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EMBEDDING THEOREMS AND INTEGRATION OPERATORS ON BERGMAN SPACES WITH EXPONENTIAL WEIGHTS

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ABSTRACT. In this article, given some positive Borel measure μ, we define two integration operators to be

$$I_\mu(f)(z) = \int_{\mathbf{D}} f(w)K(z,w)e^{-2\varphi(w)}\,d\mu(w)$$

and

$$J_\mu(f)(z) = \int_{\mathbf{D}} \big|f(w)K(z,w)\big|e^{-2\varphi(w)}\,d\mu(w).$$

We characterize the boundedness and compactness of these operators from the Bergman space A^p_φ to L^q_φ for $1 < p, q < \infty$, where φ belongs to a large class $\mathcal{W}_0$, which covers those defined by Borichev, Dhuez, and Kellay in 2007. We also completely describe those μ's such that the embedding operator is bounded or compact from A^p_φ to $L^q_\varphi(d\mu)$, $0 < p, q < \infty$.

1. Introduction

Let $\mathbf{D}$ be the unit disk in the complex plane, and let dA be the normalized area measure on $\mathbf{D}$. Denote by C_0 the set of all continuous functions ρ on $\mathbf{D}$ satisfying $\lim_{|z|\to 1}\rho(z) = 0$. Suppose that ρ is a real-valued function on $\mathbf{D}$. If $\rho \in C_0$ with

$$\|\rho\|_L = \sup_{z,w\in\mathbf{D}, z\neq w} \frac{|\rho(z)-\rho(w)|}{|z-w|} < \infty,$$

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then we say that ρ belongs to the class $\mathcal{L}$. Let $\mathcal{L}_0$ consist of those $\rho \in \mathcal{L}$ with the property that for each $\varepsilon > 0$ there is a compact subset $E \subset \mathbf{D}$ with

$$\big|\rho(z) - \rho(w)\big| \le \varepsilon |z - w|$$

whenever $z, w \in \mathbf{D} \setminus E$. The class $\mathcal{W}_0$ is the family of all real-valued functions $\varphi \in C^2(\mathbf{D})$ such that

$$\Delta\varphi > 0 \qquad \text{and} \qquad \exists \rho \in \mathcal{L}_0 \quad \text{such that } \frac{1}{\sqrt{\Delta\varphi}} \simeq \rho.$$

Here and throughout, $A \simeq B$ means there exists some constant $C > 0$, independent of the variables being considered, such that $C^{-1}A \le B \le CA$.

Two classes of weight functions closely related to ours merit discussion. Precisely, Oleĭnik [11] and Oleĭnik and Perel'man [12] considered $\varphi \in C^2(\mathbf{D})$ such that $\Delta\varphi > 0$ and $\rho = \frac{1}{\sqrt{\Delta\varphi}}$, where ρ satisfies that there are constants $a, C_1, C_2 > 0$ and $C_3 \in (0, 1)$ such that

$$\big|\rho(z) - \rho(w)\big| \le C_1 |z - w| \quad \text{for all } z, w \in \mathbf{D},$$
$$\rho(z) \le C_2\big(1 - |z|\big) \quad \text{for all } z, w \in \mathbf{D},$$

and

$$\rho(w) \le \rho(z) + C_3 |z - w| \quad \text{for } z, w \in \mathbf{D}.$$

For such φ, we denote $\varphi \in \mathcal{OP}$ for short. As discussed in [8, Section 2],

$$\mathcal{W}_0 \setminus \mathcal{OP} \ne \emptyset \qquad \text{and} \qquad \mathcal{OP} \setminus \mathcal{W}_0 \ne \emptyset.$$

In 2007, Borichev, Dhuez, and Kellay [4] studied the radial weight $\varphi \in C^2(\mathbf{D})$ satisfying

$$\Delta\varphi \ge 1, \qquad \rho(r) \searrow 0 \quad \text{as } r \to 1, \qquad \lim_{r\to 1} \rho'(r) = 0.$$

Furthermore, either

$$\rho(r)(1 - r)^{-C} \text{ increases for some constant } C \text{ and } r \text{ close to } 1,$$

or

$$\lim_{r\to 1} \rho'(r) \log \frac{1}{\rho(r)} = 0.$$

Using $\mathcal{BDK}$ to denote the class of the weights satisfying Borichev, Dhuez, and Kellay's conditions, as mentioned in [8, Section 2], we have

$$\mathcal{BDK} \subset \mathcal{W}_0 \qquad \text{and} \qquad \mathcal{W}_0 \setminus \mathcal{BDK} \ne \emptyset.$$

Given $\varphi \in \mathcal{W}_0$ and $0 < p < \infty$, the space L^p_φ consists of all Lebesgue measurable functions f on $\mathbf{D}$ satisfying

$$\|f\|_{p,\varphi} = \Big(\int_{\mathbf{D}} \big|f(z)e^{-\varphi(z)}\big|^p \, dA(z)\Big)^{1/p} < \infty.$$

Let $H(\mathbf{D})$ be the set of holomorphic functions on $\mathbf{D}$. The Bergman space is defined by

$$A^p_\varphi = L^p_\varphi \cap H(\mathbf{D}).$$

For $\varphi \in \mathcal{OP}$, the Bergman space A^p_φ has been studied in [2], [3], [9], [11], and [12]. The Bergman space A^p_φ with $\varphi \in \mathcal{BDK}$ has been considered by many authors (see, e.g., [1], [4]–[6], [13], [14]).

For $\varphi \in \mathcal{W}_0$, denote by $K(\cdot,\cdot)$ the Bergman kernel for A^2_φ. As mentioned in [8, Corollary 4.2],

$$\mathcal{K} = \operatorname{span}\{K(\cdot, z) : z \in \mathbf{D}\}$$

is dense in A^p_φ under the A^p_φ-norm for all $p \geq 1$. The orthogonal projection $P : L^2_\varphi \to A^2_\varphi$ is defined by

$$Pf(z) = \int_{\mathbf{D}} f(w)K(z,w)e^{-2\varphi(w)}\, dA(w), \quad z \in \mathbf{D}.$$

Suppose that μ is a positive Borel measure on $\mathbf{D}$ (denoted as $\mu \geq 0$) satisfying the condition

$$\int_{\mathbf{D}} \big|K(z,w)\big|^2 e^{-2\varphi(w)}\, d\mu(w) < \infty \tag{1.1}$$

for all $z \in \mathbf{D}$. Then the integral operators on A^p_φ $(p \geq 1)$ can be densely defined to be

$$I_\mu(f)(z) = \int_{\mathbf{D}} f(w)K(z,w)e^{-2\varphi(w)}\, d\mu(w) \tag{1.2}$$

and

$$J_\mu(f)(z) = \int_{\mathbf{D}} \big|f(w)K(z,w)\big|e^{-2\varphi(w)}\, d\mu(w), \tag{1.3}$$

since I_μ and J_μ are well defined on $\mathcal{K}$, which follows from (1.1) and the Cauchy–Schwarz inequality. If $d\mu = dA$, then the operator I_μ is just the Bergman projection which has been studied on L^p_φ for some restricted φ and $p > 1$ (see, e.g., [2], [5], [10], [16]). In 2016, Peláez and Rättyä [15] considered the boundedness of these two operators for $d\mu = dA$ on L^p_ϕ for some different weights φ and ϕ for $p > 1$.

The purpose of this article is to study the boundedness and compactness of two types of integration operators from A^p_φ to L^q_φ for $1 < p, q < \infty$. In Section 2, we completely describe those positive Borel measures μ on $\mathbf{D}$ such that the embedding operator i is bounded (or compact) from A^p_φ to $L^q_\varphi(d\mu)$, $0 < p, q < \infty$. Section 3 is devoted to a discussion on the boundedness and compactness of these integral operators in terms of Carleson measures. We can obtain the main result as follows.

Theorem 1.1. *Let $1 < p, q < \infty$, let $\varphi \in \mathcal{W}_0$, and let $\mu \geq 0$ with hypothesis (1.1). Set $1/s = 1 - 1/q + 1/p$. Then the following statements are equivalent:*

(A) $I_\mu : A^p_\varphi \to A^q_\varphi$ *is bounded (or compact),*
(B) $J_\mu : A^p_\varphi \to L^q_\varphi$ *is bounded (or compact),*
(C) μ *is an s-Carleson measure (or vanishing s-Carleson measure).*

In what follows, we always assume that $\varphi \in \mathcal{W}_0$. We use C, C_1, C_2 and c_1, c_2 to denote positive constants whose value may change from line to line, but do not depend on the variables being considered.

2. Carleson measures

In this section, we give the characterizations on Carleson measures for Bergman spaces. We begin with some notation and preliminaries. For $z \in \mathbf{D}$ and $r > 0$, set

$$D(z,r) = \big\{w \in \mathbf{D} : |w - z| < r\big\} \qquad \text{and} \qquad D^r(z) = D\big(z, r\rho(z)\big).$$

Regarding this disk, we have the following lemma which can be found in [8, Lemmas 3.1, 3.2].

Lemma 2.1. *Let $\rho \in \mathcal{L}$ be positive. Then there exists $\alpha > 0$ with the following properties.*

(1) *There exist constants C_1 and C_2 such that*

$$C_1\rho(w) \le \rho(z) \le C_2\rho(w)$$

for $z \in \mathbf{D}$ and $w \in D^\alpha(z)$.

(2) *There exists a constant $B > 0$ such that*

$$D^r(z) \subseteq D^{Br}(w), \qquad D^r(w) \subseteq D^{Br}(z) \tag{2.1}$$

for $w \in D^r(z)$ and $0 < r \le \alpha$.

Throughout this article, we always assume α to be chosen as in Lemma 2.1. Then there is some $s > 0$ such that for $0 < r \le \alpha$, there exists a sequence $\{z_n\}_{n\ge1} \subset \mathbf{D}$ satisfying

(1) $\mathbf{D} = \bigcup_{n\ge1} D^r(z_n)$,
(2) $D^{sr}(z_n) \cap D^{sr}(z_m) = \emptyset$ for $m \ne n$.

With these two hypotheses, it is easy to check that

(3) there exists a positive integer N depending only on B, r such that

$$1 \le \sum_{k=1}^{\infty} \chi_{D^{Br}(a_k)}(z) \le N \quad \text{for } z \in \mathbf{D},$$

where χ_E is the characteristic function of set E. A sequence $\{z_n\}$ satisfying (1)–(3) will be called a *(ρ, r)-lattice*. The (ρ, r)-lattice exists (see [8, Lemma 3.2] for details).

Let $\varphi \in \mathcal{W}_0$ with $\rho \simeq \frac{1}{\sqrt{\Delta\varphi}}$. The distance d_ρ between z and w is defined by

$$d_\rho(z,w) = \inf_\gamma \int_0^1 \big|\gamma'(t)\big| \frac{dt}{\rho(\gamma(t))},$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0,1] \to \mathbf{D}$ with $\gamma(0) = z$ and $\gamma(1) = w$. Denote $K_z(\cdot) = K(\cdot, z)$, and denote by $k_{p,z}$ the normalized Bergman kernel for A^p_φ; that is, $k_{p,z} = K_z/\|K_z\|_{p,\varphi}$. We have the following lemma.

Lemma 2.2. *The Bergman kernel for A^p_φ satisfies the following properties.*

(1) *There exist positive constants σ, C such that*

$$\big|K(z,w)\big| \le C \frac{e^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)} e^{-\sigma d_\rho(z,w)} \quad \text{for } z, w \in \mathbf{D}. \tag{2.2}$$

(2) *There exist some constants* $C > 0$ *such that*

$$\bigl|K(z,w)\bigr| \geq C\frac{e^{\varphi(z)}e^{\varphi(w)}}{\rho(z)\rho(w)} \quad \text{for } d_\rho(z,w) \leq \alpha. \tag{2.3}$$

(3) *For* $0 < p < \infty$, *there is*

$$\|K_z\|_{p,\varphi} \simeq e^{\varphi(z)}\rho(z)^{\frac{2}{p}-2}, \quad z \in \mathbf{D}. \tag{2.4}$$

(4) *For* $0 < p < \infty$, $k_{p,z} \to 0$ *uniformly on any compact subsets of* $\mathbf{D}$ *as* $|z| \to 1$.

Proof. The estimates of (2.2), (2.3), and (2.4) can be found in [8, Section 3]. Since $\varphi \in \mathcal{W}_0$, there exists $r \in (0,1)$ such that $|\rho(z) - \rho(w)| \leq \varepsilon|z-w|$ for $z, w \in \mathbf{D} \setminus D(0,r)$. Letting $w \to \frac{z}{|z|}$, by $\rho \in C_0$ we have

$$\rho(z) \leq \varepsilon\bigl(1-|z|\bigr).$$

Fixing $M > 0$ with $1 + M - 2/p > 0$, (2.4) and Theorem 3.3 in [8, Theorem 3.3] show that

$$\bigl|k_{p,z}(w)\bigr| \leq Ce^{\varphi(w)}\rho(w)^{-1}\rho(z)^{1-2/p}\Bigl(\frac{\min\{\rho(z),\rho(w)\}}{|z-w|}\Bigr)^M.$$

If w is in any compact subset of $\mathbf{D}$ and $|z|$ tends to 1, there is some $C > 0$ independent of z such that

$$\bigl|k_{p,z}(w)\bigr| \leq C\rho(z)^{1+M-2/p} \leq C\bigl(1-|z|\bigr)^{1+M-2/p} \to 0.$$

The proof is completed. □

Suppose that $\mu \geq 0$. Given any $t > 0$, the t-Berezin transform of μ is defined to be

$$\widetilde{\mu}_t(z) = \int_{\mathbf{D}} \bigl|k_{t,z}(w)\bigr|^t e^{-t\varphi(w)}\,d\mu(w), \quad z \in \mathbf{D}.$$

Note that $\widetilde{\mu}_2$ is just the classical Berezin transform. For $0 < r \leq \alpha$, the average of μ at the point $z \in \mathbf{D}$ is defined as

$$\widehat{\mu}_r(z) = \mu\bigl(D^r(z)\bigr)/A\bigl(D^r(z)\bigr).$$

Lemma 2.3. *Let* $0 < p < \infty$. *There exist positive constants* α *and* C *such that, for* $0 < r \leq \alpha$ *and* $f \in H(\mathbf{D})$,

(1)

$$\bigl|f(z)e^{-\varphi(z)}\bigr|^p \leq \frac{C}{A(D^r(z))}\int_{D^r(z)}\bigl|f(w)e^{-\varphi(w)}\bigr|^p\,dA(w), \tag{2.5}$$

(2)

$$\int_{\mathbf{D}}\bigl|f(z)e^{-\varphi(z)}\bigr|^p\,d\mu(z) \leq C\int_{\mathbf{D}}\bigl|f(z)e^{-\varphi(z)}\bigr|^p\widehat{\mu}_r(z)\,dA(z) \tag{2.6}$$

for $\mu \geq 0$.

Proof. Estimate (2.5) can be found in [8, Lemma 3.3]. By (2.5) and (2.1), there is $B > 0$ such that

$$\begin{aligned}
&\int_{\mathbf{D}} \left|f(z)e^{-\varphi(z)}\right|^p d\mu(z) \\
&\quad \le C \int_{\mathbf{D}} \frac{1}{A(D^{Br}(z))} \int_{D^{Br}(z)} \left|f(w)e^{-\varphi(w)}\right|^p dA(w)\, d\mu(z) \\
&\quad \simeq \int_{\mathbf{D}} \left|f(w)e^{-\varphi(w)}\right|^p \frac{\int_{\mathbf{D}} \chi_{D^r(w)}(z)\, d\mu(z)}{A(D^r(w))}\, dA(w) \\
&\quad = \int_{\mathbf{D}} \left|f(w)e^{-\varphi(w)}\right|^p \widehat{\mu}_r(w)\, dA(w).
\end{aligned}$$

This completes the proof. ◻

Denote by L^p the usual pth-Lebesgue space, that is,

$$L^p = \left\{ f \text{ is Lebesgue measurable on } \mathbf{D} : \|f\|_{L^p} = \left(\int_{\mathbf{D}} \left|f(z)\right|^p dA(z) \right)^{1/p} < \infty \right\}.$$

Define the operator T to be

$$Tf(z) = e^{-\varphi(z)} \int_{\mathbf{D}} \left|K(z,w)\right| f(w) e^{-\varphi(w)}\, dA(w), \quad z \in \mathbf{D}. \tag{2.7}$$

We can get the boundedness of T on L^p as follows.

Lemma 2.4. *Suppose that $1 < p < \infty$. Then the operator T is bounded on L^p.*

Proof. It is trivial that T is well defined on L^p by Hölder's inequality and (2.4). For $f \in L^p$, there holds

$$\begin{aligned}
\|Tf\|_{L^p}^p &\le \int_{\mathbf{D}} e^{-p\varphi(z)} \left(\int_{\mathbf{D}} \left|K(z,w)f(w)\right| e^{-\varphi(w)}\, dA(w) \right)^p dA(z) \\
&\le \int_{\mathbf{D}} \int_{\mathbf{D}} \left|f(w)\right|^p \left|K(z,w)\right| e^{-\varphi(w)}\, dA(w) \|K_z\|_{1,\varphi}^{p-1} e^{-p\varphi(z)}\, dA(z) \\
&\le C \int_{\mathbf{D}} e^{-\varphi(z)} \int_{\mathbf{D}} \left|f(w)\right|^p \left|K(z,w)\right| e^{-\varphi(w)}\, dA(w)\, dA(z) \\
&\le C \int_{\mathbf{D}} \left|f(w)\right|^p e^{-\varphi(w)}\, dA(w) \int_{\mathbf{D}} \left|K(z,w)\right| e^{-\varphi(z)}\, dA(z) \\
&\le C \|f\|_{L^p}^p,
\end{aligned}$$

which follows from (2.4), Hölder's inequality, and Fubini's theorem. The proof is completed. ◻

Lemma 2.5. *Let $\{a_k\}_k$ be a (ρ, r)-lattice, $0 < r \le \alpha$, and let $0 < p < \infty$. For $\{\lambda_k\}_k \in l^p$, set*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k k_{a_k}(z) \rho(a_k)^{1-\frac{2}{p}}, \quad z \in \mathbf{D}.$$

Then $f \in A^p_\varphi$ and $\|f\|_{p,\varphi} \le C \|\{\lambda_k\}_k\|_{l^p}$.

Proof. From (2.4), there holds

$$\|f\|_{p,\varphi}^p \le \sum_{k=1}^{\infty} |\lambda_k|^p \rho(a_k)^{p-2} \|k_{a_k}\|_{p,\varphi}^p \simeq \sum_{k=1}^{\infty} |\lambda_k|^p$$

if $0 < p \le 1$. For $1 < p < \infty$, define $F(z) = \sum_{k=1}^{\infty} |\lambda_k| \rho(a_k)^{-\frac{2}{p}} \chi_{D^r(a_k)}(z)$. It is clear that

$$\|F\|_{L^p}^p \le \sum_{k=1}^{\infty} |\lambda_k|^p < \infty.$$

With (2.5), we get

$$\big|f(z)\big| e^{-\varphi(z)} \le C e^{-\varphi(z)} \sum_{k=1}^{\infty} |\lambda_k| \rho(a_k)^{2-\frac{2}{p}} \big|K(z,a_k)\big| e^{-\varphi(a_k)} \le CTF(z),$$

where T is defined as in (2.7). By Lemma 2.4, we see that

$$\|f\|_{p,\varphi} \le C\|TF\|_{L^p} \le C\|F\|_{L^p} \le C\big\|\{\lambda_k\}_k\big\|_{l^p}.$$

This completes the proof. ∎

Carleson measures have been extensively applied to various problems in holomorphic function spaces. In the setting of classical Bergman spaces, Carleson measures have been well studied (see, e.g., [17], [18]). As in [7], we will introduce Carleson measures for the weighted Bergman space A^p_φ.

Let $0 < p, q < \infty$, and let $\mu \ge 0$. We call μ a (p,q)-*Carleson measure* if the embedding operator $i : A^p_\varphi \to L^q_\varphi(d\mu)$ is bounded, where $L^q_\varphi(d\mu)$ consists of all μ-measurable functions f on $\mathbf{D}$ for which

$$\|f\|_{q,\varphi,\mu} = \Big(\int_{\mathbf{D}} \big|f(z)e^{-\varphi(z)}\big|^q \, d\mu(z)\Big)^{1/q} < \infty.$$

Also, we call μ a *vanishing* (p,q)-*Carleson measure* if

$$\lim_{j\to\infty} \int_{\mathbf{D}} \big|f_j(z)e^{-\varphi(z)}\big|^q \, d\mu(z) = 0$$

whenever $\{f_j\}_{j=1}^{\infty}$ is a bounded sequence in A^p_φ that converges to 0 uniformly on any compact subset of $\mathbf{D}$ as $j \to \infty$.

Similar to the proof in [7, Section 3], we can characterize (vanishing) (p,q)-Carleson measures for all possible $0 < p, q < \infty$ in terms of Berezin transforms and average functions, which follows from Lemmas 2.1–2.5.

Theorem 2.6. *Let $0 < p \le q < \infty$, and let $\mu \ge 0$. Set $s = p/q$. Then the following statements are equivalent:*

(A) *μ is a (p,q)-Carleson measure,*
(B) *$\widetilde{\mu}_t \rho^{2(1-1/s)}$ is bounded on $\mathbf{D}$ for some (or any) $t > 0$,*
(C) *$\widehat{\mu}_\delta \rho^{2(1-1/s)}$ is bounded on $\mathbf{D}$ for some (or any) $\delta \in (0,\alpha]$,*

(D) $\{\widehat{\mu}_r(a_k)\rho(a_k)^{2(1-1/s)}\}_{k=1}^{\infty}$ *is bounded for some (or any) (ρ, r)-lattice* $\{a_k\}_{k=1}^{\infty}$, $r \in (0, \alpha]$. *Furthermore,*

$$\|i\|^q_{A^p_\varphi \to L^q_\varphi(d\mu)} \simeq \|\widetilde{\mu}_t \rho^{2(1-1/s)}\|_{L^\infty} \simeq \|\widehat{\mu}_\delta \rho^{2(1-1/s)}\|_{L^\infty} \simeq \left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2(1-1/s)}\right\}_k\right\|_{l^\infty}.$$

Theorem 2.7. *Let $0 < p \le q < \infty$, and let $\mu \ge 0$. Set $s = p/q$. Then the following statements are equivalent:*

(A) *μ is a vanishing (p, q)-Carleson measure,*
(B) *$\widetilde{\mu}_t(z)\rho(z)^{2(1-1/s)} \to 0$ as $z \to \infty$ for some (or any) $t > 0$,*
(C) *$\widehat{\mu}_\delta(z)\rho(z)^{2(1-1/s)} \to 0$ as $z \to \infty$ for some (or any) $\delta \in (0, \alpha]$,*
(D) *$\widehat{\mu}_r(a_k)\rho(a_k)^{2(1-1/s)} \to 0$ as $k \to \infty$ for some (or any) (ρ, r)-lattice $\{a_k\}_{k=1}^{\infty}$, $r \in (0, \alpha]$.*

Theorem 2.8. *Let $0 < q < p < \infty$, and let $\mu \ge 0$. Set $s = p/q$, and let s' denote the conjugate index of s. Then the following statements are equivalent:*

(A) *μ is a (p, q)-Carleson measure,*
(B) *μ is a vanishing (p, q)-Carleson measure,*
(C) *$\widetilde{\mu}_t \in L^{s'}$ for some (or any) $t > 0$,*
(D) *$\widehat{\mu}_\delta \in L^{s'}$ for some (or any) $\delta \in (0, \alpha]$,*
(E) *$\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/s'}\}_{k=1}^{\infty} \in l^{s'}$ for some (or any) (ρ, r)-lattice $\{a_k\}_{k=1}^{\infty}$, $r \in (0, \alpha]$ and furthermore,*

$$\|i\|^q_{A^p_\varphi \to L^q_\varphi(d\mu)} \simeq \|\widetilde{\mu}_t\|_{L^{s'}} \simeq \|\widehat{\mu}_\delta\|_{L^{s'}} \simeq \left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/s'}\right\}_k\right\|_{l^{s'}}.$$

Remark 2.9. For $\varphi \in \mathcal{BDK}$, Pau and Peláez [13] considered the embedding operator from A^p_φ to $L^q_\varphi(d\mu)$. The theorems above generalize their results. By Theorems 2.6, 2.7, and 2.8, we show that (vanishing) (p, q)-Carleson measures depend only on the value of p/q. For simplicity, we call them (vanishing) p/q-Carleson measures instead of (vanishing) (p, q)-Carleson measures, and we denote

$$\|\mu\|_{p/q} = \|i\|_{A^{p/q}_\varphi \to L^1_\varphi(d\mu)}.$$

3. Integration operators

Recall that $K(\cdot, \cdot)$ is the Bergman kernel for A^2_φ. Given $\mu \ge 0$ with hypothesis (1.1), the integral operators I_μ and J_μ as in (1.2) and (1.3) can be densely defined on A^p_φ for $p > 1$. In this section, we focus on discussing the boundedness and compactness of these two operators from A^p_φ to L^p_φ for all $1 < p, q < \infty$ in terms of Carleson measures, and we obtain Theorem 1.1 as our main result. To prove it, we will divide Theorem 1.1 into two separate theorems.

In the case of $p > q$, we need Khintchine's inequality. Let γ_k be the Rademacher function defined by

$$\gamma_0(t) = \begin{cases} 1 & \text{if } 0 \le t - [t] < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le t - [t] < 1 \end{cases}$$

and $\gamma_k(t)=\gamma_0(2^k t)$ for $k=1,2,\dots$, where $[t]$ denotes the largest integer less than or equal to t. For $0<p<\infty$, there exist two positive constants C_1 and C_2 depending only on p such that

$$C_1\Big(\sum_{k=1}^m |b_k|^2\Big)^{p/2}\le\int_0^1\Big|\sum_{k=1}^m b_k\gamma_k(t)\Big|^p\,dt\le C_2\Big(\sum_{k=1}^m |b_k|^2\Big)^{p/2}$$

for all $m\ge 1$ and complex numbers $b_1,b_2,\dots,b_m$.

Theorem 3.1. *Let $1<p,q<\infty$, and let $\mu\ge 0$ with hypothesis (1.1). Set $1/s=1-1/q+1/p$. Then the following statements are equivalent:*

(A) *$I_\mu: A^p_\varphi\to A^q_\varphi$ is bounded;*
(B) *$J_\mu: A^p_\varphi\to L^q_\varphi$ is bounded;*
(C) *μ is an s-Carleson measure and furthermore,*

$$\|I_\mu\|_{A^p_\varphi\to A^q_\varphi}\simeq\|J_\mu\|_{A^p_\varphi\to L^q_\varphi}\simeq\|\mu\|_s. \tag{3.1}$$

Proof. The implication (B) $\Rightarrow$ (A) is trivial and

$$\|I_\mu\|_{A^p_\varphi\to A^q_\varphi}\le\|J_\mu\|_{A^p_\varphi\to L^q_\varphi}. \tag{3.2}$$

We only need to show that (A) $\Rightarrow$ (C) and (C) $\Rightarrow$ (B).

First, we deal with the case $p\le q$. To prove (A) $\Rightarrow$ (C), we assume that $I_\mu: A^p_\varphi\to A^q_\varphi$ is bounded. For any $z\in\mathbf{D}$, by (2.5) and (2.4) we have

$$\begin{aligned}\widetilde{\mu}_2(z)\rho(z)^{2(p-q)/pq}&\le C\rho(z)^{2/q}\big|I_\mu k_{p,z}(z)\big|e^{-\varphi(z)}\\&\le C\Big(\int_{D^\alpha(z)}\big|I_\mu k_{p,z}(w)e^{-\varphi(w)}\big|^q\,dA(w)\Big)^{1/q}\\&\le C\|I_\mu\|_{A^p_\varphi\to A^q_\varphi}\|k_{p,z}\|_{p,\varphi}.\end{aligned}$$

This shows that

$$\sup_{z\in\mathbf{D}}\widetilde{\mu}_2(z)\rho(z)^{2(p-q)/pq}\le C\|I_\mu\|_{A^p_\varphi\to A^q_\varphi}. \tag{3.3}$$

Note that $s\le 1$ and that $1-1/s=1/q-1/p=(p-q)/pq$. By Theorem 2.6 and (3.3), μ is an s-Carleson measure and

$$\|\mu\|_s\le C\|I_\mu\|_{A^p_\varphi\to A^q_\varphi}. \tag{3.4}$$

To show (C) $\Rightarrow$ (B), we suppose that μ is an s-Carleson measure. Since $s\le 1$, Theorem 2.6 tells us that $\widehat{\mu}_\delta\rho^{2(p-q)/pq}$ is bounded on $\mathbf{D}$. We claim that there is some positive constant C such that

$$\big\|J_\mu(f)\big\|^q_{q,\varphi}\le C\int_{\mathbf{D}}\big|f(w)\big|^q e^{-q\varphi(w)}\widehat{\mu}_\delta(w)^q\,dA(w) \tag{3.5}$$

for $f\in A^p_\varphi$. In fact, using (2.5), Hölder's inequality, and (2.4), we obtain

$$\begin{aligned}&\big|J_\mu f(z)\big|^q e^{-q\varphi(z)}\\&\quad\le C\Big(\int_{\mathbf{D}}\widehat{\mu}_\delta(w)\big|f(w)\big|\big|K(w,z)\big|e^{-2\varphi(w)}e^{-\varphi(z)}\,dA(w)\Big)^q\end{aligned}$$

$$\begin{aligned}&\le C\int_{\mathbf{D}}\big|f(w)\big|^q e^{-q\varphi(w)}\widehat{\mu}_\delta(w)^q\big|K(w,z)e^{-\varphi(w)}e^{-\varphi(z)}\big|\,dA(w)\\ &\quad\times\Big(\int_{\mathbf{D}}\big|K(w,z)e^{-\varphi(w)}e^{-\varphi(z)}\big|\,dA(w)\Big)^{q/q'}\\ &\le C\int_{\mathbf{D}}\big|f(w)\big|^q e^{-q\varphi(w)}\widehat{\mu}_\delta(w)^q\big|K(w,z)e^{-\varphi(w)}e^{-\varphi(z)}\big|\,dA(w).\end{aligned}$$

Integrating both sides above and applying Fubini's theorem and (2.4), we get (3.5). Since $p\le q$, (3.5) and (2.5) imply that

$$\begin{aligned}\|J_\mu f\|_{q,\varphi}^q&\le C\int_{\mathbf{D}}\big|f(w)\big|^p e^{-p\varphi(w)}\widehat{\mu}_\delta(w)^q\big(\rho(w)^{-2/p}\|f\|_{p,\varphi}\big)^{q-p}\,dA(w)\\ &\le C\|\widehat{\mu}_\delta\rho^{2(p-q)/pq}\|_{L^\infty}^q\|f\|_{p,\varphi}^q\\ &\simeq\|\mu\|_s^q\|f\|_{p,\varphi}^q.\end{aligned}$$

Therefore, J_μ is bounded from A^p_φ to L^q_φ and

$$\|J_\mu\|_{A^p_\varphi\to L^q_\varphi}\le C\|\mu\|_s. \tag{3.6}$$

For $p>q$, suppose that $I_\mu:A^p_\varphi\to A^q_\varphi$ is bounded. For any (ρ,r)-lattice $\{a_k\}_k$ and sequence $\{\lambda_k\}_k\in l^p$, set f as

$$f_t(z)=\sum_{k=1}^\infty\lambda_k\gamma_k(t)k_{a_k}(z)\rho(a_k)^{1-2/p},$$

where $0<r\le\alpha$. Lemma 2.5 shows that $f\in A^p_\varphi$ with $\|f_t\|_{p,\varphi}\le C\|\{\lambda_k\}_k\|_{l^p}$. The boundedness of I_μ gives

$$\big\|I_\mu(f_t)\big\|_{q,\varphi}^q\le\|I_\mu\|^q_{A^p_\varphi\to A^q_\varphi}\|f_t\|^q_{p,\varphi}\le C\|I_\mu\|^q_{A^p_\varphi\to A^q_\varphi}\big\|\{\lambda_k\}_k\big\|^q_{l^p}.$$

Note that $I_\mu(k_{a_k})\in H(\mathbf{D})$. Fubini's theorem, Khintchine's inequality, and (2.5) yield

$$\begin{aligned}&\int_0^1\big\|I_\mu(f_t)\big\|_{q,\varphi}^q\,dt\\ &\quad=\int_0^1\int_{\mathbf{D}}\Big|\sum_{k=1}^\infty\lambda_k\gamma_k(t)\rho(a_k)^{1-2/p}I_\mu(k_{a_k})(z)\Big|^q e^{-q\varphi(z)}\,dA(z)\,dt\\ &\quad=\int_{\mathbf{D}}\Big(\int_0^1\Big|\sum_{k=1}^\infty\lambda_k\gamma_k(t)\rho(a_k)^{1-2/p}I_\mu(k_{a_k})(z)\Big|^q\,dt\Big)e^{-q\varphi(z)}\,dA(z)\\ &\quad\ge C\int_{\mathbf{D}}\Big(\sum_{k=1}^\infty|\lambda_k|^2\rho(a_k)^{2-4/p}\big|I_\mu(k_{a_k})(z)\big|^2\Big)^{q/2}e^{-q\varphi(z)}\,dA(z)\\ &\quad=C\sum_{j=1}^\infty\int_{D^r(a_j)}\Big(\sum_{k=1}^\infty|\lambda_k|^2\rho(a_k)^{2-4/p}\big|I_\mu(k_{a_k})(z)\big|^2\Big)^{q/2}e^{-q\varphi(z)}\,dA(z)\\ &\quad\ge C\sum_{j=1}^\infty|\lambda_j|^q\rho(a_j)^{2+q-2q/p}\big|I_\mu(k_{a_j})(a_j)\big|^q e^{-q\varphi(a_j)}\end{aligned}$$

$$\geq C\sum_{j=1}^{\infty}|\lambda_j|^q\rho(a_j)^{2+2q-2q/p}\Big|\int_{D^r(a_j)}\big|K(w,a_j)\big|^2e^{-2\varphi(w)}\,d\mu(w)\Big|^q e^{-2q\varphi(a_j)}$$
$$\geq C\sum_{j=1}^{\infty}|\lambda_j|^q\widehat{\mu}_r(a_j)^q\rho(a_j)^{2-2q/p},$$

where the last inequality follows from (2.3). Take $\beta_j=|\lambda_j|^q$. Then $\{\beta_j\}_{j=1}^{\infty}\in l^{p/q}$ with $p/q>1$, and

$$\begin{aligned}\sum_{j=1}^{\infty}\beta_j\widehat{\mu}_r(a_j)^q\rho(a_j)^{2-2q/p}&\leq C\|I_\mu\|^q_{A^p_\varphi\to A^q_\varphi}\big\|\{\lambda_j\}_j\big\|^q_{l^p}\\&=C\|I_\mu\|^q_{A^p_\varphi\to A^q_\varphi}\big\|\{\beta_j\}_j\big\|_{l^{p/q}}.\end{aligned}$$

The duality argument shows that $\{\widehat{\mu}_r(a_j)^q\rho(a_j)^{2-2q/p}\}_{j=1}^{\infty}\in l^{p/p-q}$, and

$$\big\|\big\{\widehat{\mu}_r(a_j)^q\rho(a_j)^{2-2q/p}\big\}_j\big\|_{l^{p/(p-q)}}\leq C\|I_\mu\|^q_{A^p_\varphi\to A^q_\varphi}.$$

This gives

$$\big\|\big\{\widehat{\mu}_r(a_j)\rho(a_j)^{2(p-q)/pq}\big\}_j\big\|_{l^{pq/(p-q)}}\leq C\|I_\mu\|_{A^p_\varphi\to A^q_\varphi}. \tag{3.7}$$

Note that the conjugate index of $pq/(p-q)$ is s. From Theorem 2.8 and (3.7), we know that μ is an s-Carleson measure and that (3.4) is true.

Assuming that μ is an s-Carleson measure, Theorem 2.8 gives $\widehat{\mu}_\delta\in L^{pq/(p-q)}$ for some $\delta\in(0,\alpha]$. Since $p/q>1$, (3.5) and the Hölder's inequality imply that

$$\begin{aligned}\|J_\mu f\|^q_{q,\varphi}&\leq C\Big\{\int_{\mathbf{D}}\big(\big|f(w)\big|^qe^{-q\varphi(w)}\big)^{p/q}\,dA(w)\Big\}^{q/p}\Big\{\int_{\mathbf{D}}\widehat{\mu}_\delta(w)^{pq/(p-q)}\,dA(w)\Big\}^{(p-q)/p}\\&\leq C\|\widehat{\mu}_\delta\|^q_{L^{pq/(p-q)}}\|f\|^q_{p,\varphi}\end{aligned}$$

for $f\in A^p_\varphi$. Hence, J_μ is bounded from A^p_φ to L^q_φ and (3.6) holds, which tells us that (C) $\Rightarrow$ (B) for $p>q$. The estimate (3.1) comes from (3.2), (3.4), and (3.6). The proof is completed. ☐

Theorem 3.2. *Let $1<p,q<\infty$, and let $\mu\geq 0$ with hypothesis (1.1). Set $1/s=1-1/q+1/p$. Then the following statements are equivalent:*

(A) *$I_\mu: A^p_\varphi\to A^q_\varphi$ is compact,*
(B) *$J_\mu: A^p_\varphi\to L^q_\varphi$ is compact,*
(C) *μ is a vanishing s-Carleson measure.*

Proof. It is easy to check that (B) $\Rightarrow$ (A). Suppose that I_μ is compact from A^p_φ to A^q_φ. If $p>q$, then Theorem 3.1 implies that μ is an s-Carleson measure, where $s>1$. By Theorem 2.8, μ is also a vanishing s-Carleson measure. If $p\leq q$, then $s\leq 1$. Similar to the proof in Theorem 3.1, by Lemma 2.2(4), there is

$$\begin{aligned}\widetilde{\mu}_2(z)\rho(z)^{2(p-q)/pq}&\leq C\Big(\int_{D^\alpha(z)}\big|I_\mu(k_{p,z})(w)e^{-\varphi(w)}\big|^q\,dA(w)\Big)^{1/q}\\&\leq C\big\|I_\mu(k_{p,z})\big\|_{q,\varphi}\to 0\end{aligned}$$

as $|z| \to 1$. Hence,

$$\lim_{z\to\infty} \widetilde{\mu}_2(z)\rho(z)^{2(p-q)/pq} = 0.$$

Theorem 2.7 shows that μ is a vanishing s-Carleson measure.

To show that (C) $\Rightarrow$ (B), we assume that statement (C) is true. Given $R \in (0,1)$, μ_R is defined by

$$\mu_R(E) = \mu\big(E \cap \overline{D(0,R)}\big) \quad \text{for } E \subseteq \mathbf{D} \text{ measurable.}$$

It is easy to check that $\mu - \mu_R \geq 0$ and that J_{μ_R} is compact from A^p_φ to L^q_φ. By Theorems 2.7 and 2.8 and (3.6), we have

$$\|J_\mu - J_{\mu_R}\|_{A^p_\varphi \to L^q_\varphi} \leq C\|\mu - \mu_R\|_s \to 0$$

as $R \to \infty$. Therefore, J_μ is compact from A^p_φ to L^q_φ. The proof is completed. $\square$

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