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ON THE MODULUS OF DISJOINTNESS-PRESERVING OPERATORS AND *b*-*AM*-COMPACT OPERATORS ON BANACH LATTICES

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Abstract. We study several properties of the modulus of order bounded disjointness-preserving operators. We show that, if T is an order bounded disjointness-preserving operator, then T and $|T|$ have the same compactness property for several types of compactness. Finally, we characterize Banach lattices having b-AM-compact (resp., AM-compact) operators defined between them as having a modulus that is b-AM-compact (resp., AM-compact).

1. Introduction

In this article, our primary focus is on the properties of the class of disjointness-preserving operators and the class of b-AM-compact operators. Various authors have studied disjointness-preserving operators. In order to read the recent research on order bounded disjointness-preserving operators see, for example, [7], [10], and [12]. Meyer proved that an order bounded disjointness-preserving operator $T : E \to F$ between two Archimedean Riesz spaces has a modulus that is a lattice homomorphism and that $|T||x| = |Tx|$ for all $x \in E$. Another important result related to order bounded disjointness-preserving operator is the polar decomposition theorem (see [8, Theorem 7]). In this paper, we prove a simplified version of the polar decomposition of disjointness-preserving operators on Banach lattices. This version is used with Meyer's theorem in order to prove several new results

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about order bounded disjointness-preserving operators. The b-AM-compact operators were introduced by Aqzzouz and H'michane in [4]. Those authors also studied the duality problem (see [3]).

Aliprantis and Burkinshaw showed that every weakly compact operator from an AL-space into a KB-space has a weakly compact modulus (see [1, Theorem 5.35]). Schmidt proved that every weakly compact operator from an AM-space into a Dedekind complete AM-space with unit has a weakly compact modulus. A similar result for the class of compact operators is due to Krengel. We study this problem for the class of b-AM-compact (AM-compact) operators. More results for the class of AM-compact operators and b-AM-compact operators can be found in [2], [9], and [5].

Before we state our results, we need to fix some notation and recall some definitions. Let E and F be two vector lattices (Riesz spaces), let $x, y \in E$ with $x \leq y$, and let the order interval $[x, y]$ be the subset of E defined by $[x, y] = \{z \in E : x \leq z \leq y\}$. A subset of E is called *order bounded* if it is included in an order interval. Let $T : E \to F$ be an operator between two Riesz spaces E and F. Note that T is considered order bounded if it maps order bounded subsets of E to order bounded subsets of F. By E' and E'' we will denote the topological dual and topological bidual of E, respectively. The vector space $E^\sim$ of all order bounded linear functionals on E is called the *order dual* of E. The vector space $E^{\sim\sim} = (E^\sim)^\sim$ will denote the order bidual of E. The b-order bounded subsets are the sets that are order bounded in $E^{\sim\sim}$. Note also that T is b-order bounded if it maps b-order bounded subsets of E to b-order bounded subsets of F. The algebraic adjoint of T will be denoted by $T' : F' \to E'$, and its order adjoint will be denoted by $T^\sim : F^\sim \to E^\sim$. A vector lattice E is considered to be discrete if it admits a complete disjoint system of discrete elements, where we say a nonzero element $x \in E$ is discrete whenever the ideal generated by x coincides with the vector subspace generated by x. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$, if $|x| \leq |y|$, then we have $\|x\| \leq \|y\|$. A norm of Banach lattice $(E, \|\cdot\|)$ is order-continuous if for each net $(x_\alpha)_{\alpha \in \Lambda}$ such that $x_\alpha \downarrow 0$, (i.e. (x_α) is decreasing and $\inf\{x_\alpha : \alpha \in \Lambda\} = 0$) we have $\|x_\alpha\| \to 0$. A Banach lattice E is said to be a Kantorovich–Banach space (KB-space) whenever every increasing norm bounded sequence of E^+ is norm-convergent. If E is a Banach lattice, and $x \in E^+$, then the principal ideal I_x generated by x is

$$I_x = \big\{y \in E : \exists \lambda > 0 \text{ with } |y| \leq \lambda x\big\},$$

and thus I_x under the norm $\|\cdot\|_\infty$, defined by

$$\|y\|_\infty = \inf\big\{\lambda > 0 : |y| \leq \lambda x\big\}, \quad y \in I_x,$$

is an AM-space with the unit x, whose closed unit ball is the order interval $[-x, x]$. Let $T : E \to X$ be an operator between Banach lattice E and Banach space X. Then T is order weakly compact (resp., b-weakly compact) if it maps an order bounded (resp., b-order bounded) subset of E to relatively weak compact subset of X. Thus T is AM-compact (resp., b-AM-compact) if it maps an order bounded (resp., b-order bounded) subset of E to relatively compact subset of X.

By $K(E,X), AM(E,X)$, and $AM_b(E,X)$ we denote the collection of compact, AM-compact, and b-AM-compact operators, respectively. Clearly we have

$$K(E,X) \subset AM_b(E,X) \subset AM(E,X).$$

For an operator $T : E \to F$ between two Riesz spaces we say that its modulus $|T|$ exists whenever

$$|T| := T \vee (-T)$$

exists. By using [1, Theorem 1.18], for Riesz spaces E and F whenever F is Dedekind-complete, each order bounded operator $T : E \to F$ satisfies the following statement:

$$|T|(x) = \sup\big\{|Ty| : |y| \le x\big\},$$

for each $x \in E^+$. We refer to [1] and [11] for any unexplained terms from Banach lattice theory.

2. Main results

2.1. **On the modulus of disjointness-preserving operators.** In this section, we study and prove some new results about disjointness-preserving operators. Recall that an operator $T : E \to F$ between two Riesz spaces is called disjointness-preserving if $Tx \perp Ty$ for all $x, y \in E$ satisfying $x \perp y$. By Meyer's theorem [11, Theorem 3.1.4], we know that, if an order bounded operator $T : E \to F$ between two Archimedean Riesz spaces preserves disjointness, then its modulus exists, and

$$|T|\big(|x|\big) = \big|T\big(|x|\big)\big| = |Tx|$$

holds for all $x \in E$.

In the following theorem, we prove an extension of the Krengel–Synnatzschke theorem in the case of disjointness-preserving operators (for another proof of the same, see [6, Lemma 2.6]).

Theorem 2.1. *If $T : E \to F$ is an order bounded disjointness-preserving operator between two Archimedean Riesz spaces, then*

$$|T^\sim| = |T|^\sim.$$

Proof. Obviously, $|T^\sim| \le |T|^\sim$ holds, so it is enough to prove that $|T|^\sim \le |T^\sim|$. Let $0 \le f \in F^\sim$, and let $x \in E^+$. By using Meyer's theorem (see [1, Theorem 2.40] and [1, Lemma 1.75]), we have

$$\begin{aligned}\big\langle |T|^\sim f, x\big\rangle &= \big\langle f, |T|x\big\rangle \\ &= \big\langle f, |Tx|\big\rangle \\ &\le \big\langle |T^\sim| f, x\big\rangle,\end{aligned}$$

and so $|T|^\sim f \le |T^\sim| f$. Thus $|T|^\sim \le |T^\sim|$, which completes the proof. □

Corollary 2.2. *If $T : E \to F$ is an order bounded disjointness-preserving operator between two Banach lattices, then $|T'| = |T|'$.*

In the following theorem, we prove a simplified version of the polar decomposition theorem, which asserts that we can write an order bounded disjointness-preserving operator as the product of a continuous operator times a lattice homomorphism.

Theorem 2.3 (Polar decomposition theorem [8, Theorem 7])**.** *Let $T : E \to F$ be an order bounded disjointness-preserving operator between two Banach lattices. Then there exists a continuous operator $U : Z \to Z$ such that $T = U|T|$. Where $Z = \mathcal{B}(|T|(E))$, Z is the band generated by $|T|(E)$. Moreover, $|U| = I$.*

Proof. By using [1, Theorem 3.46(3)], Z is a Banach sublattice of F. Since $T(E) \subset Z$, then, without loss of generality, we assume that $F = Z$. Thus by [11, Corollary 3.1.19] and its proof there exist positive operators $U_1, U_2 : Z \to Z$ such that $U_1 + U_2 = I$, and $T = (U_1 - U_2)|T|$. Since Z is a Banach lattice, then U_1 and U_2 are continuous. Thus if we set $U = U_1 - U_2$, then $U : Z \to Z$ is a continuous operator, and $T = U|T|$. In addition, $|U| = U_1 + U_2 = I$. □

As a corollary, we have the following theorem.

Theorem 2.4. *Let $T : E \to F$ be an order bounded disjointness-preserving operator between two Banach lattices, and assume that $\{x_n\}$ is a sequence in E. The following assertions are true:*

(a) *$\{Tx_n\}$ is norm-(weak)-convergent if $\{|T|(x_n)\}$ is norm-(weak)-convergent;*
(b) *$\{|T|(x_n)\}$ is norm-convergent if $\{Tx_n\}$ is norm-convergent;*
(c) $\ker(T) = \ker(|T|)$*;*
(d) *T has closed range if and only if range of $|T|$ is closed; and*
(e) *T is invertible if and only if $|T|$ is invertible.*

Proof.

(a) Operator T is an order bounded disjointness-preserving operator, so by Theorem 2.3 we have $T = U|T|$, where U is a continuous operator on $\mathcal{B}(|T|(E))$. Assume that $\{|T|(x_n)\}$ is norm-(weak)-convergent to x. Therefore $\{U|T|(x_n)\}$ is norm-(weak)-convergent to $U(x)$. In other words, $\{T(x_n)\}$ is norm-(weak)-convergent to $U(x)$.
(b) Assume that $\{Tx_n\}$ is norm-convergent. Thus $\{Tx_n\}$ is a Cauchy sequence. By the following equality,

$$\big\||T|x_n - |T|x_m\big\| = \|Tx_n - Tx_m\|,$$

and we conclude that $\{|T|(x_n)\}$ is also a Cauchy sequence. Since F is a Banach space, then $\{|T|(x_n)\}$ is norm-convergent.
(c) For each $x \in E$ we have $||T|x| = |T||x| = |Tx|$. Therefore $\||T|x\| = \|Tx\|$, for each $x \in E$. Consequently, $x \in \ker(|T|)$ if and only if $x \in \ker(T)$.
(d) Let $T(E)$ be closed. We prove that $|T|(E)$ is closed. Assume that $y \in \overline{|T|(E)}$, so there exists a sequence $\{x_n\}$ in E such that $\{|T|x_n\}$ is norm-convergent to y. By using Part (a), the sequence $\{Tx_n\}$ is norm-convergent. So there exists some $z \in F$ such that $\{Tx_n\}$ is norm-convergent to z. It follows from our hypothesis that $z = Tx$ for some $x \in E$. Hence

$\|T(x_n - x)\| \to 0$. Thus from

$$\big\||T|(x_n - x)\big\| = \big\|T(x_n - x)\big\|$$

and the uniqueness of limit, we conclude that $y = |T|x \in |T|(E)$. Conversely, let $|T|(E)$ be closed, and let $z \in \overline{T(E)}$. So there exists a sequence $\{x_n\} \subset E$ such that $\|Tx_n - z\| \to 0$. Hence by using Part (b), we conclude that $\{|T|x_n\}$ is norm-convergent. Since $|T|(E)$ is closed, we see that, for some $x \in E$, $\{|T|x_n\}$ is norm-convergent to $|T|x$. Therefore $\{U|T|x_n\}$ is norm-convergent to $U|T|x$; that is, $\{Tx_n\}$ is norm-convergent to Tx. Thus from the uniqueness of limit we have $z = Tx \in T(E)$, and therefore $T(E)$ is closed.

(e) Let T be invertible. It follows from Part (c) that $|T|$ is injective. It is easy to see that $|T|$ is surjective so $|T|$ is invertible. Conversely, let $|T|$ be invertible. By using Part (c) and part (d), we conclude that T is injective and that $T(E)$ is closed. Since $|T|$ is a lattice isomorphism, then $|T|^{-1}$ is positive (see [1, Theorem 2.15]). Hence we easily obtain that $|T|'$ and $(|T|')^{-1}$ are positive; therefore, by the same theorem and from $|T'| = |T|'$ we see that $|T'|$ is a lattice isomorphism. Thus T' is disjointness-preserving. Therefore, by applying Part (c) to T' instead of T, we have

$$\ker(T') = \ker\big(|T'|\big) = \ker\big(|T|'\big) = \{0\}.$$

Thus $T(E) = \overline{T(E)} = {}^{\perp}(\ker(|T|')) = F$, and T is invertible. For another proof of this part, see [6, Proposition 2.7]. $\square$

Corollary 2.5 (see [8, Corollary 1])**.** *Let $T : E \to F$ be an invertible order bounded disjointness-preserving operator between two Banach lattices. Then there exists a continuous operator $W : Z \to Z$ such that $|T| = WT$, where $Z = \mathcal{B}(|T|(E))$.*

Proof. By using the polar decomposition theorem there exists a continuous operator $U : Z \to Z$ such that $T = U|T|$. On the other hand, from Part (e) of Theorem 2.4 we see that $|T|$ is invertible. Therefore, $T|T|^{-1} = U$, and U is invertible. Let $W = U^{-1}$. Consequently, $|T| = W(U|T|) = WT$. $\square$

Corollary 2.6. *Let $T : E \to F$ be an invertible order bounded disjointness-preserving operator between two Banach lattices. For a sequence $\{x_n\}$ in E we observe that $\{Tx_n\}$ is weakly convergent if and only if $\{|T|(x_n)\}$ is weakly convergent.*

Recall that the solid hull of a subset A of Riesz space E is the smallest solid set including A and is exactly the set

$$\mathrm{Sol}(A) := \big\{x \in E : \exists y \in A \text{ with } |x| \le |y|\big\}.$$

Proposition 2.7. *Let $T : E \to F$ be an order bounded disjointness-preserving operator between two Archimedean Riesz spaces, and let $A \subset E$. Then we have*

$$\mathrm{Sol}\big(T(A)\big) = \mathrm{Sol}\big(|T|(A)\big).$$

Proof. Since $|T(x)| = ||T|(x)|$ for each $x \in E$, we have

$$\begin{aligned}
\mathrm{Sol}\big(T(A)\big) &= \big\{x \in F : \exists y \in T(A) \text{ with } |x| \leq |y|\big\}, \\
&= \big\{x \in F : \exists z \in A \text{ with } |x| \leq \big|T(z)\big|\big\}, \\
&= \big\{x \in F : \exists z \in A \text{ with } |x| \leq \big||T|(z)\big|\big\}, \\
&= \big\{x \in F : \exists y \in |T|(A) \text{ with } |x| \leq |y|\big\}, \\
&= \mathrm{Sol}\big(|T|(A)\big),
\end{aligned}$$

which completes the proof. □

Recall that a continuous operator $T : X \to E$ from a Banach space to a Banach lattice is semicompact whenever for each $\epsilon > 0$ there exists some $u \in E^+$ satisfying

$$\big\|\big(|Tx| - u\big)^+\big\| < \epsilon$$

for all $x \in X$ with $\|x\| \leq 1$. In addition, a continuous operator $T : E \to X$ from a Banach lattice to a Banach space is said to be M-weakly compact if $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint sequence $\{x_n\}$ of E. Similarly, a continuous operator $T : X \to E$ from a Banach space to a Banach lattice is said to be L-weakly compact whenever $\lim \|y_n\| = 0$ holds for every disjoint sequence $\{y_n\}$ in the solid hull of $T(U)$, where U is the closed unit ball of the Banach space X. Also note that if $T : E \to F$ is an order bounded disjointness-preserving operator between two Banach lattices, then $||T|x| = |Tx|$ for each $x \in E$, and so $\||T|x\| = \|Tx\|$ for each $x \in E$. A continuous operator $T : E \to X$ is b-AM-compact (resp., AM-compact) if and only if for each $0 \leq x'' \in E''$ (resp., $0 \leq x \in E$) the restriction of T to $E \cap I_{x''}$ (resp., I_x) is compact (see [4, Proposition 2.5]). We are now ready to prove the main result of this section.

Theorem 2.8. *Let $T : E \to F$ be an order bounded disjointness-preserving operator between two Banach lattices. Then $|T|$ exists, and the following assertions are true:*

(a) *T is order weakly compact if and only if $|T|$ is;*
(b) *$|T|$ is b-weakly compact if and only if T is;*
(c) *$|T|$ is b-AM-compact if and only if T is;*
(d) *$|T|$ is AM-compact if and only if T is;*
(e) *$|T|$ is compact if and only if T is;*
(f) *$|T|$ is Dunford–Pettis if and only if T is;*
(g) *$|T|$ is semicompact if and only if T is;*
(h) *$|T|$ is M-weakly compact if and only if T is;*
(i) *if T or $|T|$ is L-weakly compact then both of them are M-weakly compact and L-weakly compact; and*
(j) *T is weakly compact if $|T|$ is. Moreover, the converse is true if T is invertible.*

Proof. The existence of a modulus of T is a well-known result by Meyer [11, Theorem 3.1.4].

(a) Assume that $|T|$ is order weakly compact; we prove that T is order weakly compact. Let $\{x_n\} \subset E^+$ be a weakly null order bounded sequence. Since $|T|$ is order weakly compact, so $\||T|(x_n)\| \to 0$ by using [11, Corollary 3.4.9]. There exists a continuous operator $U : \mathcal{B}(|T|(E)) \to \mathcal{B}(|T|(E))$ such that $T = U|T|$, by using polar decomposition theorem. It follows from continuity of U that $\|U|T|(x_n)\| \to 0$. In other words, $\|T(x_n)\| \to 0$. Therefore by same corollary T is order weakly compact. For the converse, see [3, Theorem 2.2].

(b) Let $\{x_n\}$ be a b-order bounded disjoint sequence of positive elements in E. For each $n \in \mathbb{N}$ we have,

$$\big\||T|x_n\big\| = \|Tx_n\|.$$

In other words, $\|Tx_n\| \to 0$ if and only if $\||T|(x_n)\| \to 0$. Hence from [2, Proposition 1] we conclude that $|T|$ is b-weakly compact if and only if T is b-weakly compact. The same method can be used to prove parts (f) and (h).

(c) Assume that T is b-AM-compact; we prove that $|T|$ is b-AM-compact. Let $\{x_n\}$ be a b-order bounded sequence in E such that $\{|T|x_n\}$ is weakly convergent to x for some $x \in F$. By using [4, Proposition 2.6], it is sufficient to prove that $|T|x_n$ is norm-convergent to x. Since $\{|T|x_n\}$ is weakly convergent, by using part (a) of Theorem 2.4, $\{Tx_n\}$ is weakly convergent to some $y \in F$. Then T is b-AM-compact, so Tx_n is norm-convergent to y. Therefore $\{Tx_n\}$ is a Cauchy sequence. So from

$$\big\||T|x_n - |T|x_m\big\| = \|Tx_n - Tx_m\|,$$

it holds that $\{|T|x_n\}$ is a Cauchy sequence. Therefore $\{|T|x_n\}$ is norm-convergent to some $z \in F$. It follows from the uniqueness of weak limit that $z = x$. So $|T|x_n$ is norm-convergent to x, and that completes the proof.

Conversely, assume that $|T|$ is b-AM-compact. It is sufficient to prove that, for each $0 \leq x'' \in E''$, if $Y = I_{x''} \cap E$, then the restriction of T to Y is a compact operator. So let $0 \leq x'' \in E''$ be fixed and let $Y = I_{x''} \cap E$. Since $|T|$ is b-AM-compact, the restriction of $|T|$ to Y is a compact operator. On the other hand, by using polar decomposition theorem, there exists a continuous operator $U : \mathcal{B}(|T|(E)) \to \mathcal{B}(|T|(E))$ such that $T = U|T|$. So it follows from continuity of U that the restriction of $U|T|$ to Y is compact. In other words, the restriction of T to Y is compact, and the proof is complete.

(d) The proof of AM-compactness of $|T|$ is similar to the proof of b-AM-compactness whenever T is AM-compact. So we just prove that if $|T|$ is AM-compact, then T is also AM-compact. It is sufficient to show that for each $x \in E^+$ the restriction of T to I_x is a compact operator. Since $|T|$ is AM-compact, the restriction of $|T|$ to I_x is a compact operator. Now by the continuity of U that is given by the polar decomposition theorem, the restriction of $U|T|$ to I_x is a compact operator (i.e., the restriction of T to I_x is a compact operator), and the proof is complete.

(e) This is a consequence of part (a) and part (b) of Theorem 2.4. For a proof, see [12, Proposition 1.9].
(f) See the proof of part (b).
(g) We know $|Tx| = ||T|x|$ for each $x \in E$. So,

$$\big\|\big(|Tx| - u\big)^+\big\| = \big\|\big(||T|x| - u\big)^+\big\|,$$

for each $x, u \in E$. This ends the proof.
(h) See the proof of part (b).
(i) From Proposition 2.7 we conclude that $\mathrm{Sol}(T(U)) = \mathrm{Sol}(|T|(U))$, where U is the closed unit ball of X. Therefore T is L-weakly compact if and only if $|T|$ is. So both T and $|T|$ are L-weakly compact. We know that $|T|$ is a lattice homomorphism, so the result follows from [1, Exercise 4(a), p. 336] and from part (h).
(j) This follows from part (a) of Theorem 2.4. Assume that T is invertible; then the converse follows from Corollary 2.6. □

2.2. **On the modulus of b-AM-compact and AM-compact operators.** In this section, we prove a theorem that characterizes Banach lattices such that each b-AM-compact (resp., AM-compact) operator between them has a modulus that is b-AM-compact (resp., AM-compact). The proof of the first part employs the method used in the proof of [1, Theorem 5.7]. We start this section with an example of a compact operator (therefore b-AM-compact and AM-compact) whose modulus exists but is neither b-AM-compact nor AM-compact.

Example 2.9. For this example, we assume all hypotheses and definitions in [1, Example 5.6]. Then $T : E \to E$ is a norm bounded operator, which is defined as follows:

$$T(x_1, x_2, \ldots) = (\alpha_1 T_1 x_1, \alpha_2 T_2 x_2, \ldots)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots) \in \ell^\infty$ is fixed. If $\lim \alpha_n = 0$, then T is a compact operator.

(a) If we put $\alpha_n = 2^{-\frac{n}{3}}$, then T is a compact operator and also an AM-compact and b-AM-compact operator, but its modulus does not exist.
(b) If we set $\alpha_n = 2^{-\frac{n}{2}}$, then T is a compact operator, and its modulus exists but is not a compact operator. Moreover, we assert that $|T|$ is not b-AM-compact. Indeed the norm bounded sequence $\{\widehat{x}_n\}$, which was constructed as follows, is also b-order bounded. For each n, fix $x_n \in E_n$ with $\|x_n\| = 1$ and $\||T_n|(x_n)\| = 2^{\frac{n}{2}}$. Let $\widehat{x}_n$ denote the element of E whose nth component is x_n and every other zero. Thus $\|\widehat{x}_n\| = 1$. Let $\widehat{x} = (|x_1|, |x_2|, \ldots)$, and we have

$$\widehat{x} \in E'' = ({E_1}'' \oplus {E_2}'' \oplus \ldots)_\infty.$$

Therefore $\{\widehat{x}_n\} \subset [-\widehat{x}, \widehat{x}]$ so $\{\widehat{x}_n\}$ is a b-order bounded sequence. We know that for $n > m$,

$$\begin{aligned}\big\||T|\widehat{x}_n - |T|\widehat{x}_m\big\| &= \big\|\big(0, \ldots, 0, -\alpha_m |T| x_m, 0, \ldots, 0, \alpha_n |T| x_n, 0, 0, \ldots\big)\big\| \\ &= 1;\end{aligned}$$

thus $|T|$ is neither a b-AM-compact operator nor a compact operator. On the other hand, since E has order continuous norm and is a discrete Banach lattice, then $|T|$ is AM-compact by using [4, Lemma 2.2].

(c) Next we replace E with $F = (E_1 \oplus E_2 \oplus \ldots)_\infty$, and we define $T : F \to F$ as we have above. If we then put $\alpha_n = 2^{-\frac{n}{2}}$, we obtain that T is a compact operator (and also an AM-compact and b-AM-compact operator) and that $|T|$ exists. Since

$$\{\widehat{x}_n\} \subset [-\widehat{x}, \widehat{x}] \subset F,$$

then $\{\widehat{x}_n\}$ is an order bounded subset of F. In a similar manner we can show that $|T|$ is not AM-compact and therefore that it is neither a b-AM-compact nor a compact operator.

Theorem 2.10. *Let $T : E \to F$ be a b-AM-compact (resp., AM-compact) operator between two Banach lattices if either:*

(a) *F is an AM-space, or*

(b) *E is an AL-space, and F is a discrete KB-space.*

Then T has a b-AM-compact (resp., AM-compact) modulus that is given by the Riesz–Kantorovich formula,

$$|T|x = \sup\{Ty : y \in E, |y| \leq x\}.$$

In addition, the set of all b-AM-compact (resp., AM-compact) operators from E to F with the r-norm is a Banach lattice.

Proof.

(a) Let F be an AM-space, and for $x \in E^+$ we write

$$A_x = \{Ty : y \in E, |y| \leq x\} = T[-x, x].$$

Thus A_x is totally bounded; according to [1, Theorem 4.30], we know that $|T|x = \sup A_x$ exists in F. Hence $|T|x$ exists for each $x \in E^+$; therefore $|T|$ exists.

First, let T be a b-AM-compact operator, and then let B be a b-order bounded subset of E. There is some $\tilde{x} \in E''$ such that $B \subset [-\tilde{x}, \tilde{x}]$. Let $S = [-\tilde{x}, \tilde{x}] \cap E$. Hence $B \subset S$. Since S is b-order bounded, then $T(S)$ is totally bounded in F. If D denotes all suprema of finite subsets of $T(S)$, then, by [1, Theorem 4.30], D is totally bounded. For each $x \in S^+ = S \cap E^+$, let A_x be defined as above. Thus by [1, Theorem 4.30] we have $|T|x = \sup A_x \in \overline{D}$. Hence $|T|(S^+) \subset \overline{D}$ shows that $|T|(S^+)$ is totally bounded; therefore $|T|(S)$ is totally bounded. Furthermore, $|T|(B)$ is relatively compact; that is, $|T|$ is b-AM-compact.

On the other hand, let T be an AM-compact operator. Again for each $x \in E^+$ the set A_x, as defined above, is totally bounded. Let B be an order bounded subset of E; therefore, there is some $x \in E$ such that $B \subset [-x, x]$. Set $S = [-x, x]$. Similar to the above argument, $|T|(S)$ is totally bounded. So $|T|(B)$ is relatively compact; that is, $|T|$ is AM-compact.

(b) By using [1, Theorem 4.75] and the fact that E is AL-space and that F is KB-space, we see that $|T|$ exists. Now by using [4, Proposition 2.9(3)], we have that $|T|$ is b-AM-compact. To prove that the vector space of all b-AM-compact (resp. AM-compact) operators from E into F is a Banach lattice, one can repeat the arguments in the proof of [1, Theorem 4.74]. $\square$

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