Ann. Funct. Anal. 10 (2019), no. 4, 460–471
https://doi.org/10.1215/20088752-2019-0002
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

BANACH FUNCTION SPACES ON LOCALLY COMPACT GROUPS

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Communicated by D. Leung

Abstract. We study *Banach function spaces (BFS)* on locally compact topological groups. We focus on totally bounded sets in BFS. We obtain the Riesz–Kolmogorov compactness theorem as well as the Sudakov theorem.

1. Introduction

The family of *Banach function spaces (BFS)* is a huge class of function spaces. In particular, this family contains classical Lebesgue spaces and Orlicz spaces, as well as Lorentz spaces. On the other hand, the so-called *nonstandard function spaces*—such as, for example, variable exponent Lebesgue spaces and grand Lebesgue spaces—are members of the family BFS. The theory of function spaces with nonstandard growth has many applications, such as in the areas of electrorheological fluids (see [22]), image restoration (see [18]), and differential equations with nonstandard growth.

In this article we investigate relatively compact sets in Banach function spaces. In the classical L^p-spaces, the relatively compact sets are characterized by the celebrated Riesz–Kolmogorov theorem (see, e.g., [6], [16], [21], [23]–[25]; see also [13] for a historical account). The aim of this paper is to give a characterization of precompact sets in BFS on locally compact groups. Moreover, we provide a

Received Sep. 20, 2018; Accepted Jan. 9, 2019.
First published online Oct. 23, 2019.
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2010 *Mathematics Subject Classification.* Primary 46B50; Secondary 46E30, 28C10, 46B42.
Keywords. Banach function spaces, topological groups, compactness, Riesz–Kolmogorov theorem, Sudakov theorem.

detailed discussion of the relatively compact sets in BFS on connected groups, and we obtain the Sudakov theorem.

Let us mention some generalizations of the Riesz–Kolmogorov theorem. For instance, various articles (see, e.g., [9], [17], [15], [12]) contain the characterizations of precompact sets in $L^p(X,\varrho,\mu)$, where (X,ϱ,μ) is a metric measure space. Furthermore, Weil [26] showed the compactness theorem in $L^p(G)$, where G is a locally compact group. Pego [19] (see also [7], [8]) stated the Riesz–Kolmogorov theorem for $p=2$ in terms of the Fourier transform. More recently, Riesz–Kolmogorov-type theorems have been extended to variable exponent Lebesgue spaces $L^{p(\cdot)}$ (see [20] for the Euclidean case and [10], [1] for the case of metric measure spaces). Moreover, some compactness criteria were studied in BFS defined on Euclidean space (see [3]) and in BFS defined on metric measure spaces (see [11]). Roughly speaking, the Riesz–Kolmogorov theorem says that the family $\mathcal{F}\subset L^p$ is relatively compact if and only if $\mathcal{F}$ is bounded, has uniform L^p-decay, and is L^p-equicontinuous. It was discovered by Sudakov [23] that the boundedness condition was redundant. In the setting of metric measure spaces the Sudakov theorem has been proved in [12] (see also [14] for discussion of Sudakov's result in the case of $\mathbb{R}^n$).

The remainder of the present article is structured as follows. In Section 2 we introduce the required norms and Banach function spaces, and we recall basic facts about analysis on topological groups. Moreover, this section contains some auxiliary results. The characterization of relatively compact sets in BFS on locally compact groups is given in Section 3. We prove the Sudakov-type theorem in Section 4.

2. Preliminaries

2.1. Banach function spaces. We begin by recalling some notation and basic facts about Banach function spaces. Most of the properties for these spaces can be found in the book by Bennett and Sharpley [2].

Definition 2.1. Let (Ω,μ) be a σ-finite, complete measure space. A normed space $(X,\|\cdot\|_X)$ with $X\subset L^0(\Omega,\mu)$, where by $L^0(\Omega,\mu)$ we denote the space of measurable functions, is called a *Banach function space* if the following conditions are satisfied:

- (A1) if $f\in X$, then $\||f|\|_X=\|f\|_X$;
- (A2) if $0\le g\le f$, then $\|g\|_X\le\|f\|_X$;
- (A3) if $0\le f_n\uparrow f$, then $\|f_n\|_X\uparrow\|f\|_X$;
- (A4) if $\mu(A)<\infty$, then $\chi_A\in X$;
- (A5) if $\mu(A)<\infty$, then there exists a constant $C_X(A)$ such that $\int_A|f|\,d\mu\le C_X(A)\|f\|_X$.

It is well known that the Banach function space $(X,\|\cdot\|_X)$ is a Banach space. Examples of Banach function spaces are the classical Lebesgue spaces L^p, the Orlicz spaces L_Φ, the Lorentz spaces $L_{p,q}$, the grand Lebesgue space $L^{p)}$, and the variable exponent Lebesgue spaces $L^{p(\cdot)}$ (see [5]). We need to recall the following definitions.

Definition 2.2. We say that the Banach function space X has *absolutely continuous norm* if, for every $f \in X$ and every sequence $A_n \subset \Omega$ such that $\chi_{A_n} \to 0$ almost everywhere, we have $\|f\chi_{A_n}\|_X \to 0$.

Definition 2.3. We say that the Banach function space X is *rearrangement-invariant* if $\|f\|_X = \|g\|_X$ whenever $f, g \in L^0(\Omega, \mu)$ have the same distribution functions (i.e., $\mu_f = \mu_g$).

We need the following density lemma.

Lemma 2.4. *Assume that G is a locally compact Hausdorff space with σ-finite Radon measure μ. Let $X = X(G, \mu)$ be a Banach function space with absolutely continuous norm. Then we have the following.*

(i) *For any $f \in X$ and $\varepsilon > 0$ there exists a compact set K such that $\|f\chi_{G\setminus K}\|_X \le \varepsilon$.*
(ii) *The set $C_c(G)$ is dense in X, where $C_c(G)$ stands for the set of all continuous functions on G with compact support.*

Proof. The proof consists of two parts.

(i) Let $f \in X$, and let $\varepsilon > 0$. Since the measure is σ-finite, there exist mutually disjoint sets A_i with finite measures such that $G = \bigcup_{i=1}^{\infty} A_i$. Furthermore, since the norm is absolutely continuous, there exists $\delta > 0$ such that $\|f\chi_E\|_X \le \varepsilon/2$ whenever $\mu(E) < \delta$. By regularity of the measure, for any i there exists a compact set $K_i \subseteq A_i$ such that $\mu(A_i \setminus K_i) < \frac{\delta}{2^i}$. Let $B_n = \bigcup_{i=1}^{n} A_i$. Then $\chi_{G\setminus B_n} \downarrow 0$. Hence, there exists N such that

$$\|f\chi_{G\setminus B_N}\|_X \le \varepsilon/2.$$

Next, taking $K = \bigcup_{i=1}^{N} K_i$, we get

$$\mu(B_N \setminus K) \le \mu\Big(\bigcup_{i=1}^{N} A_i \setminus K_i\Big) \le \sum_{i=1}^{N} \mu(A_i \setminus K_i) < \delta.$$

Thus, finally we obtain

$$\|f\chi_{G\setminus K}\|_X \le \|f\chi_{G\setminus B_N}\|_X + \|f\chi_{B_N\setminus K}\|_X \le \varepsilon.$$

(ii) We divide this part of the proof into two steps.

Step 1. We prove that the set of simple functions with compact support is dense in X. Let $\varepsilon > 0$, and let $f \in X$. We can assume that $f \ge 0$. From (i) there exists a compact set $K \subseteq G$ such that $\|f\chi_{G\setminus K}\|_X \le \varepsilon/2$. Let f_n be a sequence of simple functions such that $f_n \uparrow f\chi_K$. Hence, $f\chi_K \ge f\chi_K - f_n \downarrow 0$ and since the norm is absolutely continuous, there exists N such that

$$\|f\chi_K - f_N\|_X \le \varepsilon/2.$$

Therefore, we obtain

$$\|f - f_N\|_X \le \|f\chi_{G\setminus K}\|_X + \|f\chi_K - f_N\|_X \le \varepsilon,$$

which proves that the set of simple functions with compact support is dense in X.

Step 2. We prove that the set $C_c(G)$ is dense in X. From the previous step it is enough to show that every simple function with compact support can be approximated in the norm $\|.\|_X$ by a continuous function with compact support. Let us fix $\varepsilon > 0$ and simple function $f = \sum_{i=1}^n c_i\chi_{A_i}$ with compact support. Since $\bigcup_{i=1}^n \overline{A_i}$ is compact and the space G is locally compact, there exists an open and precompact set V such that $\bigcup_{i=1}^n \overline{A_i} \subseteq V$. Since we deal with Radon measure, $\mu(V) < \infty$ and thus $\chi_V \in X$. Therefore, since the norm is absolutely continuous, we can find $\delta > 0$ such that $\|\chi_V\chi_E\|_X < \frac{\varepsilon}{2n\max_i|c_i|}$ whenever $\mu(E) < \delta$. By virtue of regularity of the measure, for any i there exist a compact set K_i and an open set V_i such that $K_i \subseteq A_i \subseteq V_i \subset V$ and $\mu(V_i \setminus K_i) < \delta$. Furthermore, the Urysohn's lemma yields a continuous function $g_i : G \to [0,1]$ such that $g_i|_{K_i} \equiv 1$ and $g_i|_{G\setminus V_i} \equiv 0$. Next, we define the map

$$g = \sum_{i=1}^n c_i g_i.$$

We easily see that $g \in C_c(G)$. From the above considerations, we have

$$\begin{aligned} \|f-g\|_X &= \Big\|\sum_{i=1}^n c_i\chi_{A_i} - \sum_{i=1}^n c_i g_i\Big\|_X \\ &\le \sum_{i=1}^n |c_i|\|\chi_{A_i} - g_i\|_X \\ &= \sum_{i=1}^n |c_i|\big\|\chi_{A_i\setminus K_i}(1-g_i) - \chi_{V_i\setminus A_i}g_i\big\|_X \\ &\le \sum_{i=1}^n |c_i|\big(\|\chi_{A_i\setminus K_i}\|_X + \|\chi_{V_i\setminus A_i}\|_X\big) \\ &= \sum_{i=1}^n |c_i|\big(\|\chi_V\chi_{A_i\setminus K_i}\|_X + \|\chi_V\chi_{V_i\setminus A_i}\|_X\big) \le \varepsilon, \end{aligned}$$

which finishes the proof of the lemma. □

2.2. Basic tools from harmonic analysis. We use standard notation from harmonic analysis (see [4]). Let us fix a locally compact group G. By e we will denote the identity of G. Moreover, we denote by μ the unique left invariant Haar measure on G. If f is a function on the topological group G and $y \in G$, then we define the left and right translations of f through y by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = f(xy).$$

The following proposition will be needed in our further proofs.

Proposition 2.5. *Assume that G is a locally compact topological group with left-invariant σ-finite Haar measure μ. Let $X = X(G,\mu)$ be a rearrangement-invariant Banach function space. If $f \in X$ and $h \in G$, then $L_h f \in X$ and $\|L_h f\|_X = \|f\|_X$.*

Proof. Due to the invariance of the measure, for any $h \in G$ we have

$$\begin{aligned}\mu_f(\lambda) &= \mu\big(h\{x \in G : |f(x)| > \lambda\}\big)\\ &= \mu\big(\{x \in G : |f(h^{-1}x)| > \lambda\}\big)\\ &= \mu_{L_h f}(\lambda).\end{aligned}$$

Hence, since the space is rearrangement-invariant, the proof follows. □

3. Compactness

In this section we show the sufficient and necessary conditions for compactness in Banach function spaces. We will start with the following result.

Theorem 3.1. *Assume that G is a locally compact topological group with left-invariant σ-finite Haar measure μ. Let $X = X(G,\mu)$ be a rearrangement-invariant Banach function space. If the family $\mathcal{F} \subseteq X$ satisfies the following conditions:*

(a) *$\mathcal{F}$ is bounded in X—that is, $\exists_{C>0}$*

$$\sup_{f\in\mathcal{F}} \|f\|_X \le C,$$

(b) *we have*

$$\lim_{h\to e} \sup_{f\in\mathcal{F}} \|L_h f - f\|_X = 0,$$

(c) *for any $\varepsilon > 0$ there exists a compact set $K \subset G$ such that*

$$\sup_{f\in\mathcal{F}} \|f\chi_{G\setminus K}\|_X \le \varepsilon,$$

then the family $\mathcal{F}$ is totally bounded in X.

Remark 3.2. Instead of the assumption that X is rearrangement-invariant, we can suppose that, for each $f \in \mathcal{F}$, the map $h \mapsto \|L_h f - f\|_X$ is measurable.

Proof of Theorem 3.1. Let us fix $\varepsilon > 0$. Then there exists a compact set $K \subseteq G$ such that

$$\sup_{f\in\mathcal{F}} \|f\chi_{G\setminus K}\|_X \le \varepsilon/4. \tag{1}$$

Furthermore, there exists a symmetric neighborhood U of the unit such that

$$\sup_{h\in U} \sup_{f\in\mathcal{F}} \|L_h f - f\|_X \le \varepsilon/12.$$

Let J be the so-called *Dirac net* (see, e.g., [4])—that is, non-negative function $J \in C_c(G)$ such that $\int_G J(x)\,d\mu(x) = 1$ and support $F := \operatorname{spt} J \subset U$. Then, since the space is rearrangement-invariant, by the generalized Minkowski inequality, we have

$$\begin{aligned}\big\|(J*f)-f\big\|_X &= \Big\|\int_G J(y)L_yf(\cdot)\,d\mu(y)-\int_G J(y)f(\cdot)\,d\mu(y)\Big\|_X\\ &= \Big\|\int_G J(y)\chi_F(y)\big(L_yf(\cdot)(y)-f(\cdot)\big)\,d\mu(y)\Big\|_X\\ &\le \int_G J(y)\chi_F(y)\big\|L_yf(\cdot)-f(\cdot)\big\|_X\,d\mu(y)\\ &\le \sup_{h\in U}\|L_hf-f\|_X\int_G J(y)\,d\mu(y)\\ &= \sup_{h\in U}\|L_hf-f\|_X\le \varepsilon/12.\end{aligned}$$

Hence, the inequality

$$\big\|(J*f)-f\big\|_X\le \varepsilon/12, \tag{2}$$

holds. Next, we will prove that the family

$$\tilde{\mathcal{F}}_K=\big\{(J*f)|_K : f\in\mathcal{F}\big\}$$

satisfies the assumptions of the Arzelà–Ascoli theorem in $C(K)$.

Step 1. We prove that the family $\tilde{\mathcal{F}}_K$ is bounded. Since the measure μ is left-invariant and by the very definition of the modular function Δ (see [4]) we have

$$\begin{aligned}\big|(J*f)\big|_K(x)\big| &\le \int_G\big|J(y)L_yf(x)\big|\,d\mu(y)\\ &= \int_G\big|J(y)\chi_F(y)f(y^{-1}x)\big|\,d\mu(y)\\ &\le \sup_{z\in F}J(z)\int_G L_x\big(\big|\chi_F(x\cdot)f(\cdot^{-1})\big|\big)(y)\,d\mu(y)\\ &= \sup_{z\in F}J(z)\int_G\big|\chi_F(xy)f(y^{-1})\big|\,d\mu(y)\\ &= \sup_{z\in F}J(z)\int_G\big|\chi_F(xy^{-1})f(y)\Delta(y^{-1})\big|\,d\mu(y)\\ &\le \sup_{z\in F}J(z)\int_G\big|\chi_{F^{-1}K}(y)f(y)\Delta(y^{-1})\big|\,d\mu(y)\\ &\le \sup_{z\in F}J(z)\sup_{z\in F^{-1}K}\Delta(z^{-1})\int_G\big|\chi_{F^{-1}K}(y)f(y)\big|\,d\mu(y)\\ &\le \sup_{z\in F}J(z)\sup_{z\in F^{-1}K}\Delta(z^{-1})\int_{F^{-1}K}\big|f(y)\big|\,d\mu(y)\\ &\le \sup_{z\in F}J(z)\sup_{z\in F^{-1}K}\Delta(z^{-1})C_X(F^{-1}K)\|f\|_X\\ &\le \sup_{z\in F}J(z)\sup_{z\in F^{-1}K}\Delta(z^{-1})C_X(F^{-1}K)C.\end{aligned}$$

Since the modular function $\Delta : G\to\mathbb{R}^{\times}_{>0}$ is a continuous group homomorphism and the set $F^{-1}K$ is compact, we have that $\sup_{z\in F^{-1}K}\Delta(z^{-1})$ is finite. Therefore, we get

$$\sup_{f\in\mathcal{F}}\sup_{x\in K}\big|(J*f)\big|_K(x)\big| < \infty.$$

Step 2. We prove equicontinuity of the family $\tilde{\mathcal{F}}_K$. Let us fix any precompact neighborhood of the unit U_0. Then, for $h \in U_0$ we get

$$\begin{aligned}
&\big|(J*f)\big|_K(h^{-1}x) - (J*f)\big|_K(x)\big| \\
&\quad= \Big|\int_G J(y)L_yf(h^{-1}x)\,d\mu(y) - \int_G J(y)L_yf(x)\,d\mu(y)\Big| \\
&\quad= \Big|\int_G J(h^{-1}xy)f(y^{-1})\,d\mu(y) - \int_G J(xy)f(y^{-1})\,d\mu(y)\Big| \\
&\quad\le \int_G \big|\big(J(h^{-1}xy) - J(xy)\big)f(y^{-1})\big|\,d\mu(y) \\
&\quad= \int_G \big|\big(J(h^{-1}xy^{-1}) - J(xy^{-1})\big)f(y)\Delta(y^{-1})\big|\,d\mu(y) \\
&\quad= \int_G \chi_{F^{-1}\overline{U_0}K}(y)\big|\big(J(h^{-1}xy^{-1}) - J(xy^{-1})\big)f(y)\Delta(y^{-1})\big|\,d\mu(y) \\
&\quad\le \sup_{z\in F^{-1}\overline{U_0}K}\Delta(z^{-1})\sup_{z\in G}\big|L_hJ(z) - J(z)\big|\int_G \chi_{F^{-1}\overline{U_0}K}(y)\big|f(y)\big|\,d\mu(y) \\
&\quad= \sup_{z\in F^{-1}\overline{U_0}K}\Delta(z^{-1})\sup_{z\in G}\big|L_hJ(z) - J(z)\big|\int_{F^{-1}\overline{U_0}K}\big|f(y)\big|\,d\mu(y) \\
&\quad\le \sup_{z\in F^{-1}\overline{U_0}K}\Delta(z^{-1})\sup_{z\in G}\big|L_hJ(z) - J(z)\big|C_X(F^{-1}\overline{U_0}K)\big\|f(y)\big\|_X \\
&\quad\le \sup_{z\in F^{-1}\overline{U_0}K}\Delta(z^{-1})\sup_{z\in G}\big|L_hJ(z) - J(z)\big|C_X(F^{-1}\overline{U_0}K)C.
\end{aligned}$$

Since $J \in C_c(G)$, for every $\varepsilon_0 > 0$ there is a unit neighborhood $V_0 \subseteq U_0$ such that

$$\sup_{h\in V_0}\sup_{z\in G}\big|L_hJ(z) - J(z)\big| \le \frac{\varepsilon_0}{\sup_{z\in F^{-1}\overline{U_0}K}\Delta(z^{-1})C_X(F^{-1}\overline{U_0}K)C}.$$

Therefore, for any $h \in V_0$ and $f \in \mathcal{F}$

$$\big|(J*f)\big|_K(h^{-1}x) - (J*f)\big|_K(x)\big| \le \varepsilon_0,$$

and it means that the family $\tilde{\mathcal{F}}_K$ is equicontinuous.

Hence, due to the Arzelà–Ascoli theorem, $\tilde{\mathcal{F}}_K$ is compact in $C(K)$. Thus, we can choose $f_1, f_2, \dots, f_n \in \mathcal{F}$ such that the sequence $\{(J*f_k)|_K : f \in \mathcal{F}\}_{k=1,\dots,n}$ is an $\frac{\varepsilon}{3\|\chi_K\|_X}$-net in $\tilde{\mathcal{F}}_K$, which means that

$$\tilde{\mathcal{F}}_K \subset \bigcup_{k=1}^n B\Big((J*f_k)|_K, \frac{\varepsilon}{3\|\chi_K\|_X}\Big).$$

We will show that $\{f_k\}_{k=1,\dots,n} \subset \mathcal{F}$ is an ε-net in $\mathcal{F}$. By (1) and (2) we have

$$\begin{aligned}
\|f - f_i\|_X &\le \big\|(f - f_i)\chi_{G\setminus K}\big\|_X + \big\|(f - f_i)\chi_K\big\|_X \\
&\le \|f\chi_{G\setminus K}\|_X + \|f_i\chi_{G\setminus K}\|_X
\end{aligned}$$

$$+\|f-J*f\|_X+\big\|(J*f-J*f_i)\chi_K\big\|_X+\|J*fi-f_i\|_X$$
$$\le\varepsilon/4+\varepsilon/4+\varepsilon/12+\varepsilon/3+\varepsilon/12=\varepsilon,$$

which completes the proof of Theorem 3.1. □

We now show the reciprocal of Theorem 3.1. In order to do so, we need some additional assumptions on Banach function space X.

Theorem 3.3. *Assume that G is a locally compact topological group with left-invariant σ-finite Haar measure μ. Let $X = X(G,\mu)$ be a rearrangement-invariant Banach function space with absolutely continuous norm. If the family $\mathcal{F}\subseteq X$ is totally bounded, then*

(a) *$\mathcal{F}$ is bounded in X;*
(b) *we have*

$$\lim_{h\to e}\sup_{f\in\mathcal{F}}\|L_hf-f\|_X=0;$$

(c) *for any $\varepsilon>0$ there exists a compact set K such that*

$$\sup_{f\in\mathcal{F}}\|f\chi_{G\setminus K}\|_X\le\varepsilon.$$

Remark 3.4. If we do not assume that a Banach function space is rearrangement-invariant, then the above theorem does not hold. Indeed, taking for example the weighted Lebesgue space defined on $\mathbb{R}^n$ one can easily construct the counterexample.

Proof of Theorem 3.3. If $\mathcal{F}$ is totally bounded, then there exists a finite $\varepsilon/4$-net, that is, $f_1,\dots,f_n\in\mathcal{F}$, such that

$$\mathcal{F}\subset\bigcup_{i=1}^{n}B(f_i,\varepsilon/4). \tag{3}$$

We will prove that conditions (a), (b), and (c) hold.

(a) It is obviously satisfied.

(b) Let $f\in\mathcal{F}$. Then by (3) there exists i such that $\|f-f_i\|_X\le\varepsilon/4$. Furthermore, by virtue of Lemma 2.4, there exists $g_i\in C_c(G)$ such that $\|f_i-g_i\|_X\le\varepsilon/6$. Next, let U be any precompact neighborhood of the unit, and let $K=\bigcup_{i=1}^{n}\operatorname{spt}g_i$. Then there exists a unit neighborhood $V_i\subseteq U$ such that for $h\in V_i$

$$\sup_{x\in G}\big|L_hg_i(x)-g_i(x)\big|\le\frac{\varepsilon}{6C_{X'}(U^{-1}K)},$$

where X' stands for the associate space of X. Subsequently, we introduce the neighborhood of the unit $V=\bigcap_{i=1}^{n}V_i$. By the Lorentz–Luxemburg theorem (see [2, p. 10]), for every $h\in V$ we have

$$\|L_hg_i-g_i\|_X=\sup_{\|\phi\|_{X'}\le1}\int_G\big|\phi(x)\big|\big|L_hg_i(x)-g_i(x)\big|\,d\mu(x)$$
$$=\sup_{\|\phi\|_{X'}\le1}\int_G\big|\phi(x)\big|\chi_{V^{-1}K}(x)\big|L_hg_i(x)-g_i(x)\big|\,d\mu(x)$$

$$
\begin{aligned}
&\le \sup_{h\in V}\sup_{x\in G}\bigl|L_h g_i(x)-g_i(x)\bigr| \sup_{\|\phi\|_{X'}\le 1}\int_G \bigl|\phi(x)\bigr|\chi_{V^{-1}K}(x)\,d\mu(x)\\
&\le \sup_{h\in V}\sup_{x\in G}\bigl|L_h g_i(x)-g_i(x)\bigr| \sup_{\|\phi\|_{X'}\le 1}\int_G \bigl|\phi(x)\bigr|\chi_{U^{-1}K}(x)\,d\mu(x)\\
&\le \frac{\varepsilon}{6C_{X'}(U^{-1}K)} \sup_{\|\phi\|_{X'}\le 1}\int_{U^{-1}K} \bigl|\phi(x)\bigr|\,d\mu(x)\\
&\le \varepsilon/6.
\end{aligned}
$$

Hence, due to Proposition 2.5, for any $h \in V$ we have

$$
\begin{aligned}
\|L_h f - f\|_X &\le \bigl\|L_h(f-f_i)\bigr\|_X + \bigl\|L_h(f_i-g_i)\bigr\|_X + \|L_h g_i - g_i\|_X\\
&\quad + \|g_i - f_i\|_X + \|f_i - f\|_X\\
&\le \varepsilon/4 + \varepsilon/6 + \varepsilon/6 + \varepsilon/6 + \varepsilon/4 = \varepsilon.
\end{aligned}
$$

(c) Let $f \in \mathcal{F}$, and let i be such that $\|f - f_i\|_X \le \varepsilon/4$. From Lemma 2.4 there exists a compact set K_i such that $\|f_i\chi_{G\setminus K_i}\|_X \le \frac{3}{4}\varepsilon$. Therefore, if we introduce the following compact set $K = \bigcup_{i=1}^n K_i$, then we have

$$
\begin{aligned}
\|f\chi_{G\setminus K}\|_X &\le \|f_i\chi_{G\setminus K}\|_X + \|f_i\chi_{G\setminus K} - f\chi_{G\setminus K}\|_X\\
&\le \|f_i\chi_{G\setminus K_i}\|_X + \|f_i - f\|_X \le \varepsilon,
\end{aligned}
$$

which completes the proof. □

4. Sudakov-type theorem

The following theorem is an extension of the Sudakov theorem to the setting of Banach function spaces on locally compact topological groups. It shows that boundedness condition (a) in Theorem 3.1 is redundant for rearrangement-invariant Banach function spaces defined on connected groups.

Theorem 4.1. *Assume that G is a locally compact, connected, and not compact topological group with left-invariant σ-finite Haar measure μ. Let $X = X(G,\mu)$ be a rearrangement-invariant Banach function space. If the family $\mathcal{F} \subseteq X$ satisfies the conditions*

(i)

$$
\lim_{h\to e}\sup_{f\in\mathcal{F}} \|L_h f - f\|_X = 0,
$$

(ii) *for any $\varepsilon > 0$ there exists a compact set $K \subset G$ such that*

$$
\sup_{f\in\mathcal{F}} \|f\chi_{G\setminus K}\|_X \le \varepsilon,
$$

then the family $\mathcal{F}$ is bounded in X.

Example 4.2. Let G be a compact topological group, and let $p \in [1,\infty)$. Then the family $\mathcal{F} = \{n : n \in \mathbb{N}\} \subset L^p(G)$ satisfies conditions (i) and (ii) in Theorem 4.1, but $\sup_{f\in\mathcal{F}} \|f\|_{L^p(G)} = \infty$.

Example 4.3. Let $p \in [1,\infty)$, and let $\mathcal{G} = \{n\chi_{\{0\}\times S^1} : n \in \mathbb{N}\} \subset L^p(\mathbb{Z}\times S^1)$. The family $\mathcal{G}$ obviously satisfies (i) and (ii) in Theorem 4.1 and $\sup_{f\in\mathcal{G}} \|f\|_{L^p(\mathbb{Z}\times S^1)} = \infty$.

Proof. There exist an open neighborhood of the unit U and a compact set K such that

$$\sup_{h\in U}\sup_{f\in\mathcal{F}} \|L_h f - f\|_X \le 1, \qquad \sup_{f\in\mathcal{F}} \|f\chi_{G\setminus K}\|_X \le 1.$$

Since G is not compact, there exists $y \in G\setminus K$. Therefore, the set $V = U \cap (y^{-1}(G\setminus K))$ is an open neighborhood of the unit. Let $W = V\cap V^{-1}$. Then, due to the compactness of K, there exist $x_1,\dots,x_n \in K$ such that $K \subset \bigcup_{i=1}^n x_iW$. By Proposition 2.5, for any $f\in\mathcal{F}$, $h\in W$, $z,x\in G$ we have

$$\begin{aligned}\big\|(L_zL_hf - L_zf)\chi_{xW}\big\|_X &= \big\|L_z\big((L_hf-f)L_{z^{-1}}\chi_{xW}\big)\big\|_X\\ &= \big\|(L_hf-f)L_{z^{-1}}\chi_{xW}\big\|_X\\ &\le \big\|(L_hf-f)\big\|_X \le 1.\end{aligned}$$

Hence,

$$\big\|L_z(f)\chi_{xW}\big\|_X \le 1 + \big\|L_zL_h(f)\chi_{xW}\big\|_X. \tag{4}$$

It is known that a locally compact group is connected if and only if any neighborhood of the unit generates the group. Thus, since G is connected, the neighborhood W generates the group G, that is, $\langle W\rangle = G$. Therefore, for any i there exist $y_{i,1},\dots,y_{i,n_i}\in W$ such that $x_i = y_{i,1},\dots,y_{i,n_i}$. Now, iterating inequality (4) we have

$$\begin{aligned}\|f\chi_{x_iW}\|_X &\le 1 + \big\|L_{y_{i,1}}(f)\chi_{x_iW}\big\|_X\\ &\le 2 + \big\|L_{y_{i,1}}L_{y_{i,2}}(f)\chi_{x_iW}\big\|_X\\ &\le n_i + \big\|L_{y_{i,1}y_{i,2},\dots,y_{i,n_i}}(f)\chi_{x_iW}\big\|_X\\ &= n_i + \big\|L_{x_i}(f)\chi_{x_iW}\big\|_X\\ &= n_i + \big\|L_{x_i}(f\chi_W)\big\|_X\\ &= n_i + \|f\chi_W\|_X.\end{aligned}$$

Furthermore, since $y = z_1,\dots,z_m$ for some $z_1,\dots,z_m\in W$, we get

$$\begin{aligned}\|f\chi_W\|_X &\le m + \big\|L_{y^{-1}}(f)\chi_W\big\|_X\\ &= m + \|f\chi_{yW}\|_X\\ &\le m + \|f\chi_{G\setminus K}\|_X,\end{aligned}$$

where the last inequality follows from the inclusion $yW \subset G\setminus K$. Finally, thanks to the covering $K\subset \bigcup_{i=1}^n x_iW$, for any $f\in\mathcal{F}$ we obtain

$$\begin{aligned}\|f\|_X &\le \|f\chi_{G\setminus K}\|_X + \|f\chi_K\|_X\\ &\le \|f\chi_{G\setminus K}\|_X + \sum_{i=1}^n \|f\chi_{x_iW}\|_X\end{aligned}$$

$$\leq \|f\chi_{G\setminus K}\|_X + \sum_{i=1}^{n}\big(n_i + m + \|f\chi_{G\setminus K}\|_X\big)$$

$$\leq 1 + \sum_{i=1}^{n} n_i + mn + n,$$

and this finishes the proof of the Theorem 4.1. □

Combining Theorem 3.1 with Theorems 3.3 and 4.1, we get the following characterization of relatively compact sets.

Theorem 4.4. *Assume that G is a locally compact, connected, and not compact topological group with left-invariant σ-finite Haar measure μ. Let $X = X(G,\mu)$ be a rearrangement-invariant Banach function space with absolutely continuous norm. The family $\mathcal{F} \subseteq X$ is totally bounded if and only if*

(i)

$$\lim_{h\to e}\sup_{f\in\mathcal{F}} \|L_h f - f\|_X = 0;$$

(ii) *for any $\varepsilon > 0$ there exists a compact set $K \subset G$ such that*

$$\sup_{f\in\mathcal{F}} \|f\chi_{G\setminus K}\|_X \leq \varepsilon.$$

Acknowledgments. The authors would like to thank the referees for many useful suggestions and comments.

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