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THE APPROXIMATE HYPERPLANE SERIES PROPERTY AND RELATED PROPERTIES

MARÍA D. ACOSTA,[1] RICHARD MARTIN ARON,[2] and FRANCISCO JAVIER GARCÍA-PACHECO[3*]

In beloved memory of Professor Antonio Aizpuru

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Abstract. We show that the *approximate hyperplane series property* (AHSp) is stable under finite ℓ_p-sums ($1 \leq p < \infty$). As a consequence, we obtain that the class of spaces Y such that the pair (ℓ_1, Y) has the Bishop–Phelps–Bollobás property for operators is stable under finite ℓ_p-sums for $1 \leq p < \infty$. We also deduce that every Banach space of dimension at least 2 can be equivalently renormed to have the AHSp but to fail Lindenstrauss' property β. We also show that every infinite-dimensional Banach space admitting an equivalent strictly convex norm also admits such an equivalent norm failing the AHSp.

1. Introduction

Our main objectives here are to examine the stability properties of the *approximate hyperplane series property* (AHSp) and the behavior of this property under equivalent renormings. This section is devoted to basic definitions and a review of known results related to the AHSp and to the Bishop–Phelps–Bollobás property. All Banach spaces throughout this manuscript will be considered real or complex since all the results and definitions work for both cases.

For a Banach space X, as usual, B_X and S_X denote the closed unit ball and the unit sphere of X, respectively. We will write X^* for the topological dual of X. By a *convex series* we mean a series of nonnegative real numbers whose sum is

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*Corresponding author.
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equal to 1. The approximate hyperplane series property was originally studied in 2008. The following is an equivalent formulation of this property.

Definition 1.1 ([1, Remark 3.2]). Let X be a Banach space. We say that X satisfies the AHSp if for every $\varepsilon > 0$ there exist $\gamma_X(\varepsilon) > 0$ and $\eta_X(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\gamma_X(\varepsilon) = 0$ such that, for every sequence $\{x_n\}$ in S_X and every convex series $\sum_n \alpha_n$ with

$$\Big\|\sum_{k=1}^{\infty}\alpha_k x_k\Big\| > 1-\eta_X(\varepsilon),$$

there are a subset $A \subseteq \mathbb{N}$ with $\sum_{k\in A}\alpha_k > 1-\gamma_X(\varepsilon)$, an element $x^* \in S_{X^*}$, and $\{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.

It will be helpful to use the following characterization of the AHSp.

Proposition 1.2. *Let X be a Banach space. The following are equivalent.*

(a) *The space X has the AHSp.*
(b) *For every $\varepsilon > 0$ there exist $\gamma_X(\varepsilon) > 0$ and $\eta_X(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\gamma_X(\varepsilon) = 0$ such that, for every sequence $\{x_n\}$ in B_X and every convex series $\sum_n \alpha_n$ with $\|\sum_{k=1}^{\infty}\alpha_k x_k\| > 1-\eta_X(\varepsilon)$, there are a subset $A \subseteq \mathbb{N}$ with $\sum_{k\in A}\alpha_k > 1-\gamma_X(\varepsilon)$, an element $x^* \in S_{X^*}$, and $\{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.*
(c) *For every $0 < \varepsilon < 1$ there exists $0 < \eta < \varepsilon$ such that, for any sequence $\{x_n\}$ in B_X and every convex series $\sum_n \alpha_n$ with $\|\sum_{k=1}^{\infty}\alpha_k x_k\| > 1-\eta$, there are a subset $A \subset \mathbb{N}$ with $\sum_{k\in A}\alpha_k > 1-\varepsilon$, an element $x^* \in S_{X^*}$, and $\{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.*
(d) *This is the same as (c) but for any sequence $\{x_n\}$ in S_X.*

It is known that many Banach spaces have this property.

Theorem 1.3 ([1, Propositions 3.5, 3.6, 3.7])**.** *Let X be a Banach space. Then X has the AHSp if*

(1) *X is finite-dimensional,*
(2) *$X = \mathcal{C}(K)$ for some compact Hausdorff topological space K,*
(3) *$X = \mathsf{L}^1(\mu)$ for some σ-finite measure μ.*

Uniformly convex spaces also have the AHSp. From Theorem 1.3 it follows that the converse does not hold. However, we have the following result.

Theorem 1.4 ([1, Propositions 3.8, 3.9])**.** *Let X be a Banach space. The following conditions are equivalent:*

(1) *X is uniformly convex,*
(2) *X is strictly convex and satisfies the AHSp.*

The AHSp was introduced as a useful tool to prove extensions of the Bishop–Phelps–Bollobás theorem for the Banach space of *continuous linear operators* $\mathcal{L}(X,Y)$ between Banach spaces X and Y.

Definition 1.5 ([1, Definition 1.1]). Given two Banach spaces (both real or complex) X and Y, the pair (X,Y) has the *Bishop–Phelps–Bollobás property* (BPBp) for operators if for every $\varepsilon > 0$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for any

$S \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist $T \in S_{\mathcal{L}(X,Y)}$ and $u_0 \in S_X$ satisfying the following conditions:

$$\|T(u_0)\| = 1, \qquad \|u_0 - x_0\| < \varepsilon, \qquad \text{and} \qquad \|T - S\| < \varepsilon.$$

Roughly speaking, a pair of Banach spaces (X, Y) has the BPBp for operators if any pair (T, x_0) of an operator $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that $\|Tx_0\|$ is close to 1 can be approximated by a new pair of elements (S, z) in the product $S_{\mathcal{L}(X,Y)} \times S_X$ such that S attains its norm at z. The utility of the two concepts, AHSp and BPBp, is evident from the following result.

Theorem 1.6 ([1, Theorem 4.1]). *Let Y be a Banach space. The following conditions are equivalent:*

1. *the pair (ℓ_1, Y) has the BPBp,*
2. *Y satisfies the AHSp.*

Another related and helpful concept is *Lindenstrauss's property β*, which was introduced in [10] as another means of studying the denseness of norm-attaining operators. For our purpose, the following definition is worth mentioning.

Definition 1.7 ([10, Proposition 3]). A Banach space Y is said to have *property β* (of Lindenstrauss) if there are two sets $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$ and $0 \leq \rho < 1$ such that the following conditions hold:

1. $y_i^*(y_i) = 1$ for every $i \in I$,
2. $|y_i^*(y_j)| \leq \rho < 1$ for any $i, j \in I, i \neq j$,
3. $\|y\| = \sup\{|y_i^*(y)| : i \in I\}$ for every $y \in Y$.

Theorem 1.8 ([1, Theorem 2.2]). *Let Y be a Banach space. If Y satisfies property β of Lindenstrauss, then the pair (X, Y) has the BPBp for every Banach space X.*

Relying on [13, Theorem 1.8], the following is a consequence of the previous result.

Corollary 1.9 ([1, Corollary 2.3]). *Let X be a Banach space. Then we have the following.*

1. *If X has property β of Lindenstrauss, then X has the AHSp.*
2. *There exists an equivalent norm on X that satisfies the AHSp.*

We finish this Introduction with an outline of the main results of this note. In the upcoming section, we prove that the AHSp is stable under finite ℓ_p-sums ($1 \leq p < \infty$). As a consequence, we also obtain the following two results. Every Banach space of dimension at least 2 admits an equivalent norm having the AHSp but failing property β of Lindenstrauss. Every infinite-dimensional Banach space admitting a strictly convex equivalent norm also admits such a strictly convex equivalent norm failing the AHSp.

2. Stability of the approximate hyperplane series property

Our aim is to show that the AHSp is preserved by finite ℓ_p-sums. In order to do this, the first step we take is to show the inheritance of the AHSp to ℓ_p-summands for $1 \le p \le \infty$. It should be mentioned that the following result is already known for $p = 1$ and $p = \infty$ (see [3, Propositions 2.4 and 2.7] and [1, Theorem 4.1]).

Proposition 2.1. *Let X be a Banach space, and let $1 \le p < \infty$. If $X = M \oplus_p N$ has the AHSp, then both M and N also have it. In this case, with obvious notation, $\eta_M(\varepsilon)$ can be chosen to equal $\eta_X(\varepsilon/2)$.*

Proof. Assume that $1 \le p < \infty$ and that $X = M \oplus_p N$ has the AHSp. Let $0 < \varepsilon < 1$, $\{x_n\}$ be a sequence in S_M, and consider a convex series $\sum_n \alpha_n$ satisfying

$$\Big\|\sum_{n=1}^{\infty} \alpha_n x_n\Big\| > 1 - \eta_X\Big(\frac{\varepsilon}{2}\Big).$$

By hypothesis, there exist $A \subseteq \mathbb{N}$, $x^* \in S_{X^*}$, and $\{z_k : k \in A\} \subset (x^*)^{-1}(1) \cap B_X$ such that

$$\sum_{n \in A} \alpha_n > 1 - \frac{\varepsilon}{2} > 1 - \varepsilon \qquad \text{and} \qquad \|z_k - x_k\| < \frac{\varepsilon}{2} \quad \text{for all } k \in A.$$

For every $k \in A$, we can write $z_k = m_k + n_k$, where $m_k \in M$ and $n_k \in N$. Suppose that $m_k = 0$ for some $k \in A$. Then

$$2^{1/p} = \|n_k - x_k\| = \|z_k - x_k\| < \frac{\varepsilon}{2},$$

which contradicts our assumption on ε. Hence $m_k \neq 0$ for every $k \in A$. Observe also that for every $k \in A$, we have

$$\begin{aligned}
\Big\|x_k - \frac{m_k}{\|m_k\|}\Big\| &\le \|x_k - m_k\| + \Big\|m_k - \frac{m_k}{\|m_k\|}\Big\| \\
&\le \|x_k - m_k\| + \big|1 - \|m_k\|\big| \\
&= \|x_k - m_k\| + \big|\|x_k\| - \|m_k\|\big| \\
&\le 2\|x_k - m_k\| \\
&\le 2\|x_k - z_k\| \\
&< \varepsilon.
\end{aligned}$$

Recall that $X^* = M^* \oplus_q N^*$, with q being the conjugate exponent of p. So we can write $x^* = m^* + n^*$, where $m^* \in B_{M^*}$ and $n^* \in B_{N^*}$. If $m^* = 0$, then for every $k \in A$ we have

$$1 = \operatorname{Re} x^*(z_k) = \operatorname{Re} n^*(n_k) \le \|n_k\| < \|z_k\| = 1,$$

which is impossible.

Finally, for every $k \in A$ we have

$$\begin{aligned}
1 &= \operatorname{Re} x^*(z_k) \\
&= \operatorname{Re} m^*(m_k) + \operatorname{Re} n^*(n_k) \\
&\le \|m^*\|\|m_k\| + \|n^*\|\|n_k\| \\
&\le \big\|\big(\|m^*\|, \|n^*\|\big)\big\|_q \big\|\big(\|m_k\|, \|n_k\|\big)\big\|_p \\
&= \|x^*\|\|z_k\| \\
&= 1.
\end{aligned}$$

Since $\|m^*\|\|m_k\| > 0$ for every $k \in A$, we deduce that $m^*(m_k) = \operatorname{Re} m^*(m_k) = \|m^*\|\|m_k\|$ for every $k \in A$. Thus, we have proved that M has the AHSp. □

Before approaching the converse to Proposition 2.1, we would like to point out that the AHSp is not an inherited property, as shown in the next remark.

Remark 2.2. Let X be a nonreflexive, strictly convex Banach space. We know by Theorem 1.4 that X does not have the AHSp. Now, let Γ be an index set so that X can be regarded as an isometric subspace of $\ell_\infty(\Gamma)$. Since $\ell_\infty(\Gamma)$ has property β, it satisfies the AHSp by virtue of [1, Theorems 2.2, 4.1].

It is time now to take care of the converse to Proposition 2.1 for $1 \leq p < \infty$. Due to the necessity of employing different proofs, we will prove the cases $p = 1$ and $1 < p < \infty$. Let us begin with the case $p = 1$.

Theorem 2.3. *Let X be a Banach space. If $X = M \oplus_1 N$ and if M and N both have the AHSp, then so does X.*

Proof. Let us fix $0 < \varepsilon < 1$. We write $\varepsilon' = \varepsilon/5$. By assumption there is $0 < \eta' < \varepsilon'/3$ such that condition (c) in Proposition 1.2 is satisfied for M and N with (ε', η'), simultaneously. We take $\eta = \frac{(\eta')^2\varepsilon'}{6(1+\varepsilon'+\varepsilon'\eta')}$. In order to prove that X satisfies the AHSp, we will check that condition (d) in Proposition 1.2 is satisfied for (ε, η).

Assume that $\{x_n\}$ is a sequence in S_X and that $\sum_n \alpha_n$ is a convex series such that $\|\sum_{n=1}^\infty \alpha_n x_n\| > 1 - \eta$. If P and Q denote the canonical projections from X onto M and N, respectively, then

$$\begin{aligned}
1 - \eta &< \Big\|\sum_{n=1}^\infty \alpha_n x_n\Big\| \\
&= \Big\|\sum_{n=1}^\infty \alpha_n P(x_n)\Big\| + \Big\|\sum_{n=1}^\infty \alpha_n Q(x_n)\Big\| \\
&\leq \sum_{n=1}^\infty \alpha_n \|P(x_n)\| + \Big\|\sum_{n=1}^\infty \alpha_n Q(x_n)\Big\| \\
&\leq \sum_{n=1}^\infty \alpha_n \|P(x_n)\| + \sum_{n=1}^\infty \alpha_n \|Q(x_n)\| \\
&= \sum_{n=1}^\infty \alpha_n \|x_n\| \\
&= 1. \qquad (2.1)
\end{aligned}$$

As a consequence, we obtain

$$\begin{cases} \|\sum_{n=1}^\infty \alpha_n P(x_n)\| \geq \sum_{n=1}^\infty \alpha_n \|P(x_n)\| - \eta, \\ \|\sum_{n=1}^\infty \alpha_n Q(x_n)\| \geq \sum_{n=1}^\infty \alpha_n \|Q(x_n)\| - \eta. \end{cases} \qquad (2.2)$$

For simplicity, we will denote $r_n := \|P(x_n)\|$, $s_n := \|Q(x_n)\|$, $r := \sum_{n=1}^\infty \alpha_n r_n$, and $s := \sum_{n=1}^\infty \alpha_n s_n$. Note that $r + s = 1 = r_n + s_n$ for every $n \in \mathbb{N}$.

Notice that it is trivially satisfied that

$$\frac{\eta'\varepsilon'}{3(1+\varepsilon'+\varepsilon'\eta')} < \frac{\eta'\varepsilon'(3(1+\varepsilon')+2\varepsilon'\eta')}{(1+\varepsilon')3(1+\varepsilon'+\varepsilon'\eta')}.$$

So we can choose a real number a such that

$$\frac{\eta'\varepsilon'}{3(1+\varepsilon'+\varepsilon'\eta')} < a < \frac{\eta'\varepsilon'(3(1+\varepsilon')+2\varepsilon'\eta')}{(1+\varepsilon')3(1+\varepsilon'+\varepsilon'\eta')}. \tag{2.3}$$

In order to prove the result, we will distinguish three cases.

Case 1. Assume that $r \leq a$.

Let $C = \{k \in \mathbb{N} : r_k < \varepsilon/5\}$. Then

$$\frac{\varepsilon}{5}\sum_{k\in\mathbb{N}\setminus C}\alpha_k \leq \sum_{k\in\mathbb{N}\setminus C}\alpha_k r_k \leq \sum_{k=1}^{\infty}\alpha_k r_k = r \leq a,$$

so

$$\sum_{k\in\mathbb{N}\setminus C}\alpha_k \leq \frac{5a}{\varepsilon} \qquad \text{and} \qquad \sum_{k\in C}\alpha_k \geq 1-\frac{5a}{\varepsilon}. \tag{2.4}$$

On the other hand, from inequality (2.1) we obtain that

$$\sum_{k=1}^{\infty}\alpha_k s_k \geq 1-\eta-\sum_{k=1}^{\infty}\alpha_k r_k \geq 1-\eta-a. \tag{2.5}$$

By combining (2.2) and (2.5) we obtain that

$$\Big\|\sum_{k=1}^{\infty}\alpha_k Q(x_k)\Big\| \geq \sum_{k=1}^{\infty}\alpha_k s_k - \eta \geq 1-2\eta-a.$$

As a consequence, in view of (2.4) and (2.3) we deduce that

$$\begin{aligned}\Big\|\sum_{k\in C}\alpha_k Q(x_k)\Big\| &\geq 1-2\eta-a-\sum_{k\in\mathbb{N}\setminus C}\alpha_k s_k\\ &\geq 1-2\eta-a-\sum_{k\in\mathbb{N}\setminus C}\alpha_k\\ &\geq 1-2\eta-a-\frac{5a}{\varepsilon}\\ &> 1-\eta'.\end{aligned}$$

Since N has the AHSp, by Proposition 1.2, there is a set $D \subseteq C$ such that $\sum_{k\in D}\alpha_k > 1-\varepsilon'$, and there exists $\{v_k : k \in D\} \subseteq S_N$ so that there is $n^* \in S_{N^*}$ with $n^*(v_k) = 1$ and $\|v_k - Q(x_k)\| < \varepsilon'$ for all $k \in D$. Since $D \subseteq C$, for every $k \in D$, we define $m_k := r_k m_0$ for an arbitrary $m_0 \in S_M$. Note that if $k \in D \subseteq C$, then $r_k < \varepsilon/5$ so $s_k > 1-\varepsilon/5$. Take $n_k = s_k v_k$ for $k \in D$. Then for $k \in D$, we have

$$\big\|n_k-Q(x_k)\big\| = \big\|s_k v_k-Q(x_k)\big\| \leq \|s_k v_k-v_k\|+\big\|v_k-Q(x_k)\big\| \leq 1-s_k+\varepsilon' < \frac{\varepsilon}{5}+\varepsilon'.$$

Hence the element $y_k := m_k + n_k$, for $k \in D$, satisfies $\|y_k\| = r_k + s_k\|v_k\| = r_k + s_k = 1$. By the choice of ε', for all $k \in D$ we have

$$\begin{aligned}\|y_k - x_k\| &= \big\|P(y_k - x_k)\big\| + \big\|Q(y_k - x_k)\big\| \\ &= \big\|m_k - P(x_k)\big\| + \big\|n_k - Q(x_k)\big\| \\ &\le 2r_k + \frac{\varepsilon}{5} + \varepsilon' \\ &< 3\frac{\varepsilon}{5} + \frac{\varepsilon}{5} \\ &< \varepsilon.\end{aligned}$$

If we choose an element $m^* \in S_{M^*}$ such that $m^*(m_0) = 1$, then $m^*(m_k) = r_k$ for all $k \in D$, and the element $x^* = m^* + n^* \in S_{X^*}$ verifies that

$$x^*(y_k) = m^*(r_k m_0) + n^*(s_k v_k) = r_k + s_k = \|x_k\| = 1.$$

Finally, $\sum_{k \in D} \alpha_k > 1 - \varepsilon' > 1 - \varepsilon$.

Case 2. Assume that $s \le a$.

If we assume this, then we may proceed analogously to the case 1 since M also satisfies the AHSp.

Case 3. Assume now that $r, s > a$.

First, we apply the fact that M has the AHSp. In view of equation (2.2), there is $m^* \in S_{M^*}$ such that

$$\operatorname{Re} m^*\Big(\sum_{k=1}^{\infty} \alpha_k P(x_k)\Big) = \Big\|\sum_{k=1}^{\infty} \alpha_k P(x_k)\Big\| \ge r - \eta.$$

Let $A_1 := \{k \in \mathbb{N} : r_k \neq 0$ and let $\operatorname{Re} m^*(\frac{P(x_k)}{r_k}) > 1 - \eta'/2\}$.

Since

$$\begin{aligned} r - \eta &\le \sum_{k \in A_1} \alpha_k \operatorname{Re} m^*\big(P(x_k)\big) + \sum_{k \in \mathbb{N}\setminus A_1} \alpha_k \operatorname{Re} m^*\big(P(x_k)\big) \\ &\le \sum_{k \in A_1} \alpha_k r_k + \sum_{k \in \mathbb{N}\setminus A_1} \alpha_k r_k \Big(1 - \frac{\eta'}{2}\Big) \\ &= \sum_{k=1}^{\infty} \alpha_k r_k - \frac{\eta'}{2} \sum_{k \in \mathbb{N}\setminus A_1} \alpha_k r_k \\ &= r - \frac{\eta'}{2} \sum_{k \in \mathbb{N}\setminus A_1} \alpha_k r_k, \end{aligned}$$

we deduce that $\frac{\eta'}{2} \sum_{k \in \mathbb{N}\setminus A_1} \alpha_k r_k \le \eta$, and in view of (2.3) and the definition of η, we get $\sum_{k \in \mathbb{N}\setminus A_1} \alpha_k r_k \le 2\eta/\eta' < a < r$. Hence, $\sum_{k \in A_1} \alpha_k \ge \sum_{k \in A_1} \alpha_k r_k > 0$.

Now we define the sets L_1 and C_1 as follows:

$$L_1 := \Big\{k \in \mathbb{N}\setminus A_1 : r_k \le \frac{\varepsilon'}{3}\Big\}, \qquad C_1 := \Big\{k \in \mathbb{N}\setminus A_1 : r_k > \frac{\varepsilon'}{3}\Big\}.$$

Then

$$\frac{\varepsilon'\eta'}{6}\sum_{k\in C_1}\alpha_k < \frac{\eta'}{2}\sum_{k\in C_1}\alpha_k r_k \le \frac{\eta'}{2}\sum_{k\in\mathbb{N}\setminus A_1}\alpha_k r_k \le \eta,$$

and so

$$\sum_{k\in C_1}\alpha_k < \frac{6\eta}{\eta'\varepsilon'}. \tag{2.6}$$

Next, take $B_1 := A_1 \cup L_1$. Note that

$$L_1 = \{k\in\mathbb{N} : r_k = 0\} \cup \Big\{k\in\mathbb{N} : r_k \neq 0, \operatorname{Re} m^*\Big(\frac{P(x_k)}{r_k}\Big) \le 1-\frac{\eta'}{2} \text{ and } r_k \le \frac{\varepsilon'}{3}\Big\}.$$

From equation (2.6) and the choice of η, we have

$$\sum_{k\in B_1}\alpha_k > 1-\frac{6\eta}{\eta'\varepsilon'} > 1-\varepsilon'. \tag{2.7}$$

It is clearly satisfied that

$$\Big\|\sum_{k\in A_1}\frac{\alpha_k}{\sum_{j\in A_1}\alpha_j}\frac{P(x_k)}{r_k}\Big\| \ge \operatorname{Re} m^*\Big(\sum_{k\in A_1}\frac{\alpha_k}{\sum_{j\in A_1}\alpha_j}\frac{P(x_k)}{r_k}\Big) \ge 1-\frac{\eta'}{2} > 1-\eta'.$$

Taking into consideration that M has the AHSp, there is a set $E_1 \subseteq A_1$ such that

$$\sum_{k\in E_1}\alpha_k > (1-\varepsilon')\sum_{k\in A_1}\alpha_k > 0 \tag{2.8}$$

and there exist $\{m_k : k\in E_1\} \subseteq S_M$ and $m_2^* \in S_{M^*}$ with $m_2^*(m_k) = 1$ and $\|m_k - \frac{P(x_k)}{r_k}\| < \varepsilon'$ for all $k\in E_1$. In particular, $E_1 \neq \varnothing$, and there is $m_0 \in S_M$ such that $m_2^*(m_0) = 1$. Let us write $D_1 = E_1 \cup L_1$. For every $k \in D_1$, since $E_1 \subset A_1$ and $A_1 \cap L_1 = \varnothing$, we can define

$$u_k := \begin{cases} r_k m_0 & \text{if } k\in L_1,\\ r_k m_k & \text{if } k\in E_1.\end{cases}$$

Note that

$$m_2^*(u_k) = r_k = \|u_k\| \quad \text{for all } k\in D_1. \tag{2.9}$$

Also, if $k\in L_1$, then $\|u_k - P(x_k)\| \le 2r_k \le 2(\varepsilon'/3) < \varepsilon'$, and if $k\in E_1$, then $\|u_k - P(x_k)\| < r_k\varepsilon' \le \varepsilon'$. That is,

$$\big\|u_k - P(x_k)\big\| < \varepsilon' \quad \text{for all } k\in D_1. \tag{2.10}$$

Notice that from (2.8) we have

$$\begin{aligned}\sum_{k\in D_1}\alpha_k &> (1-\varepsilon')\sum_{k\in A_1}\alpha_k + \sum_{k\in L_1}\alpha_k\\ &\ge (1-\varepsilon')\sum_{k\in B_1}\alpha_k\\ &> (1-\varepsilon')^2 \quad (\text{by } (2.7)).\end{aligned} \tag{2.11}$$

Next, repeating this argument, equation (2.2) implies that there is $n^* \in S_{N^*}$ such that

$$\operatorname{Re} n^*\Big(\sum_{k=1}^{\infty} \alpha_k Q(x_k)\Big) = \Big\|\sum_{k=1}^{\infty} \alpha_k Q(x_k)\Big\| \geq \sum_{k=1}^{\infty} \alpha_k s_k - \eta.$$

Then we can proceed as above, and by using the fact that N has the AHSp, we deduce that there is a subset $D_2 \subset \mathbb{N}$, $\{v_k : k \in D_2\} \subset N$, and an element $n_2^* \in S_{N^*}$ satisfying the following conditions:

$$\begin{aligned} &\sum_{k \in D_2} \alpha_k > (1-\varepsilon')^2, \qquad n_2^*(v_k) = \|v_k\| = s_k, \qquad \text{and} \\ &\big\|v_k - Q(x_k)\big\| < \varepsilon' \quad \text{for all } k \in D_2. \end{aligned} \tag{2.12}$$

Let $D := D_1 \cap D_2$. By using the choice of ε', we clearly obtain

$$\begin{aligned} \sum_{k \in D} \alpha_k &\geq \sum_{k \in D_1} \alpha_k - \sum_{k \in \mathbb{N} \setminus D_2} \alpha_k \\ &> (1-\varepsilon')^2 - \big(1-(1-\varepsilon')^2\big) \quad \text{(by (2.11) and (2.12))} \\ &= 1 - 4\varepsilon' + 2(\varepsilon')^2 \\ &> 1 - \varepsilon. \end{aligned}$$

Now, for $k \in D$, let $y_k := u_k + v_k \in S_X$. As a consequence of (2.10) and (2.12), we deduce that

$$\|y_k - x_k\| \leq \big\|u_k - P(x_k)\big\| + \big\|v_k - Q(x_k)\big\| \leq 2\varepsilon' < \varepsilon.$$

Finally, in view of (2.9) and (2.12), the element $x^* = m_2^* + n_2^* \in S_{X^*}$ verifies that

$$x^*(y_k) = m_2^*(u_k) + n_2^*(v_k) = r_k + s_k = 1 \quad \text{for all } k \in D.$$

This completes the proof that the ℓ_1 sum of a finite number of spaces having the AHSp also has the AHSp. $\square$

Before stating and proving the case $1 < p < \infty$, we need a couple of elementary lemmas.

Lemma 2.4. *Let a, b, p be nonnegative real numbers such that $p \geq 1$ and such that $a^p + b^p \leq 1$. Let $M_{a,b} := (1-b^p)^{\frac{1}{p}} - a$. Then,*

$$x^p \leq p\Big(\big((a+x)^p + b^p\big)^{\frac{1}{p}} - (a^p + b^p)^{\frac{1}{p}}\Big) \quad \textit{for all } x \in [0, M_{a,b}].$$

Proof. If $a = b = 0$, then the above inequality is clearly satisfied. Otherwise define the function

$$\begin{aligned} f : \quad [0, M_{a,b}] \quad &\to \quad \mathbb{R} \\ x \quad &\mapsto \quad \big((a+x)^p + b^p\big)^{\frac{1}{p}} - (a^p + b^p)^{\frac{1}{p}} - \tfrac{x^p}{p}. \end{aligned}$$

We deduce the result from the two facts that $f(0) = 0$ and

$$f'(x) = \frac{(a+x)^{p-1}}{\left((a+x)^p + b^p\right)^{\frac{p-1}{p}}} - x^{p-1} \geq (a+x)^{p-1} - x^{p-1} \geq 0$$

for all $x \in [0, M_{a,b}]$. □

Since ℓ_p^2 ($\mathbb{R}^2$ with the ℓ_p-norm) is uniformly convex for $1 < p < \infty$, Theorem 1.4 may be applied. Consequently, we have the following.

Lemma 2.5. *For $1 < p < \infty$, ℓ_p^2 satisfies the following condition. Given any $0 < \varepsilon < 1$, there is $0 < \eta < \varepsilon$ such that, for every sequence $(r_k, s_k)_{k\in\mathbb{N}} \subset S_{\ell_p^2}$ and for every convex series $\sum_{n\geq 1} \alpha_n$ with*

$$\Big\|\sum_{k=1}^{\infty} \alpha_k (r_k, s_k)\Big\|_p > 1 - \eta,$$

there is a subset $A \subset \mathbb{N}$ with $\sum_{k\in A} \alpha_k > 1 - \varepsilon$ and some element $(r, s) \in S_{\ell_p^2}$ satisfying $|r - r_k| < \varepsilon$ and $|s - s_k| < \varepsilon$ for every $k \in A$.

Now we take care of the case $1 < p < \infty$. Several other stability properties will be obtained from the following result.

Theorem 2.6. *Let X be a Banach space, and let $1 < p < \infty$. If $X = M \oplus_p N$ and M and N both have AHSp, then so does X.*

Proof. We can clearly assume that $M \neq \{0\} \neq N$. For arbitrary $\varepsilon \in (0, 1)$, fix any $0 < \varepsilon' < \varepsilon/5$ and choose η' so that (d) of Proposition 1.2 applies for both M and N. Next, let

$$0 < \varepsilon_0 < \min\Big\{\Big(\frac{\varepsilon}{5}\Big)^{p+1}, \frac{(\eta')^p}{2^{p+2}p}, \Big(\frac{(\eta')^p}{4p}\Big)^{p+1}\Big\} \tag{2.13}$$

and choose η_0 as in Lemma 2.5 for ℓ_p^2.

We will begin the process of checking that X has the AHSp by applying (d) of Proposition 1.2. In order to use Proposition 1.2(d) to prove that X has the AHSp, we will rely on the parameter η_0 that was chosen above (note that $0 < \eta_0 < \varepsilon_0$). So, assume that $\sum_{n\geq 1} \alpha_n$ is a convex series and that $(x_k)_{k\in\mathbb{N}} \subset S_X$ is a sequence such that $\| \sum_{k=1}^{\infty} \alpha_k x_k\| > 1 - \eta_0$.

We will denote by P and Q the canonical projections from X onto M and N, respectively. Then we have

$$\begin{aligned}
1 - \eta_0 &< \Big\|\sum_{k=1}^{\infty} \alpha_k x_k\Big\| \\
&= \Big(\Big\|\sum_{k=1}^{\infty} \alpha_k P(x_k)\Big\|^p + \Big\|\sum_{k=1}^{\infty} \alpha_k Q(x_k)\Big\|^p\Big)^{\frac{1}{p}} \\
&\leq \Big(\Big(\sum_{k=1}^{\infty} \alpha_k \big\|P(x_k)\big\|\Big)^p + \Big(\sum_{k=1}^{\infty} \alpha_k \big\|Q(x_k)\big\|\Big)^p\Big)^{\frac{1}{p}} \\
&= \Big\|\sum_{k=1}^{\infty} \alpha_k \big(\|P(x_k)\|, \|Q(x_k)\|\big)\Big\|_p,
\end{aligned}$$

where the last summation is viewed as an element of ℓ_p^2. From Lemma 2.5 applied to the sequence $((\|P(x_k)\|, \|Q(x_k)\|))_{k\in\mathbb{N}}$, we have an element $(r,s) \in \mathbb{R}^2$ with $r^p + s^p = 1$ and $r, s \geq 0$ as well as a subset $A \subset \mathbb{N}$ with

$$\sum_{k\in A} \alpha_k > 1 - \varepsilon_0 > 0 \tag{2.14}$$

so that, for all $k \in A$, we have

$$\big|\|P(x_k)\| - r\big| < \varepsilon_0 \qquad \text{and} \qquad \big|\|Q(x_k)\| - s\big| < \varepsilon_0. \tag{2.15}$$

Now fix arbitrary elements $m_0 \in S_M$ and $n_0 \in S_N$, and define the following sequences:

$$m_k := \begin{cases} P(x_k) & \text{if } k \notin A, \\ \frac{rP(x_k)}{\|P(x_k)\|} & \text{if } k \in A \text{ and } P(x_k) \neq 0, \\ rm_0 & \text{if } k \in A \text{ and } P(x_k) = 0 \end{cases}$$

and

$$n_k := \begin{cases} Q(x_k) & \text{if } k \notin A, \\ \frac{sQ(x_k)}{\|Q(x_k)\|} & \text{if } k \in A \text{ and } Q(x_k) \neq 0, \\ sn_0 & \text{if } k \in A \text{ and } Q(x_k) = 0. \end{cases}$$

Next, define $y_k := m_k + n_k$ for all $k \in \mathbb{N}$. It is clear that $(y_k)_{k\in\mathbb{N}} \subset S_X$, and in view of (2.15), we have

$$\|y_k - x_k\| \leq 2^{\frac{1}{p}}\varepsilon_0 < 2\varepsilon_0 \tag{2.16}$$

for all $k \in \mathbb{N}$. Note that

$$\Big\|\sum_{k=1}^{\infty} \alpha_k x_k\Big\| - \Big\|\sum_{k=1}^{\infty} \alpha_k y_k\Big\| \leq \sum_{k=1}^{\infty} \alpha_k \|x_k - y_k\| \leq 2\varepsilon_0.$$

By bearing in mind (2.14) and the above chain of inequalities, we have

$$\begin{aligned} \Big\|\sum_{k\in A} \alpha_k y_k\Big\| &> \Big\|\sum_{k=1}^{\infty} \alpha_k y_k\Big\| - \varepsilon_0 \\ &\geq \Big\|\sum_{k=1}^{\infty} \alpha_k x_k\Big\| - 2\varepsilon_0 - \varepsilon_0 \\ &> 1 - \eta_0 - 3\varepsilon_0 \\ &> 1 - 4\varepsilon_0. \end{aligned}$$

We set $\beta_k := \frac{\alpha_k}{\sum_{j\in A}\alpha_j}$ for every $k \in A$, so that $\sum_{k\in A}\beta_k$ is a convex series. The series $\sum_{k\in A}\beta_k y_k$ satisfies that

$$\begin{aligned} 1 - 4\varepsilon_0 < \Big\|\sum_{k\in A} \alpha_k y_k\Big\| &= \Big(\sum_{k\in A}\alpha_k\Big)\Big\|\sum_{k\in A}\beta_k y_k\Big\| \leq \Big\|\sum_{k\in A}\beta_k y_k\Big\| \\ &= \Big(\Big\|\sum_{k\in A}\beta_k m_k\Big\|^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}&\le \Big(\Big(\sum_{k\in A}\beta_k\|m_k\|\Big)^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}}\\&\le \Big(r^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}}\\&\le (r^p+s^p)^{\frac{1}{p}} = 1.\end{aligned}$$

From the above chain of inequalities, we know that

$$\Big(r^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}} < \Big(\Big\|\sum_{k\in A}\beta_k m_k\Big\|^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}} + 4\varepsilon_0, \tag{2.17}$$

$$1 = (r^p+s^p)^{\frac{1}{p}} < \Big(r^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}} + 4\varepsilon_0,$$

and

$$r^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p \le 1. \tag{2.18}$$

Now we apply Lemma 2.4 to $a := \|\sum_{k\in A}\beta_k m_k\|$ and $b := \|\sum_{k\in A}\beta_k n_k\|$. Note that $t := r - a \in [0, M_{a,b}]$, and by virtue of (2.18) we have $(a+t)^p + b^p \le 1$. By combining Lemma 2.4 and (2.17), we deduce that

$$\begin{aligned}\Big(r - \Big\|\sum_{k\in A}\beta_k m_k\Big\|\Big)^p &= t^p \le p\Big(\big((a+t)^p+b^p\big)^{\frac{1}{p}} - (a^p+b^p)^{\frac{1}{p}}\Big)\\&= p\Big(\Big(r^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}}\\&\quad - \Big(\Big\|\sum_{k\in A}\beta_k m_k\Big\|^p + \Big\|\sum_{k\in A}\beta_k n_k\Big\|^p\Big)^{\frac{1}{p}}\Big)\\&< 4p\varepsilon_0.\end{aligned}$$

Hence

$$\Big\|\sum_{k\in A}\beta_k m_k\Big\| > r - (4p\varepsilon_0)^{\frac{1}{p}}. \tag{2.19}$$

By proceeding in a similar way, we also deduce that

$$\Big\|\sum_{k\in A}\beta_k n_k\Big\| > s - (4p\varepsilon_0)^{\frac{1}{p}}. \tag{2.20}$$

Next, we will consider the following three possibilities.

Case (a): $r < \varepsilon_0^{\frac{1}{p+1}}$.

In this case $s \ge s^p = 1 - r^p > 1 - \varepsilon_0^{\frac{p}{p+1}} > 1/2$ since $r^p + s^p = 1$. From (2.20) and (2.13), we have

$$\Big\|\sum_{k\in A}\beta_k \frac{n_k}{s}\Big\| > 1 - \frac{(4p\varepsilon_0)^{\frac{1}{p}}}{s} > 1 - 2(4p\varepsilon_0)^{\frac{1}{p}} > 1 - \eta'.$$

By using our hypothesis that N has the AHSp, we know that there exists $C \subset A$, $n^* \in S_{N^*}$ (which can be seen as an element of S_{X^*}), and a set $\{v_k : k \in C\} \subset S_N$ such that

$$\begin{aligned} &\sum_{k\in C} \beta_k > 1-\varepsilon', \qquad \left\| v_k - \frac{n_k}{s} \right\| < \varepsilon' \qquad \text{and} \\ &n^*(v_k) = 1 \quad \text{for all } k \in C. \end{aligned} \tag{2.21}$$

Using (2.21) and (2.14) and by our initial choice of constant ε_0, we obtain

$$\sum_{k\in C} \alpha_k = \sum_{k\in C} \beta_k \sum_{k\in A} \alpha_k > (1-\varepsilon')(1-\varepsilon_0) > 1-(\varepsilon'+\varepsilon_0) > 1-\varepsilon.$$

For every $k \in C$, we have

$$\begin{aligned} \|x_k - v_k\| &\le \|x_k - y_k\| + \|y_k - v_k\| \\ &\le 2\varepsilon_0 + \|m_k\| + \|n_k - v_k\| \quad (\text{by } (2.16)) \\ &\le 2\varepsilon_0 + r + \left\| n_k - \frac{n_k}{s} \right\| + \left\| \frac{n_k}{s} - v_k \right\| \\ &< 2\varepsilon_0 + \varepsilon_0^{\frac{1}{p+1}} + \varepsilon' + \|n_k\| \left| 1 - \frac{1}{s} \right| \\ &= 2\varepsilon_0 + \varepsilon_0^{\frac{1}{p+1}} + \varepsilon' + 1 - s \\ &< 2\varepsilon_0 + \varepsilon_0^{\frac{1}{p+1}} + \varepsilon' + \varepsilon_0^{\frac{1}{p+1}} \\ &< \varepsilon. \end{aligned}$$

Case (b): $s < \varepsilon_0^{\frac{1}{p+1}}$.

We proceed here in the same way as Case (a) above, by using the assumption that M has the AHSp.

Case (c): $\varepsilon_0^{\frac{1}{p+1}} \le r, s$.

From (2.19) and (2.13), we deduce that

$$\left\| \sum_{k\in A} \beta_k \frac{m_k}{r} \right\| > 1 - \frac{(4p\varepsilon_0)^{\frac{1}{p}}}{r} \ge 1 - \frac{(4p\varepsilon_0)^{\frac{1}{p}}}{\varepsilon_0^{\frac{1}{p+1}}} > 1-\eta'.$$

Since M has the AHSp, there are $B \subset A$, $\{u_k : k \in B\} \subset S_M$ and $m^* \in S_{M^*}$ satisfying

$$\begin{aligned} &\sum_{k\in B} \beta_k > 1-\varepsilon', \qquad m^*(u_k) = 1, \qquad \text{and} \\ &\left\| u_k - \frac{m_k}{r} \right\| < \varepsilon' \quad \text{for all } k \in B. \end{aligned} \tag{2.22}$$

In view of (2.20) and reasoning as before, we deduce that $\| \sum_{k\in A} \beta_k \frac{n_k}{s} \| > 1-\eta'$. Hence, since N has the AHSp, there are $C \subset A$, $\{v_k : k \in C\} \subset S_N$ and $n^* \in S_{N^*}$

satisfying

$$\begin{aligned} &\sum_{k\in C}\beta_k > 1-\varepsilon', \qquad n^*(v_k)=1 \qquad \text{and} \\ &\left\| v_k - \frac{n_k}{s}\right\| < \varepsilon' \quad \text{for all } k\in C. \end{aligned} \tag{2.23}$$

Taking $D := B\cap C$ and bearing (2.22) and (2.23) in mind, we see that

$$\sum_{k\in D}\beta_k \geq \sum_{k\in A}\beta_k - \sum_{k\in A\setminus B}\beta_k - \sum_{k\in A\setminus C}\beta_k = 1 - \sum_{k\in A\setminus B}\beta_k - \sum_{k\in A\setminus C}\beta_k > 1-2\varepsilon'.$$

Hence by (2.14)

$$\sum_{k\in D}\alpha_k = \sum_{k\in D}\beta_k\sum_{k\in A}\alpha_k > (1-2\varepsilon')(1-\varepsilon_0) > 1-(2\varepsilon'+\varepsilon_0) > 1-\varepsilon.$$

For every $k\in D$, the element $ru_k + sv_k \in S_X$ verifies that

$$\|ru_k + sv_k - y_k\| \leq \|ru_k - m_k\| + \|sv_k - n_k\| < r\varepsilon' + s\varepsilon' \leq 2\varepsilon',$$

in accordance with (2.22) and (2.23). Therefore, by taking into consideration (2.16), for every $k\in D$ we have

$$\|(ru_k + sv_k) - x_k\| \leq \|ru_k + sv_k - y_k\| + \|y_k - x_k\| \leq 2\varepsilon' + 2\varepsilon_0 < \varepsilon.$$

Finally, if $(\alpha,\beta)\in\mathbb{R}^2$ is the unique element satisfying $\alpha^q+\beta^q=1$ with $\alpha r+\beta s = 1$, then the element $\alpha m^* + \beta n^* \in S_{X^*}$ satisfies

$$(\alpha m^* + \beta n^*)(ru_k + sv_k) = \alpha r + \beta s = 1,$$

for every $k\in D$. □

The statement of Theorem 2.6 remains true when $p=\infty$ due to [3, Proposition 2.4] and [1, Theorem 4.1]. At the very end of this article, we will argue that the AHSp is not stable under infinite ℓ_p-sums for $1<p<\infty$. (This fact is already known for infinite c_0-sums, ℓ_1-sums, and ℓ_∞-sums in view of [3, Corollary 4.6]).

We now show how Theorem 2.6 can be used to obtain equivalent renormings involving the AHSp.

Corollary 2.7. *Let X be a Banach space.*

(1) *If $\dim(X)>1$, then X can be equivalently renormed to have the AHSp but not the property β of Lindenstrauss.*
(2) *If X is infinite-dimensional and admits an equivalent strictly convex norm, then X admits an equivalent strictly convex renorming that fails the AHSp.*

Proof. (1) Let $x\in S_X$, and consider any closed subspace M of X such that $X = \mathbb{K}x\oplus M$ ($\mathbb{K}$ is the base scalar field). Since every Banach space can be equivalently renormed to have the AHSp (see Theorem 1.9), we can assume without loss of generality that M has the ASHp. This means, by Theorem 2.6, that $\mathbb{K}x\oplus_2 M$ has the AHSp. Now, to see that $\mathbb{K}x\oplus_2 M$ does not verify property β, we observe two things. By virtue of [12, Proposition 3.3], the unit sphere of a Banach space having property β has no locally uniformly rotund points. The element x is a locally uniformly rotund point of the unit ball of $\mathbb{K}x \oplus_2 M$ (see, for instance, [2, Proposition 2.1]).

(2) We will distinguish two cases. If X is super-reflexive, then there exists an infinite-dimensional closed separable subspace Y of X that is complemented (see [11, Proposition 1]). Next, Y is reflexive, so Y^* is separable too. By [9, Theorem 1], there exists an equivalent renorming on Y^* that is uniformly Gateaux-smooth but lacks asymptotic normal structure. By [8], we have that this equivalent norm on Y^* is not uniformly Fréchet-smooth. Since Y is reflexive, that equivalent norm on Y^* is a dual norm, whose predual norm on Y is strictly convex but not uniformly convex. By [1, Proposition 3.9], we have that this equivalent norm on Y fails to have the AHSp. The complement of Y in X is also reflexive, so it can be equivalently renormed to be strictly convex (see [5, Proposition VII.2.1]). Finally, take the ℓ_2-sum of Y and its complement with their corresponding new norms and apply Proposition 2.1.

In the case that X is not super-reflexive, there is no need to renorm because of [1, Proposition 3.9] and [6]. □

Notice that the class of Banach spaces admitting an equivalent strictly convex norm is very large. However, there are examples of Banach spaces that do not belong to this class (see, e.g., [5, Corollary II.7.13]).

Our final purpose is to deduce some stability properties of the BPBp in the case where the domain space is ℓ_1. In order to accomplish this, we introduce the following notion. Given a real or complex Banach space X, we say that Y has *property* $\mathcal{P}_X$ if the pair (X, Y) has the BPBp for operators. In view of the stability result for $1 \leq p \leq \infty$ (see Theorem 2.3, Theorem 2.6, [3, Proposition 2.4], and [1, Theorem 4.1]), we obtain the following result.

Corollary 2.8. *The property $\mathcal{P}_{\ell_1}$ is stable under finite ℓ_p-sums for $1 \leq p \leq \infty$.*

Question 2.9. Given an arbitrary Banach space X, is the property $\mathcal{P}_X$ stable under finite ℓ_p-sums for $1 \leq p < \infty$?

For $p = \infty$, the property $\mathcal{P}_X$ is stable under finite ℓ_∞-sums (see [3, Proposition 2.4]). It is also known that in general $\mathcal{P}_X$ is neither stable under infinite ℓ_p-sums for $1 \leq p < \infty$ in view of the Bishop–Phelps–Bollobás theorem (see [4]) and the counterexample given in [7, Appendix] nor under c_0-sums and ℓ_∞-sums (see [3, Corollary 4.4]).

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References

1. M. D. Acosta, R. M. Aron, D. García, and M. Maestre, *The Bishop-Phelps-Bollobás Theorem for operators*, J. Funct. Anal. **254** (2008), no. 11, 2780–2799. Zbl 1152.46006. MR2414220. DOI 10.1016/j.jfa.2008.02.014. 296, 297, 298, 299, 308, 309

2. A. Aizpuru and F. J. García-Pacheco, *L^2-summand vectors in Banach spaces*, Proc. Amer. Math. Soc. **134** (2006), no. 7, 2109–2115. Zbl 1094.46007. MR2215781. DOI 10.1090/S0002-9939-06-08243-8. 309
3. R. M. Aron, Y. S. Choi, S. K. Kim, H. J. Lee, and M. Martín, *The Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6085–6101. Zbl 1331.46008. MR3356930. DOI 10.1090/S0002-9947-2015-06551-9. 298, 308, 309
4. B. Bollobás, *An extension to the theorem of Bishop and Phelps*, Bull. London. Math. Soc. **2** (1970), 181–182. Zbl 0217.45104. MR0267380. DOI 10.1112/blms/2.2.181. 309
5. R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64**, Longman Scientific, New York, 1993. Zbl 0782.46019. MR1211634. 309
6. P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel J. Math. **13** (1972), 281–288. Zbl 0259.46012. MR0336297. 309
7. W. T. Gowers, *Symmetric block bases of sequences with large average growth*, Israel J. Math. **69** (1990), no. 2, 129–151. Zbl 0721.46010. MR1045369. DOI 10.1007/BF02937300. 309
8. M. A. Khamsi, *Uniform smoothness implies super-normal structure property*, Nonlinear Anal. **19** (1992), no. 11, 1063–1069. Zbl 0792.46008. MR1194145. DOI 10.1016/0362-546X(92)90124-W. 309
9. M. Johanis and J. Rychtář, *On uniformly Gateaux smooth norms and normal structure*, Proc. Amer. Math. Soc. **135** (2007), no. 5, 1511–1514. Zbl 1119.46019. MR2276661. DOI 10.1090/S0002-9939-06-08819-8. 309
10. J. Lindenstrauss, *On operators which attain their norms*, Israel J. Math. **1** (1963), 139–148. Zbl 0127.06704. MR0160094. DOI 10.1007/BF02759700. 297
11. J. Lindenstrauss, *On nonseparable reflexive Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 967–970. Zbl 0156.36403. MR0205040. DOI 10.1090/S0002-9904-1966-11606-3. 309
12. J. P. Moreno, *Geometry of Banach spaces with (α, ϵ)-Property or (β, ϵ)-Property*, Rocky Mountain J. Math. **27** (1997), no. 1, 241–256. Zbl 0896.46009. MR1453101. DOI 10.1216/rmjm/1181071959. 308
13. J. R. Partington, *Norm attaining operators*, Israel J. Math. **43** (1982), no. 3, 273–276. Zbl 0507.46008. MR0689984. DOI 10.1007/BF02761947. 297

[1]Department of Mathematical Analysis, University of Granada, 18071 Granada, Spain.
E-mail address: dacosta@ugr.es

[2]Department of Mathematics, Kent State University, Kent OH 44242, USA.
E-mail address: aron@math.kent.edu

[3]Department of Mathematics, University of Cádiz, 11519 Puerto Real, Spain.
E-mail address: garcia.pacheco@uca.es