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DUALITY FOR INCREASING CONVEX FUNCTIONALS WITH COUNTABLY MANY MARGINAL CONSTRAINTS

DANIEL BARTL,[1] PATRICK CHERIDITO,[2]
MICHAEL KUPPER,[1] and LUDOVIC TANGPI[3]

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Abstract. In this work we derive a convex dual representation for increasing convex functionals on a space of real-valued Borel measurable functions defined on a countable product of metric spaces. Our main assumption is that the functionals fulfill marginal constraints satisfying a certain tightness condition. In the special case where the marginal constraints are given by expectations or maxima of expectations, we obtain linear and sublinear versions of Kantorovich's transport duality and the recently discovered martingale transport duality on products of countably many metric spaces.

1. Introduction

We consider an increasing convex functional $\phi : B_b \to \mathbb{R}$, where B_b is the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ defined on a countable product of metric spaces $X = \prod_n X_n$. Under the assumption that there exist certain mappings ϕ_n defined on the bounded Borel measurable functions $g_n : X_n \to \mathbb{R}_+$, such that

$$\phi(f) \le \sum_n \phi_n(g_n) \quad \text{whenever } f(x) \le \sum_n g_n(x_n) \text{ for all } x \in X,$$

we show that ϕ can be represented as

$$\phi(f) = \max_{\mu \in ca^+} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \big) \quad \text{for all } f \in C_b, \tag{1.1}$$

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where ca^+ is the set of finite Borel measures, C_b is the set of bounded continuous functions $f : X \to \mathbb{R}$, $\langle f, \mu \rangle$ is the integral $\int f \, d\mu$, and $\phi^*_{C_b}$ is the convex conjugate defined by

$$\phi^*_{C_b}(\mu) := \sup_{f \in C_b} \big(\langle f, \mu \rangle - \phi(f) \big).$$

We also provide equivalent conditions under which the representation (1.1) extends to all bounded upper semicontinuous functions $f : X \to \mathbb{R}$. In the special case where the mappings ϕ_n are linear, our arguments can be generalized to cover functionals ϕ that are defined on spaces of unbounded functions $f : X \to \mathbb{R}$. This yields variants of the representation (1.1) for unbounded continuous and upper semicontinuous functions $f : X \to \mathbb{R}$.

As an application we derive versions of Kantorovich's transport duality and the recently discovered martingale transport duality in the case where the state space is a countable product of metric spaces. A standard Monge–Kantorovich transport problem consists in finding a probability measure on the product of two metric spaces with fixed marginals that minimizes the expectation of a given cost function. It is a linear optimization problem whose dual has the form of a subreplication problem (which, after changing the sign, becomes a superreplication problem). Kantorovich [14] first showed that there is no duality gap between the two problems under compactness and continuity assumptions. Since then, the result has been generalized in various directions (see, e.g., [1], [16], [17] for an overview). We establish linear and sublinear versions of Kantorovich's duality for countable products of metric spaces and lower semicontinuous cost functions (corresponding to upper semicontinuous functions $f : X \to \mathbb{R}$ in our setup). It has been shown that in the case where the state space is a finite product of Polish spaces, Kantorovich's duality even holds for Borel measurable cost functions (see, e.g., [3], [4], [15]). However, we provide a counterexample illustrating that this is no longer true if the state space is a countable product of compact metric spaces.

Martingale transport duality was discovered by [2] and [11] in the context of model-independent finance, the authors noting that the superreplication problem in the presence of liquid markets for European call and put options can be viewed as the analogue of a transport problem in which the optimization is carried out over the set of all martingale measures. While [2] considers a discrete-time model with finitely many marginal distributions, [11] studies a continuous-time model with just two marginal distributions. In the present article, we obtain a martingale transport duality for countably many time periods and equally many marginal constraints (for martingale transport in continuous time, see, e.g., [8], [12], and the references therein). Standard martingale transport duality describes a situation where a financial asset can be traded dynamically without transaction costs and any European derivative can efficiently be replicated with a static investment in European call and put options. From our general results, we obtain a sublinear generalization of the martingale transport duality corresponding to proportional transaction costs and incomplete markets of European call and put options. This extends the duality of [7] to a setup with countably many time periods and markets for European options with all maturities.

Our proofs differ from the standard arguments used in establishing Kantorovich duality and martingale transport duality in that they view the subreplication (or superreplication) problem as the primal problem and they use the Daniell–Stone theorem to deduce that increasing convex functionals on certain function spaces have a max-representation with countably additive measures if they are continuous from above under pointwise decreasing sequences.

The rest of this paper has the following organization. In Section 2, we derive two general representation results for increasing convex functionals satisfying countably many tight marginal constraints. In Section 3, we focus on the special cases where the constraints are linear and sublinear. In Section 4, we derive linear and sublinear versions of Kantorovich's transport duality and the martingale transport duality for countably many marginal constraints.

2. Main representation results

Let (X_n) be a countable (finite or countably infinite) family of metric spaces, and consider the product topology on $X = \prod_n X_n$. Denote by C_b, U_b, and B_b all bounded functions $f : X \to \mathbb{R}$ that are continuous, upper semicontinuous, or Borel measurable, respectively. Similarly, let $C_{b,n}$, $U_{b,n}$, and $B_{b,n}$ be all bounded functions $f : X_n \to \mathbb{R}$ that are continuous, upper semicontinuous, or Borel measurable, respectively. By ca^+ we denote all finite Borel measures on X, and by ca^+_n we denote all finite Borel measures on X_n. For a measure $\mu \in ca^+$, we denote by μ_n the nth marginal distribution; that is, $\mu_n := \mu \circ \pi_n^{-1}$, where $\pi_n : X \to X_n$ is the projection on the nth coordinate $x \mapsto \pi_n(x) := x_n$. For a sequence $g_n \in B^+_{b,n}$, where $B^+_{b,n}$ is the set of all bounded Borel measurable functions $f : X_n \to \mathbb{R}_+$, we define $\oplus g := \sum_n g_n \circ \pi_n : X \to \mathbb{R}_+ \cup \{+\infty\}$. When we write $f_j \downarrow f$, we mean that f_j is a decreasing sequence of functions that converges pointwise to f.

Our goal in this section is to derive a dual representation for an increasing convex functional $\phi : B_b \to \mathbb{R}$, where by "increasing" we mean that $\phi(f) \geq \phi(g)$ whenever $f \geq g$ and the second inequality is understood pointwise. For every n, let $\phi_n : B^+_{b,n} \to \mathbb{R}_+$ be a mapping satisfying the following tightness condition: for all $m, \varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists a compact set $K_n \subseteq X_n$ such that

$$\phi_n(m 1_{K_n^c}) \leq \varepsilon. \tag{2.1}$$

(In the special case where ϕ_n is given by $\phi_n(f) = \sup_{\nu \in \mathcal{P}_n} \int f \, d\nu$ for a set of Borel probability measures $\mathcal{P}_n$ on X_n, (2.1) means that $\mathcal{P}_n$ is tight in the standard sense (see, e.g., [5]). A related condition for convex risk measures was introduced in [10].) We use the notation $\langle f, \mu \rangle := \int f \, d\mu$, and define the convex conjugate

$$\phi^*_{C_b} : ca^+ \to \mathbb{R} \cup \{+\infty\} \quad \text{by } \phi^*_{C_b}(\mu) := \sup_{f \in C_b} \big(\langle f, \mu \rangle - \phi(f) \big).$$

Then the following holds.

Theorem 2.1. *Let $\phi : B_b \to \mathbb{R}$ be an increasing convex functional satisfying $\phi(f) \leq \sum_n \phi_n(g_n)$ for all $f \in B_b$ and $g_n \in B^+_{b,n}$ such that $f \leq \oplus g$. Then*

$$\phi(f) = \max_{\mu \in ca^+} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \big) \quad \text{for all } f \in C_b.$$

Proof. Fix $f \in C_b$, and let (f_j) be a sequence in C_b such that $f_j \downarrow 0$. Since $\alpha \mapsto \phi(\alpha f)$ is a real-valued convex function on $\mathbb{R}$, it is continuous, and so, for a given constant $\varepsilon > 0$, one can choose $\alpha \in (0,1)$ small enough such that

$$(1-\alpha)\phi\Big(\frac{f}{1-\alpha}\Big) - \phi(f) \leq \varepsilon.$$

By assumption, there exist compact sets $K_n \subseteq X_n$ such that $\sum_n \phi_n(g_n) \leq \varepsilon$, where

$$g_n := \frac{2}{\alpha}\|f_1\|_\infty 1_{K_n^c}.$$

By Tychonoff's theorem, $K := \prod_n K_n \subseteq X$ is compact. Since the function

$$\tilde{\phi}(\cdot) := \phi(\cdot + f) - \phi(f) : B_b \to \mathbb{R}$$

is convex, one has

$$\tilde{\phi}(f_j) \leq \frac{\tilde{\phi}(2f_j 1_K) + \tilde{\phi}(2f_1 1_{K^c})}{2}.$$

By Dini's lemma, $f_j \to 0$ uniformly on the compact K. Then, since $\lim_{\alpha\to 0}\tilde{\phi}(\alpha 1) = 0$, it follows by monotonicity that $\tilde{\phi}(2f_j 1_K) \to 0$. On the other hand, one obtains from $\frac{2}{\alpha} f_1 1_{K^c} \leq \oplus g$ that

$$\phi\Big(\frac{2}{\alpha} f_1 1_{K^c}\Big) \leq \sum_n \phi_n(g_n) \leq \varepsilon,$$

and therefore that

$$\tilde{\phi}(2f_1 1_{K^c}) \leq \alpha\phi\Big(\frac{2}{\alpha} f_1 1_{K^c}\Big) + (1-\alpha)\phi\Big(\frac{f}{1-\alpha}\Big) - \phi(f) \leq 2\varepsilon.$$

This shows $\phi(f+f_j) \downarrow \phi(f)$. By the Hahn–Banach extension theorem, there exists a positive linear functional $\psi : C_b \to \mathbb{R}$ such that

$$\psi(g) \leq \tilde{\phi}(g) = \phi(f+g) - \phi(f) \quad \text{for all } g \in C_b.$$

Since $\psi(g_j) \downarrow 0$ for every sequence (g_j) in C_b satisfying $g_j \downarrow 0$, one obtains from the Daniell–Stone theorem (see, e.g., [9, Theorem 4.5.2]) that there exists a $\nu \in ca^+$ such that $\psi(g) = \langle g, \nu\rangle$ for all $g \in C_b$. It follows that $\phi(f) + \phi^*_{C_b}(\nu) \leq \langle f, \nu\rangle$, which together with $\phi(f) \geq \sup_{\mu\in ca^+}(\langle f,\mu\rangle - \phi^*_{C_b}(\mu))$ yields

$$\phi(f) = \max_{\mu \in ca^+}\big(\langle f,\mu\rangle - \phi^*_{C_b}(\mu)\big).$$

□

The next result gives conditions under which the dual representation of Theorem 2.1 extends to the set of bounded upper semicontinuous functions U_b. We call a subset Λ of ca^+ sequentially compact if every sequence in Λ has a subsequence that converges to some $\mu \in \Lambda$ with respect to the topology $\sigma(ca^+, C_b)$.

Theorem 2.2. *Let $\phi : B_b \to \mathbb{R}$ be an increasing convex functional satisfying the assumption of Theorem 2.1. Then the lower level sets*

$$\Lambda_a := \big\{\mu \in ca^+ : \phi^*_{C_b}(\mu) \leq a\big\}, \quad a \in \mathbb{R},$$

are sequentially compact, and the following are equivalent:

(i) $\phi(f) = \max_{\mu \in ca^+}(\langle f, \mu\rangle - \phi^*_{C_b}(\mu))$ *for all* $f \in U_b$,
(ii) $\phi(f_j) \downarrow \phi(f)$ *for all* $f \in U_b$ *and every sequence* (f_j) *in* C_b *satisfying* $f_j \downarrow f$,
(iii) $\phi(f) = \inf_{g \in C_b, g \geq f} \phi(g)$ *for all* $f \in U_b$,
(iv) $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) := \sup_{f \in U_b}(\langle f, \mu\rangle - \phi(f))$ *for all* $\mu \in ca^+$.

Proof. It is clear that, for all $a \in \mathbb{R}$, Λ_a is $\sigma(ca^+, C_b)$-closed. Moreover, for all $\mu \in ca^+$,

$$\phi^*_{C_b}(\mu) \geq \sup_{x \in \mathbb{R}_+} \big(\langle x1, \mu\rangle - \phi(x1)\big) = \gamma\big(\langle 1, \mu\rangle\big),$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ is the increasing convex function given by

$$\gamma(y) := \sup_{x \in \mathbb{R}_+} \big(xy - \phi(x1)\big).$$

Since ϕ is real-valued, γ has the property $\lim_{y \to +\infty} \gamma(y)/y = +\infty$ from which it follows that the right-continuous inverse $\gamma^{-1} : \mathbb{R} \to \mathbb{R}_+$ given by

$$\gamma^{-1}(x) := \sup\big\{y \in \mathbb{R}_+ : \gamma(y) \leq x\big\} \quad \text{with } \sup \emptyset := 0$$

is increasing and satisfies $\lim_{x \to +\infty} \gamma^{-1}(x)/x = 0$. For every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ such that $(a+1)/m \leq \varepsilon$ and compact sets $K_n \subseteq X_n$ so that $\sum_n \phi_n(m1_{K_n^c}) \leq 1$. Since $m1_{K^c} \leq \oplus g$ for the compact $K := \prod_n K_n$ and $g_n := m1_{K_n^c}$, one has $\phi(m1_{K^c}) \leq \sum_n \phi_n(m1_{K_n^c}) \leq 1$. Moreover, the product topology on X is metrizable, and $m1_{K^c}$ is lower semicontinuous. Therefore, there exists a sequence (g_j) in C_b such that $g_j \uparrow m1_{K^c}$. Since $\phi(g_j) \leq \phi(m1_{K^c}) \leq 1$, one has, for all $\mu \in \Lambda_a$,

$$m\mu(K^c) = \sup_j \langle g_j, \mu\rangle \leq \sup_j \big(\langle g_j, \mu\rangle - \phi(g_j) + 1\big) \leq \phi^*_{C_b}(\mu) + 1 \leq a + 1.$$

In particular, $\mu(K^c) \leq \varepsilon$, and $\mu(X) = \langle 1, \mu\rangle \leq \gamma^{-1}(\phi^*_{C_b}(\mu)) \leq \gamma^{-1}(a)$. Now one obtains from the first half of Prokhorov's theorem (see, e.g., Theorem 5.1 in [5]) that Λ_a is sequentially compact.

(i) $\Rightarrow$ (ii): Fix $f \in U_b$, and assume that (f_j) is a sequence in C_b such that $f_j \downarrow f$. If (i) holds, there exists a sequence (μ_j) in ca^+ such that

$$\begin{aligned}\phi(f_j) = \langle f_j, \mu_j\rangle - \phi^*_{C_b}(\mu_j) &\leq \|f_1\|_\infty \langle 1, \mu_j\rangle - \phi^*_{C_b}(\mu_j) \\ &\leq \|f_1\|_\infty \gamma^{-1}\big(\phi^*_{C_b}(\mu_j)\big) - \phi^*_{C_b}(\mu_j).\end{aligned}$$

It follows that (μ_j) is in Λ_a for some $a \in \mathbb{R}$ large enough. Therefore, after possibly passing to a subsequence, μ_j converges to a measure $\mu \in \Lambda_a$ in $\sigma(ca^+, C_b)$. Clearly, $\phi^*_{C_b}$ is $\sigma(ca^+, C_b)$-lower semicontinuous, and so

$$\phi^*_{C_b}(\mu) \leq \liminf_j \phi^*_{C_b}(\mu_j).$$

Moreover, for every $\varepsilon > 0$, there is a k such that $\langle f_k, \mu\rangle \leq \langle f, \mu\rangle + \varepsilon$. Now choose $j \geq k$ such that $\langle f_k, \mu_j\rangle \leq \langle f_k, \mu\rangle + \varepsilon$. Then

$$\langle f_j, \mu_j\rangle \leq \langle f_k, \mu_j\rangle \leq \langle f_k, \mu\rangle + \varepsilon \leq \langle f, \mu\rangle + 2\varepsilon.$$

It follows that $\limsup_j \langle f_j, \mu_j\rangle \leq \langle f, \mu\rangle$, and therefore that

$$\lim_j \phi(f_j) = \lim_j \big(\langle f_j, \mu_j\rangle - \phi^*_{C_b}(\mu_j)\big) \leq \langle f, \mu\rangle - \phi^*_{C_b}(\mu) \leq \phi(f),$$

showing that $\phi(f_j) \downarrow \phi(f)$.

(ii) $\Rightarrow$ (iii): This follows from the fact that, for every $f \in U_b$, there exists a sequence (f_j) in C_b such that $f_j \downarrow f$.

(iii) $\Rightarrow$ (vi): It is immediate from the definitions that $\phi^*_{U_b} \geq \phi^*_{C_b}$. On the other hand, if (iii) holds, then, for every $f \in U_b$, there is a sequence (f_j) in C_b such that $f_j \geq f$ and $\phi(f_j) \downarrow \phi(f)$. In particular,

$$\sup_j \big(\langle f_j, \mu\rangle - \phi(f_j)\big) \geq \langle f, \mu\rangle - \phi(f)$$

from which one obtains $\phi^*_{C_b} \geq \phi^*_{U_b}$.

(iv) $\Rightarrow$ (i): Fix $f \in U_b$. It is a direct consequence of the definition of $\phi^*_{U_b}$ that

$$\phi(f) \geq \sup_{\mu \in ca^+} \big(\langle f, \mu\rangle - \phi^*_{U_b}(\mu)\big) = \sup_{\mu \in ca^+} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big).$$

On the other hand, there exists a sequence (f_j) in C_b such that $f_j \downarrow f$. Since

$$\langle f_j, \mu\rangle \leq \langle f_1, \mu\rangle \leq \|f_1\|_\infty \langle 1, \mu\rangle \leq \|f_1\|_\infty \gamma^{-1}\big(\phi^*_{C_b}(\mu)\big),$$

it follows from Theorem 2.1 that one can choose $a \in \mathbb{R}$ large enough such that

$$\phi(f_j) = \langle f_j, \mu_j\rangle - \phi^*_{C_b}(\mu_j)$$

for a sequence (μ_j) in the sequentially compact set Λ_a. After passing to a subsequence, μ_j converges to a μ in $\sigma(ca^+, C_b)$. Then it follows as above that

$$\phi(f) \leq \lim_j \phi(f_j) = \lim_j \big(\langle f_j, \mu_j\rangle - \phi^*_{C_b}(\mu_j)\big) \leq \langle f, \mu\rangle - \phi^*_{C_b}(\mu)$$

from which one obtains $\phi(f) = \max_{\mu \in ca^+}(\langle f, \mu\rangle - \phi^*_{C_b}(\mu))$. □

3. Linear and Sublinear marginal constraints

In this section we assume the X_n to be Polish spaces, and we assume the mappings $\phi_n : B^+_{b,n} \to \mathbb{R}$ to be of the form

$$\phi_n(g) = \sup_{\nu_n \in \mathcal{P}_n} \langle g, \nu_n\rangle,$$

where $\mathcal{P}_n$ is a nonempty convex $\sigma(ca^+_n, C_{b,n})$-compact set of Borel probability measures on X_n. Then all ϕ_n are increasing and sublinear. Moreover, they have the translation property

$$\phi_n(g + m) = \phi_n(g) + m, \qquad g \in B_{b,n}, m \in \mathbb{R},$$

and it follows from Prokhorov's theorem that they satisfy the tightness condition (2.1) (see, e.g., [5]). By $\mathcal{P}$ we denote the set of Borel probability measures μ on the product $X = \prod_n X_n$ whose marginal distributions $\mu_n := \mu \circ \pi_n^{-1}$ are in $\mathcal{P}_n$ for all n. Under these circumstances the following holds.

Proposition 3.1. *Let $\phi : B_b \to \mathbb{R}$ be an increasing convex functional satisfying*

$$\phi(f) \leq m + \sum_n \phi_n(g_n) \tag{3.1}$$

whenever $f \le m + \oplus g$ *for some* $m \in \mathbb{R}$ *and* $g_n \in B^+_{b,n}$*. Then*

$$\phi(f) = \max_{\mu \in \mathcal{P}} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \big) \quad \text{for all } f \in C_b. \tag{3.2}$$

If, in addition, $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ *for all* $\mu \in \mathcal{P}$*, the representation (3.2) extends to all* $f \in U_b$*.*

Proof. One obtains from Theorem 2.1 that

$$\phi(f) = \max_{\mu \in ca^+} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \big) \quad \text{for all } f \in C_b,$$

and Theorem 2.2 implies that the representation holds for all $f \in U_b$ if $\phi^*_{C_b} = \phi^*_{U_b}$. Therefore, the proposition follows if we can show that $\phi^*_{C_b}(\mu) = +\infty$ for all $\mu \in ca^+ \setminus \mathcal{P}$. To do that, fix a $\mu \in ca^+ \setminus \mathcal{P}$. If it is not a probability measure, then

$$\phi^*_{C_b}(\mu) \ge \sup_{m \in \mathbb{R}} \big(\langle m, \mu \rangle - \phi(m) \big) \ge \sup_{m \in \mathbb{R}} \big(\langle m, \mu \rangle - m \big) = +\infty.$$

On the other hand, if μ is a probability measure, but does not belong to $\mathcal{P}$, one obtains from the Hahn–Banach separation theorem that there exist n and $g_n \in C_{b,n}$ such that $\langle g_n, \mu_n \rangle > \phi_n(g_n)$. Moreover, since ϕ_n has the translation property, g_n can be shifted until it is nonnegative. Then

$$\phi(m g_n \circ \pi_n) \le \phi_n(m g_n) = m \phi_n(g_n) \quad \text{for all } m \in \mathbb{R}_+,$$

and therefore

$$\begin{aligned} \phi^*_{C_b}(\mu) &\ge \sup_{m \in \mathbb{R}_+} \big(\langle m g_n \circ \pi_n, \mu \rangle - \phi(m g_n \circ \pi_n) \big) \ge \sup_{m \in \mathbb{R}_+} m \big(\langle g_n, \mu_n \rangle - \phi_n(g_n) \big) \\ &= +\infty. \end{aligned}$$

□

In the next step we concentrate on the special case where every $\mathcal{P}_n$ consists of just one Borel probability measure ν_n on X_n. Then the mappings ϕ_n are of the form $\phi_n(g) = \langle g, \nu_n \rangle$. In particular, they are linear, and the representation (3.2) can be extended to unbounded functions f.

Let us denote by $\mathcal{P}(\nu)$ the set of all Borel probabilities on X with marginals $\mu_n = \nu_n$. Furthermore, let B be the space of all Borel measurable functions $f : X \to \mathbb{R}$, let U be the subset of upper semicontinuous functions $f : X \to \mathbb{R}$, and let B^+_n be the set of all Borel measurable functions $f : X_n \to \mathbb{R}_+$. Consider the following sets:

$$\begin{aligned} G(\nu) &:= \Big\{ \oplus g : (g_n) \in \prod_n B^+_n \text{ such that } \sum_n \langle g_n, \nu_n \rangle < +\infty \Big\}, \\ B(\nu) &:= \big\{ f \in B : |f| \le \oplus g \text{ for some } \oplus g \in G(\nu) \big\}, \\ U(\nu) &:= \big\{ f \in U : f^+ \in B_b \text{ and } f^- \in B(\nu) \big\}. \end{aligned}$$

Note that $G(\nu)$ is not contained in $B(\nu)$ since a function $\oplus g \in G(\nu)$ can take on the value $+\infty$. But one has $\langle \oplus g, \mu \rangle = \sum_n \langle g_n, \nu_n \rangle < +\infty$ for all $\oplus g \in G(\nu)$ and $\mu \in \mathcal{P}(\nu)$. This shows that $G(\nu)$ is contained in $L^1(\mu)$, and every $\oplus g \in G(\nu)$ is finite μ-almost surely.

Proposition 3.2. *Let $\phi : B(\nu) \to \mathbb{R}$ be increasing and convex such that*

$$\phi(f) \le m + \sum_n \langle g_n, \nu_n \rangle \tag{3.3}$$

if $f \le m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\nu)$. Moreover, assume that

$$\phi^*_{C_b}(\mu) = \phi^*_{U(\nu)}(\mu) := \sup_{f \in U(\nu)} \big(\langle f, \mu \rangle - \phi(f)\big) \quad \text{for all } \mu \in \mathcal{P}(\nu).$$

Then

$$\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\big) \quad \text{for all } f \in B(\nu) \cap \big(U(\nu) + G(\nu)\big).$$

Proof. By Proposition 3.1, one has

$$\phi(f) = \max_{\mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\big) \quad \text{for all } f \in C_b.$$

Furthermore, for given $f \in U(\nu)$, there exists a sequence (f_j) in C_b such that $f_j \downarrow f$, and it follows as in the proof of (iv) $\Rightarrow$ (i) in Theorem 2.2 that there exists a $\mu \in \mathcal{P}(\nu)$ such that $\phi(f) \le \langle f, \mu \rangle - \phi^*_{C_b}(\mu)$. Since, on the other hand,

$$\phi(f) \ge \sup_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{U(\nu)}(\mu)\big) = \sup_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\big),$$

one obtains

$$\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\big).$$

Next, notice that it follows as in the proof of Theorem 2.1 from the Hahn–Banach extension theorem that

$$\phi(f) = \max_{\psi \in B'(\nu)} \big(\psi(f) - \phi^*(\psi)\big) \quad \text{for all } f \in B(\nu),$$

where $B'(\nu)$ is the algebraic dual of $B(\nu)$ and $\phi^*(\psi) := \sup_{f \in B(\nu)} (\psi(f) - \phi(f))$, $\psi \in B'(\nu)$. For $\psi \in B'(\nu)$ with $\phi^*(\psi) < +\infty$, one has, for all $\oplus g \in G(\nu) \cap B(\nu)$,

$$\psi(\oplus g) - \sum_n \langle g_n, \nu_n \rangle \le \psi(\oplus g) - \phi(\oplus g) \le \phi^*(\psi) < +\infty,$$

and therefore $\psi(\oplus g) \le \sum_n \langle g_n, \nu_n \rangle$. On the other hand, if one sets $g_n^N := g_n \wedge N$ for $n \le N$ and $g_n^N := 0$ for $n > N$, then

$$\psi(N^2 - \oplus g^N) \le N^2 - \sum_{n=1}^N \langle g_n \wedge N, \nu_n \rangle$$

from which one obtains

$$\psi(\oplus g) \ge \lim_N \psi(\oplus g^N) \ge \lim_N \sum_{n=1}^N \langle g_n \wedge N, \nu_n \rangle = \sum_n \langle g_n, \nu_n \rangle.$$

This shows that $\psi(\oplus g) = \sum_n \langle g_n, \nu_n \rangle$ for all $\oplus g \in G(\nu) \cap B(\nu)$, and, as a result,

$$\phi(f - \oplus g) = \max_{\psi \in B'(\nu)} \big(\psi(f - \oplus g) - \phi^*(\psi)\big) = \phi(f) - \sum_n \langle g_n, \nu_n \rangle$$

for all $f \in B(\nu)$ and $\oplus g \in G(\nu)$. Finally, let $f \in B(\nu)$ be of the form $f = \oplus g + h$ for $\oplus g \in G(\nu)$ and $h \in U(\nu)$. Then $f - \oplus g \in U(\nu)$ and $\oplus g \in G(\nu) \cap B(\nu)$. Therefore,

$$\begin{aligned}\phi(f) - \sum_n \langle g_n, \nu_n\rangle = \phi(f - \oplus g) &= \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f - \oplus g, \mu\rangle - \phi^*_{C_b}(\mu)\big) \\ &= \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) - \sum_n \langle g_n, \nu_n\rangle,\end{aligned}$$

and hence, $\phi(f) = \max_{\mu \in \mathcal{P}(\nu)}(\langle f, \mu\rangle - \phi^*_{C_b}(\mu))$. □

4. Generalized (martingale) transport dualities

In this section we derive generalizations of Kantorovich's transport duality and the more recently introduced martingale transport duality.

4.1. **Generalized transport dualities.** As in Section 3, let X_n be Polish spaces. We first study the case where a probability measure ν_n is given on each X_n. For given $f \in B(\nu)$, consider the minimization problem

$$\phi(f) := \inf\Big\{m + \sum_n \langle g_n, \nu_n\rangle : m \in \mathbb{R}, \oplus g \in G(\nu) \text{ such that } m + \oplus g \geq f\Big\}. \quad (4.1)$$

Remark 4.1. Up to a different sign, (4.1) can be viewed as a generalized version of the dual of a transport problem. A standard transport problem in the sense of Kantorovich consists in finding a Borel probability measure μ on the product of two metric spaces $X_1 \times X_2$ with given marginals ν_1 and ν_2 that minimizes the expectation $\mathbb{E}^\mu c$ of a cost function $c : X_1 \times X_2 \to \mathbb{R}$. The (negative of the) dual problem is a minimization problem of the form

$$\inf \sum_{n=1}^{2} \langle g_n, \nu_n\rangle, \quad (4.2)$$

where the infimum is taken over all $g_n \in L^1(\nu_n)$ such that $\oplus g \geq f := -c$. To relate (4.1) to (4.2) more closely, note that $\oplus g^1 - \oplus g^2$ is well defined for all $\oplus g^1 \in G(\nu)$ and $\oplus g^2 \in G(\nu) \cap B(\nu)$, and so, instead of (4.1), we could have defined $\phi(f)$ equivalently as

$$\inf\Big\{\sum_n \langle g^1_n - g^2_n, \nu_n\rangle : \begin{array}{l}\oplus g^1 \in G(\nu), \oplus g^2 \in G(\nu) \cap B(\nu) \\ \text{such that } \oplus g^1 - \oplus g^2 \geq f\end{array}\Big\}.$$

Indeed, it is clear that the above infimum minorizes $\phi(f)$. On the other hand, since $\lim_{N \to +\infty} \sum_{n=1}^N \langle g^2_n \wedge N, \nu_n\rangle = \sum_n \langle g^2_n, \nu_n\rangle$, it cannot be strictly smaller.

As a consequence of the results in Section 3, one obtains the following version of Kantorovich's transport duality with countably many marginal distributions.

Corollary 4.2. *We have the following:*

$$\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \langle f, \mu\rangle \quad \textit{for all } f \in B(\nu) \cap \big(U(\nu) + G(\nu)\big). \quad (4.3)$$

Proof. Clearly, $\phi(f) < +\infty$ for all $f \in B(\nu)$. On the other hand, since $\mathcal{P}(\nu)$ is nonempty (it contains the product measure $\otimes_n \nu_n$), one has

$$m + \sum_n \langle g_n, \nu_n \rangle \geq \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle > -\infty$$

for all $m \in \mathbb{R}$, $\oplus g \in G(\nu)$ and $f \in B(\nu)$ such that $m + \oplus g \geq f$. It follows that $\phi : B(\nu) \to \mathbb{R}$ is an increasing sublinear functional satisfying

$$\phi(f) \geq \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \text{for all } f \in B(\nu).$$

In particular, $\phi(0) = 0$, and $\phi^*_{C_b}(\mu) = \phi^*_{U(\nu)}(\mu) = 0$ for all $\mu \in \mathcal{P}(\nu)$. Therefore, the duality (4.3) follows from Proposition 3.2. □

Remark 4.3. If X is a finite product of Polish spaces, it can be shown that

$$\phi(f) = \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \text{for all } f \in B_b$$

(see, e.g., [3], [4], [15]). But for countably infinite products, there may arise a duality gap; that is, it may happen that

$$\phi(f) > \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \text{for some } f \in B_b.$$

For instance, if X is the product of $X_n = \{0, 1\}$, $n \in \mathbb{N}$, and $\nu_n = \frac{1}{2}(\delta_0 + \delta_1)$ for all n, then $f := \liminf_n \pi_n$ belongs to B_b, and it follows from Fatou's lemma that

$$\langle f, \mu \rangle \leq \liminf_n \langle \pi_n, \mu \rangle = \frac{1}{2} \quad \text{for all } \mu \in \mathcal{P}(\nu).$$

On the other hand, assume that $f \leq m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\nu)$. Since

$$\frac{1}{2} \sum_n \big(g_n(0) + g_n(1)\big) = \sum_n \langle g_n, \nu_n \rangle < +\infty,$$

one has $\sum_n g_n(x_n) < +\infty$ for all $x \in X$, and therefore

$$\inf_{k \in \mathbb{N}} \min_{(y_1, \dots, y_k) \in \{0,1\}^k} \Big(\sum_{n \leq k} g_n(y_n) + \sum_{n > k} g_n(x_n) \Big) = \sum_n \min_{y_n \in \{0,1\}} g_n(y_n) \leq \sum_n \langle g_n, \nu_n \rangle.$$

Consequently,

$$1 = \inf_{k \in \mathbb{N}} \min_{(y_1, \dots, y_k) \in \{0,1\}^k} f(y_1, \dots, y_k, 1, 1, \dots) \leq m + \sum_n \langle g_n, \nu_n \rangle$$

from which it follows that $\phi(f) \geq 1$.

In the more general case where the $\phi_n : B_{b,n} \to \mathbb{R}$ are sublinear functionals given by

$$\phi_n(g) = \sup_{\nu_n \in \mathcal{P}_n} \langle g, \nu_n \rangle$$

for nonempty convex $\sigma(ca^+_n, C_{b,n})$-compact sets of Borel probability measures $\mathcal{P}_n$ on X_n, we obtain a generalized Kantorovich duality with countably many sets of marginal distributions. As in Section 3, $\mathcal{P}$ denotes the set of probability distributions such that $\mu_n \in \mathcal{P}_n$ for all n. Compared to Corollary 4.2, one has to modify the definition of ϕ slightly:

$$\phi(f) := \inf\Big\{m + \sum_n \phi_n(g_n) : m \in \mathbb{R}, g_n \in B^+_{b,n} \text{ such that } m + \oplus g \geq f\Big\}. \quad (4.4)$$

Then an application of Proposition 3.1 and essentially the same arguments as in the proof of Corollary 4.2 yield the following duality.

Corollary 4.4. *We have the following:*

$$\phi(f) = \max_{\mu \in \mathcal{P}} \langle f, \mu\rangle \quad \textit{for all } f \in U_b. \quad (4.5)$$

Proof. As in the proof of Corollary 4.2, it is easy to see that $\phi : B_b \to \mathbb{R}$ is an increasing sublinear functional such that

$$\phi(f) \geq \sup_{\mu \in \mathcal{P}} \langle f, \mu\rangle \quad \text{for all } f \in B_b.$$

Since $\mathcal{P}$ is nonempty (it contains all product measures $\otimes_n \nu_n$ for $\nu_n \in \mathcal{P}_n$), it follows that $\phi(0) = 0$ and $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) = 0$ for all $\mu \in \mathcal{P}$. Hence (4.5) follows from Proposition 3.1. □

4.2. **Generalized martingale transport dualities.** Next, we derive linear and sublinear versions of the martingale transport duality with countably many marginal constraints. Let X_n be nonempty closed subsets of $\mathbb{R}^d$, and model the discounted prices of d financial assets by $S_0 := s_0 \in \mathbb{R}^d$ and $S_n(x) := x_n$, $x \in X = \prod_n X_n$. The corresponding filtration is given by $\mathcal{F}_n := \sigma(S_j : j \leq n)$.

We first assume that each space X_n carries a single Borel probability measure ν_n. Moreover, we suppose that money can be lent and borrowed at the same interest rate, and European options with general discounted payoffs $g_n \in B^+_n$ can be bought at price $\langle g_n, \nu_n\rangle$ (we suppose they either exist as structured products or they can be synthesized by investing in more standard options; see, e.g., [6] for the form of ν_n if European call options exist with maturity n and all strikes). A function $\oplus g \in G(\nu)$ then corresponds to a static option portfolio costing $\sum_n \langle g_n, \nu_n\rangle$. In addition, the underlying assets can be traded dynamically. The set $\mathcal{H}$ of dynamic trading strategies consists of all finite sequences $h_1, \ldots, h_N$ such that each h_n is an $\mathbb{R}^d$-valued $\mathcal{F}_{n-1}$-measurable function on X. An $h \in \mathcal{H}$ generates gains of the form

$$(h \cdot S)_N := \sum_{n=1}^N h_n \cdot (S_n - S_{n-1}).$$

A triple $(m, \oplus g, h) \in \Theta := \mathbb{R} \times G(\nu) \times \mathcal{H}$ describes a semistatic trading strategy with cost $m + \sum_n \langle g_n, \nu_n\rangle$ and outcome $m + \oplus g + (h \cdot S)_N$.

A strategy $(m, \oplus g, h) \in \Theta$ is said to be a model-independent arbitrage if

$$m + \sum_n \langle g_n, \nu_n\rangle \leq 0 \qquad \text{and} \qquad m + \oplus g + (h \cdot S)_N > 0.$$

Similarly, we call a strategy $(m, \oplus g, h) \in \Theta$ a uniform arbitrage if

$$m + \sum_n \langle g_n, \nu_n \rangle < 0 \qquad \text{and} \qquad m + \oplus g + (h \cdot S)_N \geq 0.$$

Consider the superhedging functional

$$\phi(f) := \inf\Big\{ m + \sum_n \langle g_n, \nu_n \rangle : \begin{array}{l} (m, \oplus g, h) \in \Theta \text{ such that} \\ m + \oplus g + (h \cdot S)_N \geq f \end{array} \Big\}, \tag{4.6}$$

and denote by $\mathcal{M}(\nu)$ the set of probability measures $\mu \in \mathcal{P}(\nu)$ under which S is a d-dimensional martingale.

Remark 4.5. The static part of a semistatic strategy in Θ consists of a cash position and a portfolio of options with nonnegative payoffs. But one could extend the set of strategies to include portfolios with outcomes $\oplus g^1 - \oplus g^2 + (h \cdot S)_N$ and prices $\sum_n \langle g^1_n - g^2_n, \nu_n \rangle$ for $g^1 \in G(\nu)$, $g^2 \in G(\nu) \cap B(\nu)$, and $h \in \mathcal{H}$. It follows as in Remark 4.1 that this would not change the superhedging functional (4.6), the definition of a model-independent arbitrage, or the definition of a uniform arbitrage.

The following corollary extends the superhedging duality of [2] to a model with countably many time periods, and contains a model-independent fundamental theorem of asset pricing as a consequence. For $x \in X_n \subseteq \mathbb{R}^d$, denote by $|x|$ the Euclidean norm of x.

Corollary 4.6. *Assume that* $\int_{X_n} |x| \, d\nu_n(x) < +\infty$ *for all* n. *Then the following are equivalent:*

(i) *there is no model-independent arbitrage,*
(ii) *there is no uniform arbitrage,*
(iii) $\mathcal{M}(\nu) \neq \emptyset$.

Moreover, if (i)–(iii) hold, then

$$\phi(f) = \max_{\mu \in \mathcal{M}(\nu)} \langle f, \mu \rangle \quad \text{for all } f \in B(\nu) \cap \big(U(\nu) + G(\nu)\big). \tag{4.7}$$

Proof. It is clear that (i) implies (ii) since, for every uniform arbitrage $(m, \oplus g, h)$, there exists an $\varepsilon > 0$ such that $(m + \varepsilon, \oplus g, h)$ is a model-independent arbitrage.

Furthermore, if (iii) holds, there exists a μ in $\mathcal{M}(\nu)$. Let $(m, \oplus g, h) \in \Theta$ be a strategy such that $m + \oplus g + (h \cdot S)_N > 0$. Then $\mathbb{E}^\mu (h \cdot S)^-_N \leq m^+ + \sum_n \langle g_n, \nu_n \rangle < +\infty$, and it follows that $(h \cdot S)_n$, $n = 1, \ldots, N$, is a martingale under μ (see, e.g., [13]). In particular, $\mathbb{E}^\mu (h \cdot S)_N = 0$, and therefore

$$m + \sum_n \langle g_n, \nu_n \rangle = \big\langle m + \oplus g + (h \cdot S)_N, \mu \big\rangle > 0.$$

So there is no model-independent arbitrage, showing that (i) is satisfied.

Now let us assume (ii). Then $\phi : B(\nu) \to \mathbb{R} \cup \{-\infty\}$ is an increasing sublinear functional with the property that $\phi(f) \leq m + \sum_{n \geq 1} \langle g_n, \nu_n \rangle$ whenever $f \leq m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\nu)$. If there is no uniform arbitrage, one has

$\phi(0) = 0$ from which it follows by subadditivity that $\phi(f) > -\infty$ for all $f \in B(\nu)$. Moreover, if

$$m + \oplus g + (h \cdot S)_N \geq f$$

for $(m, \oplus g, h) \in \Theta$ and $f \in B(\nu)$, one has. for all $\mu \in \mathcal{M}(\nu)$,

$$\mathbb{E}^\mu (h \cdot S)_N^- \leq m^+ + \sum_n \langle g_n, \nu_n \rangle + \langle f^-, \mu \rangle < +\infty.$$

It follows as above that $\mathbb{E}^\mu (h \cdot S)_N = 0$, and therefore $m + \sum_n \langle g_n, \nu_n \rangle \geq \langle f, \mu \rangle$. This implies that $\phi(f) \geq \langle f, \mu \rangle$, and, consequently, that $\phi^*_{C_b}(\mu) = \phi^*_{U(\nu)}(\mu) = 0$ for all $\mu \in \mathcal{M}(\nu)$. Therefore, if we can show that

$$\phi^*_{C_b}(\mu) = +\infty \quad \text{for all } \mu \in \mathcal{P}(\nu) \setminus \mathcal{M}(\nu), \tag{4.8}$$

then we obtain from Proposition 3.2 that (4.7) holds, which in turn implies that $\mathcal{M}(\nu)$ cannot be empty.

To show (4.8), let $\mu \in \mathcal{P}(\nu)$. If $\mathbb{E}^\mu S_1 = s_0$ and $\mathbb{E}^\mu [v(x_1, \ldots, x_n) \cdot (x_{n+1} - x_n)] = 0$ for all $n \geq 1$ and every bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}^d$, then S is a martingale under μ, and therefore $\mu \in \mathcal{M}(\nu)$. Hence, for $\mu \in \mathcal{P}(\nu) \setminus \mathcal{M}(\nu)$, there must exist a continuous function $f \in B(\nu)$ with $\langle f, \mu \rangle > 0$ such that f is either of the form $f(x) = v \cdot (x_1 - s_0)$ for a vector $v \in \mathbb{R}^d$ or $f(x) = v(x_1, \ldots, x_n) \cdot (x_{n+1} - x_n)$ for some $n \geq 1$ and a bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}^d$. For $k \in \mathbb{N}$, $f^k := f \wedge k$ is bounded above and $f^k_k := f^k \vee (-k)$ bounded. By monotonicity, then, $\phi(f^k) \leq \phi(f) \leq 0$. Moreover,

$$f^k_k(x) = f^k(x) + \big(k + f(x)\big)^- \leq f^k(x) + w^k(x),$$

where

$$w^k(x) := \big(c|x_n| - k/2\big)^+ + \big(c|x_{n+1}| - k/2\big)^+$$

and $c \in \mathbb{R}_+$ is a bound on $|v|$. Since w^k is in $G(\nu)$, we have

$$\phi(w^k) \leq \int_{X_n} \big(c|x_n| - k/2\big)^+ d\nu_n(x_n) + \int_{X_{n+1}} \big(c|x_{n+1}| - k/2\big)^+ d\nu_{n+1}(x_{n+1}) \to 0$$

for $k \to +\infty$. Thus, for k large enough, one obtains from monotonicity and subadditivity that

$$\langle f^k_k, \mu \rangle - \phi(f^k_k) \geq \langle f^k, \mu \rangle - \phi(f^k) - \phi(w^k) \geq \langle f^k, \mu \rangle - \phi(w^k) > 0,$$

and, as a result, that $\phi^*_{C_b}(\mu) = +\infty$. □

Now, we extend the setting of Corollary 4.6 by adding friction and incompleteness. To simplify the presentation, we assume that each X_n is a nonempty closed subset of $\mathbb{R}^d_+$. As above, $S_0 = s_0 \in \mathbb{R}^+_d$, $S_n(x) = x_n$, $x \in X$, and the set of dynamic trading strategies $\mathcal{H}$ is given by all finite sequences $h_1, \ldots, h_N$ of $\mathcal{F}_{n-1}$-measurable mappings $h_n : X \to \mathbb{R}^d$. But now we assume that dynamic trading incurs proportional transaction costs. If the bid and ask prices of asset i are given by $(1 - \varepsilon_i) S^i_n$ and $(1 + \varepsilon_i) S^i_n$ for a constant $\varepsilon_i \geq 0$, a strategy $h \in \mathcal{H}$ leads to an outcome of

$$h(S) := \sum_{n=1}^N \sum_{i=1}^d h^i_n (S^i_n - S^i_{n-1}) - \varepsilon_i |h^i_n - h^i_{n-1}| S^i_{n-1}, \quad \text{where } h^i_0 := 0.$$

(We assume there are no initial asset holdings; consequently, there is a transaction cost at time 0. On the other hand, asset holdings at time N are valued at $h_N \cdot S_N$, and do not have to be converted into cash.) Similarly, a European option with payoff $g_n \in B_n^+$ at time n is assumed to cost

$$\phi_n(g_n) = \sup_{\nu_n \in \mathcal{P}_n} \langle g_n, \nu_n \rangle,$$

where $\mathcal{P}_n$ is a nonempty convex $\sigma(ca_n^+, C_{b,n})$-compact set of Borel probability measures on X_n (nonlinear prices $\phi_n(g_n)$ may arise if, e.g., not enough liquidly traded vanilla options exist to exactly replicate the payoffs g_n, or there are positive bid-ask spreads in the vanilla options market; see, e.g., [7]). Compared to the frictionless case, we now have to require a little bit more integrability of the option portfolio. As in Section 3, we denote by $\mathcal{P}$ the set of all Borel probability measures μ on $X = \prod_n X_n$ with marginal distributions in $\mathcal{P}_n$. We introduce the sets

$$G(\mathcal{P}) := \Big\{ \oplus g : (g_n) \in \prod_n B_n^+ \text{ such that } \sum_n \phi_n(g_n) < +\infty \Big\},$$
$$B(\mathcal{P}) := \big\{ f \in B : |f| \le \oplus g \text{ for some } \oplus g \in G(\mathcal{P}) \big\},$$

and consider option portfolios with payoffs $\oplus g$ for functions $g_n \in B_n^+$ such that $\sum_n \phi_n(g_n) < +\infty$. We still denote the set of corresponding strategies $(m, \oplus g, h)$ by Θ. The corresponding superhedging functional is given by

$$\phi(f) := \Big\{ m + \sum_n \phi_n(g_n) : \begin{array}{l} (m, \oplus g, h) \in \Theta \text{ such that} \\ m + \oplus g + h(S) \ge f \end{array} \Big\}. \tag{4.9}$$

A model-independent arbitrage now consists of a strategy $(m, \oplus g, h) \in \Theta$ such that

$$m + \sum_n \phi_n(g_n) \le 0 \qquad \text{and} \qquad m + \oplus g + h(S) > 0,$$

and a uniform arbitrage of a strategy $(m, \oplus g, h) \in \Theta$ satisfying

$$m + \sum_n \phi_n(g_n) < 0 \qquad \text{and} \qquad m + \oplus g + h(S) \ge 0.$$

The set of martingale measures has to be extended to the set $\mathcal{M}(\mathcal{P})$ of all measures $\mu \in \mathcal{P}$ satisfying

$$(1 - \varepsilon_i) S_n^i \le \mathbb{E}^\mu[S_N^i \mid \mathcal{F}_n] \le (1 + \varepsilon_i) S_n^i \quad \text{for all } i, N \text{ and } n \le N. \tag{4.10}$$

The following is a variant of Corollary 4.6 with friction and incompleteness. It extends the duality result of [7] to the case of countably many time periods and European options with all maturities.

Corollary 4.7. *Assume that* $\lim_{k \to +\infty} \sup_{\nu_n \in \mathcal{P}_n} \int_{X_n} (|x| - k)^+ \, d\nu_n(x) = 0$ *for all* n. *Then the following are equivalent:*

(i) *there is no model-independent arbitrage,*
(ii) *there is no uniform arbitrage,*
(iii) $\mathcal{M}(\mathcal{P}) \ne \emptyset$.

Moreover, if (i)–(iii) hold, then

$$\phi(f) = \max_{\mu \in \mathcal{M}(\mathcal{P})} \langle f, \mu \rangle \quad \textit{for all } f \in U_b. \tag{4.11}$$

Proof. As in the proof of Corollary 4.6, the implication (i) ⇒ (ii) is straightforward since the existence of a uniform arbitrage implies the existence of a model-independent arbitrage.

If (iii) holds, then there exists a μ in $\mathcal{M}(\mathcal{P})$, and so if $(m, \oplus g, h) \in \Theta$ is a strategy with $m + \oplus g + h(S) > 0$, then

$$\mathbb{E}^\mu h(S)^- \le m^+ + \langle \oplus g, \mu \rangle \le m^+ + \sum_n \phi_n(g_n) < +\infty.$$

Moreover, for all i,

$$\begin{aligned}
&\sum_{n=1}^N h_n^i (S_n^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \\
&\quad = \sum_{n=1}^N \sum_{k=1}^n (h_k^i - h_{k-1}^i)(S_n^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \\
&\quad = \sum_{k=1}^N (h_k^i - h_{k-1}^i)(S_N^i - S_{k-1}^i) - \varepsilon_i |h_k^i - h_{k-1}^i| S_{k-1}^i.
\end{aligned}$$

Denote $\tilde{S}_n^i = \mathbb{E}^\mu[S_N^i \mid \mathcal{F}_n]$ and

$$Y_n = \sum_{k=1}^n \sum_{i=1}^d (h_k^i - h_{k-1}^i)(\tilde{S}_n^i - S_{k-1}^i) - \varepsilon_i |h_k^i - h_{k-1}^i| S_{k-1}^i \quad \text{with } Y_0 = 0.$$

Then $Y_N = h(S)$, and if the conditional expectation is understood in the general sense of [13], we have

$$\begin{aligned}
&\mathbb{E}^\mu[Y_n \mid \mathcal{F}_{n-1}] - Y_{n-1} \\
&\quad = \sum_{i=1}^d \mathbb{E}^\mu\big[(h_n^i - h_{n-1}^i)(\tilde{S}_n^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \mid \mathcal{F}_{n-1}\big] \\
&\quad = \sum_{i=1}^d (h_n^i - h_{n-1}^i)(\tilde{S}_{n-1}^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \le 0.
\end{aligned}$$

This shows that Y_n is of the form $Y_n = M_n - A_n$, where M_n is a generalized μ-martingale starting at 0, and

$$A_n = \sum_{k=1}^n Y_{k-1} - \mathbb{E}^\mu[Y_k \mid \mathcal{F}_{k-1}],$$

a predictable increasing process. Since $\mathbb{E}^\mu M_N^- \le \mathbb{E}^\mu Y_N^- = \mathbb{E}^\mu h(S)^- < +\infty$, one obtains from [13] that (M_n) is a true μ-martingale. In particular, $h(S) = M_N - A_N$

is μ-integrable with $\mathbb{E}^\mu h(S) \le 0$. Therefore,

$$m + \sum_n \phi_n(g_n) \ge m + \sum_n \langle g_n, \nu_n \rangle \ge \mathbb{E}^\mu \big[m + \oplus g + h(S) \big] > 0,$$

which shows that $(m, \oplus g, h)$ cannot be a model-independent arbitrage.

Finally, let us assume (ii). Then it follows as in the proof of Corollary 4.2 that ϕ is a real-valued increasing convex functional on $B(\mathcal{P})$ such that $\phi(0) = 0$ and $\phi(f) \le m + \sum_n \phi_n(g_n)$ whenever $f \le m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\mathcal{P})$. Moreover, if

$$m + \oplus g + h(S) \ge f$$

for a strategy $(m, \oplus g, h) \in \Theta$ and $f \in B(\mathcal{P})$, one has, for all $\mu \in \mathcal{M}(\mathcal{P})$,

$$\mathbb{E}^\mu h(S)^- \le m^+ + \sum_n \langle g_n, \nu_n \rangle + \langle f^-, \mu \rangle \le m^+ + \sum_n \phi_n(g_n) + \langle f^-, \mu \rangle < +\infty.$$

So it follows as above that $\mathbb{E}^\mu h(S) \le 0$, and therefore $m + \sum_n \phi_n(g_n) \ge \langle f, \mu \rangle$. This implies that $\phi(f) \ge \langle f, \mu \rangle$, and, consequently, $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) = 0$ for all $\mu \in \mathcal{M}(\mathcal{P})$. It remains to show that

$$\phi^*_{C_b}(\mu) = +\infty \quad \text{for } \mu \in \mathcal{P} \setminus \mathcal{M}(\mathcal{P}). \tag{4.12}$$

Then Proposition 3.1 implies (4.11) and thereby also (iii).

To show (4.12), fix $\mu \in \mathcal{P}$. If

$$(1 - \varepsilon_i) s_0^i \le \mathbb{E}^\mu x_N^i \le (1 + \varepsilon_i) s_0^i$$

as well as

$$\mathbb{E}^\mu \big[v(x_1, \dots, x_n) \big(x_N^i - (1 + \varepsilon_i) x_n^i \big) \big] \le 0$$

and

$$\mathbb{E}^\mu \big[v(x_1, \dots, x_n) \big((1 - \varepsilon_i) x_n^i - x_N^i \big) \big] \le 0,$$

for all i, N, $n \le N$ and every bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}_+$, then

$$(1 - \varepsilon_i) S_n^i \le \mathbb{E}^\mu [S_N^i \mid \mathcal{F}_n] \le (1 + \varepsilon_i) S_n^i \quad \text{for all } i, N, \text{ and } n \le N.$$

Thus, for $\mu \in \mathcal{P} \setminus \mathcal{M}(\mathcal{P})$, there exists an f with $\langle f, \mu \rangle > 0$, where f is of the form $f(x) = x_N^i - (1 + \varepsilon_i) s_0^i$, $f(x) = (1 - \varepsilon_i) s_0^i - x_N^i$, $f(x) = v(x_1, \dots, x_n)(x_N^i - (1 + \varepsilon_i) x_n^i)$, or $f(x) = v(x_1, \dots, x_n)((1 - \varepsilon_i) x_n^i - x_N^i)$ for a bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}_+$. For $k \in \mathbb{N}$, define $f^k := f \wedge k$ and $f^k_k := f^k \vee (-k)$. By monotonicity, one has $\phi(f^k) \le \phi(f) \le 0$. Moreover,

$$f^k_k(x) = f^k(x) + \big(k + f(x) \big)^- \le f^k(x) + \big(c|x_n^i| - k/2 \big)^+ + \big(c|x_N^i| - k/2 \big)^+$$

for $c \in \mathbb{R}_+$ large enough. Since $w^k(x) := (c|x_n^i| - k/2)^+ + (c|x_N^i| - k/2)^+$ belongs to $G(\mathcal{P})$, one gets

$$\phi(w^k) \le \phi_n\big((c|x_n| - k/2)^+ \big) + \phi_N\big((c|x_N| - k/2)^+ \big) \to 0 \quad \text{for } k \to +\infty$$

by our assumption on $\mathcal{P}_n$. Hence, for k large enough, one has

$$\langle f_k^k, \mu\rangle - \phi(f_k^k) \geq \langle f^k, \mu\rangle - \phi(f^k) - \phi(w^k) \geq \langle f^k, \mu\rangle - \phi(w^k) > 0,$$

and therefore $\phi^*_{C_b}(\mu) = +\infty$. □

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[1]Department of Mathematics, University of Konstanz, 78464 Konstanz, Germany.
E-mail address: daniel.bartl@uni-konstanz.de; kupper@uni-konstanz.de

[2]Operations Research and Financial Engineering (ORFE), Princeton University, Princeton, NJ 08544, USA.
E-mail address: dito@princeton.edu

[3]Faculty of Mathematics, University of Vienna, 1090 Vienna, Austria.
E-mail address: ludovic.tangpi@univie.ac.at