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TRIANGULAR SUMMABILITY AND LEBESGUE POINTS OF 2-DIMENSIONAL FOURIER TRANSFORMS

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Abstract. We consider the triangular θ-summability of 2-dimensional Fourier transforms. Under some conditions on θ, we show that the triangular θ-means of a function f belonging to the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^2)$ converge to f at each modified strong Lebesgue point. The same holds for a weaker version of Lebesgue points for the so-called *modified Lebesgue points* of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ whenever $1 < p < \infty$. Some special cases of the θ-summation are considered, such as the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski, and Riesz summations.

1. Introduction

For the Fejér means of an integrable function f, the classical theorem of Lebesgue [11, p. 274] says that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x) = f(x)$$

at each Lebesgue point of f (thus almost everywhere), where $s_k f$ denotes the kth partial sum of the Fourier series of the 1-dimensional function f.

A general method of summation, the so-called *θ-summation method*, which is generated by a single function θ and which includes the well-known Fejér, Riesz, Weierstrass, Abel (and so forth) summability methods, is studied intensively in

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the literature (see, e.g., Butzer and Nessel [3]; Trigub and Belinsky [15]; Gát [5], [6]; Goginava [7]–[9]; Simon [13]; Persson, Tephnadze, and Wall [12]; and Weisz [16]). The triangular means of 2-dimensional Fourier transforms generated by the θ-summation are defined by

$$\sigma_T^\theta f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\Big(\frac{|u|+|v|}{T}\Big)\widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv.$$

Berens, Li, and Xu in [1] and [2] have proved that $\sigma_T^\theta f \to f$ almost everywhere for the Riesz summability (i.e., if $\theta(v) := \max((1-|v|)^\beta, 0)$, $0<\beta<\infty$), where $f \in L_1(\mathbb{R}^2)$. Szili and Vértesi [14] considered the triangular Fejér summability (when $\theta(t) = \max(1-|t|, 0)$). Recently, we [17] generalized this convergence result and gave a common proof for several different θ's, such as for the Weierstrass, Abel, Picar, Bessel, Fejér, de La Vallée-Poussin, Rogosinski, and Riesz summations. However, the set of convergence is not yet known.

In this article, we generalize the preceding convergence result for Wiener amalgam spaces that are much larger spaces than the $L_1(\mathbb{R}^2)$-space. Moreover, we characterize the set of convergence. We introduce the concept of modified Lebesgue points and modified strong Lebesgue points. It is verified in [18] that almost every point is a modified Lebesgue point and a modified strong Lebesgue point of $f \in L_1(\mathbb{R}^2)$ or $f \in W(L_1,\ell_\infty)(\mathbb{R}^2)$. Here $W(L_p,\ell_q)(\mathbb{R}^2)$ denotes the Wiener amalgam space. Under some conditions on θ, we show that the triangular θ-means of a function $f \in W(L_1,\ell_\infty)(\mathbb{R}^2)$ converge to f at each modified strong Lebesgue point. The same result holds for the modified Lebesgue points of $f \in W(L_p,\ell_\infty)(\mathbb{R}^2)$ whenever $1<p<\infty$.

2. Wiener amalgam spaces

We briefly write $L_p(\mathbb{R}^2)$ instead of the $L_p(\mathbb{R}^2,\lambda)$-space equipped with the norm

$$\|f\|_p := \Big(\int_{\mathbb{R}^2} \big|f(x)\big|^p\,d\lambda(x)\Big)^{1/p} \quad (1 \le p < \infty)$$

with the usual modification for $p=\infty$, where λ is the Lebesgue measure. These spaces are generalized as follows. A measurable function f belongs to the *Wiener amalgam space* $W(L_p,\ell_q)(\mathbb{R}^2)$ $(1 \le p,q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \Big(\sum_{k\in\mathbb{Z}^2} \big\|f(\cdot+k)\big\|^q_{L_p[0,1)^2}\Big)^{1/q} < \infty$$

with the obvious modification for $q=\infty$.

It is easy to see that $W(L_p,\ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and the following continuous embeddings hold true:

$$W(L_{p_1},\ell_q)(\mathbb{R}^2) \supset W(L_{p_2},\ell_q)(\mathbb{R}^2) \quad (p_1 \le p_2)$$

and

$$W(L_p,\ell_{q_1})(\mathbb{R}^2) \subset W(L_p,\ell_{q_2})(\mathbb{R}^2) \quad (q_1 \le q_2),$$

$(1 \le p_1,p_2,q_1,q_2 \le \infty)$. Thus

$$W(L_\infty,\ell_1)(\mathbb{R}^2) \subset L_p(\mathbb{R}^2) \subset W(L_1,\ell_\infty)(\mathbb{R}^2) \quad (1 \le p \le \infty).$$

In this paper, the constants C and C_p may vary from line to line, the constants C_p depending only on p.

3. The kernel functions

The *Fourier transform* of $f \in L_1(\mathbb{R}^2)$ is given by

$$\widehat{f}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(u,v)e^{-\imath(xu+yv)}\,du\,dv \quad (x,y\in\mathbb{R}),$$

where $\imath = \sqrt{-1}$. Suppose first that $f \in L_p(\mathbb{R}^2)$ for some $1 \le p \le 2$. The Fourier inversion formula

$$f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv \quad \big(x,y\in\mathbb{R}, \widehat{f}\in L_1(\mathbb{R}^2)\big)$$

motivates the definition of the *triangular Dirichlet integral* $s_t f$ $(t>0)$:

$$\begin{aligned} s_t f(x,y) &:= \frac{1}{2\pi}\int_{\mathbb{R}^2} 1_{\{|u|+|v|\le t\}}\widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} f(x-u,y-v)D_t(u,v)\,du\,dv, \end{aligned}$$

where 1_H denotes the characteristic function of the set H and the *Dirichlet kernel* is defined by

$$D_t(x,y) := \int_{\mathbb{R}^2} 1_{\{|u|+|v|\le t\}}e^{\imath(xu+yv)}\,du\,dv = 4\frac{\cos(yt)-\cos(xt)}{(x-y)(x+y)}$$

(see Weisz [17]). Obviously, $|D_t| \le Ct^2$.

It is known (see, e.g., Grafakos [10] or [16]) that, for $f \in L_p(\mathbb{R}^2)$, $1<p<\infty$,

$$\lim_{T\to\infty} s_T f = f \quad \text{in the } L_p(\mathbb{R}^2)\text{-norm and almost everywhere}$$

This convergence does not hold for $p=1$. However, using a summability method, we can generalize these results. We consider a general summability method, the so-called *triangular θ-summation* defined by a function $\theta:\mathbb{R}_+\to\mathbb{R}$. This summation contains all well-known summability methods, such as the triangular Fejér, Riesz, Weierstrass, Abel, Picard, and Bessel summations.

Suppose that θ is continuous on $\mathbb{R}_+$, the support of θ is $[0,c]$ for some $0<c\le\infty$, and θ is differentiable on $(0,c)$. Suppose further that

$$\theta(0)=1, \qquad \int_0^\infty (t\vee 1)^2\big|\theta'(t)\big|\,dt<\infty, \qquad \lim_{t\to\infty} t^2\theta(t)=0, \tag{3.1}$$

where $\vee$ denotes the maximum and $\wedge$ denotes the minimum.

For $T>0$, the *triangular θ-means* of a function $f\in L_p(\mathbb{R}^2)$ $(1\le p\le 2)$ are defined by

$$\sigma_T^\theta f(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2}\theta\Big(\frac{|u|+|v|}{T}\Big)\widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv.$$

It is easy to see that

$$\sigma_T^\theta f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x-u,y-v) K_T^\theta(u,v)\,du\,dv, \tag{3.2}$$

where the *triangular θ-kernel* is given by

$$\begin{aligned} K_T^\theta(x,y) &:= \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\Big(\frac{|u|+|v|}{T}\Big) e^{\imath(xu+yv)}\,du\,dv \\ &= \frac{-1}{2\pi T}\int_{\mathbb{R}^2}\int_{|u|+|v|}^\infty \theta'\Big(\frac{t}{T}\Big)\,dt e^{\imath(xu+yv)}\,du\,dv \\ &= \frac{-1}{2\pi T}\int_0^\infty \theta'\Big(\frac{t}{T}\Big) D_t(x,y)\,dt. \end{aligned} \tag{3.3}$$

Hence

$$\sigma_T^\theta f(x,y) = \frac{-1}{T}\int_0^\infty \theta'\Big(\frac{t}{T}\Big) s_t f(x,y)\,dt.$$

Note that for the triangular Fejér means (i.e., for $\theta(t)=\max((1-|t|),0)$), we get the usual definition

$$\sigma_T^\theta f(x,y) = \frac{1}{T}\int_0^T s_t f(x,y)\,dt.$$

We may suppose that $x>y>0$. The next two lemmas were proved in Weisz [17].

Lemma 3.1. *If*

$$\Big|\int_0^\infty \theta'(t)\cos(tu)\,dt\Big| \le Cu^{-\alpha}, \qquad \Big|\int_0^\infty \theta'(t)t\sin(tu)\,dt\Big| \le Cu^{-\alpha} \tag{3.4}$$

for some $0<\alpha<\infty$*, then*

$$\big|K_T^\theta(x,y)\big| \le CT^2, \tag{3.5}$$

$$\big|K_T^\theta(x,y)\big| \le C(x-y)^{-3/2}(x+y)^{-1/2}, \tag{3.6}$$

$$\big|K_T^\theta(x,y)\big| \le CT^{-\alpha}(x-y)^{-1}(x+y)^{-1}y^{-\alpha}, \tag{3.7}$$

$$\big|K_T^\theta(x,y)\big| \le CT^{1-\alpha}(x+y)^{-1}y^{-\alpha}. \tag{3.8}$$

Lemma 3.2. *If (3.4) is satisfied for some* $0<\alpha<\infty$*, then* $\int_{\mathbb{R}^2}|K_T^\theta|\,d\lambda \le C$ $(T\in\mathbb{R}_+)$.

Now we can extend the definition of the triangular θ-means $\sigma_T^\theta f$ with the formula (3.2) to all $f\in W(L_1,\ell_\infty)(\mathbb{R}^2)$.

4. Modified Lebesgue points

The term $L_p^{\mathrm{loc}}(\mathbb{R}^2)$ $(1\le p<\infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable. For $f\in L_p^{\mathrm{loc}}(\mathbb{R}^2)$, the *Hardy–Littlewood maximal function* is defined by

$$M_p f(x,y) := \sup_{h>0}\Big(\frac{1}{4h^2}\int_{-h}^h\int_{-h}^h \big|f(x-s,y-t)\big|^p\,ds\,dt\Big)^{1/p}.$$

We are going to generalize the Hardy–Littlewood maximal function. For some $\tau > 0$ and $f \in L_p^{\mathrm{loc}}(\mathbb{R}^2)$, let

$$\mathcal{M}_p^{(1)} f(x,y) := \sup_{i,j\in\mathbb{N},h>0} 2^{-\tau(i+j)} \Big(\frac{1}{4\cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{-2^j h}^{2^j h} \big| f(x-s,y-t) \big|^p \, ds \, dt \Big)^{1/p},$$

$$\mathcal{M}_p^{(2)} f(x,y) := \sup_{i,j\in\mathbb{N},h>0} 2^{-\tau(i+j)} \Big(\frac{1}{4\cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{s-2^j h}^{s+2^j h} \big| f(x-s,y-t) \big|^p \, dt \, ds \Big)^{1/p}$$

and

$$\mathcal{M}_p f(x,y) := \mathcal{M}_p^{(1)} f(x,y) + \mathcal{M}_p^{(2)} f(x,y).$$

For $p = 1$, we write simply Mf, $\mathcal{M}^{(1)} f$, $\mathcal{M}^{(2)} f$, and $\mathcal{M} f$, respectively. All the results of this section were proved by the author in [18].

Theorem 4.1. *For $1 \le p < \infty$, we have*

$$\sup_{\rho>0} \rho \lambda(\mathcal{M}_p f > \rho)^{1/p} \le C \|f\|_p \quad \big(f \in L_p(\mathbb{R}^2)\big),$$

$$\|\mathcal{M}_p f\|_r \le C_r \|f\|_r \quad \big(f \in L_r(\mathbb{R}^2), p < r \le \infty\big)$$

and

$$\sup_{k\in\mathbb{Z}^d} \sup_{\rho>0} \rho \lambda(1_{[k,k+1)} \mathcal{M}_p f > \rho)^{1/p} \le C \|f\|_{W(L_p,\ell_\infty)} \quad \big(f \in W(L_p,\ell_\infty)(\mathbb{R}^2)\big),$$

$$\|\mathcal{M}_p f\|_{W(L_r,\ell_\infty)} \le C_r \|f\|_{W(L_r,\ell_\infty)} \quad \big(f \in W(L_r,\ell_\infty)(\mathbb{R}^2), p < r \le \infty\big).$$

A point $(x,y) \in \mathbb{R}^2$ is called a *p-Lebesgue point* (or a Lebesgue point of order p) of $f \in L_p^{\mathrm{loc}}(\mathbb{R}^2)$ if

$$\lim_{h\to 0} \Big(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \big| f(x-s,y-t) - f(x,y) \big|^p \, ds \, dt \Big)^{1/p} = 0.$$

It is known (see, e.g., Feichtinger and Weisz [4]) that almost every point $(x,y) \in \mathbb{R}^2$ is a p-Lebesgue point of $f \in L_p^{\mathrm{loc}}(\mathbb{R}^2)$ $(1 \le p < \infty)$.

Starting from the maximal function $\mathcal{M}_p^{(1)} f$ and $\mathcal{M}_p^{(2)} f$, we introduce

$$U_{r,p}^{(1)} f(x,y) := \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r, 2^j h<r}} 2^{-\tau(i+j)} \Big(\frac{1}{4\cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{-2^j h}^{2^j h} \big| f(x-s,y-t) - f(x,y) \big|^p \, ds \, dt \Big)^{1/p},$$

$$U_{r,p}^{(2)} f(x,y) := \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r, 2^j h<r}} 2^{-\tau(i+j)} \Big(\frac{1}{4\cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{s-2^j h}^{s+2^j h} \big| f(x-s,y-t) - f(x,y) \big|^p \, dt \, ds \Big)^{1/p},$$

and in case $p=1$ we write simply $U_r^{(1)}f$ and $U_r^{(2)}f$. We say that a point $(x,y)\in\mathbb{R}^2$ is a *modified p-Lebesgue point* (or a modified Lebesgue point of order p) of $f\in L_p^{\mathrm{loc}}(\mathbb{R}^2)$ $(1\le p<\infty)$ if, for all $\tau>0$,

$$\lim_{r\to 0} U_{r,p}^{(1)} f(x,y)=0.$$

If, in addition,

$$\lim_{r\to 0} U_{r,p}^{(2)} f(x,y)=0,$$

then we say that $(x,y)\in\mathbb{R}^2$ is a *modified strong p-Lebesgue point* (or a modified strong Lebesgue point of order p). If $p=1$, then we call the points *modified Lebesgue points* or *modified strong Lebesgue points*. Obviously, every modified (strong) p-Lebesgue point is a modified (strong) Lebesgue point.

Theorem 4.2. *Almost every point $(x,y)\in\mathbb{R}^2$ is a modified p-Lebesgue point and a modified strong p-Lebesgue point of $f\in L_p^{\mathrm{loc}}(\mathbb{R}^2)$ $(1\le p<\infty)$.*

5. Pointwise convergence of the summability means

Now we prove that the triangular summability means $\sigma_T^\theta f$ converge to f at each modified strong Lebesgue point.

Theorem 5.1. *Suppose that (3.4) is satisfied for some $0<\alpha<\infty$ and that $f\in W(L_1,\ell_\infty)(\mathbb{R}^2)$. If (x,y) is a modified strong Lebesgue point of f and $\mathcal{M}f(x,y)$ is finite, then*

$$\lim_{T\to\infty}\sigma_T^\theta f(x,y)=f(x,y).$$

Proof. Let $\theta_0(s,t):=\theta(|s|+|t|)$. The first equation of (3.3) implies that

$$K_T^\theta(s,t):=T^2\widehat{\theta_0}(Ts,Tt).$$

The function θ_0 is integrable because

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}^2}\theta\big(|s|+|t|\big)\,ds\,dt&=\int_{(0,\infty)^2}\theta(s+t)\,ds\,dt=\int_0^\infty\int_0^x\theta(x)\,dx\,dy\\&=\int_0^\infty x\theta(x)\,dx=\frac{c^2\theta(c)}{2}-\frac{1}{2}\int_0^c x^2\theta'(x)\,dx.\end{aligned}$$

Since $\widehat{\theta_0}\in L_1(\mathbb{R}^2)$ by Lemma 3.2, the Fourier inversion formula yields

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}K_T^\theta(s,t)\,ds\,dt=\frac{1}{2\pi}\int_{\mathbb{R}^2}\widehat{\theta_0}(s,t)\,ds\,dt=\theta(0)=1.$$

Thus

$$\big|\sigma_T^\theta f(x,y)-f(x,y)\big|\le\frac{1}{2\pi}\int_{\mathbb{R}^2}\big|f(x-s,y-t)-f(x,y)\big|\big|K_T^\theta(s,t)\big|\,ds\,dt. \tag{5.1}$$

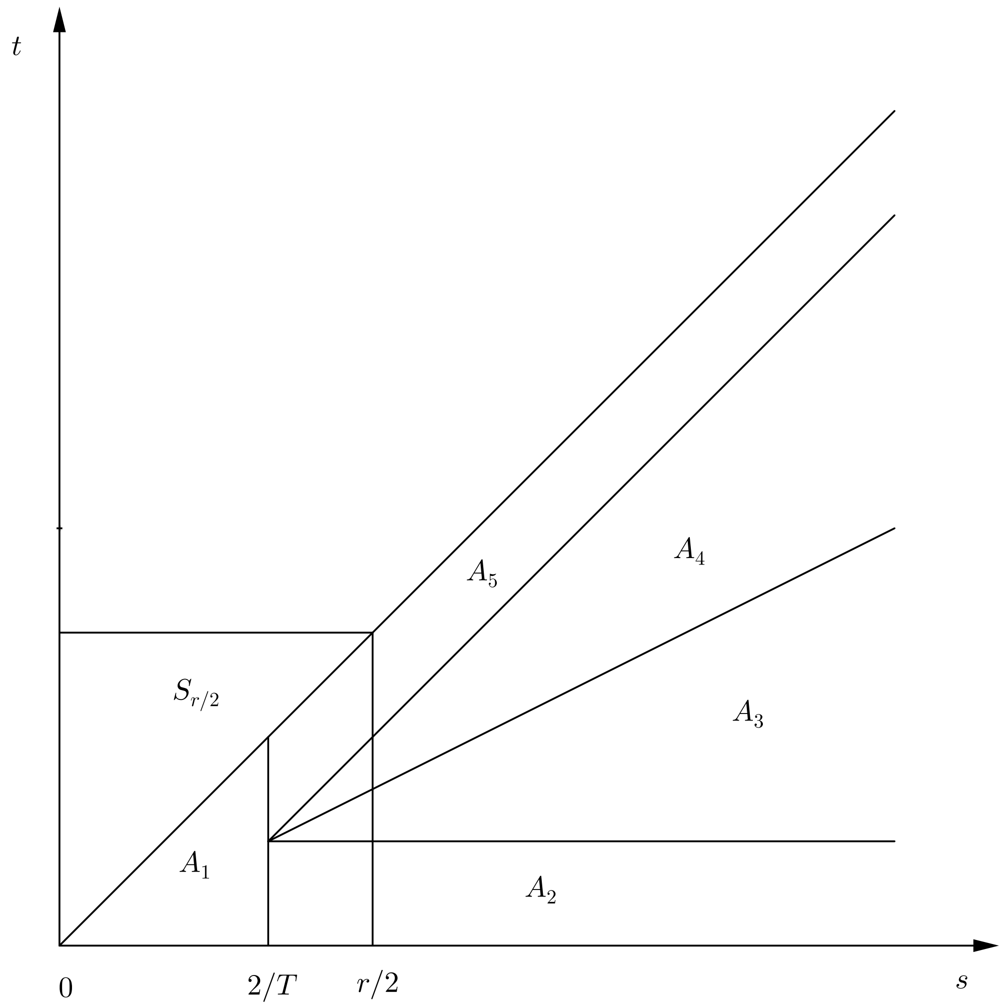

FIGURE 1. The sets A_i.

It is enough to integrate over the sets of Figure 1:

$$\begin{aligned}
A_1 &:= \big\{(s,t) : 0 < s \le 2/T, 0 < t < s\big\},\\
A_2 &:= \big\{(s,t) : s > 2/T, 0 < t \le 1/T\big\},\\
A_3 &:= \big\{(s,t) : s > 2/T, 1/T < t \le s/2\big\},\\
A_4 &:= \big\{(s,t) : s > 2/T, s/2 < t \le s - 1/T\big\},\\
A_5 &:= \big\{(s,t) : s > 2/T, s - 1/T < t \le s\big\}.
\end{aligned}$$

Let $\tau < \alpha/2 \wedge 1$. Since (x,y) is a modified strong Lebesgue point of f, we can fix a number $r < 1$ such that

$$U_r f(x,y) < \epsilon.$$

Let us denote the square $[0,r/2]\times[0,r/2]$ by $S_{r/2}$, and let $2/T<r/2$. We will integrate the right-hand side of (5.1) over the sets

$$\bigcup_{i=1}^{5}(A_i\cap S_{r/2}) \qquad \text{and} \qquad \bigcup_{i=1}^{5}(A_i\cap S_{r/2}^c),$$

where S^c denotes the complement of the set S. Since $A_1\subset S_{r/2}$, we obtain by (3.5) that

$$\begin{aligned}
&\int_{A_1}\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right|\,ds\,dt\\
&\quad\le CT^2\int_0^{2/T}\int_0^{2/T}\left|f(x-s,y-t)-f(x,y)\right|\,ds\,dt\le CU_r^{(1)}f(x,y)<C\epsilon.
\end{aligned}$$

Let us denote by r_0 the largest number i for which $r/2\le 2^{i+1}/T<r$. By (3.6),

$$\begin{aligned}
&\int_{A_2\cap S_{r/2}}\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right|\,ds\,dt\\
&\quad\le C\sum_{i=1}^{r_0}\Big(\frac{2^i}{T}-\frac{1}{T}\Big)^{-3/2}\Big(\frac{2^i}{T}\Big)^{-1/2}\\
&\qquad\times\int_{2^i/T}^{2^{i+1}/T}\int_0^{1/T}\left|f(x-s,y-t)-f(x,y)\right|\,ds\,dt\\
&\quad\le C\sum_{i=1}^{r_0}2^{(\tau-1)i}2^{-\tau i}\Big(\frac{T^2}{2^i}\Big)\int_{2^i/T}^{2^{i+1}/T}\int_0^{1/T}\left|f(x-s,y-t)-f(x,y)\right|\,ds\,dt\\
&\quad\le C\sum_{i=1}^{r_0}2^{(\tau-1)i}U_r^{(1)}f(x,y)<C\epsilon.
\end{aligned}$$

Since $s-t>s/2$ on the set A_3, we get from (3.7) that

$$\left|K_T^\theta(s,t)\right|\le CT^{-\alpha}(s-t)^{-1-\beta}t^{-\alpha+\beta-1}\le CT^{-\alpha}s^{-1-\beta}t^{-\alpha+\beta-1} \tag{5.2}$$

for some $0\le\beta\le1$. Let us choose $\beta=\alpha/2$ if $0<\alpha\le2$ and $\beta=1$ if $\alpha>2$. Then

$$\begin{aligned}
&\int_{A_3\cap S_{r/2}}\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right|\,ds\,dt\\
&\quad\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1}T^{-\alpha}\Big(\frac{2^i}{T}\Big)^{-1-\beta}\Big(\frac{2^j}{T}\Big)^{-\alpha+\beta-1}\\
&\qquad\times\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right|\,ds\,dt\\
&\quad\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1}2^{(\tau-\beta)i}2^{(\tau-\alpha+\beta)j}
\end{aligned}$$

$$\times 2^{-\tau(i+j)}\Big(\frac{T^2}{2^{i+j}}\Big)\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\big|f(x-s,y-t)-f(x,y)\big|\,ds\,dt$$
$$\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1}2^{(\tau-\beta)i}2^{(\tau-\alpha+\beta)j}U_r^{(1)}f(x,y)<C\epsilon.$$

We have $t>s/2$ on A_4; hence (3.7) implies

$$\big|K_T^\theta(s,t)\big|\le CT^{-\alpha}(s-t)^{-1-\beta}s^{-\alpha+\beta-1}, \tag{5.3}$$

and so

$$\int_{A_4\cap S_{r/2}}\big|f(x-s,y-t)-f(x,y)\big|\big|K_T^\theta(s,t)\big|\,ds\,dt$$
$$\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1}T^{-\alpha}\Big(\frac{2^i}{T}\Big)^{-\alpha+\beta-1}\Big(\frac{2^j}{T}\Big)^{-1-\beta}$$
$$\times\int_{2^i/T}^{2^{i+1}/T}\int_{s-2^{j+1}/T}^{s-2^j/T}\big|f(x-s,y-t)-f(x,y)\big|\,dt\,ds$$
$$\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1}2^{(\tau-\alpha+\beta)i}2^{(\tau-\beta)j}$$
$$\times 2^{-\tau(i+j)}\Big(\frac{T^2}{2^{i+j}}\Big)\int_{2^i/T}^{2^{i+1}/T}\int_{s-2^{j+1}/T}^{s-2^j/T}\big|f(x-s,y-t)-f(x,y)\big|\,dt\,ds$$
$$\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1}2^{(\tau-\alpha+\beta)i}2^{(\tau-\beta)j}U_r^{(2)}f(x,y)<C\epsilon,$$

where β is chosen as before.

We get from (3.8) that

$$\big|K_T^\theta(s,t)\big|z\le CT^{1-\alpha}s^{-1-\alpha} \tag{5.4}$$

on the set A_5. This implies that

$$\int_{A_5\cap S_{r/2}}\big|f(x-s,y-t)-f(x,y)\big|\big|K_T^\theta(s,t)\big|\,ds\,dt$$
$$\le C\sum_{i=1}^{r_0}T^{1-\alpha}\Big(\frac{2^i}{T}\Big)^{-1-\alpha}\int_{2^i/T}^{2^{i+1}/T}\int_{s-1/T}^{s}\big|f(x-s,y-t)-f(x,y)\big|\,dt\,ds$$
$$\le C\sum_{i=1}^{r_0}2^{(\tau-\alpha)i}2^{-\tau i}\Big(\frac{T^2}{2^i}\Big)\int_{2^i/T}^{2^{i+1}/T}\int_{s-1/T}^{s}\big|f(x-s,y-t)-f(x,y)\big|\,dt\,ds$$
$$\le C\sum_{i=1}^{r_0}2^{(\tau-\alpha)i}U_r^{(2)}f(x,y)<C\epsilon.$$

Similarly, we can show that

$$\begin{aligned}
&\int_{A_2\cap S^c_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\, dt\\
&\quad\le C\sum_{i=r_0}^{\infty} 2^{(\tau-1)i}\mathcal{M}^{(1)}f(x,y)+C\sum_{i=r_0}^{\infty}2^{-i}f(x,y)\\
&\quad\le C2^{(\tau-1)r_0}\mathcal{M}^{(1)}f(x,y)+C2^{-r_0}f(x,y)\\
&\quad\le C(Tr)^{\tau-1}\mathcal{M}^{(1)}f(x,y)+C(Tr)^{-1}f(x,y)\to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{A_3\cap S^c_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\, dt\\
&\quad\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\tau-\beta)i}2^{(\tau-\alpha+\beta)j}\mathcal{M}^{(1)}f(x,y)+C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1}2^{-\beta i}2^{(-\alpha+\beta)j}f(x,y)\\
&\quad\le C2^{(\tau-\beta)r_0}\mathcal{M}^{(1)}f(x,y)+C2^{-\beta r_0}f(x,y)\to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{A_4\cap S^c_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\, dt\\
&\quad\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\tau-\alpha+\beta)i}2^{(\tau-\beta)j}\mathcal{M}^{(2)}f(x,y)+C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1}2^{(-\alpha+\beta)i}2^{-\beta j}f(x,y)\\
&\quad\le C2^{(\tau-\alpha+\beta)r_0}\mathcal{M}^{(2)}f(x,y)+C2^{-\beta r_0}f(x,y)\to 0
\end{aligned}$$

as $T\to\infty$. Finally,

$$\begin{aligned}
&\int_{A_5\cap S^c_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\, dt\\
&\quad\le C\sum_{i=r_0}^{\infty} 2^{(\tau-\alpha)i}\mathcal{M}^{(2)}f(x,y)+C\sum_{i=r_0}^{\infty}2^{-\alpha i}f(x,y)\\
&\quad\le C2^{(\tau-\alpha)r_0}\mathcal{M}^{(2)}f(x,y)+C2^{-\alpha r_0}f(x,y)\to 0
\end{aligned}$$

as $T\to\infty$. Note that $A_1\cap S^c_{r/2}=\emptyset$. This completes the proof of the theorem. □

Since by Theorems 4.1 and 4.2 almost every point is a modified strong Lebesgue point and the maximal operator $\mathcal{M}f$ is almost everywhere finite in case $f\in W(L_1,\ell_\infty)(\mathbb{R}^2)$, Theorem 5.1 implies the following corollary.

Corollary 5.2. *Suppose that (3.4) is satisfied for some* $0<\alpha<\infty$ *and that* $f\in W(L_1,\ell_\infty)(\mathbb{R}^2)$. *Then*

$$\lim_{T\to\infty}\sigma^\theta_T f(x,y)=f(x,y)\quad a.e.$$

If $1\le\alpha<\infty$, then in Theorem 5.1 we can omit the condition that $\mathcal{M}f(x,y)$ is finite.

Theorem 5.3. *Suppose that (3.4) is satisfied for some $1 \leq \alpha < \infty$ and that $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$. If (x,y) is a modified strong Lebesgue point of f, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y).$$

Proof. Taking into account the proof of Theorem 5.1, we have to show only that

$$\int_{\bigcup_{i=2}^5 (A_i \cap S_{r/2}^c)} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| \, ds\, dt \to 0$$

as $T \to \infty$. For small $\delta > 0$, let us introduce the sets

$$\begin{aligned} B_1 &:= \left\{(s,t) : s > r/2, 0 < t \leq \delta\right\}, \\ B_2 &:= \left\{(s,t) : s > r/2, \delta < t \leq s-\delta\right\}, \\ B_3 &:= \left\{(s,t) : s > r/2, s-\delta < t \leq s\right\}. \end{aligned}$$

Then we have to integrate over these three sets. On B_3, we use estimation (5.4) to obtain

$$\begin{aligned} &\int_{B_3} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| \, ds\, dt \\ &\quad\leq CT^{1-\alpha} \sum_{i=0}^{N_0-1} (i\vee 1)^{-1-\alpha} \int_i^{i+1} \int_{s-\delta}^{s} \left|f(x-s,y-t)\right| \, ds\, dt \\ &\qquad + CT^{1-\alpha} \sum_{i=N_0}^{\infty} i^{-1-\alpha} \int_i^{i+1} \int_{s-\delta}^{s} \left|f(x-s,y-t)\right| \, ds\, dt \\ &\quad\leq CT^{1-\alpha} \|f 1_{\{(s,t): r/2 < s < N_0, s-\delta < t \leq s\}}\|_{W(L_1,\ell_\infty)} + CT^{1-\alpha} N_0^{-\alpha} \|f\|_{W(L_1,\ell_\infty)} < \epsilon. \end{aligned}$$

The second term is less than ϵ if N_0 is large enough, and the first term is less than ϵ if δ is small enough. The rest of the proof works for all $0 < \alpha < \infty$. Indeed, by (3.6),

$$\begin{aligned} &\int_{B_1} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| \, ds\, dt \\ &\quad\leq C \sum_{i=0}^{N_0-1} (i\vee 1)^{-2} \int_i^{i+1} \int_0^{\delta} \left|f(x-s,y-t)\right| \, ds\, dt \\ &\qquad + C \sum_{i=N_0}^{\infty} i^{-2} \int_i^{i+1} \int_0^{\delta} \left|f(x-s,y-t)\right| \, ds\, dt \\ &\quad\leq C\|f 1_{[r/2,N_0]\times[0,\delta]}\|_{W(L_1,\ell_\infty)} + CN_0^{-\alpha} \|f\|_{W(L_1,\ell_\infty)} < \epsilon \end{aligned}$$

if N_0 is large enough and δ is small enough. Moreover, by (5.2) and (5.3),

$$\begin{aligned} &\int_{B_2\cap A_3} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| \, ds\, dt \\ &\quad\leq CT^{-\alpha} \sum_{i=0}^{\infty} (i\vee 1)^{-1-\beta} \delta^{-\alpha+\beta-1} \int_i^{i+1} \int_\delta^{1} \left|f(x-s,y-t)\right| \, ds\, dt \end{aligned}$$

$$+ CT^{-\alpha} \sum_{i=1}^{\infty} \sum_{j=1}^{i} i^{-1-\beta} j^{-\alpha+\beta-1} \int_i^{i+1} \int_j^{j+1} \left| f(x-s, y-t) \right| ds\, dt$$
$$\leq CT^{-\alpha} \| f \|_{W(L_1, \ell_\infty)} \to 0,$$

and

$$\int_{B_2 \cap A_4} \left| f(x-s, y-t) \right| \left| K_T^\theta(s,t) \right| ds\, dt$$
$$\leq CT^{-\alpha} \sum_{i=0}^{\infty} \sum_{j=0}^{i} (i \vee 1)^{-\alpha+\beta-1} (j \vee 1)^{-1-\beta} \int_i^{i+1} \int_{s-j-1}^{s-j} \left| f(x-s, y-t) \right| ds\, dt$$
$$\leq CT^{-\alpha} \| f \|_{W(L_1, \ell_\infty)} \to 0$$

as $T \to \infty$, where β is chosen as in Theorem 5.1. □

In the next theorem we do not need the maximal operator $\mathcal{M}^{(2)} f$.

Theorem 5.4. *Suppose that (3.4) is satisfied for some $0 < \alpha < \infty$ and that $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ with $1/\alpha \vee 1 < p < \infty$. If (x,y) is a modified p-Lebesgue point of f and $\mathcal{M}_p^{(1)} f(x,y)$ is finite, then*

$$\lim_{T \to \infty} \sigma_T^\theta f(x,y) = f(x,y).$$

Proof. Now let $\tau < (\alpha - 1/p)/2 \wedge 1$. Since (x,y) is a modified p-Lebesgue point of f, we can fix a number r such that

$$U_{r,p}^{(1)} f(x,y) < \epsilon.$$

Since

$$U_{r,1}^{(1)} f \leq U_{r,p}^{(1)} f \qquad \text{and} \qquad \mathcal{M}^{(1)} f \leq \mathcal{M}_p^{(1)} f,$$

we can prove in the same way as in Theorem 5.1 that

$$\int_{A_i} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| ds\, dt < C\epsilon$$

for $i = 1, 2, 3$ if T is large enough.

Then we have to consider the integral (5.1) over the sets A_4 and A_5 only. By (5.3) with $\beta = 0$ and Hölder's inequality,

$$\int_{A_4 \cap S_{r/2}} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| ds\, dt$$
$$\leq \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| 1_{A_4}\, dt\, ds$$
$$\leq \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \Big(\int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left| f(x-s, y-t) - f(x,y) \right|^p dt\, ds \Big)^{1/p}$$
$$\times \Big(\int_{2^i/T}^{2^{i+1}/T} \int_{2^{i-1}/T}^{s-1/T} T^{-\alpha q} s^{(-\alpha-1)q} (s-t)^{-q} 1_{A_4}\, dt\, ds \Big)^{1/q}.$$

Since $q > 1$, we have

$$\int_{2^i/T}^{2^{i+1}/T} \int_{2^{i-1}/T}^{s-1/T} T^{-\alpha q} s^{(-\alpha-1)q} (s-t)^{-q} \, dt \, ds$$
$$\leq CT^{-\alpha q} \Big(\frac{2^i}{T}\Big)^{(-\alpha-1)q+1} \Big(\frac{1}{T}\Big)^{-q+1} \leq C \Big(\frac{T}{2^i}\Big)^{2q-2} 2^{i(q-1-\alpha q)}.$$

Thus

$$\int_{A_4 \cap S_{r/2}} \big|f(x-s,y-t) - f(x,y)\big| \big|K_T^\theta(s,t)\big| \, ds \, dt$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\tau-(\alpha-1/p)/2)(i+j)}$$
$$\times 2^{-\tau(i+j)} \Big(\frac{T^2}{2^{i+j}} \int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \big|f(x-s,y-t) - f(x,y)\big|^p \, dt \, ds\Big)^{1/p}$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\tau-(\alpha-1/p)/2)(i+j)} U_{r,p}^{(1)} f(x,y) < C\epsilon.$$

Using (5.4), we get in the same way that

$$\int_{A_5 \cap S_{r/2}} \big|f(x-s,y-t) - f(x,y)\big| \big|K_T^\theta(s,t)\big| \, ds \, dt$$
$$\leq \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \Big(\int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \big|f(x-s,y-t) - f(x,y)\big|^p \, dt \, ds\Big)^{1/p}$$
$$\times \Big(\int_{2^i/T}^{2^{i+1}/T} \int_{s-1/T}^{s} T^{(1-\alpha)q} s^{(-\alpha-1)q} \, dt \, ds\Big)^{1/q}$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \Big(\int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \big|f(x-s,y-t) - f(x,y)\big|^p \, dt \, ds\Big)^{1/p}$$
$$\times T^{1-\alpha-1/q} \Big(\frac{2^i}{T}\Big)^{-\alpha-1+1/q}$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\tau-(\alpha-1/p)/2)(i+j)}$$
$$\times 2^{-\tau(i+j)} \Big(\frac{T^2}{2^{i+j}} \int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \big|f(x-s,y-t) - f(x,y)\big|^p \, dt \, ds\Big)^{1/p}$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\tau-(\alpha-1/p)/2)(i+j)} U_{r,p}^{(1)} f(x,y) < C\epsilon.$$

Similarly,

$$\begin{aligned}
\int_{A_4\cap S^c_{r/2}} & \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\, dt \\
&\leq C_p \sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{(\tau-(\alpha-1/p)/2)(i+j)}\mathcal{M}^{(1)}_p f(x,y) \\
&\quad + C_p \sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{-(\alpha-1/p)(i+j)/2} f(x,y) \\
&\leq C_p 2^{(2\tau-(\alpha-1/p))r_0}\mathcal{M}^{(1)}_p f(x,y) + C_p 2^{-(\alpha-1/p)r_0} f(x,y) \to 0
\end{aligned}$$

as $T\to\infty$. It is easy to see that the same holds for the integral over the set $A_5\cap S^c_{r/2}$, which finishes the proof of the theorem. □

With the same proof as in Theorem 5.3, we can see that the finiteness of $\mathcal{M}^{(1)}_p f(x,y)$ can be omitted.

Theorem 5.5. *Suppose that (3.4) is satisfied for some $1\leq\alpha<\infty$ and that $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$ $(1<p<\infty)$. If (x,y) is a modified p-Lebesgue point of f, then*

$$\lim_{T\to\infty}\sigma^\theta_T f(x,y) = f(x,y).$$

6. Applications to various summability methods

In this section, we list some summability methods that were also considered in [3]. The elementary computations in the examples below are left to the reader.

Example 6.1 (Fejér summation). Let

$$\theta(t) := \begin{cases} 1-|t| & \text{if } |t|\leq 1,\\ 0 & \text{if } |t|>1.\end{cases}$$

Example 6.2 (de La Vallée-Poussin summation). Let

$$\theta(t) = \begin{cases} 1 & \text{if } |t|\leq 1/2,\\ -2|t|+2 & \text{if } 1/2<|t|\leq 1,\\ 0 & \text{if } |t|>1.\end{cases}$$

Example 6.3 (Jackson-de La Vallée-Poussin summation). Let

$$\theta(t) = \begin{cases} 1-3t^2/2+3|t|^3/4 & \text{if } |t|\leq 1,\\ (2-|t|)^3/4 & \text{if } 1<|t|\leq 2,\\ 0 & \text{if } |t|>2.\end{cases}$$

The next example generalizes Examples 6.1, 6.2, and 6.3.

Example 6.4. Let $0=\alpha_0<\alpha_1<\cdots<\alpha_m$ and $\beta_0,\ldots,\beta_m$ $(m\in\mathbb{N})$ be real numbers, $\beta_0=1$, $\beta_m=0$. Suppose that θ is even, that $\theta(\alpha_j)=\beta_j$ $(j=0,1,\ldots,m)$, that $\theta(t)=0$ for $t\geq\alpha_m$, and that θ is a polynomial on the interval $[\alpha_{j-1},\alpha_j]$ $(j=1,\ldots,m)$.

Example 6.5 (Rogosinski summation). Let

$$\theta(t) = \begin{cases} \cos \pi t/2 & \text{if } |t| \leq 1 + 2j, \\ 0 & \text{if } |t| > 1 + 2j, \end{cases} \qquad (j \in \mathbb{N}).$$

Example 6.6 (Weierstrass summation). Let $\theta(t) = e^{-|t|^\gamma}$ for some $1 \leq \gamma < \infty$. Note that if $\gamma = 1$, then we obtain the *Abel summation*.

Example 6.7. Let $\theta(t) = e^{-(1+|t|^q)^\gamma}$ $(t \in \mathbb{R}, 1 \leq q < \infty, 0 < \gamma < \infty)$.

Example 6.8 (Picard and Bessel summations). Let $\theta(t) = (1 + |t|^\gamma)^{-\delta}$ $(0 < \delta < \infty, 1 \leq \gamma < \infty, \gamma\delta > 2)$.

Example 6.9 (Riesz summation). Let

$$\theta(t) = \begin{cases} (1 - |t|^\gamma)^\delta & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| > 1 \end{cases}$$

for some $0 < \delta < \infty, 1 \leq \gamma < \infty$.

By an easy computation we get that the conditions (3.1) and (3.4) are satisfied for examples 6.1–6.5 and for example 6.9 if $1 \leq \delta, \gamma < \infty$ with $\alpha = 1$. Moreover, examples 6.6–6.8 satisfy (3.1) and (3.4) with $\alpha = 2$ and example 6.9 with $\alpha = \delta$ if $0 < \delta \leq 1 \leq \gamma < \infty$.

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