zw-t}{1-tzw}$.

Lemma 2.2 ([9, Lemma 3.3])**.** *For* $0 < t < 1$, $\theta = \frac{zw-t}{1-tzw}$. *Then*

$$\left\{\dots, f_2 = (1+t\theta)w^2, f_1 = (1+t\theta)w, e_0 = (1+t\theta), e_1 = (1+t\theta)z, \dots\right\}$$

is an orthogonal basis of $\mathcal{K}_\theta = H^2 \ominus \frac{zw-t}{1-tzw} H^2$.

It is easy to check that $\|e_n\| = \|f_n\| = \sqrt{1-t^2}$. We have the following corollary.

Corollary 2.3. *On* $\mathcal{K}_\theta = H^2 \ominus \frac{zw-t}{1-tzw} H^2$, *we have that*

(1) $S_z e_n = e_{n+1}$, $n = 0, 1, \dots$.
(2) $S_z f_1 = te_0$. $S_z f_{n+1} = tf_n$, $n = 1, 2, \dots$.
(3) $S_z^* e_0 = tf_1$, $S_z^* e_n = e_{n-1}$, $n = 1, 2, \dots$.
(4) $S_z^* f_n = tf_{n+1}$, $n = 1, 2, \dots$.

Proof. Item (1) is obvious. For (2),

$$\begin{aligned} S_z f_1 &= P_\theta (1+t\theta) zw \\ &= P_\theta (zw) \\ &= P_\theta\big(t + \theta(1-tzw)\big) \\ &= tP_\theta 1 \\ &= tP_\theta (1+t\theta) \\ &= te_0, \\ S_z f_{n+1} &= P_\theta (1+t\theta) zw^{n+1} \\ &= P_\theta (zw^{n+1}) \\ &= P_\theta\big(t + \theta(1-tzw)\big) w^n \\ &= tP_\theta w^n \\ &= tP_\theta (1+t\theta) w^n \\ &= tf_n. \end{aligned}$$

The equalities (3) and (4) follow from (1) and (2) easily. □

Theorem 2.4. *For* $\theta = \frac{zw-t}{1-tzw}$ *with* $0 < t < 1$, S_z *is irreducible on* $\mathcal{K}_\theta$.

Proof. Let K be a reducing subspace of S_z and pick a nonzero function

$$h=\sum_{i=0}^{\infty}x_ie_i+\sum_{i=1}^{\infty}y_if_i\in K.$$

Then

$$S_zS_z^*h=S_z\Big(x_0tf_1+\sum_{i=0}^{\infty}x_{i+1}e_i+\sum_{i=1}^{\infty}y_itf_{i+1}\Big)=x_0t^2e_0+\sum_{i=1}^{\infty}x_ie_i+t^2\sum_{i=1}^{\infty}y_if_i,$$

and

$$S_z^*S_zh=S_z^*\Big(\sum_{i=0}^{\infty}x_ie_{i+1}+y_1te_0+\sum_{i=1}^{\infty}y_{i+1}tf_i\Big)=\sum_{i=0}^{\infty}x_ie_i+t^2\sum_{i=1}^{\infty}y_if_i.$$

Therefore, we have

$$\sum_{i=0}^{\infty}x_ie_i,\sum_{i=1}^{\infty}y_if_i\in K,$$

and hence $x_0e_0\in K$.

If there is j such that $x_j\neq 0$, then let x_n be the first x_j such that $x_j\neq 0$.

$$\begin{aligned}S_z^{n+1}S_z^{*n+1}(x_ne_n+\cdots)&=S_z^{n+1}(x_ntf_1+x_{n+1}e_0+\cdots)\\&=x_nt^2e_n+x_{n+1}e_{n+1}+\cdots\end{aligned}$$

We obtain $e_n\in K$ and therefore $K=\mathcal{K}_\theta$.

If all $x_n=0$, which means that $K\subset\bigvee\{f_1,f_2,\ldots\}$, then by considering $K^\perp\supseteq\bigvee\{e_0,e_1,\ldots\}$, we get

$$K^\perp=\mathcal{K}_\theta,$$

where the symbol $\bigvee$ denotes the closed linear span in the corresponding space. This completes the proof. □

By the symmetry of z and w, S_w is also irreducible on $\mathcal{K}_\theta=H^2\ominus\frac{zw-t}{1-tzw}H^2$.

Corollary 2.5. *For $\theta=\frac{zw-t}{1-tzw}$ with $0<t<1$, $S_{\varphi_{\lambda_1}(z)}$ and $S_{\varphi_{\lambda_2}(w)}$ are both irreducible on $\mathcal{K}_\theta$.*

Proof. By the symmetry of z and w, it suffices that we prove for $S_{\varphi_{\lambda_1}(z)}$. Since $\varphi_{\lambda_1}(z)=\sum_{n=0}^{\infty}c_nz^n$ is convergent absolutely on $\overline{\mathbb{D}}$, we have that $\sum_{n=0}^{\infty}|c_n|<\infty$. For $f\in\mathcal{K}_\theta$, it follows from $\sum_{n=0}^{\infty}\|c_nz^nf\|<\infty$ that $P_\theta(\sum_{n=0}^{\infty}c_nz^nf)=\sum_{n=0}^{\infty}c_nP_\theta z^nf$, and hence

$$S_{\varphi_{\lambda_1}(z)}=\sum_{j=0}^{\infty}c_nS_z^n.$$

Therefore, the reducing subspace for S_z also reduces $S_{\varphi_{\lambda_1}(z)}$. Let $\varphi_{\lambda_1}^{-1}(z)=\sum_{n=0}^{\infty}d_nz^n$ be a power-series expansion of $\varphi_{\lambda_1}^{-1}$. Then it is also not hard to see

$$z=\sum_{j=0}^{\infty}d_n\big(\varphi_{\lambda_1}(z)\big)^n.$$

It follows that S_z and $S_{\varphi_{\lambda_1}(z)}$ have the same reducing subspace and hence that $S_{\varphi_{\lambda_1}(z)}$ is irreducible. □

In the following, we will consider the case for $\theta = \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}$. We denote the one-variable Hardy space in z and w by H_z^2 and H_w^2, respectively. The following facts come from work by Bickel and Gorkin and by Bickel and Liaw.

Lemma 2.6 ([3], [4])**.** *For* $\theta = \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}$, $0 < t < 1$, $\mathcal{K}_\theta$ *can be decomposed as*

$$\mathcal{K}_\theta = gH_z^2 \oplus fH_w^2,$$

where $g = \gamma\frac{z-1}{p}$, $f = \delta\frac{w-1}{p}$, *and* $\gamma^2 = 1-t$, $\delta^2 = t$. *Moreover, for* $f_j(w) \in H_w^2$ *and* $g_j(z) \in H_z^2$, $j = 1, 2$, *we have*

$$\big\langle f_1(w)f, f_2(w)f\big\rangle_{\mathcal{K}_\theta} = \langle f_1, f_2\rangle_{H_w^2}, \qquad \big\langle g_1(z)g, g_2(z)g\big\rangle_{\mathcal{K}_\theta} = \langle g_1, g_2\rangle_{H_z^2},$$

where $\langle\cdot,\cdot\rangle$ *means the inner product in the corresponding space.*

For simplicity, we can assume that $\gamma = \sqrt{1-t}$, $\delta = \sqrt{t}$. In the following, we write $\mathcal{S}_1 = gH_z^2$ and $\mathcal{S}_2 = fH_w^2$, respectively. For a bounded analytic function φ on $\mathbb{D}$, T_φ denotes the Toeplitz operator on Hardy space $H^2(\mathbb{D})$ in one variable. The following calculations are key for the proof.

Lemma 2.7. *For* $\theta = \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}$ *with* $0 < t < 1$, $g_0 \in H_z^2$ *and* $f_0 \in H_w^2$, *the following hold:*

(1) $S_z^*(f_0 f) = \frac{1-t}{1-tw} f_0 f$,
(2) $S_z^*(g_0 g) = -g_0(0)\frac{\sqrt{t(1-t)}}{1-tw} f + (T_z^* g_0)g$,
(3) $S_z(f_0 f) = -\sqrt{t(1-t)}f_0(t)g + (T^*_{\frac{1-t}{1-tw}} f_0)f$.

Proof. (1) By Lemma 4.2 in [3], we have $S_z^* f = \frac{1-t}{1-tw} f$. Therefore, $S_z^*(f_0 f) = f_0 S_z^* f = \frac{1-t}{1-tw} f_0 f$.

(2) The definitions of g and f give

$$\begin{aligned} S_z^* g &= \frac{g - g(0,w)}{z} \\ &= \frac{\gamma}{z}\Big(\frac{z-1}{p} - \frac{-1}{p(0,w)}\Big) \\ &= \frac{\gamma}{z}\frac{zp(0,w) + (p - p(0,w))}{pp(0,w)} \\ &= \gamma\frac{\delta(w-1)}{p}\frac{(1-tw)+(t-1)}{\delta(w-1)(1-tw)} \\ &= -\frac{\sqrt{t(1-t)}}{1-tw} f. \end{aligned}$$

For $g_0 \in H^2_z$, we have

$$\begin{aligned} S_z^*(g_0 g) &= \frac{g_0 g - g_0(0) g(0,w)}{z} \\ &= g_0(0)\frac{g - g(0,w)}{z} + g\frac{g_0 - g_0(0)}{z} \\ &= -g_0(0)\frac{\sqrt{t(1-t)}}{1-tw} f + (T_z^* g_0) g. \end{aligned}$$

The formula in (3) comes from the article [3]. However, for the reader's convenience, we include the calculations here.

$$\begin{aligned} S_z\big(f_0(w)f\big) &= \sum_{k=0}^{\infty} \langle z f_0(w) f, z^k g\rangle z^k g + \sum_{k=0}^{\infty} \langle z f_0(w) f, w^k f\rangle w^k f \\ &= \langle f_0(w) f, S_z^* g\rangle g + \sum_{k=0}^{\infty} \langle f_0(w) f, w^k S_z^* f\rangle w^k f \\ &= -\sqrt{t(1-t)}\Big\langle f_0(w), \frac{1}{1-tw}\Big\rangle_{H^2_w} g + \sum_{k=0}^{\infty} \Big\langle f_0(w), w^k \frac{1-t}{1-tw}\Big\rangle_{H^2_w} w^k f \\ &= -\sqrt{t(1-t)} f_0(t) g + (T^*_{\frac{1-t}{1-tw}} f_0) f. \end{aligned}$$

In particular, we have $S_z f = -\sqrt{t(1-t)}g + (1-t)f$. □

The following lemma is of interest by itself.

Lemma 2.8. *Let $\phi(w) = \frac{b+dw}{1+cw}$ such that $d \neq bc$ and $|c| < 1$. We have*

$$\bigvee\{\phi^n : n \geq 0\} = H^2_w.$$

Proof. We first find $x_0, x_1 \in \mathbb{C}$ such that

$$x_0 + x_1\phi = \frac{1}{1+cw},$$

which is equivalent to solving the equation

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

Note that since $d \neq bc$, the determinant of the matrix is nonzero, so equation (2.1) has a solution, which shows that

$$\frac{1}{1+cw} \in \bigvee\{\phi^n : n \geq 0\}.$$

By the same argument, we can obtain

$$\frac{w}{1+cw} \in \bigvee\{\phi^n : n \geq 0\}.$$

Assume that for a fixed positive integer n, we have proved that

$$\frac{1}{1+cw}, \frac{w}{1+cw}, \dots, \frac{1}{(1+cw)^{n-1}}, \frac{w}{(1+cw)^{n-1}}, \dots, \frac{w^{n-1}}{(1+cw)^{n-1}} \in \bigvee\{\phi^n : n \geq 0\}.$$

For $k = 0, 1, \dots, n$, we want to find $x_0, x_1, \dots, x_n \in \mathbb{C}$ such that

$$x_0 \frac{1}{(1+cw)^{n-1}} + x_1 \frac{w}{(1+cw)^{n-1}} + \dots + x_{n-1} \frac{w^{n-1}}{(1+cw)^{n-1}} + x_n \phi^n = \frac{w^k}{(1+cw)^n},$$

which is equivalent to solving

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & b^n \\ c & 1 & 0 & \cdots & C_n^1 b^{n-1} d \\ 0 & c & 1 & \cdots & C_n^2 b^{n-2} d^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c & d^n \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.2}$$

By some calculations, it is not hard to see that the determinant of the matrix is $(d - bc)^n$, so the equation (2.2) has a solution. Therefore,

$$\frac{w^k}{(1+cw)^n} \in \bigvee\{\phi^n : n \geq 0\}, \quad k = 0, 1, \dots, n. \tag{2.3}$$

It follows from (2.3) that for an arbitrary positive integer n,

$$\frac{\partial^n}{\partial(-c)^n} \frac{1}{1+cw} \in \bigvee\{\phi^n : n \geq 0\}.$$

Let $F \in H_w^2$ such that $F \perp \bigvee\{\phi^n : n \geq 0\}$, then

$$F^{(n)}(-\bar{c}) = \left\langle F, \frac{\partial^n}{\partial(-c)^n} \frac{1}{1+cw} \right\rangle = 0.$$

It follows that $F = 0$, which completes the proof. $\square$

Corollary 2.9. *For $0 < t < 1$ and fixed nonnegative integer n_0, we have*

$$\bigvee\left\{\left(\frac{1}{1-tw}\right)^n : n \geq n_0\right\} = H_w^2.$$

Proof. Note that for any $f \in H_w^2$, we have $(1-tw)^{n_0} f \in H_w^2$. By Lemma 2.8, there exists a sequence of polynomials $\{p_n\}_{n=1}^\infty$ such that $\{p_n(\frac{1}{1-tw})\}_{n=1}^\infty$ converges to $(1-tw)^{n_0} f$ in H_w^2. Since $(\frac{1}{1-tw})^{n_0}$ is a bounded analytic function, we obtain that $\{(\frac{1}{1-tw})^{n_0} p_n(\phi)\}_{n=1}^\infty$ converges to f in H_w^2. Therefore $f \in \bigvee\{(\frac{1}{1-tw})^n : n \geq n_0\}$ and this completes the proof. $\square$

For a function $h \in \mathcal{K}_\theta$, let $[h]$ be the smallest reducing subspace for S_z that contains h. We have the following lemma.

Lemma 2.10. *Let $\theta = \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}$, $0 < t < 1$, and g, f be as in Lemma 2.6. Then $[g] = \mathcal{K}_\theta$ and $[f] = \mathcal{K}_\theta$.*

Proof. It is easy to see that

$$z^k g \in [g], \quad k = 0, 1, \dots.$$

By Lemma 2.7, $S_z^* g = \frac{\sqrt{t(1-t)}}{1-tw} f \in [g]$, which means that

$$\frac{f}{1-tw} \in [g].$$

For $n = 0, 1, \dots,$

$$S_z^{*n}\Big(\frac{f}{1-tw}\Big) = \Big(\frac{1-t}{1-tw}\Big)^n \frac{f}{1-tw} \in [g].$$

Combining with Corollary 2.9 and Lemma 2.6, we know that

$$\bigvee\Big\{\Big(\frac{1}{1-tw}\Big)^n f : n \geq 1\Big\} = fH_w^2 \subseteq [g],$$

and hence $[g] = \mathcal{K}_\theta$.

Since

$$S_z f = -\sqrt{t(1-t)}g + (1-t)f \in [f],$$

we have $g \in [f]$, and hence $[f] = \mathcal{K}_\theta$, and this completes the proof. □

Let $P_{\mathcal{S}_1}$ and $P_{\mathcal{S}_2}$ be the orthogonal projection from $\mathcal{K}_\theta$ onto $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.

Corollary 2.11. *If K is a nontrivial reducing subspace for S_z, then $P_{\mathcal{S}_1}K \neq 0$ and $P_{\mathcal{S}_2}K \neq 0$.*

Proof. If $P_{\mathcal{S}_2}K = 0$, then $K \subset \mathcal{S}_1$, and $f \in \mathcal{K}_\theta \ominus K$. Then $\mathcal{K}_\theta \ominus K = \mathcal{K}_\theta$. This is a contradiction. By a similar argument, we obtain that $P_{\mathcal{S}_1}K \neq 0$. □

Lemma 2.12. *Let $\theta = \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}$, $0 < t < 1$, and K be a reducing subspace for S_z. Then either* $\operatorname{clos} P_{\mathcal{S}_1}K = gH_z^2$ *or* $\operatorname{clos} P_{\mathcal{S}_1}(\mathcal{K}_\theta \ominus K) = gH_z^2$, *where* clos *denotes the norm closure in $\mathcal{K}_\theta$.*

Proof. If $K = \{0\}$ or $\mathcal{K}_\theta$, it is obvious that $\operatorname{clos} P_{\mathcal{S}_1}K = gH_z^2$ or $\operatorname{clos} P_{\mathcal{S}_1}(\mathcal{K}_\theta \ominus K) = gH_z^2$.

Now for a nontrivial reducing subspace K, let $h = g_0(z)g + f_0(w)f \in K$. Then it follows from Corollary 2.7 that

$$\begin{aligned} S_z^* h &= -g_0(0)\frac{\sqrt{t(1-t)}}{1-tw}f + (T_z^* g_0)g + \frac{1-t}{1-tw}f_0(w)f \\ &= (T_z^* g_0)g + f_1(w)f, \end{aligned}$$

where

$$f_1(w) = -g_0(0)\frac{\sqrt{t(1-t)}}{1-tw} + \frac{1-t}{1-tw}f_0(w). \tag{2.4}$$

Then,

$$\begin{aligned} S_z^{*2}h &= S_z^*\big((T_z^*g_0)g + f_1(w)f\big) \\ &= -(T_z^*g_0)(0)\frac{\sqrt{t(1-t)}}{1-tw}f + (T_z^{*2}g_0)g + \frac{1-t}{1-tw}f_1(w)f \\ &= (T_z^{*2}g_0)g + f_2(w)f \end{aligned}$$

for some $f_2 \in H_w^2$. Then we have

$$\begin{aligned} S_zS_z^*h &= S_z\big((T_z^*g_0)g + f_1(w)f\big) \\ &= \big(g_0(z) - g_0(0)\big)g - \sqrt{t(1-t)}f_1(t)g + (T^*_{\frac{1-t}{1-tw}}f_1)f. \end{aligned}$$

Since $g_0(z)g \in P_{\mathcal{S}_1}K$, we have

$$-\sqrt{t(1-t)}f_1(t)g - g_0(0)g \in P_{\mathcal{S}_1}K. \tag{2.5}$$

Claim. *For a nontrivial reducing subspace K, either $g \in P_{\mathcal{S}_1}K$ or $g \in P_{\mathcal{S}_1}(\mathcal{K}_\theta \ominus K)$.*

Proof of Claim. If there exist $h(z,w) = g_0(z)g + f_0(w)f \in K$ such that

$$-\sqrt{t(1-t)}f_1(t) - g_0(0) \neq 0,$$

where f_1 is defined as (2.4), then by (2.5), $g \in P_{\mathcal{S}_1}K$. Otherwise, if for every $h(z,w) = g_0(z)g + f_0(w)f \in K$, we have

$$-\sqrt{t(1-t)}f_1(t) - g_0(0) = 0,$$

and by a calculation, then we know that

$$g_0(0) + \sqrt{t(1-t)}f_0(t) = 0.$$

Then,

$$\begin{aligned} h(0,t) &= -g_0(0)\sqrt{1-t}\frac{-1}{1-t^2} + \sqrt{t}(t-1)f_0(t)\frac{-1}{1-t^2} \\ &= 0 \end{aligned} \tag{2.6}$$

for every $h \in K$.

Let $K_{(\lambda_1,\lambda_2)}$ be the reproducing kernel for $\mathcal{K}_\theta$ at $(\lambda_1,\lambda_2) \in \mathbb{D}^2$. By (2.6), for every $h \in K$, $\langle h, K_{(0,t)}\rangle = h(0,t) = 0$, which means that

$$K_{(0,t)} \in \mathcal{K}_\theta \ominus K.$$

Note that

$$K_{(\lambda_1,\lambda_2)} = \frac{\overline{g(\lambda_1,\lambda_2)}g}{1-\overline{\lambda}_1 z} + \frac{\overline{f(\lambda_1,\lambda_2)}f}{1-\overline{\lambda}_2 w}.$$

Then we have

$$K_{(0,t)} = \overline{g(0,t)}g + \frac{\overline{f(0,t)}f}{1-tw} \in \mathcal{K}_\theta \ominus K.$$

Since $g(0,t) \neq 0$, we get that $g \in P_{\mathcal{S}_1}(\mathcal{K}_\theta \ominus K)$. This finishes the proof of the claim. □

In what follows, without loss of generality, we assume that $g \in P_{\mathcal{S}_1}(K)$. Then there exists $f_0(w) \in H_w^2$ such that

$$g + f_0(w)f \in K.$$

Hence

$$S_z\big(g + f_0(w)f\big) = zg - \sqrt{t(1-t)}f_0(t)g + (T^*_{\frac{1-t}{1-tw}} f_0)f \in K,$$

and then

$$zg - \sqrt{t(1-t)}f_0(t)g \in P_{\mathcal{S}_1}(K),$$

so therefore, $zg \in P_{\mathcal{S}_1}(K)$. By induction, we have $z^k g \in P_{\mathcal{S}_1}(K)$, $k = 0, 1, \ldots,$ which implies that

$$gH_z^2 = \operatorname{clos} P_{\mathcal{S}_1}(K).$$

This ends the proof of Lemma 2.12. □

Lemma 2.13. *Let* $\theta = \frac{zw - tz - (1-t)w}{1 - tw - (1-t)z}$, $0 < t < 1$, *and let* K *be a nontrivial reducing subspace for* S_z. *If* $\operatorname{clos} P_{\mathcal{S}_1} K = gH_z^2$, *then* $\operatorname{clos} P_{\mathcal{S}_2} K = fH_w^2$.

Proof. If $g \in K$, then the proof is done. Otherwise, by the proof of Lemma 2.12, we can assume that $g \in P_{\mathcal{S}_1} K$, so there exists $f_0 \in H_w^2$, $f_0 \neq 0$ such that $h = g + f_0(w)f \in K$. Then

$$\begin{aligned} S_z h &= zg - \sqrt{t(1-t)}f_0(t)g - (t-1)(T^*_{\frac{1}{1-tw}} f_0)f \\ &= g_1(z)g - (t-1)(T^*_{\frac{1}{1-tw}} f_0)f \in K, \end{aligned} \tag{2.7}$$

and hence $(T^*_{\frac{1}{1-tw}} f_0)f \in P_{\mathcal{S}_2} K$. For any nonnegative integer n, applying S_z^n on h, we can obtain that

$$(T^{*n}_{\frac{1}{1-tw}} f_0)f \in P_{\mathcal{S}_2} K, \quad n = 0, 1, 2, \ldots.$$

Since

$$\begin{aligned} S_z^* h &= -\frac{\sqrt{t(1-t)}}{1-tw} f + \frac{1-t}{1-tw} f_0(w) f \\ &= f_1(w) f \in K, \end{aligned}$$

it follows that

$$f_1(w)f \in P_{\mathcal{S}_2} K,$$

where $f_1(w) = -\frac{\sqrt{t(1-t)}}{1-tw} + \frac{1-t}{1-tw} f_0(w)$. Again for any nonnegative integer k, we can apply S_z^{*k} on h to get

$$S_z^{*k} h = \Big(\frac{1-t}{1-tw}\Big)^{k-1} f_1(w) f,$$

and hence $(\frac{1-t}{1-tw})^{k-1} f_1(w) f \in P_{\mathcal{S}_2} K$, $k = 1, 2, \ldots$.

If $f_1 = 0$, then $f_0(w) = \sqrt{\frac{t}{1-t}}$ and $h = g + \sqrt{\frac{t}{1-t}} f \in K$. By the formula (2.7), we have

$$S_z h = (z - t)g + \sqrt{t(1-t)} f \in K.$$

It follows that $S_z h - (1-t)h = (z-1)g \in K$. Now for any nonnegative integer n, we have $S_z^n(z-1)g = z^n(z-1)g \in K$. Since $z-1$ is an outer function, combining with Lemma 2.10, we obtain that $K = \mathcal{K}_\theta$, which is a contradiction. In the following, we assume that $f_1 \neq 0$.

Claim. *We have* $\bigvee\{T^{*n}_{\frac{1}{1-tw}} f_0, (\frac{1}{1-tw})^k f_1(w) : n, k = 0, 1, 2, \ldots\} = H^2_w$.

Proof of Claim. Recall that for $0 < t < 1$, $\varphi_t(w) = \frac{t-w}{1-tw} = \frac{1}{t} + \frac{t-\frac{1}{t}}{1-tw}$. It follows that

$$T^*_{\varphi_t} = \frac{1}{t} I + \Big(t - \frac{1}{t}\Big) T^*_{\frac{1}{1-tw}};$$

hence every $T^*_{\frac{1}{1-tw}}$-invariant subspace is also $T^*_{\varphi_t}$-invariant. By the same argument as in Corollary 2.5, we know that every $T^*_{\varphi_t}$-invariant subspace is T^*_w-invariant. It is easy to see that the subspace $\bigvee\{(T^{*n}_{\frac{1}{1-tw}} f_0), n = 0, 1, 2, \ldots\}$ is a $T^*_{\frac{1}{1-tw}}$-invariant subspace, and therefore it is T^*_w-invariant. By Beurling's theorem (see [7]), there exists a one-variable inner function $\eta(w)$ with $\eta(0) \neq 0$ such that

$$\bigvee\big\{(T^{*n}_{\frac{1}{1-tw}} f_0), n = 0, 1, 2, \ldots\big\} = H^2_w \ominus \eta(w) w^\alpha H^2_w$$

for some nonnegative integer α.

Let $\phi(w) = \eta(w) w^\alpha \psi(w)$, $\psi(w) \in H^2_w$ such that

$$\phi(w) \perp \Big(\frac{1}{1-tw}\Big)^k f_1(w)$$

for $k = 0, 1, 2 \ldots$. By Corollary 2.9, we have that

$$\phi(w) \perp f_1(w) w^k$$

for $k = 0, 1, 2, \ldots$. Hence

$$\big\langle \eta(w)\psi(w), w^k f_1 \big\rangle = 0$$

for $k = -\alpha, \ldots, 0, 1, \ldots$. Since $f_0(w) \in H^2_w \ominus \eta(w) w^\alpha H^2_w$, we have

$$T^{*n}_w f_0 \in H^2_w \ominus \eta(w) w^\alpha H^2_w,$$

and hence

$$\phi(w) \perp T^{*n}_w f_0(w)$$

for $n = 0, 1, 2, \ldots$. Note that

$$(1-tw) f_1 = -\sqrt{t(1-t)} + (1-t) f_0,$$

we have

$$\begin{aligned} 0 &= \big\langle \eta(w) w^{\alpha+1} \psi(w), f_0 \big\rangle \\ &= \Big\langle \eta(w) w^{\alpha+1} \psi(w), \frac{(1-tw) f_1 + \sqrt{t(1-t)}}{1-t} \Big\rangle. \end{aligned}$$

Therefore $\langle \eta(w) w^{\alpha+1} \psi(w), f_1 \rangle = 0$. By induction, we obtain

$$\langle \eta w^k \psi, f_1 \rangle = 0,$$

for $k \in \mathbb{Z}$. Since $f_1 \neq 0$, we get that $\eta\psi = 0$, and therefore $\phi = 0$. This completes the proof of the Claim and the conclusion follows easily. □

This completes the proof of Lemma 2.13. □

Lemma 2.14. *Both S_z and S_w are both irreducible on $H^2 \ominus \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}H^2$, $0<t<1$.*

Proof. Let K be a nontrivial reducing subspace for S_z, and we assume that

$$\operatorname{clos} P_{\mathcal{S}_1}K = gH_z^2 \qquad \text{and} \qquad \operatorname{clos} P_{\mathcal{S}_2}K = fH_w^2.$$

There is $g_0(w) \in H_w^2$ such that

$$h = f + g_0(z)g \in K.$$

Then we have that

$$S_z h = -\sqrt{t(1-t)}g + (1-t)f + zg_0 g,$$

and

$$\begin{aligned} S_z^* S_z h &= -\sqrt{t(1-t)}\Big(-\frac{\sqrt{t(1-t)}}{1-tw}f\Big) + (1-t)\Big(\frac{1-t}{1-tw}\Big)f + g_0 g \\ &= \frac{1-t}{1-tw}f + g_0 g. \end{aligned}$$

It follows that

$$\frac{1-w}{1-tw}f = S_z^* S_z h - h \in K.$$

Applying S_z^{*n} to $\frac{1-w}{1-tw}f$, we get that

$$\Big\{(1-w)\Big(\frac{1}{1-tw}\Big)^n f : n = 1, 2, \ldots\Big\} \subseteq K.$$

By Corollary 2.9, we have

$$(1-w)H_w^2 f \subseteq K.$$

Since $1-w$ is an outer function, $fH_w^2 \subseteq K$. Therefore S_z is irreducible and the proof is completed. □

By the same argument as in Corollary 2.5, we have the following corollary.

Corollary 2.15. *For $\theta = \frac{zw-tz-(1-t)w}{1-tw-(1-t)z}$ with $0<t<1$, both $S_{\varphi_{\lambda_1}(z)}$ and $S_{\varphi_{\lambda_2}(w)}$ are irreducible on $\mathcal{K}_\theta$.*

Now we can prove the main theorem.

Proof of Theorem 1.1. For the inner function $\theta(z,w) = \frac{zw+\bar{b}w+\bar{c}z+\bar{d}}{1+bz+cw+dzw}$, it is easy to see that θ is the product of two one-variable inner functions if and only if $d = bc$. If $d \neq bc$, then by combining Lemma 2.1, Corollary 2.5, and Corollary 2.15 we know that S_z is irreducible. If θ is the product of two one-variable inner functions, it is also easy to see that S_z is reducible. This finishes the proof. □

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