Banach J. Math. Anal. 11 (2017), no. 1, 188–206
http://dx.doi.org/10.1215/17358787-3784616
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

ℓ_p-MAXIMAL REGULARITY FOR A CLASS OF FRACTIONAL DIFFERENCE EQUATIONS ON UMD SPACES: THE CASE $1 < \alpha \leq 2$

CARLOS LIZAMA[1*] and MARINA MURILLO-ARCILA[2]

Communicated by C. Badea

Abstract. By using Blunck's operator-valued Fourier multiplier theorem, we completely characterize the existence and uniqueness of solutions in Lebesgue sequence spaces for a discrete version of the Cauchy problem with fractional order $1 < \alpha \leq 2$. This characterization is given solely in spectral terms on the data of the problem, whenever the underlying Banach space belongs to the UMD-class.

1. Introduction

Our concern in this article is the ℓ_p-maximal regularity of solutions for the abstract nonhomogeneous Cauchy problem of fractional order

$$\begin{cases} \Delta^\alpha u(n) = Tu(n) + f(n), \quad n \in \mathbb{Z}_+, n \geq 2, 1 < \alpha \leq 2; \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < p < \infty$, T is a bounded linear operator defined on a Banach space X and $f : \mathbb{Z}_+ \to X$ is given. Here, the discrete fractional operator Δ^α corresponds

Received Oct. 12, 2015; Accepted Mar. 11, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A10; Secondary 47A50, 43A22, 35R20, 35R11, 34A33.
Keywords. maximal regularity, Lebesgue sequence spaces, UMD Banach spaces, R-boundedness, lattice models.

to sampling, by means of the Poisson distribution, of the Riemann–Liouville fractional derivative D_t^α of order α on $\mathbb{R}_+$

$$\Delta^\alpha u(n) = \int_0^\infty \frac{t^{n-2}}{(n-2)!} e^{-t} D_t^\alpha u(t)\, dt, \quad n \in \mathbb{Z}_+, n \geq 2.$$

(See [19, Theorem 3.5], where this remarkable connection between discrete and continuous fractional operators was recently proved.) Note that (1.1) is an abstract way to describe the modeling of classes of fractional integro-differential equations in discrete time. There are many situations where this type of mixed equations appear. In the unidimensional case, some of them are called *lattice models* in the literature, for instance, the discrete Nagumo equation (see [14, p. 250]). A different example is given by the nonconvolution equation

$$\Delta^\alpha u(n,x) = \int k(x,s) u(n,s)\, ds + f(n,x), \quad n \in \mathbb{N}_0, x \in \Omega \subset \mathbb{R}^N,$$

where the kernel k is a complex-valued measurable function and f is a suitable forcing term. It admits the form (1.1), where

$$Tf(x) = \int k(x,s) f(s)\, ds$$

is a bounded operator. In the nonfractional case, such equations arise in a variety of contexts. From a numerical point of view, our analysis refers to schemes that are discretized only in time. For instance, Strikwerda and Lee [23] discussed the accuracy of the fractional step projection method for the incompressible Navier–Stokes equations restricting the analysis to schemes that are discretized only in time. From another point of view, one-step time-discrete equations naturally appear in some fields of physics (see [9]) and in fracture mechanics and biology (see [20, Section 5]). Cardiac cells provide another suitable context (see [17]). A recent rich source of examples is provided by the master equation for an animal in behavioral ecology (see formulas (24) and (25) in [22]). Note that in such examples the operator T may also be unbounded.

On the other hand, the study of the existence and qualitative properties of discrete solutions for fractional difference equations started over the last number of years (see, e.g., [2], [6], [7]). However, most of these studies refer only to scalar situations, that is, when $X = \mathbb{R}$ or $\mathbb{C}$. The study of abstract models in general Banach spaces that includes the analysis of mixed partial differential equations and integral equations is a very recent and promising area of research (see [19], [18], [24]). We note that because of the nature of this area of investigation, the notion of fractional difference may vary.

Maximal regularity is an important tool in the investigation of the existence and uniqueness of solutions to evolution equations. For a recent review of this topic in the context of discrete models on Banach spaces, see the monograph [3] and references therein. Although research in this area has been done, there are many interesting questions related to the study of fractional difference equations that remain unanswered. In [18], the maximal regularity property on Lebesgue sequence spaces was recently studied for problem (1.1) when $0 < \alpha \leq 1$. However, this study was left open for any other values of α.

The main objective of this article is to provide a complete answer to this open problem. We have succeeded in solving it in the full range $1 < \alpha \leq 2$ by means of a *characterization* of maximal regularity for the solutions of the equation (1.1) in Lebesgue vector-valued spaces defined on the set $\mathbb{Z}_+$. In order to solve this problem, we will introduce a special sequence of bounded operators, called *α-resolvent families*, which will play a central role in the representation of the solution of the problem (1.1). Then, we use Blunck's operator-valued multiplier theorem (see [3, Section 2.4], [10, Theorem 1.3]) in order to obtain the desired characterization. One remarkable fact is that such a characterization is obtained solely in terms of the data of the problem. More precisely, for $1 < \alpha \leq 2$, $p > 1$ and X a UMD space, suppose that $\{z^{2-\alpha}(z-1)^{\alpha}\}_{\{|z|=1, z\neq 1\}} \subset \rho(T)$ holds, where $T \in B(X)$ and where $\rho(T)$ denotes the resolvent set of T. Then the following assertions are equivalent.

(i) Equation (1.1) has ℓ_p-maximal regularity.
(ii) The set

$$\left\{ z^{2-\alpha}(z-1)^{\alpha}\left(z^{2-\alpha}(z-1)^{\alpha} - T\right)^{-1} : |z| = 1, z \neq 1 \right\} \text{ is } R\text{-bounded.}$$

Compared with [18], this result is different. Here, the set $\Omega_2 := \{(z^{2-\alpha}(z-1)^{\alpha})\}_{|z|=1, z\neq 1}$ must lie in the resolvent set of T instead of the set $\Omega_1 := \{(z^{1-\alpha}(z-1)^{\alpha})\}_{|z|=1, z\neq 1}$ corresponding to the case $0 < \alpha \leq 1$. This last set has, in a certain sense, dual geometry compared with Ω_2. In other words, whereas the set Ω_1 lies mainly in the left-hand side of the complex plane for values of α near to 1, we have that the set Ω_2 lies mainly in the right-hand side of the complex plane for values of α near to 2 (see [18, Figures 1 and 2] and Figures 1 and 2 below). The transition between both geometries has a jump in the border case $\alpha = 1$, although they have a symmetry with respect to the imaginary axis. Concerning the method of proof of our main result (Theorem 4.2), we want to point out that we consider in the present article a different class of sequences of bounded operators than those considered in [18, Definition 3.1] (see Definition 3.1 below). This is due to the consideration of two initial values in equation (1.1) instead of only one. In the case $\alpha = 2$, Definition 3.1 can be compared with the notion of a discrete-time cosine function. In [18], the case $\alpha = 1$ corresponds to the concept of a discrete-time semigroup (powers of a bounded operator). In this way, the representation of the solution in the cases $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$ varies (compare [18, Theorem 3.7] with Theorem 3.8 below). This representation, in the case $1 < \alpha \leq 2$, was difficult to obtain and hence the results of the present article were not considered in our earlier one. Therefore, the present article can be considered as a companion work to the published paper [18].

We point out that characterizations of maximal regularity for evolution equations using methods of operator-valued Fourier multiplier theorems has already been studied (see, e.g., [3]). For instance, in [12] and [11], Bu used Fourier multipliers to characterize the Lebesgue maximal regularity of fractional evolution equations in compact intervals. The corresponding study in Hölder spaces was done by Ponce in [21].

Our article is organized as follows. In Section 2, we introduce some basic concepts related to the study of fractional differences that will be needed later. In Section 3, we introduce a special sequence of bounded operators that we call an *α-resolvent sequence*, denoted by $S_\alpha(n)$, which will play a very important role in the study of ℓ_p-maximal regularity. Then, we provide an explicit representation of the solution for the fractional difference equation (1.1) with initial values $u(0) = x$ and $u(1) = y$, namely,

$$u(n) = S_\alpha(n)u(0) + (S_\alpha * h_\alpha)(n-1)\big[u(1) - u(0)\big] + (S_\alpha * h_\alpha * f)(n-2), \quad n \geq 2$$

(see Theorem 3.8 below). Here h_α is defined by the sequence $h_\alpha(n) = (\alpha-1)^n$. It is interesting to observe that in case of $\alpha = 2$, the resolvent sequence $S_2(n)$ coincides with the notion of a discrete-time cosine function introduced by Chojnacki [13], who studied it also in the context of UMD-spaces. Finally, in Section 4, we show our main result: Theorem 4.2. There, we prove the above-mentioned characterization of ℓ_p-maximal regularity. A simple criterion in the special case of Hilbert space is also provided. This is given only in terms of a spectral property of a normal operator T. Namely, we show that if $T \in \mathcal{B}(H)$ is a normal operator defined on a Hilbert space H and

$$\sigma(T) \subset \{z \in \mathbb{C} : |z| > 2^\alpha\},$$

then the equation (1.1) has ℓ^p-maximal regularity. We finish this article with a concrete example on a nonconvolution integral equation arising in the study of numerical methods on polygonal domains, highlighting the role of the fractional parameter in the treatment of additive perturbations for the given equation.

2. Preliminaries

In this section, we recall some necessary concepts related to UMD spaces, R-boundedness, fractional differences, and operator-valued Fourier multipliers. (See also the recent monograph [3].)

From now on, given a a real number, we denote by $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$ and $s(\mathbb{N}_a; X)$ the vector space consisting of all vector-valued sequences $f : \mathbb{N}_a \to X$. We recall that the forward Euler operator $\Delta_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is defined by

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

For each $m \in \mathbb{N}_2$, we define recursively the mth order forward difference operator $\Delta_a^m : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a.$$

In particular, we have $(\Delta_0^1 f)(n) = f(n+1) - f(n), n \in \mathbb{N}_0$. The following definition of fractional sum was formally introduced in [19], after previous work of Abdeljawad and Atici [2] and Atici and Eloe [6], [7].

Definition 2.1. Let $\alpha > 0$ be given, and let $f : \mathbb{N}_0 \to X$. We define the fractional sum of order α as

$$\Delta^{-\alpha} f(n) = \sum_{k=0}^{n} k^{\alpha}(n-k) f(k), \quad n \in \mathbb{N}_0, \tag{2.1}$$

where

$$k^{\alpha}(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0.$$

Concerning the development of discrete fractional calculus, we observe that Holm in [16] is among the first authors who employed the technique of Laplace transform for discrete fractional calculus in the arena of fractional difference equations. In [15] Goodrich studied the existence of positive solutions and geometrical properties. Applications of discrete fractional calculus for several biological and physical problems have been studied in [8].

Now, we recall from [19] the discrete analogous concept of the definition of a fractional derivative in the sense of Riemann–Liouville (see also [6]). In that paper, we show that concept's strong connection, by means of the Poisson distribution, with the Riemann–Liouville fractional derivative on $\mathbb{R}_+$. We refer the reader also to the recent papers [1] and [18], where that concept's usefulness is shown in different contexts of research.

Definition 2.2. The fractional difference operator of order $\alpha > 0$ (in the sense of Riemann–Liouville) is defined by

$$\Delta^{\alpha} f(n) := \Delta_0^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

In other words, to a given vector-valued sequence, first fractional summation and then integer difference are applied. We also recall the concept of finite convolution $*$ of two sequences $f(n)$ and $g(n)$:

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j) g(j), \quad n \in \mathbb{N}_0.$$

The discrete-time Fourier transform (DTFT) of a vector-valued sequence $f \in s(\mathbb{Z}; X)$ is given by

$$\widehat{f}(z) := \sum_{j=-\infty}^{\infty} z^{-j} f(j), \quad \text{where } z = e^{it}, t \in (-\pi, \pi),$$

whenever it exists. We now recall the definition of the UMD class of Banach spaces. (For more details, see [4, Sections III.4.3–III.4.5].)

Definition 2.3. A Banach space X is said to have the *unconditional martingale difference* (UMD) property if for each $p \in (1, \infty)$ there exists a constant $C_p >$

0 such that, for any martingale $(f_n)_{n\geq 0} \subset L^p(\Omega,\Sigma,\mu;X)$, any choice of signs $(\xi_n)_{n\geq 0} \subset \{-1,1\}$, and any $N \in \mathbb{Z}_+$, the following estimate holds:

$$\Big\| f_0 + \sum_{n=1}^{N} \xi_n (f_n - f_{n-1}) \Big\|_{L^p(\Omega,\Sigma,\mu;X)} \leq C_p \|f_N\|_{L^p(\Omega,\Sigma,\mu;X)}.$$

To end this section, we recall the Fourier multiplier theorem for operator-valued symbols that provides necessary and sufficient conditions for the R-boundedness property due to Blunck in [10]. We will first need the notion of an R-bounded set.

Definition 2.4. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called *R-bounded* if there is a constant $c \geq 0$ such that

$$\big\|(T_1x_1,\ldots,T_nx_n)\big\|_R \leq c\big\|(x_1,\ldots,x_n)\big\|_R, \tag{2.2}$$

for all $T_1,\ldots,T_n \in \mathcal{T}$, $x_1,\ldots,x_n \in X$, $n \in \mathbb{N}$, where

$$\big\|(x_1,\ldots,x_n)\big\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \Big\| \sum_{j=1}^{n} \epsilon_j x_j \Big\|, \quad x_1,\ldots,x_n \in X.$$

Let now $\mathbb{T} := (-\pi,\pi) \setminus \{0\}$.

Theorem 2.5 ([10, Theorem 1.3]). *Let $p \in (1,\infty)$, and let X be a UMD space. Let $M : \mathbb{T} \to \mathcal{B}(X)$ be differentiable and such that the set*

$$\big\{M(t), (z-1)(z+1)M'(t) : z = e^{it}, t \in \mathbb{T}\big\}$$

is R-bounded. Then there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z};X))$ such that

$$\widehat{(T_M f)}(z) = M(t)\widehat{f}(z), \quad \textit{for all } z = e^{it}, t \in \mathbb{T}. \tag{2.3}$$

The converse of Blunck's theorem also holds without any restriction on the Banach space X, as follows.

Theorem 2.6 ([10, Proposition 1.4]). *Let $p \in (1,\infty)$, and let X be a Banach space. Let $M : \mathbb{T} \to \mathcal{B}(X)$ be an operator-valued function. Suppose that there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z};X))$ such that the identity (2.3) holds. Then the set*

$$\big\{M(t) : t \in \mathbb{T}\big\}$$

is R-bounded.

3. Resolvent sequences: $1 < \alpha \leq 2$

Let $T \in \mathcal{B}(X)$ be given. In this section, we introduce an operator-theoretical method to study the linear fractional difference equation

$$\Delta^\alpha u(n) = Tu(n) + f(n), \quad n \in \mathbb{N}, \tag{3.1}$$

with initial conditions $u(0) = x$, $u(1) = y \in X$ and $1 < \alpha \leq 2$. We observe that the case $0 < \alpha \leq 1$ was previously studied in [18]. Therefore, our analyses in this article complement the results given in [18] and provide new insights in the case $1 < \alpha \leq 2$.

Definition 3.1. Let T be a bounded operator defined on a Banach space X, and let $\alpha > 1$. We call T the *generator of an α-resolvent sequence* if there exists a sequence of bounded and linear operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset B(X)$ such that the following properties are satisfied:

(i) $S_\alpha(0) = I$,
(ii) $S_\alpha(1) = I$,
(iii) $S_\alpha(n+2)-S_\alpha(n+1) = T(S_\alpha * k^{\alpha-1})(n)+k^{\alpha-1}(n+2)I-(\alpha-1)k^{\alpha-1}(n+1)I$, $n \in \mathbb{N}_0$.

In this case, $S_\alpha(n)$ is called the *α-resolvent sequence generated by T*.

Remark 3.2. Observe that (iii) can be rewritten as

$$\text{(iii)}' \qquad \begin{aligned}\Delta S_\alpha(n+1) &= T(S_\alpha * k^{\alpha-1})(n) + \Delta k^{\alpha-1}(n+1)\\ &\quad + (2-\alpha)k^{\alpha-1}(n+1), \quad n \in \mathbb{N}_0,\end{aligned}$$

and therefore this property is comparable with [18, Definition 3.1] except for the extra term $(2-\alpha)k^{\alpha-1}(n+1)$.

The following lemma follows easily from the definition.

Lemma 3.3. *If T generates an α-resolvent sequence, then it is unique.*

Proof. Let $S_\alpha(n)$ and $Q_\alpha(n)$ be two α-resolvent sequences generated by T. Let us define $P_\alpha(n) = S_\alpha(n) - Q_\alpha(n)$. Then $P_\alpha(0) = 0$, $P_\alpha(1) = 0$ and $P_\alpha(n+2) - P_\alpha(n+1) = T\sum_{j=0}^{n} k^{\alpha-1}(n-j)P_\alpha(j)$, for all $n \in \mathbb{N}_0$, which implies that $P_\alpha(n) = P_\alpha(1) = 0$ for all $n \in \mathbb{N}_0$. $\square$

Example 3.4. In the case $\alpha = 2$, we have

$$\begin{aligned} k^1(j) &= \frac{\Gamma(1+j)}{\Gamma(1)\Gamma(j+1)} = 1, \quad j \in \mathbb{N}_0.\\ S_2(n+2) - S_2(n+1) &= T(S_2 * k^1)(n) + k^1(n+2)I - k^1(n+1)I\\ &= T\sum_{j=0}^{n} S_2(j), \quad n \in \mathbb{N}_0.\end{aligned}$$

Since $S_2(0) = I$ and $S_2(1) = I$, we get

$$S_2(n) = \sum_{k=0}^{[n/2]} \binom{n}{2k} T^k.$$

Remark 3.5. Let $1 < \alpha \le 2$ be given. Suppose that for all $z \in \mathbb{C}$ with $|z| = 1$, we have $z^{2-\alpha}(z-1)^\alpha \in \rho(T)$, the resolvent set of T. Then, the following holds:

$$\hat{S}_\alpha(z) = z\big(z-(\alpha-1)\big)\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1}.$$

In particular, in the case $\alpha = 2$ we have $\hat{S}_2(z) = z(z-1)((z-1)^2 - T)^{-1}$, and therefore we obtain from [3, Proposition 1.4.2] that

$$S_2(n) = \mathcal{C}(n), \quad n \in \mathbb{N}_0,$$

where $\mathcal{C}$ is the discrete-time cosine operator sequence generated by T. From [3, Corollary 1.4.6], it follows that $\mathcal{C}$ satisfies

$$\mathcal{C}(n+m)+\mathcal{C}(n-m)=2\mathcal{C}(n)\mathcal{C}(m), \quad n,m\in\mathbb{Z}.$$

We observe that the notion of a cosine sequence of operators was first introduced by Chojnacki [13].

Now, we present the following useful lemma.

Lemma 3.6. *Let $1<\alpha\le 2$, $a:\mathbb{N}_0\to\mathbb{C}$, and $S:\mathbb{N}_0\to X$ be given. Then*

$$\begin{aligned}\Delta^\alpha(a*S)(n)=&\sum_{j=0}^{n}\Delta^\alpha S(n-j)a(j)+S(0)a(n+2)\\ &-\alpha S(0)a(n+1)+S(1)a(n+1).\end{aligned}\tag{3.2}$$

Proof. By definition, we have

$$\begin{aligned}\Delta^\alpha(a*S)(n)&=\Delta^2(\Delta^{-(2-\alpha)}a*S)(n)\\ &=\Delta^{-(2-\alpha)}(a*S)(n+2)-2\Delta^{-(2-\alpha)}(a*S)(n+1)\\ &\quad+\Delta^{-(2-\alpha)}(a*S)(n)\\ &=(k^{2-\alpha}*a*S)(n+2)-2(k^{2-\alpha}*a*S)(n+1)\\ &\quad+(k^{2-\alpha}*a*S)(n)\\ &=\sum_{j=0}^{n+2}(k^{2-\alpha}*S)(n+2-j)a(j)-2\sum_{j=0}^{n+1}(k^{2-\alpha}*S)(n+1-j)a(j)\\ &\quad+\sum_{j=0}^{n}(k^{2-\alpha}*S)(n-j)a(j).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^\alpha(a*S)(n)=&\sum_{j=0}^{n}\big[(k^{2-\alpha}*S)(n+2-j)-2(k^{2-\alpha}*S)(n+1-j)\\ &+(k^{2-\alpha}*S)(n-j)\big]a(j)+(k^{2-\alpha}*S)(1)a(n+1)\\ &+(k^{2-\alpha}*S)(0)a(n+2)-2(k^{2-\alpha}*S)(0)a(n+1)\\ =&\sum_{j=0}^{n}\Delta^2(k^{2-\alpha}*S)(n-j)a(j)+(k^{2-\alpha}*S)(1)a(n+1)\\ &+k^{2-\alpha}(0)S(0)a(n+2)-2k^{2-\alpha}(0)S(0)a(n+1).\end{aligned}$$

Hence,

$$\begin{aligned}\Delta^\alpha(a*S)(n)=&\sum_{j=0}^{n}\Delta^\alpha S(n-j)a(j)+k^{2-\alpha}(1)S(0)a(n+1)\\ &+k^{2-\alpha}(0)S(1)a(n+1)+S(0)a(n+2)-2S(0)a(n+1)\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{n} \Delta^{\alpha} S(n-j)a(j) + (2-\alpha)S(0)a(n+1) + S(1)a(n+1) \\
&\quad + S(0)a(n+2) - 2S(0)a(n+1) \\
&= \sum_{j=0}^{n} \Delta^{\alpha} S(n-j)a(j) + S(0)a(n+2) \\
&\quad - \alpha S(0)a(n+1) + S(1)a(n+1),
\end{aligned}$$

proving the lemma. □

Remark 3.7. In the case $S = S_\alpha$, Lemma 3.6 states that

$$\Delta^{\alpha}(a * S_\alpha)(n) = \sum_{j=0}^{n} \Delta^{\alpha} S_\alpha(n-j)a(j) + a(n+2) - (\alpha-1)a(n+1) \tag{3.3}$$

because $S_\alpha(0) = I$ and $S_\alpha(1) = I$ by definition.

For each $1 < \alpha \leq 2$, we define

$$h_\alpha(n) = \begin{cases} (\alpha-1)^n & n \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

The following theorem is the main result of this section.

Theorem 3.8. *Let $1 < \alpha \leq 2$ and $f : \mathbb{N} \to X$ be given. The unique solution of (3.1) with initial conditions $u(0) = x, u(1) = y$ is given by:*

$$\begin{aligned}
u(n) &= S_\alpha(n)u(0) + (S_\alpha * h_\alpha)(n-1)\big[u(1) - u(0)\big] \\
&\quad + (S_\alpha * h_\alpha * f)(n-2), \quad n \geq 2.
\end{aligned} \tag{3.5}$$

Proof. Applying the operator Δ^α to (3.5), we obtain

$$\begin{aligned}
\Delta^{\alpha} u(n) &= \Delta^{\alpha} S_\alpha(n)u(0) + \Delta^{\alpha}(S_\alpha * h_\alpha)(n-1)\big[u(1) - u(0)\big] \\
&\quad + \Delta^{\alpha}(S_\alpha * h_\alpha * f)(n-2).
\end{aligned} \tag{3.6}$$

By Definition 3.1, we get

$$\begin{aligned}
\Delta^{\alpha} S_\alpha(n) &= \Delta^{\alpha} S_\alpha(n-1) + T\Delta^{\alpha}(S_\alpha * k^{\alpha-1})(n-2) + \Delta^{\alpha} k^{\alpha-1}(n)I \\
&\quad - (\alpha-1)\Delta^{\alpha} k^{\alpha-1}(n-1)I.
\end{aligned} \tag{3.7}$$

Note that $\Delta^{\alpha} k^{\alpha-1}(n) = \Delta^2 \Delta^{2-\alpha} k^{\alpha-1}(n) = \Delta^2 k^1(n) = 0$ for all $n \in \mathbb{N}$. Then

$$\Delta^{\alpha} S_\alpha(n) = \Delta^{\alpha} S_\alpha(n-1) + T\Delta^{\alpha}(S_\alpha * k^{\alpha-1})(n-2).$$

Using Lemma 3.6, we get

$$\begin{aligned}
\Delta^{\alpha}(S_\alpha * k^{\alpha-1})(n) &= (\Delta^{\alpha} k^{\alpha-1} * S_\alpha)(n) + S_\alpha(n+2) - \alpha S_\alpha(n+1) \\
&\quad + (\alpha-1)S_\alpha(n+1) \\
&= S_\alpha(n+2) - S_\alpha(n+1) \\
&= \Delta S_\alpha(n+1).
\end{aligned}$$

Replacing the above identity in equation (3.7), we have

$$\Delta^\alpha S_\alpha(n) = \Delta^\alpha S_\alpha(n-1) + \Delta T S_\alpha(n-1), \tag{3.8}$$

or equivalently

$$\Delta\Delta^\alpha S_\alpha(n-1) = \Delta T S_\alpha(n-1).$$

We claim that $\Delta^\alpha S_\alpha(n-1) = TS_\alpha(n-1)$. Indeed, we observe that this happens if and only if $\Delta^\alpha S_\alpha(0) = TS_\alpha(0) = T$. Now, we will prove this last assertion.

By Definitions 2.1 and 2.2, we have

$$\begin{aligned}\Delta S_\alpha(n) &= \Delta^2(k^{2-\alpha} * S_\alpha)(n)\\ &= (k^{2-\alpha} * S_\alpha)(n+2) - 2(k^{2-\alpha} * S_\alpha)(n+1) + (k^{2-\alpha} * S_\alpha)(n).\end{aligned}$$

For $n = 0$:

$$\begin{aligned}\Delta S_\alpha(0) &= \Delta^2(k^{2-\alpha} * S_\alpha)(0)\\ &= (k^{2-\alpha} * S_\alpha)(2) - 2(k^{2-\alpha} * S_\alpha)(1) + (k^{2-\alpha} * S_\alpha)(0).\end{aligned} \tag{3.9}$$

Note that

$$\begin{aligned}(k^{2-\alpha} * S_\alpha)(2) &= k^{2-\alpha}(0)S_\alpha(2) + k^{2-\alpha}(1)S_\alpha(1) + k^{2-\alpha}(2)S_\alpha(0)\\ &= S_\alpha(2) + (2-\alpha)I + \frac{(3-\alpha)(2-\alpha)}{2}I\\ &= I + T + \frac{\alpha(\alpha-1)}{2}I - (\alpha-1)^2 I + (2-\alpha)I + \frac{(3-\alpha)(2-\alpha)}{2}I\\ &= T + (5-2\alpha)I,\end{aligned} \tag{3.10}$$

as well as

$$(k^{2-\alpha} * S_\alpha)(1) = (2-\alpha)I + I = (3-\alpha)I \qquad \text{and} \qquad (k^{2-\alpha} * S_\alpha)(0) = I. \tag{3.11}$$

Replacing (3.10) and (3.11) in (3.9), we get

$$\Delta^\alpha S_\alpha(0) = T + (5-2\alpha)I - 2(3-\alpha)I + I = T.$$

So, the claim is proved and we have

$$\Delta^\alpha S_\alpha(n) = TS_\alpha(n), \tag{3.12}$$

for all $n \in \mathbb{N}_0$. By Lemma 3.6,

$$\begin{aligned}\Delta^\alpha(S_\alpha * h_\alpha)(n) &= (\Delta^\alpha S_\alpha * h_\alpha)(n) + h_\alpha(n+2) - (\alpha-1)h_\alpha(n+1)\\ &= (\Delta^\alpha S_\alpha * h_\alpha)(n)\\ &= T(S_\alpha * h_\alpha)(n).\end{aligned} \tag{3.13}$$

Moreover, again using Lemma 3.6,

$$\begin{aligned}\Delta^\alpha(S_\alpha * h_\alpha * f)(n) &= \big(\Delta^\alpha(S_\alpha * h_\alpha) * f\big)(n) + (S_\alpha * h_\alpha)(0)f(n+2)\\ &\quad - \alpha(S_\alpha * h_\alpha)(0)f(n+1) + (S_\alpha * h_\alpha)(1)f(n+1)\\ &= (\Delta^\alpha S_\alpha * h_\alpha * f)(n) + f(n+2) - \alpha f(n+1) + \alpha f(n+1)\\ &= (\Delta^\alpha S_\alpha * h_\alpha * f)(n) + f(n+2)\\ &= T(S_\alpha * h_\alpha * f)(n) + f(n+2).\end{aligned} \tag{3.14}$$

Replacing equations (3.12), (3.13), and (3.14) in (3.6), we finally obtain

$$\begin{aligned}\Delta^\alpha u(n) &= T\big[S_\alpha(n)u(0) + (\Delta^\alpha S_\alpha * h_\alpha)(n-1)\big[u(1)-u(0)\big]\\ &\quad + (S_\alpha * h_\alpha * f)(n-2)\big] + f(n)\\ &= Tu(n) + f(n),\end{aligned}$$

for all $n \in \mathbb{N}_0$, proving the theorem. □

In the border case $\alpha = 2$, we have $h_2(j) = 1$ for all $j \in \mathbb{N}_0$, and hence we recover the following result proved in [3, Proposition 1.3.1] by a different method.

Corollary 3.9. *Let $T \in \mathcal{B}(X)$ be given. Then the unique solution of the equation*

$$\begin{cases}\Delta^2 u(n) = Tu(n) + f(n), & n \in \mathbb{Z}_+,\\ u(0) = x, \\ u(1) = y,\end{cases} \tag{3.15}$$

is given by

$$u(n) = S_2(n)x + (S_2 * h_2)(n-1)(y-x) + (S_2 * h_2 * f)(n-2), \quad n \geq 2,$$

where $S_2(n)$ coincides with the discrete-time cosine operator function and

$$(S_2 * h_2)(n) = \sum_{j=0}^{n} S_2(j)$$

coincides with the discrete-time sine operator function. (See [3].)

4. A characterization of maximal ℓ_p-regularity

Let $T \in \mathcal{B}(X)$ be given, and let $f : \mathbb{Z}_+ \to X$ be a vector-valued sequence. In this section, we consider the discrete-time evolution equation of fractional order

$$\begin{cases}\Delta^\alpha u(n) = Tu(n) + f(n), & n \in \mathbb{N},\\ u(0) = 0, \\ u(1) = 0,\end{cases} \tag{4.1}$$

where $1 < \alpha \leq 2$. By Theorem 3.8, the solution of equation (4.1) can be represented by

$$u(n) = (S_\alpha * h_\alpha * f)(n-2), \quad n \in \mathbb{N}, n \geq 2.$$

Note that

$$\Delta^\alpha u(n) = T(S_\alpha * h_\alpha * f)(n-2) + f(n). \tag{4.2}$$

The following definition is motivated by the case $\alpha = 2$, which, in turn, comes from [10] following the continuous case.

Definition 4.1. We say that equation (4.1) has *maximal ℓ_p-regularity* if

$$(\mathcal{K}_\alpha f)(n) = T\sum_{j=0}^{n} (S_\alpha * h_\alpha)(n-j)f(j)$$

defines a bounded operator $K_\alpha \in \mathcal{B}(\ell_p(\mathbb{Z}_+; X))$ for some $p \in (1, \infty)$.

In other words, and in view of the relation (4.2), the question is if $f \in \ell_p(\mathbb{N}_0, X)$ implies $u, \Delta^\alpha u \in \ell_p(\mathbb{N}_0, X)$.

We will consider the following hypothesis that will be denoted by $(H)_\alpha$:

$(H)_\alpha$ The operator $(z^{2-\alpha}(z-1)^\alpha - T)$ is invertible for all $|z| = 1$, $z \neq 1$.

Denote $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, and define the set

$$\Omega_\alpha := \left\{ z \in \mathbb{C} : z = (w-1)^\alpha w^{2-\alpha}, w \in \partial\mathbb{D}, w \neq 1 \right\}.$$

Some cases are illustrated in Figures 1 and 2.

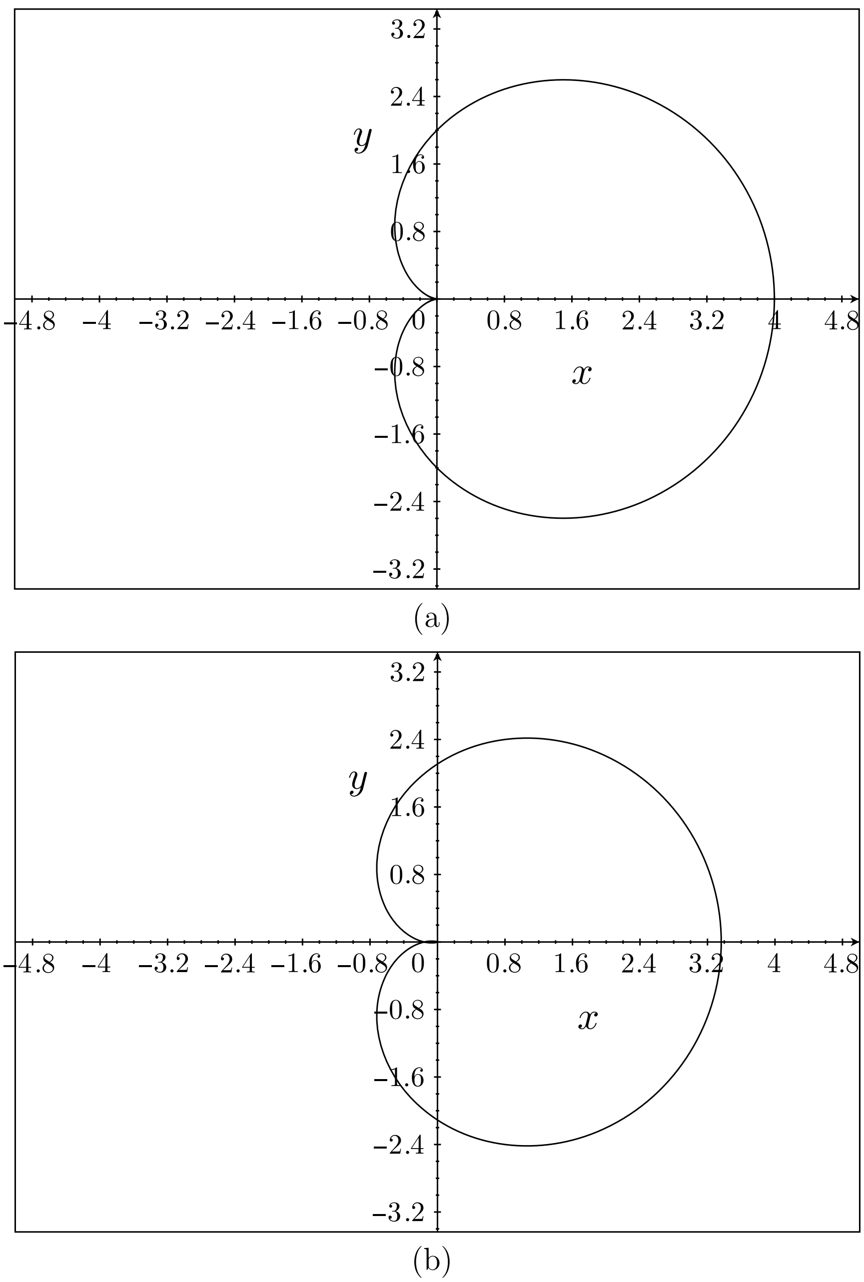
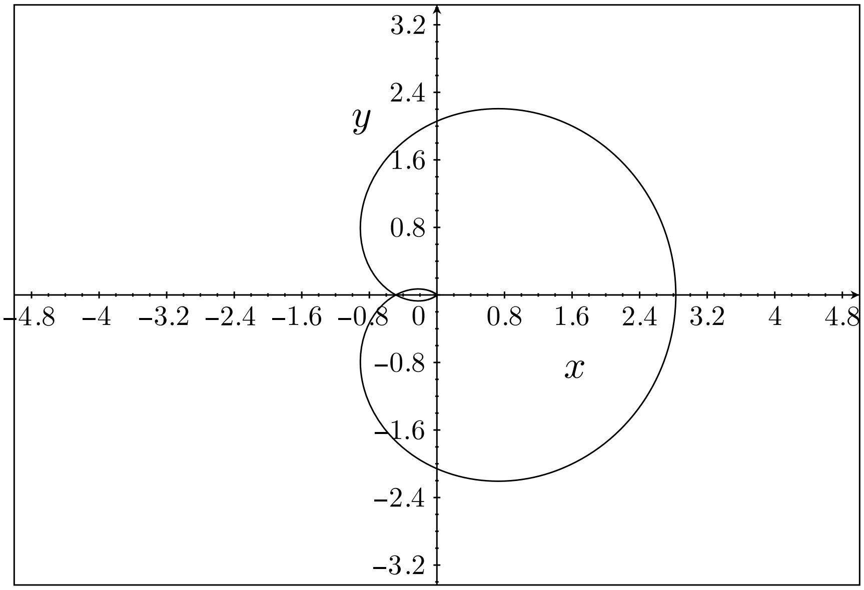
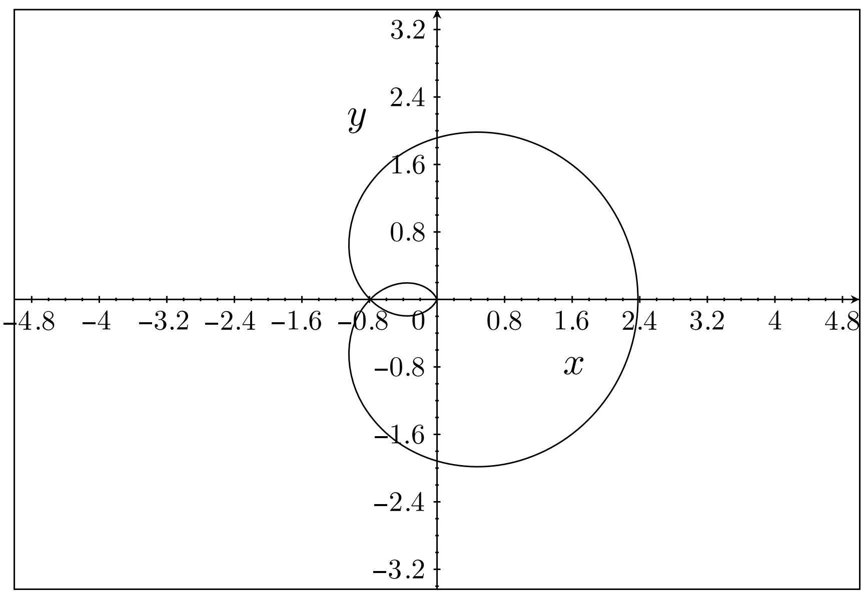


FIGURE 1. (a): $\alpha = 2$; (b): $\alpha = 1.75$.



(a)



(b)

FIGURE 2. (a): $\alpha = 1.5$; (b): $\alpha = 1.25$.

We now present our main theorem.

Theorem 4.2. *Let $1 < \alpha \leq 2$, $p > 1$, and let X be a UMD space. Let $T \in B(X)$ be given, and let us suppose that $(H)_\alpha$ holds. Then the following assertions are equivalent.*

(i) *Equation (4.1) has maximal ℓ_p-regularity.*

(ii) *The following set*

$$\left\{z^{2-\alpha}(z-1)^\alpha\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1} : |z| = 1, z \neq 1\right\},$$

is R-bounded.

Proof. By hypothesis $(H)_\alpha$, we can define $M(t) := z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$ for all $z = e^{it}, t \in (-\pi, \pi)$. Also, we define $f_\alpha(t) := e^{2it}(1-e^{-it})^\alpha$. Then $M(t) = f_\alpha(t)(f_\alpha(t) - T)^{-1}$.

Suppose (ii). Observe that $M'(t) = \frac{f'_\alpha(t)}{f_\alpha(t)}M(t) - \frac{f'_\alpha(t)}{f_\alpha(t)}M(t)^2$. Moreover, we have

$$\begin{aligned} f'_\alpha(t) &= 2if_\alpha(t) + \alpha i(1-e^{it})^{\alpha-1}e^{2it}e^{-it} \\ &= 2if_\alpha(t) + \frac{\alpha i f_\alpha(t)}{e^{it}-1} = 2if_\alpha(t) + \frac{i\alpha}{z-1}f_\alpha(t). \end{aligned}$$

Therefore,

$$\begin{aligned} (z-1)(z+1)M'(t) = \big[&2i(z-1)(z+1) + i\alpha(z+1)\big]M(t) \\ &- \big[2i(z-1)(z+1) + i\alpha(z+1)\big]M(t)^2, \end{aligned}$$

where $a_\alpha(t) := 2i(z-1)(z+1) + i\alpha(z+1)$ is bounded for $z = e^{it}, t \in (-\pi, \pi)$. We conclude from [3, Proposition 2.2.5] that the set $\{(z-1)(z+1)M'(t) : z = e^{it}, t \in (-\pi, \pi)\}$ is R-bounded. Then, by Theorem 2.5, there exists an operator $T_\alpha \in \mathcal{B}(\ell_p(\mathbb{Z}, X))$ such that

$$\widehat{(T_\alpha f)}(z) = M(t)\hat{f}(z), \quad \text{for all } z = e^{it}, t \in (-\pi, \pi) \text{ and } f \in \ell_p(\mathbb{Z}, X). \tag{4.3}$$

From the identity

$$T\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1} = z^{2-\alpha}(z-1)^\alpha\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1} - I \tag{4.4}$$

and from (4.3), we have that the left-hand side of the following identity

$$\begin{aligned} T\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1}\hat{f}(z) &= z^{2-\alpha}(z-1)^\alpha\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1}\hat{f}(z) - \hat{f}(z) \\ &= M(t)\hat{f}(e^{it}) - \hat{f}(e^{it}) \end{aligned} \tag{4.5}$$

defines a bounded operator on $\ell_p(\mathbb{Z}, X)$ given by $R_\alpha f(n) := T_\alpha f(n) - f(n), n \in \mathbb{Z}$. For $f \in \ell_p(\mathbb{Z}, X)$, we define the operator:

$$K_\alpha f(n) = \begin{cases} T(S_\alpha * h_\alpha * f)(n), & n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

By $(H)_\alpha$, Remark 3.5, and definition of h_α we have that the Z-transform of $S_\alpha * h_\alpha(z)$ is $z(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$. Then, the identity (4.5) shows that the discrete-time Fourier transform of $K_\alpha f(n-1)$ coincides with the discrete-time Fourier transform of $R_\alpha f(n)$ for all $n \in \mathbb{N}$. Therefore, $K_\alpha f(n-1) = R_\alpha f(n)$ for all $n \in \mathbb{N}$ by uniqueness. It proves (i).

Now, we suppose that (i) holds. We define the operator

$$K_\alpha f(n) = \begin{cases} \mathcal{K}_\alpha f(n), & n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where by hypothesis $\mathcal{K}_\alpha f(n) = T(S_\alpha * h_\alpha * f)(n)$, $n \in \mathbb{Z}_+$ is given in Definition 4.1. Define $T_\alpha f(n) := K_\alpha f(n-1) + f(n)$, $n \in \mathbb{Z}$. Given $z = e^{it}$, $t \in (-\pi, \pi)$, we have

$$\begin{aligned}\widehat{T_\alpha f}(z) &= \sum_{j\in\mathbb{Z}} z^{-j} T_\alpha f(j) = \sum_{j=1}^{\infty} z^{-j}\mathcal{K}_\alpha f(j-1) + \sum_{j\in\mathbb{Z}} z^{-j} f(j)\\ &= z^{-1}\sum_{j=0}^{\infty} z^{-j}\mathcal{K}_\alpha f(j) + \hat{f}(z).\end{aligned}$$

By hypothesis $(H)_\alpha$, the Z-transform of $(S_\alpha * h_\alpha)(z)$ which is equal to $z(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$ exists for $|z| = 1$, and therefore

$$\begin{aligned}\widehat{T_\alpha f}(z) &= z^{-1} T(\widehat{S_\alpha * h_\alpha})(z)\hat{f}(z) + \hat{f}(z)\\ &= T\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1}\hat{f}(z) + \hat{f}(z)\\ &= z^{2-\alpha}(z-1)^\alpha\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1}\hat{f}(z) - \hat{f}(z) + \hat{f}(z)\\ &= M(t)\hat{f}(z), \qquad (4.6)\end{aligned}$$

where we have used the identity (4.4) and where the definition of $M(t)$ is given at the beginning of the proof. An application of Theorem 2.6 shows that (ii) holds, and the proof is complete. □

Remark 4.3. Under the hypothesis that equation (4.1) has l_p-maximal regularity, we deduce that the operator $(z^{2-\alpha}(z-1)^\alpha - T)$ in $(H)_\alpha$ is always surjective. Indeed, given $x \in X$, we define

$$f(n) = \begin{cases} x & n = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, by hypothesis, we obtain that there exists $u_x \in l_p(\mathbb{Z}, X)$ such that $(z^{2-\alpha}(z-1)^\alpha - T)\widehat{u}_x(z) = \widehat{f}(z) = x$, where $z = e^{it}, t \in (-\pi, \pi)$.

Remark 4.4. In the case that X is a Hilbert space, condition (ii) can be replaced by

$$\text{(ii)}' \qquad \sup_{|z|=1, z\neq 1} \left\| (z-1)^\alpha\big(z^{2-\alpha}(z-1)^\alpha - T\big)^{-1}\right\| < \infty.$$

In the case of Hilbert spaces, the hypothesis of R-boundedness can be replaced by boundedness. In this case, we obtain an interesting alternative condition on the operator T in order to have l_p-maximal regularity.

Theorem 4.5. *Let $T \in \mathcal{B}(H)$ be a normal operator defined on a Hilbert space H, and assume that*

$$\sigma(T) \subset \big\{z \in \mathbb{C} : |z| > 2^\alpha\big\}.$$

Then for each $f \in l_p(\mathbb{Z}_+, X), p > 1$, there is a unique $u \in l_p(\mathbb{Z}_+, X)$ such that

$$\begin{cases} \Delta^\alpha u(n) = Tu(n) + f(n), \quad n \in \mathbb{N},\\ u(0) = 0,\\ u(1) = 0,\end{cases} \qquad (4.7)$$

for any $1 < \alpha \leq 2$.

Proof. We define $f_\alpha(z) = z^{2-\alpha}(z-1)^\alpha$ for $z = e^{it}, t \in (-\pi,\pi)$. Observe that

$$\begin{aligned} f_\alpha(z) &= z^{2-\alpha}(z-1)^\alpha = \Big(1-\frac{1}{z}\Big)^\alpha z^2 = (1-\cos t + i\sin t)^\alpha e^{2it+2ki\pi} \\ &= (2-2\cos t)^{\frac{\alpha}{2}} e^{i[\alpha \arctan(\frac{\sin t}{1-\cos t})+2t+2k\pi]}, \quad k \in \mathbb{Z}. \end{aligned}$$

We now consider the function $m(\alpha,t) = (2-2\cos t)^{\frac{\alpha}{2}}$ that represents the modulus of $f_\alpha(e^{it})$ as t varies on $(-\pi,\pi)$. Then

$$\sup_{t\in(-\pi,\pi)} \big|m(\alpha,t)\big| = \sup_{t\in(-\pi,\pi)} \big|(2-2\cos t)^{\frac{\alpha}{2}}\big| = 4^{\alpha/2}.$$

Since $\sigma(T) \subset \{z \in \mathbb{C} : |z| > 2^\alpha\}$, we have $f_\alpha(z) \in \Omega_\alpha$ for all $z \in \partial\mathbb{D}, z \neq 1$, and therefore condition (H_α) is satisfied. Moreover, there exists $\epsilon > 0$ such that $d(f_\alpha(z), \sigma(T)) > \epsilon > 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Since T is normal, it follows that

$$\big\|(z-1)^\alpha\big(f_\alpha(z)-T\big)^{-1}\big\| \le \frac{2}{d(f_\alpha(z),\sigma(T))} < \frac{2}{\epsilon}$$

for all $|z| = 1$, $z \neq 1$. It follows from Remark 4.4 that condition (ii) in Theorem 4.2 is satisfied, and therefore the assertion of the theorem is proved. $\square$

Remark 4.6. From the proof of the above theorem, we observe that the maximum value of the function $m(\alpha,t) = (2-2\cos t)^{\frac{\alpha}{2}}$ is attained at the points $t = \pm\pi$.

We encourage the reader to compare Theorem 4.5 with the characterization given for $0 < \alpha \le 1$ in [18, Corollary 4.5].

We finish this work with the following simple example that highlights the role of the fractional difference in a given equation when we are dealing with additive perturbations.

Example 4.7. Let $1 < \alpha \le 2$ and $\epsilon > 0$ be given. We consider the equation

$$\begin{cases} \Delta^\alpha u(n,x) = \int_0^1 \frac{k(x/t)}{t} u(n,t)\,dt + (2^\alpha+\epsilon)u(n,x) + F(n,x), \\ u(0,x) = 0, \\ u(1,x) = 0, \end{cases} \tag{4.8}$$

where $n \in \mathbb{N}_0$, $x \in [0,1]$, and

$$k(u) = \frac{1}{\pi}\Big(\frac{\sin(\pi-\sigma)}{u+u^{-1}-2\cos(\pi-\sigma)}\Big), \quad u > 0,$$

and $0 < \sigma < \pi$. The kernel k appeared in [5] associated to boundary integral equations on polygonal domains. For each $f \in C([0,1])$, we define

$$T_\alpha f(x) = \int_0^1 \frac{k(x/t)}{t} f(t)\,dt + (2^\alpha+\epsilon)f(x), \quad x \in [0,1].$$

The operator $T_\alpha \in \mathcal{B}(C([0,1]))$ corresponds to an additive perturbation of the integral operator

$$Kf(x) := \int_0^1 \frac{k(x/t)}{t} f(t)\,dt.$$

It can be shown (see [5, Section 7, (7.4)]) that the spectrum of K is $[0, 1-\sigma/\pi]$. Therefore,

$$\sigma(T_\alpha) = \Big[2^\alpha + \epsilon, 2^\alpha + \epsilon + 1 - \frac{\sigma}{\pi}\Big] \subset \big\{z \in \mathbb{C} : |z| > 2^\alpha\big\}.$$

From Theorem 4.5, we can conclude that if

$$\sum_{j=0}^{\infty}\Big(\sup_{x\in[0,1]}\big|F(j,x)\big|\Big)^2 < \infty,$$

then there exists a unique solution $u(n,x)$ of (4.8) satisfying

$$\sum_{j=0}^{\infty}\Big(\sup_{x\in[0,1]}\big|u(j,x)\big|\Big)^2 < \infty.$$

In particular, such a solution satisfies $|u(n,x)| \to 0$ as $n \to \infty$, uniformly for $x \in [0,1]$.

Let us consider the limit case $\epsilon = 0$. Observe that $T_\alpha \to K + 4I$ as $\alpha \to 2$ and that $T_\alpha \to K + 2I$ as $\alpha \to 1$. Therefore, beginning with $\alpha = 2$ and as α approaches 1, the additive perturbation of (4.8) is better in the sense that $\|T_\alpha - K\| = 2^\alpha < 4$ for $1 < \alpha < 2$.

Remark 4.8. Compared with the case $0 < \alpha \leq 1$, the obtained characterization in Theorem 4.2 is not continuous at $\alpha = 1$. This is due to the discrete character of the equation (4.1), and also to the structure used in the right-hand side of (4.1). In other words, the spectral structure obtained in Figures 1 and 2 changes according to the consideration of $\Delta^\alpha u(n) = Tu(n)$ or $\Delta^\alpha u(n) = Tu(n+1)$, for instance. It should be noted that in the last case, the use of closed linear operators instead of only bounded operators is important (see [19]) and therefore deserves further investigation. This will be done in a forthcoming work.

Acknowledgments. Lizama's work was partially supported by the Comisión Nacional de Investigación Científica y Tecnológica (CONICYT-Chile) under Fondecyt grant 1140258. Murillo-Arcila's work was supported in part by MEC Projects MTM2010-14909, MTM2013-47093-P, and by an FPU grant from the Ministry of Education, Spain.

References

1. L. Abadias and C. Lizama, *Almost automorphic mild solutions to fractional partial difference-differential equations*, Appl. Anal. **95** (2016), no. 6, 1347–1369. Zbl 06579980. MR3479007. DOI 10.1080/00036811.2015.1064521. 192
2. T. Abdeljawad and F. M. Atici, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal. **2012**, art. ID rnm406757, 13 pp. Zbl 1253.39004. MR2984037. 189, 191
3. R. P. Agarwal, C. Cuevas, and C. Lizama, *Regularity of Difference Equations on Banach Spaces*, Springer, Champagne, 2014. Zbl 1306.39001. MR3244339. DOI 10.1007/978-3-319-06447-5. 189, 190, 191, 194, 195, 198, 201
4. H. Amann, *Linear and Quasilinear Parabolic Problems, I: Abstract Linear Theory*, Monogr. Math. **89**, Birkhäuser, Boston, 1995. Zbl 0819.35001. MR1345385. DOI 10.1007/978-3-0348-9221-6. 192

5. S. Amini and I. H. Sloan, *Collocation methods for second kind integral equations with non-compact operators*, J. Integral Equations Appl. **2** (1989), no. 1, 1–30. Zbl 0703.65090. MR1033201. DOI 10.1216/JIE-1989-2-1-1. 203, 204
6. F. M. Atici and P. W. Eloe, *A transform method in discrete fractional calculus*, Int. J. Difference Equ. **2** (2007), no. 2, 165–176. MR2493595. 189, 191, 192
7. F. M. Atici and P. W. Eloe, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc. **137** (2009), no. 3, 981–989. Zbl 1166.39005. MR2457438. DOI 10.1090/S0002-9939-08-09626-3. 189, 191
8. D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, Ser. Complex. Nonlinearity Chaos **3**, World Scientific, Hackensack, NJ, 2012. Zbl 1248.26011. MR2894576. DOI 10.1142/9789814355216. 192
9. H. Bateman, *The control of an elastic fluid*, Bull. Amer. Math. Soc. (N.S.) **51** (1945), 601–646. Zbl 0061.18803. MR0014548. 189
10. S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. **146** (2001), no. 2, 157–176. Zbl 0981.39009. MR1853519. DOI 10.4064/sm146-2-3. 190, 193, 198
11. S. Bu, *Well-posedness of fractional differential equations on vector-valued function spaces*, Integral Equations Operator Theory **71** (2011), no. 2, 259–274. Zbl 1232.45014. MR2838145. DOI 10.1007/s00020-011-1895-y. 190
12. S. Bu, *Mild well-posedness of equations with fractional derivative*, Math. Nachr. **285** (2012), no. 2–3, 202–209. Zbl 1251.47013. MR2881275. DOI 10.1002/mana.201000056. 190
13. W. Chojnacki, *On operator-valued cosine sequences on UMD spaces*, Studia Math. **199** (2010), no. 3, 267–278. Zbl 1215.47036. MR2678905. DOI 10.4064/sm199-3-4. 191, 195
14. S. N. Chow, J. Mallet-Paret, and W. Shen, *Traveling waves in lattice dynamical systems*, J. Differential Equations **149** (1998), no. 2, 248–291. Zbl 0911.34050. MR1646240. DOI 10.1006/jdeq.1998.3478. 189
15. C. S. Goodrich, *Existence of a positive solution to a system of discrete fractional boundary value problems*, Appl. Math. Comput. **217** (2011), no. 9, 4740–4753. Zbl 1215.39003. MR2745153. DOI 10.1016/j.amc.2010.11.029. 192
16. M. T. Holm, *The Laplace transform in discrete fractional calculus*, Comput. Math. Appl. **62** (2011), no. 3, 1591–1601. Zbl 1228.44010. MR2824746. DOI 10.1016/j.camwa.2011.04.019. 192
17. J. P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math. **47** (1987), no. 3, 556–572. Zbl 0649.34019. MR0889639. DOI 10.1137/0147038. 189
18. C. Lizama, *ℓ_p-maximal regularity for fractional difference equations on UMD spaces*, Math. Nachr. **288** (2015), no. 17–18, 2079–2092. Zbl 1335.39003. MR3434300. DOI 10.1002/mana.201400326. 189, 190, 192, 193, 194, 203
19. C. Lizama, *The Poisson distribution, abstract fractional difference equations, and stability*, preprint, to appear in Proc. Amer. Math. Soc. 189, 191, 192, 204
20. J. Mallet-Paret, *The global structure of traveling waves in spatially discrete dynamical systems*, J. Dynam. Differential Equations **11** (1999), no. 1, 49–127. Zbl 0921.34046. MR1680459. DOI 10.1023/A:1021841618074. 189
21. R. Ponce, *Hölder continuous solutions for fractional differential equations and maximal regularity*, J. Differential Equations **255** (2013), no. 10, 3284–3304. Zbl 1321.34104. MR3093364. DOI 10.1016/j.jde.2013.07.035. 191
22. J. R. Potts, S. Harris, and L. Giuggioli, *Territorial dynamics and stable home range formation for central place foragers*, preprint, `https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3316599/` (accessed 28 October, 2016). 189
23. J. C. Strikwerda and Y. S. Lee, *The accuracy of the fractional step method*, SIAM J. Numer. Anal. **37** (1999), no. 1, 37–47. Zbl 0953.65061. MR1721264. DOI 10.1137/S0036142997326938. 189

24. G. C. Wu, D. Baleanu, Z. G. Deng, and S. D. Zeng, *Lattice fractional diffusion equation in terms of a Riesz-Caputo difference*, Phys. A **438** (2015), 335–339. MR3384291. DOI 10.1016/j.physa.2015.06.024. 189

[1]Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile.
E-mail address: carlos.lizama@usach.cl

[2]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022, València, Spain.
E-mail address: mamuar1@posgrado.upv.es