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ORDER STRUCTURE, MULTIPLIERS, AND GELFAND REPRESENTATION OF VECTOR-VALUED FUNCTION ALGEBRAS

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Abstract. We continue the study begun by the third author of C^*-Segal algebra-valued function algebras with an emphasis on the order structure. Our main result is a characterization theorem for C^*-Segal algebra-valued function algebras with an order unitization. As an intermediate step, we establish a function algebraic description of the multiplier module of arbitrary Segal algebra-valued function algebras. We also consider the Gelfand representation of these algebras in the commutative case.

Introduction

The concept of a Segal algebra originated in the work of Reiter on subalgebras of the L_1-algebra of a locally compact group (see [18]). It was generalized to arbitrary Banach algebras by Burnham in [7]. A C^**-Segal algebra* is a Banach algebra which is a dense (not necessarily self-adjoint) ideal in a C^*-algebra. Despite many important examples in analysis, such as the Schatten classes, for instance, little has been known about the general structure and properties of C^*-Segal algebras. In particular, most results have relied on the additional assumption of an approximate identity. The multiplier algebra and the bidual of self-adjoint C^*-Segal algebras were described in [2] and [11], and the form of the closed ideals of C^*-Segal algebras with an approximate identity was given in [5].

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Recently, a new approach to Segal algebras was introduced in [4] and [12], which allows a more detailed study of their structure and properties, particularly in the absence of an approximate identity (see [4] for the commutative case and [12] for the noncommutative case). The main new tools used in these papers are the notion of the "approximate ideal" and the theory of multiplier modules. The primary motivation of the present investigation was to gain a better understanding of the order structure of (not necessarily closed) ideals of C^*-algebras. Indeed, the main results of [4] and [12] are structure theorems for self-adjoint C^*-Segal algebras with an order unit whose norm coincides with the order-unit norm on the self-adjoint part. These results were extended in [13] to C^*-Segal algebras with an order unitization (i.e., to which an order unit can be added in a structure-preserving way).

In this article, we continue the work begun in [15] on C^*-Segal algebra-valued function algebras with an emphasis on the order structure. Our main result is a characterization theorem for C^*-Segal algebra-valued function algebras with an order unitization. Together with results of [13], this yields a structure theorem for these algebras. As an intermediate step, we establish a function algebraic description of the multiplier module of arbitrary Segal algebra-valued function algebras. This extends and improves previous results by Akemann, Pedersen, and Tomiyama [1] and by Candeal Haro and Lai [8]. We also discuss Gelfand representation of these algebras in the commutative case. Here, an important role is played by certain weighted-function algebras introduced by Nachbin in [16].

This paper is organized as follows. In Section 1, we review some facts about C^*-Segal algebra-valued function algebras and state our main results, Theorems 1.12 and 1.21. In Section 2, we give the proofs of these theorems. In Section 3, we consider Gelfand representation of commutative C^*-Segal algebra-valued function algebras.

1. Preliminaries and statements of the main results

In this section, we review some facts about C^*-Segal algebra-valued function algebras and state our main results, Theorems 1.12 and 1.21.

Notation. Throughout this article, let A be a Banach algebra with norm $\|\cdot\|_A$. For a bimodule D over A, we denote by $\operatorname{ann}_A(D)$ its *annihilator* in A; that is,

$$\operatorname{ann}_A(D) := \{a \in A : a \cdot x = x \cdot a = 0 \text{ for all } x \in D\}.$$

The bimodule D is called *faithful* if $\operatorname{ann}_A(D) = \{0\}$. Similarly, an ideal of A is considered faithful if it is faithful when viewed as a bimodule over A. For any two ideals I and J of A, we write $IJ := \{xy \colon x \in I \text{ and } y \in J\}$. For any two Banach spaces E and F, we denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear mappings from E into F. The equivalence of two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space is denoted by $\|\cdot\| \sim \|\cdot\|'$. For any topological space Y, we denote by $C(Y)$ the set of all continuous complex-valued functions on Y. The subsets of $C(Y)$ consisting of all bounded and vanishing-at-infinity functions are denoted by $C_b(Y)$ and $C_0(Y)$, respectively.

1.1. **Vector-valued function algebras.** Throughout this paper, let X be a locally compact Hausdorff space. We define

$$C_b(X,A) := \{f\colon X \to A : f \text{ is continuous and } t \mapsto \|f(t)\|_A \in C_b(X)\}$$

and

$$C_0(X,A) := \{f\colon X \to A : f \text{ is continuous and } t \mapsto \|f(t)\|_A \in C_0(X)\}.$$

These sets are Banach algebras under pointwise operations and the supremum norm

$$\|f\|_\infty^A := \sup_{t\in X}\|f(t)\|_A.$$

Moreover, if A is a C^*-algebra, then so are $C_b(X,A)$ and $C_0(X,A)$ with involution $f \mapsto f^*$ given by $f^*(t) := f(t)^*$ for all $t \in X$. For any $\phi \in C(X)$ and any $a \in A$, we denote

$$\phi \otimes a\colon X \to A; \qquad t \mapsto \phi(t)a \quad (t \in X).$$

The linear span of the set $\{\phi \otimes a\colon \phi \in C_0(X) \text{ and } a \in A\}$ is a dense subspace of $C_0(X,A)$. (For a standard reference on vector-valued function algebras, we refer to [17].)

1.2. C^***-Segal algebras.** Let B be a Banach algebra with norm $\|\cdot\|_B$. Recall that A is said to be a *Segal algebra* in B if it is a dense ideal of B and there exists a constant $l > 0$ such that $\|a\|_B \le l\|a\|_A$ for all $a \in A$. The following standard result of Barnes is recorded for completeness.

Lemma 1.1. *Let B be a Banach algebra containing A as a Segal algebra. Then A is a Banach B-bimodule; that is, there exists a positive constant l such that*

$$\|ax\|_A \le l\|a\|_A\|x\|_B \qquad \text{and} \qquad \|xa\|_A \le l\|a\|_A\|x\|_B$$

for all $a \in A$ and all $x \in B$.

Proof. For the proof, see [5, Theorem 2.3]. □

The following class of Segal algebras concerns our main interest in this paper.

Definition 1.2. By a *C^*-Segal algebra*, we understand a Banach algebra contained as a Segal algebra in a C^*-algebra. We call a C^*-Segal algebra *self-adjoint* if it is closed under the involution of the surrounding C^*-algebra.

The following property of $C_0(X,A)$ will be used in the rest of this article without further reference.

Proposition 1.3. *The following conditions are equivalent for a Banach algebra B:*

(a) *A is a Segal algebra in B,*
(b) *$C_0(X,A)$ is a Segal algebra in $C_0(X,B)$.*

In particular, $C_0(X,A)$ is a C^*-Segal algebra whenever A is a C^*-Segal algebra.

Proof. For the proof, see [15, Theorem 3.8]. □

1.3. **Approximate identities of C^*-Segal algebras.** Let B be a Banach algebra with norm $\|\cdot\|_B$. Recall that an *approximate identity* for B is a net $\{e_\alpha\}_{\alpha\in\Omega}$ in B such that $\|xe_\alpha - x\|_B \to 0$ and $\|e_\alpha x - x\|_B \to 0$ for every $x \in B$; it is called *bounded* if there exists a constant $l > 0$ with $\|e_\alpha\|_B \leq l$ for all $\alpha \in \Omega$.

Definition 1.4. Let A be a Segal algebra in a C^*-algebra C. The *approximate ideal* of A is the set

$$E_A := \overline{\operatorname{span}}^{\|\cdot\|_A}\{ab : a, b \in A\};$$

this is a Banach subalgebra of A which is contained as a Segal algebra in C.

Remark 1.5. Strictly speaking, this is not the definition of the approximate ideal originally given in [4] and [12], but it coincides with the original one in [12, Lemmas 2.9, 2.16].

The above terminology is motivated by the lemma below.

Lemma 1.6. *Let A be a Segal algebra in a C^*-algebra C. Then*

(i) *E_A is a closed faithful ideal of A,*
(ii) *E_A has an approximate identity,*
(iii) *$E_A = AC = CA$.*

Moreover, every closed ideal of A with an approximate identity is contained in E_A.

Proof. For the proof, see [12, Proposition 2.10, Lemma 2.16]. □

The following characterization of the existence of approximate identities for $C_0(X, A)$ is one of the main results of [15].

Proposition 1.7. *Let A be a Segal algebra in a C^*-algebra C. The following conditions are equivalent:*

(a) *A has an approximate identity,*
(b) *$C_0(X, A)$ has an approximate identity.*

Moreover, the approximate ideal of $C_0(X, A)$ satisfies $E_{C_0(X,A)} = C_0(X, E_A)$.

Proof. For the proof, see [15, Theorem 4.8]. □

1.4. **Multipliers of C^*-Segal algebras.** Let B be a Banach algebra with zero annihilator. Recall that a *double centralizer* of B is a pair (m_l, m_r) of linear mappings from B into B satisfying the condition

$$xm_l(y) = m_r(x)y$$

for all $x, y \in B$. The set $M(B)$ of all double centralizers of B is a unital Banach subalgebra of $\mathcal{L}(B) \oplus_\infty \mathcal{L}(B)^{\mathrm{op}}$. Moreover, B is canonically embedded as a faithful ideal into $M(B)$ by $x \mapsto (l_x, r_x)$, where

$$l_x(y) := xy \qquad \text{and} \qquad r_x(y) := yx$$

for all $y \in B$. Under this identification—which we will always make—every double centralizer (m_l, m_r) of B is given by a unique element m in $M(B)$ as a *multiplier*; that is,

$$m_l(x) = mx \qquad \text{and} \qquad m_r(x) = xm$$

for all $x \in B$. The Banach algebra $M(B)$ is called the *multiplier algebra* of B.

Definition 1.8. Let A be a Segal algebra in a C^*-algebra C. The *multiplier module* of A is the set

$$M_C(A) := \big\{(m_l, m_r) \in M(C) : m_l(C) \subseteq A \text{ and } m_r(C) \subseteq A\big\};$$

this is a Banach subalgebra of $\mathcal{L}(C, A) \oplus_\infty \mathcal{L}(C, A)^{\mathrm{op}}$ which is contained as a Segal algebra in $\overline{M_C(A)}^{\|\cdot\|_{M(C)}}$.

Remark 1.9. Strictly speaking, the operator norm on $\mathcal{L}(C, A)$ has to be multiplied by a constant l satisfying $\|a\|_C \le l\|a\|_A$ for all $a \in A$ in order to obtain an algebra norm on $\mathcal{L}(C, A)$.

The above terminology is motivated by the lemma below.

Lemma 1.10. *Let A be a Segal algebra in a C^*-algebra C. Then*

(i) *$M_C(A)$ is a faithful Banach C-bimodule,*
(ii) *$M_C(A)$ contains E_A as a closed faithful ideal,*
(iii) *$M_C(A)$ satisfies $E_A = M_C(A)C = CM_C(A)$.*

Moreover, if V is a faithful C-bimodule containing E_A such that $E_A = V \cdot C = C \cdot V$, then there exists a unique C-bimodule homomorphism from V into $M_C(A)$ which is the identity on E_A.

Proof. For the proof, see [12, Proposition 2.21]. □

The *strict topology* on $M_C(A)$ is defined by the seminorms

$$m \mapsto \|mc\|_A + \|cm\|_A \quad (c \in C).$$

The next lemma collects some basic properties of the strict topology for later use.

Lemma 1.11. *Let A be a Segal algebra in a C^*-algebra C. Then $M_C(A)$ equipped with the strict topology is a complete locally convex vector space such that the multiplication and the module operations (viewed as mappings of $M_C(A) \times C$ and $C \times M_C(A)$ into A) are jointly continuous on norm-bounded subsets.*

Proof. These assertions follow from [20, Theorem 3.5] and standard computations. □

The following function algebraic description of the multiplier module of $C_0(X, A)$ is the first main result of this article.

Theorem 1.12. *Let A be a Segal algebra in a C^*-algebra C. Then, up to an isometric algebra and order isomorphism, we have*

$$M_{C_0(X,C)}\big(C_0(X, A)\big) = C_{s,b}\big(X, M_C(A)\big),$$

where $C_{s,b}(X, M_C(A))$ denotes the C^*-Segal algebra of all strictly continuous and norm-bounded functions from X into $M_C(A)$.

Remark 1.13. In fact, as will become clear from the proof, this result (except for the order-theoretic part) holds for all Banach algebras which are Segal algebras in a Banach algebra with a zero annihilator.

Remark 1.14. As mentioned in the Introduction, several authors have considered the question of representing multipliers of algebras of continuous vector-valued functions by vector-valued functions:

- Akemann, Pedersen, and Tomiyama [1, Theorem 3.3] obtained the preceding theorem in the special case of C^*-algebras.
- Candeal Haro and Lai [8, Theorem 2] obtained a theorem similar to Theorem 1.12 in the setting of faithful Banach modules over a commutative Banach algebra.

Moreover, recently Khan and Alsulami [14, Theorem 16] addressed this question for certain topological modules over a commutative Banach algebra.

1.5. **Order-unit C^*-Segal algebras.** Let A be a Segal algebra in a C^*-algebra C. The *positive cone* of A is defined by

$$A_+ := A_{\mathrm{sa}} \cap C_+,$$

where A_{sa} denotes the real vector space of self-adjoint elements of A. Then A_{sa} becomes a partially ordered vector space when equipped with the relation

$$a \leq b \quad \text{if } b - a \in A_+ \quad (a, b \in A_{\mathrm{sa}}).$$

An element $u \in A_+$ is called an *order unit* for A if each $a \in A_{\mathrm{sa}}$ satisfies the relation $-lu \leq a \leq lu$ for some scalar $l > 0$. Clearly, an order-unit u for a self-adjoint A is *strictly positive* in the sense that $\tau(u) > 0$ for every nonzero positive linear functional τ on A.

Example 1.15. Let $v\colon X \to \mathbb{R}$ be an upper semicontinuous real-valued function on X such that $v(t) \geq 1$ for all $t \in X$. We define

$$C_b^v(X) := \big\{ f \in C(X) : vf \text{ is bounded on } X \big\}$$

and

$$C_0^v(X) := \big\{ f \in C(X) : vf \text{ vanishes at infinity on } X \big\}.$$

These sets are C^*-Segal algebras under pointwise operations and the norm

$$\|f\|_v := \sup_{t \in X} v(t) \big| f(t) \big|.$$

In fact, they are examples of the so-called *Nachbin algebras* (see [3], [6], [16], for instance). Clearly, the function $\frac{1}{v}$ serves as an order unit for $C_b^v(X)$ whenever v is continuous on X.

The following special C^*-Segal algebras relate to our focus here.

Definition 1.16. By an *order-unit C^*-Segal algebra*, we understand a pair (A, u), where A is a self-adjoint C^*-Segal algebra and u is an order unit for A such that

$$\|a\|_A = \|a\|_u := \inf\{ l > 0 : -lu \leq a \leq lu \}$$

for all $a \in A_{\mathrm{sa}}$.

The following structure theorem for order-unit C^*-Segal algebras is the main result of [12].

Theorem 1.17. *Let A be a Segal algebra in a C^*-algebra C. The following conditions are equivalent for a strictly positive element u of A:*

(a) *(A, u) is an order-unit C^*-Segal algebra,*
(b) *there exists a self-adjoint C-subbimodule D of $M(C)$ containing C and $1_{M(C)}$ such that $A = u^{\frac{1}{2}} D u^{\frac{1}{2}}$ and $\|u^{\frac{1}{2}}\, du^{\frac{1}{2}}\|_A = \|d\|_{M(C)}$ for all $d \in D_{\mathrm{sa}}$.*

In particular, $E_A = u^{\frac{1}{2}} C u^{\frac{1}{2}} = uC = Cu$ and $M_C(A) = u^{\frac{1}{2}} M(C) u^{\frac{1}{2}} = uM(C) = M(C)u$ whenever (A, u) is an order unit C^*-Segal algebra.

Proof. For the proof, see [12, Theorem 3.12, Corollary 3.14]. □

The following enlargement of the class of order-unit C^*-Segal algebras will be useful in our study of the order structure of $C_0(X, A)$.

Definition 1.18. Let A be a C^*-Segal algebra. By an *order unitization* for A, we mean a triplet (B, u, λ), where

(i) (B, u) is an order-unit C^*-Segal algebra,
(ii) $\lambda\colon A \to B$ is a topological algebra and order isomorphism onto its image,
(iii) the image of A under λ is a faithful ideal of B.

In what follows, we will without loss of generality (see [13, Remark 2.18]) drop the notation λ; in other words, we will identify A and $\lambda(A)$.

The next two theorems are taken from [13], to which we refer for a detailed treatment of C^*-Segal algebras with an order unitization.

Theorem 1.19. *Let A be a Segal algebra in a C^*-algebra C with an order unitization. Then there exist a closed C-subbimodule D of $M(C)$ containing C and an order-unit u for $M_C(A)$ such that $A = u^{\frac{1}{2}} D u^{\frac{1}{2}}$ and $\|u^{\frac{1}{2}}\, du^{\frac{1}{2}}\|_u = \|d\|_{M(C)}$ for all $d \in D_{\mathrm{sa}}$. Moreover, $E_A = u^{\frac{1}{2}} C u^{\frac{1}{2}} = uC = Cu$ and $M_C(A) = u^{\frac{1}{2}} M(C) u^{\frac{1}{2}} = uM(C) = M(C)u$.*

Theorem 1.20 (Universal property of the multiplier module)**.** *Let A be a C^*-Segal algebra. Suppose that A has an order unitization (B, u). Then there exist a unique topological algebra and order isomorphism ϕ from B onto its image in $M_C(A)$ which is the identity on A and a positive constant k such that $(M_C(A), k\phi(u))$ is an order unitization for A.*

Proof. For the proof, see [13, Theorems 2.15, 2.20, 2.21]. □

The following characterization of the existence of order unitizations for $C_0(X, A)$ is the second main result of this paper.

Theorem 1.21. *Let A be a Segal algebra in a C^*-algebra C. The following conditions are equivalent:*

(a) *A has an order unitization,*
(b) *$C_0(X, A)$ has an order unitization.*

Moreover, if $C_0(X, A)$ has an order unitization, then $C_{s,b}(X, M_C(A))$ is the maximal order unitization for $C_0(X, A)$ in the sense of Theorem 1.20.

Remark 1.22. This equivalence does not hold in the smaller class of order-unit C^*-Segal algebras, as the example $C_0(X,\mathbb{C}) = C_0(X)$ with X noncompact trivially shows.

Remark 1.23. Theorems 1.19 and 1.21 together immediately yield a structure theorem for $C_0(X,A)$ with A a C^*-Segal algebra having an order unitization.

2. Proofs of the main results

In this section, we give the proofs of the main results, Theorems 1.12 and 1.21, starting with the following natural result.

Proposition 2.1. *Let A be a Segal algebra in a C^*-algebra C. Then $C_{s,b}(X, M_C(A))$ is a C^*-Segal algebra under pointwise operations and the supremum norm.*

For the proof of this result, we need some further facts about the multiplier module.

Lemma 2.2. *Let A be a Segal algebra in a C^*-algebra C. Then $M_C(A)$ is a Banach $M(C)$-bimodule under the multiplication mappings*

$$(m,n) \mapsto mn \qquad \textit{and} \qquad (n,m) \mapsto nm$$

of $M_C(A) \times M(C)$ and $M(C) \times M_C(A)$ into $M_C(A)$. Moreover, the module operations are jointly strictly continuous on norm-bounded subsets of $M_C(A)$ and $M(C)$.

Proof. Clearly, the norm $\|\cdot\|_{M_C(A)}$ dominates the norm $\|\cdot\|_{M(C)}$ on $M_C(A)$. Note also that, by Lemmas 1.6(iii) and 1.10(iii), we have

$$CM_C(A)M(C) = ACM(C) = E_A = M_C(A)C = M_C(A)M(C)C$$

and

$$CM(C)M_C(A) = CM_C(A) = E_A = M(C)CA = M(C)M_C(A)C$$

so that $M_C(A)$ is an ideal of $M(C)$. Combining these two facts with [5, Theorem 2.3], we obtain the Banach $M(C)$-bimodule property of $M_C(A)$.

Consider next norm-bounded nets $\{m_\mu\}_{\mu\in\Lambda}$ and $\{n_\nu\}_{\nu\in\Gamma}$ in $M_C(A)$ and $M(C)$ with $m_\mu \to m$ and $n_\nu \to n$ strictly in $M_C(A)$ and $M(C)$, respectively. By the above reasoning, for each $c \in C$ there exist $m' \in M_C(A)$ and $n' \in M(C)$ such that

$$cm = m'c_1 \qquad \text{and} \qquad nc = c_2 n'$$

for some $c_1, c_2 \in C$. Applying the Banach $M(C)$-bimodule property of $M_C(A)$ to

$$cmn - cm_\mu n_\nu = m'(c_1 n - c_1 n_\nu) + (cm - cm_\mu)n_\nu$$

and

$$mnc - m_\mu n_\nu c = (mc_2 - m_\mu c_2)n' + m_\mu(nc - n_\nu c),$$

and taking into account parts (ii) and (iii) of Lemma 1.10 (which, in particular, imply that the norms $\|\cdot\|_A$ and $\|\cdot\|_{M_C(A)}$ are equivalent when restricted to E_A), we find that

$$\|cmn - cm_\mu n_\nu\|_A \leq \|m'\|_{M_C(A)}\|c_1 n - c_1 n_\nu\|_C + l\|cm - cm_\mu\|_A\|n_\nu\|_{M(C)}$$

and

$$\|mnc - m_\mu n_\nu c\|_A \leq l\|mc_2 - m_\mu c_2\|_A\|n'\|_{M(C)} + \|m_\mu\|_{M_C(A)}\|nc - n_\nu c\|_C$$

for some constant $l > 0$. Consequently, $m_\mu n_\nu \to mn$, and similarly, $n_\nu m_\mu \to nm$ strictly in $M_C(A)$. This establishes the joint strict continuity of the module operations on norm-bounded subsets of $M_C(A)$ and $M(C)$. □

Proof of Proposition 2.1. Lemmas 1.11 and 2.2, together with a standard completeness argument and the fact that $C_{s,b}(X, M(C))$ is a C^*-algebra (see [1, Lemma 3.2]), imply that $C_{s,b}(X, M_C(A))$ is a Banach algebra which is contained as a C^*-Segal algebra in the closed ideal of $C_{s,b}(X, M(C))$ given by $\overline{C_{s,b}(X, M_C(A))}^{\|\cdot\|_\infty^{M(C)}}$. □

The result in the lemma below was obtained by Candeal Haro and Lai.

Lemma 2.3. *Let E be a Banach space. Suppose that $T\colon C_0(X) \to C_0(X, E)$ is a bounded $C_0(X)$-module homomorphism. Then there exists $h \in C_b(X, E)$ with $\|h\|_\infty^E = \|T\|_{\mathrm{op}}$ such that $T(\phi) = \phi h$ for all $\phi \in C_0(X)$.*

Proof. For the proof, see [8, Proposition 1]. □

We are now ready to prove Theorems 1.12 and 1.21.

Proof of Theorem 1.12. Let $f \in C_{s,b}(X, M_C(A))$. In view of Lemmas 1.10(iii) and 1.11, we can define linear mappings $l_f, r_f\colon C_0(X, C) \to C_0(X, A)$ by setting

$$l_f(h) := fh \qquad \text{and} \qquad r_f(h) := hf \quad \big(h \in C_0(X, C)\big).$$

Clearly, (l_f, r_f) is a double centralizer of $C_0(X, C)$; thus there exists a unique $m_f \in M_{C_0(X,C)}(C_0(X, A))$ satisfying

$$l_f(h) = m_f h \qquad \text{and} \qquad r_f(h) = hm_f$$

for all $h \in C_0(X, C)$. By means of this, we can define an algebra homomorphism

$$\pi\colon C_{s,b}\big(X, M_C(A)\big) \to M_{C_0(X,C)}\big(C_0(X, A)\big); \qquad f \mapsto m_f.$$

Indeed, let $f, g \in C_{s,b}(X, M_C(A))$. Then, for all $h \in C_0(X, C)$, we have

$$\pi(fg)h = fgh = \pi(f)(gh) = \pi(f)\pi(g)h$$

so that $\pi(fg) = \pi(f)\pi(g)$ by Lemma 1.10(i). The linearity of π is shown in a similar way.

For the order isomorphism property of π, let $f \in C_{s,b}(X, M_C(A))$. Then, invoking Lemma 1.10(iii) and recalling that a multiplier n of C is positive if and only if c^*nc belongs to C_+ for each $c \in C$, we conclude that

$$\begin{aligned}
&f \in C_{s,b}\big(X, M_C(A)\big)_+ \\
&\quad\iff \quad f(t) \in M_C(A)_+ \quad \text{for all } t \in X \\
&\quad\iff \quad c^*f(t)c \in A_+ \quad \text{for all } c \in C \text{ and all } t \in X \\
&\quad\iff \quad h(t)^*f(t)h(t) \in A_+ \quad \text{for all } h \in C_0(X, C) \text{ and all } t \in X \\
&\quad\iff \quad h^*fh \in C_0(X, A)_+ \quad \text{for all } h \in C_0(X, C) \\
&\quad\iff \quad h^*\pi(f)h \in C_0(X, A)_+ \quad \text{for all } h \in C_0(X, C) \\
&\quad\iff \quad \pi(f) \in M_{C_0(X,C)}\big(C_0(X, A)\big)_+,
\end{aligned}$$

as desired.

For the isometric property of π, let $f \in C_{s,b}(X, M_C(A))$. Then, for each $t \in X$, we have

$$\big\{h(t) : h \in C_0(X, C), \|h\|_\infty^C \le 1\big\} = \big\{c \in C : \|c\|_C \le 1\big\},$$

whence

$$\begin{aligned}
\big\|\pi(f)\big\|_{\mathrm{op}(l)} &= \sup_{\|h\|_\infty^C \le 1} \big\|\pi(f)h\big\|_\infty^A = \sup_{\|h\|_\infty^C \le 1} \|fh\|_\infty^A = \sup_{\|h\|_\infty^C \le 1} \sup_{t\in X} \big\|f(t)h(t)\big\|_A \\
&= \sup_{t\in X} \sup_{\|h\|_\infty^C \le 1} \big\|f(t)h(t)\big\|_A = \sup_{t\in X} \sup_{\|c\|_C \le 1} \big\|f(t)c\big\|_A = \sup_{t\in X} \big\|f(t)\big\|_{\mathrm{op}(l)}
\end{aligned}$$

and

$$\begin{aligned}
\big\|\pi(f)\big\|_{\mathrm{op}(r)} &= \sup_{\|h\|_\infty^C \le 1} \big\|h\pi(f)\big\|_\infty^A = \sup_{\|h\|_\infty^C \le 1} \|hf\|_\infty^A = \sup_{\|h\|_\infty^C \le 1} \sup_{t\in X} \big\|h(t)f(t)\big\|_A \\
&= \sup_{t\in X} \sup_{\|h\|_\infty^C \le 1} \big\|h(t)f(t)\big\|_A = \sup_{t\in X} \sup_{\|c\|_C \le 1} \big\|cf(t)\big\|_A = \sup_{t\in X} \big\|f(t)\big\|_{\mathrm{op}(r)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\big\|\pi(f)\big\| &= \max\big\{\big\|\pi(f)\big\|_{\mathrm{op}(l)}, \big\|\pi(f)\big\|_{\mathrm{op}(r)}\big\} = \max\big\{\sup_{t\in X}\big\|f(t)\big\|_{\mathrm{op}(l)}, \sup_{t\in X}\big\|f(t)\big\|_{\mathrm{op}(r)}\big\} \\
&= \sup_{t\in X} \max\big\{\big\|f(t)\big\|_{\mathrm{op}(l)}, \big\|f(t)\big\|_{\mathrm{op}(r)}\big\} = \sup_{t\in X}\big\|f(t)\big\|_{M_C(A)} = \|f\|_\infty^{M_C(A)},
\end{aligned}$$

as desired.

For the surjectivity of π, let $m \in M_{C_0(X,C)}(C_0(X, A))$. Then, for each $c \in C$, we can define linear mappings $T_l^c, T_r^c \colon C_0(X) \to C_0(X, A)$ by setting

$$T_l^c(\phi) := m(\phi \otimes c) \qquad \text{and} \qquad T_r^c(\phi) := (\phi \otimes c)m \quad \big(\phi \in C_0(X)\big).$$

Clearly, the operator norms of T_l^c and T_r^c are bounded above by $\|m\|\|c\|_C$. Further, since for every pair $\phi, \psi \in C_0(X)$ and for all $h \in C_0(X, C)$ we have

$$h\big(m(\phi\psi \otimes c)\big) = (hm)(\phi\psi \otimes c) = \phi(hm)(\psi \otimes c) = h\big(\phi\big(m(\psi \otimes c)\big)\big)$$

and

$$\big((\phi\psi \otimes c)m\big)h = (\phi\psi \otimes c)(mh) = \phi(\psi \otimes c)(mh) = \big(\phi\big((\psi \otimes c)m\big)\big)h,$$

it follows from Lemma 1.10(i) that T_l^c and T_r^c are $C_0(X)$-module homomorphisms. Thus, applying Lemma 2.3, we can find $h_l^c, h_r^c \in C_b(X, A)$ with

$$\|h_l^c\|_\infty^A = \|T_l^c\|_{\mathrm{op}} \qquad \text{and} \qquad \|h_r^c\|_\infty^A = \|T_r^c\|_{\mathrm{op}}$$

such that

$$T_l^c(\phi) = \phi h_l^c \qquad \text{and} \qquad T_r^c(\phi) = \phi h_r^c$$

for all $\phi \in C_0(X)$. On the basis of these considerations, we obtain strong operator continuous mappings $s_l, s_r \colon X \to \mathcal{L}(C, A)$ by setting

$$s_l(t)(c) := h_l^c(t) \qquad \text{and} \qquad s_r(t)(c) := h_r^c(t) \quad (t \in X, c \in C).$$

For $t \in X$, choose $\phi_t \in C_0(X)$ with $\phi_t(t) = 1$. Then, for every pair $c, d \in C$, we have

$$\begin{aligned} cs_l(t)(d) &= ch_l^d(t) = cT_l^d(\phi_t)(t) = (\phi_t \otimes c)(t)\big(m(\phi_t \otimes d)\big)(t) \\ &= \big((\phi_t \otimes c)m\big)(t)(\phi_t \otimes d)(t) = T_r^c(\phi_t)(t)d = h_r^c(t)d = s_r(t)(c)d, \end{aligned}$$

whence there exists a unique $m_t \in M_C(A)$ satisfying

$$s_l(t)(c) = m_t c \qquad \text{and} \qquad s_r(t)(c) = cm_t$$

for all $c \in C$. Consequently, we can define a strictly continuous mapping

$$f \colon X \to M_C(A); \qquad t \mapsto m_t \quad (t \in X).$$

Moreover, since for all $t \in X$ we have

$$\big\|f(t)\big\|_{\mathrm{op}(l)} = \big\|s_l(t)\big\|_{\mathrm{op}} \le \sup_{\|c\|_C \le 1} \|h_l^c\|_\infty^A = \sup_{\|c\|_C \le 1} \|T_l^c\|_{\mathrm{op}} \le \sup_{\|c\|_C \le 1} \|m\|\|c\|_C = \|m\|$$

and

$$\big\|f(t)\big\|_{\mathrm{op}(r)} = \big\|s_r(t)\big\|_{\mathrm{op}} \le \sup_{\|c\|_C \le 1} \|h_r^c\|_\infty^A = \sup_{\|c\|_C \le 1} \|T_r^c\|_{\mathrm{op}} \le \sup_{\|c\|_C \le 1} \|m\|\|c\|_C = \|m\|,$$

it follows that the supremum norm of f is bounded above by $\|m\|$, and hence f belongs to $C_{s,b}(X, M_C(A))$. Take $\phi \in C_0(X)$, and take $c \in C$. Then, for all $t \in X$, we have

$$\begin{aligned} \big(\pi(f)(\phi \otimes c)\big)(t) &= \big(f(\phi \otimes c)\big)(t) = f(t)\phi(t)c = m_t\phi(t)c \\ &= \phi(t)s_l(t)(c) = \phi(t)h_l^c(t) = T_l^c(\phi)(t) = \big(m(\phi \otimes c)\big)(t) \end{aligned}$$

so that $\pi(f) = m$ by Lemma 1.10(i) and the density of the linear span of the set $\{\psi \otimes d \colon \psi \in C_0(X) \text{ and } d \in C\}$ in $C_0(X, C)$.

With these considerations, the proof of the theorem is complete. □

Proof of Theorem 1.21. Theorems 1.12 and 1.20 allow us to reduce the proof to showing that the following two statements are equivalent:

(a) $M_C(A)$ is an order unitization for A,
(b) $C_{s,b}(X, M_C(A))$ is an order unitization for $C_0(X, A)$.

Note also that $M_C(A)$ is self-adjoint if and only if $C_{s,b}(X, M_C(A))$ is self-adjoint because the involution on every self-adjoint C^*-Segal algebra is continuous (see [19, Chapter IV, Theorem 4.1.15]).

(a) $\Longrightarrow$ (b): Suppose that $(M_C(A), u)$ is an order unitization for A. Then, for all $f \in C_{s,b}(X, M_C(A))_{\mathrm{sa}}$, we have

$$\begin{aligned}\|f\|_\infty^{M_C(A)} &= \sup_{t\in X}\big\|f(t)\big\|_{M_C(A)} = \sup_{t\in X}\big\|f(t)\big\|_u \\ &= \inf\big\{l > 0 : -lu \le f(t) \le lu \text{ for all } t \in X\big\} \\ &= \inf\big\{l > 0 : -l(1\otimes u) \le f \le l(1\otimes u)\big\} \\ &= \|f\|_{1\otimes u}\end{aligned}$$

so that $(C_{s,b}(X, M_C(A)), 1\otimes u)$ is an order-unit C^*-Segal algebra. Moreover, since for each $f \in C_0(X, A)$ we have

$$\|f\|_\infty^A = \sup_{t\in X}\big\|f(t)\big\|_A \sim \sup_{t\in X}\big\|f(t)\big\|_{M_C(A)} = \|f\|_\infty^{M_C(A)},$$

it follows that $C_0(X, A)$ is closed in $C_{s,b}(X, M_C(A))$. Finally, the fact that $C_0(X, A)$ is a faithful ideal of $C_{s,b}(X, M_C(A))$ is standard. The proof is now complete.

(b) $\Longrightarrow$ (a): Suppose that $(C_{s,b}(X, M_C(A)), f)$ is an order unitization for $C_0(X, A)$. Then, for all $t \in X$, we have

$$1\otimes f(t) \in C_{s,b}\big(X, M_C(A)\big)_+,$$

whence

$$\begin{aligned}1\otimes f(t) &\le \big\|1\otimes f(t)\big\|_f f = \big\|1\otimes f(t)\big\|_\infty^{M_C(A)} f \\ &= \big\|f(t)\big\|_{M_C(A)} f \le \|f\|_\infty^{M_C(A)} f \\ &= \|f\|_f f = f.\end{aligned}$$

Consequently, for every pair $t, s \in X$, we have

$$f(t) = \big(1\otimes f(t)\big)(s) \le f(s) = \big(1\otimes f(s)\big)(t) \le f(t),$$

which implies that the function f is constant on X. Let us denote this constant by u. Then, for all $m \in M_C(A)_{\mathrm{sa}}$, we have

$$\begin{aligned}\|m\|_{M_C(A)} &= \|1\otimes m\|_\infty^{M_C(A)} = \|1\otimes m\|_f = \inf\{l > 0 : -lf \le 1\otimes m \le lf\} \\ &= \inf\{l > 0 : -lu \le m \le lu\} \\ &= \|m\|_u\end{aligned}$$

so that $(M_C(A), u)$ is an order-unit C^*-Segal algebra. Moreover, since for each $a \in A$ and for all $\phi \in C_0(X)$ with $\|\phi\|_\infty = 1$ we have

$$\|a\|_A = \|\phi\otimes a\|_\infty^A \sim \|\phi\otimes a\|_\infty^{M_C(A)} = \|a\|_{M_C(A)},$$

it follows that A is closed in $M_C(A)$. Finally, the fact that A is a faithful ideal of $M_C(A)$ is standard. The proof is now complete. □

3. Gelfand representation of $C_0(X, A)$

In this section, we consider functional representation of commutative C^*-Segal algebra-valued function algebras. Our approach is the one introduced in [4, Section 4]. For the remainder of this article, we assume that the Banach algebra A is commutative.

Notation. For the Banach algebra A, we denote by $\Delta(A)$ its *Gelfand space*. For $a \in A$, we denote by $\widehat{a}$ its *Gelfand transform*. By the *Gelfand transformation* on A, we mean the mapping $a \mapsto \widehat{a}$ from A onto the *Gelfand transform algebra* of A denoted by $\widehat{A}$. Recall that $\Delta(A)$ is a locally compact Hausdorff space and that $\widehat{A}$ is a subalgebra of $C_0(\Delta(A))$.

Let us now consider the Banach algebra $C_0(X, A)$. Here and throughout, we identify the Gelfand space of $C_0(X, A)$ with the Cartesian product of X and $\Delta(A)$ via the inverse of the canonical homeomorphism

$$X \times \Delta(A) \ni (t, \tau) \mapsto \delta_{t,\tau} \in \Delta\big(C_0(X, A)\big),$$

where $\delta_{t,\tau}(f) := \tau(f(t))$ for all $f \in C_0(X, A)$ (see, e.g., [10, Theorem 4]). Following [9], we define a mapping $\widehat{v}\colon X \times \Delta(A) \to \mathbb{R}$ by setting

$$\widehat{v}(t, \tau) := \frac{1}{\|\delta_{t,\tau}\|} \quad \big((t, \tau) \in X \times \Delta(A)\big),$$

where $\|\delta_{t,\tau}\|$ denotes the dual norm of $\delta_{t,\tau}$. The next lemma collects some elementary properties of this mapping.

Lemma 3.1. *The mapping $\widehat{v}$ is upper semicontinuous on $X \times \Delta(A)$ with*

$$\widehat{v}(t, \tau) = \frac{1}{\|\tau\|} \geq 1$$

for all $(t, \tau) \in X \times \Delta(A)$.

Proof. Let $(t, \tau) \in X \times \Delta(A)$. Since

$$\big\{f(t) : f \in C_0(X, A), \|f\|_\infty^A \leq 1\big\} = \big\{a \in A : \|a\|_A \leq 1\big\},$$

we obtain

$$\|\delta_{t,\tau}\| = \sup_{\|f\|_\infty^A \leq 1} \big|\delta_{t,\tau}(f)\big| = \sup_{\|f\|_\infty^A \leq 1} \big|\tau\big(f(t)\big)\big| = \sup_{\|a\|_A \leq 1} \big|\tau(a)\big| = \|\tau\|,$$

and hence $\widehat{v}(t, \tau) = 1/\|\tau\|$ as desired. The other assertions follow from [4, Lemma 4.2]. □

With this result in hand, we can represent the Banach algebra $C_0(X, A)$ by means of the Nachbin algebras introduced in Example 1.15. Clearly, for every pair $(t, \tau) \in X \times \Delta(A)$, we have

$$\widehat{v}(t, \tau) = \sup\big\{l > 0 : l\big|\widehat{f}(t, \tau)\big| \leq \|f\|_\infty^A \text{ for all } f \in C_0(X, A)\big\}.$$

Therefore, besides being a subalgebra of $C_0(X \times \Delta(A))$, $C_0(X, A)\widehat{}$ is also a subalgebra of $C_b^{\widehat{v}}(X \times \Delta(A))$. Moreover, since each $f \in C_0(X, A)$ satisfies

$$\|\widehat{f}\|_\infty \leq \|\widehat{f}\|_{\widehat{v}} \leq \|f\|_\infty^A,$$

the Gelfand transformation is a contractive algebra homomorphism from $C_0(X,A)$ into $C_b^{\widehat{v}}(X \times \Delta(A))$. Our interest here is in the isometric case.

Definition 3.2. The Banach algebra A is called a *weighted uniform algebra* if

$$\|a\|_A = \sup_{\tau \in \Delta(A)} \frac{1}{\|\tau\|} \left|\widehat{a}(\tau)\right|$$

for all $a \in A$.

The importance of weighted uniform algebras in the study of commutative C^*-Segal algebras with an order unitization is explained by the next theorem.

Theorem 3.3. *Let A be a C^*-Segal algebra. Then A has an order unitization if and only if A is a weighted uniform algebra with $\tau \mapsto \|\tau\|$ continuous on $\Delta(A)$.*

Proof. This is immediate from [4, Lemma 5.4, Theorems 5.7 and 5.24]. □

The next result together with Lemma 3.1 and Theorem 3.3 yields a simple proof for commutative C^*-Segal algebras of the equivalence obtained in Theorem 1.21.

Proposition 3.4. *The following conditions are equivalent for the Banach algebra A:*

(a) *A is a weighted uniform algebra,*
(b) *$C_0(X,A)$ is a weighted uniform algebra.*

Proof. Suppose that A is a weighted uniform algebra. Then, for all $f \in C_0(X,A)$, we have

$$\begin{aligned}\|\widehat{f}\|_{\widehat{v}} &= \sup_{(t,\tau)\in X\times\Delta(A)} \widehat{v}(t,\tau)\left|\widehat{f}(t,\tau)\right| = \sup_{(t,\tau)\in X\times\Delta(A)} \frac{1}{\|\delta_{t,\tau}\|}\left|\widehat{f}(\delta_{t,\tau})\right| \\ &= \sup_{(t,\tau)\in X\times\Delta(A)} \frac{1}{\|\tau\|}\left|\widehat{f(t)}(\tau)\right| = \sup_{t\in X}\left\|f(t)\right\|_A = \|f\|_\infty^A,\end{aligned}$$

whence $C_0(X,A)$ is a weighted uniform algebra. The converse implication is established similarly by considering the supremum norm of $\phi \otimes a$ with $\phi \in C_0(X)$ and $a \in A$. □

The following theorem is the vector-valued counterpart of the main result of [4].

Theorem 3.5. *Let A be a C^*-Segal algebra with an order unitization. Then, up to an isometric algebra and order isomorphism, we have*

$$E_{C_0(X,A)} = C_0^{\widehat{v}}\big(X \times \Delta(A)\big) \qquad \text{and} \qquad M_{C_0(X,C)}\big(C_0(X,A)\big) = C_b^{\widehat{v}}\big(X \times \Delta(A)\big).$$

Moreover, $C_0(X,A)$ is, isometrically, algebra- and order-isomorphic to a closed subalgebra of $C_b^{\widehat{v}}(X \times \Delta(A))$.

Proof. This is a direct consequence of Theorem 3.3 and Proposition 3.4, together with [4, Corollaries 4.18, 4.24]. □

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