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ON THE STABILITY OF THE ORTHOGONALITY EQUATION AND THE ORTHOGONALITY-PRESERVING PROPERTY WITH TWO UNKNOWN FUNCTIONS

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Abstract. For two unknown functions f, g, the equation

$$\big\langle f(x) \bigm| g(y)\big\rangle = \langle x \mid y\rangle$$

and its stability as well as the approximate orthogonality-preserving property

$$x \perp y \implies fx \perp^{\varepsilon} gy$$

are considered.

1. Introduction

The orthogonality equation and the related orthogonality-preserving property have been intensively studied recently in connection with functional analysis and operator theory, as well as functional equations (see [15]). In the present article, we consider both of these topics in a generalized setting—that is, with two unknown mappings.

Throughout, we use X, Y to mean real or complex inner-product spaces. The scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ sometimes will be restricted to $\mathbb{C}$. Moreover, for some results, the completeness of X, Y will be additionally assumed. For linear mappings, we will write fx, gy, and so on, instead of $f(x)$, $g(y)$.

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1.1. **Orthogonality equation.** The term *orthogonality equation* usually means the one with a single unknown function $f\colon X \to Y$,

$$\big\langle f(x) \bigm| f(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X. \tag{1.1}$$

It is not difficult to observe that f is a solution of (1.1) if and only if it is a linear isometry—and hence injective but not necessarily surjective (if so f is called a *unitary* mapping). The above equation was generalized in [6] by introducing two unknown functions $f, g\colon X \to Y$ so that we have

$$\big\langle f(x) \bigm| g(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X. \tag{1.2}$$

A description of solutions of (1.2), as well as some related topics, can be found in the authors' previous papers [6], [13].

It is obvious that a pair (f, g) satisfies (1.2) whenever the pair (g, f) does, and hence f and g share their necessary properties. It can be also noticed (see [6] for details and examples) that if (f, g) is a solution of (1.2), then both mappings f, g are injective but need not be surjective or linear. In particular, none of the f, g need satisfy (1.1). On the other hand, a pair of linear isometries—that is, solutions of (1.1)—need not be a solution of (1.2). Actually, a pair (f, g) consisting of two different solutions of (1.1) cannot be a solution of (1.2). Similarly, if only one of the mappings f or g is a surjective solution of (1.1), then (1.2) does not hold unless $g = f$. Surjectivity (or some approximate surjectivity) of one of mappings f and g has strong consequences.

Theorem 1.1 ([6, Lemmas 1.5, 1.6, Theorem 1.7])**.** *Suppose that (f, g) solves (1.2). If $f(X)^\perp = \{0\}$, then g is linear; if f is surjective, then both f and g are linear and $g(X)^\perp = \{0\}$. Moreover, if X and Y are Hilbert spaces (completeness being essential) and f is surjective, then both mappings f and g are linear, continuous, and bijective.*

1.2. **Orthogonality-preserving property.** For inner product spaces X, Y and a single linear mapping $f\colon X \to Y$ one can consider the *orthogonality-preserving property*

$$x \perp y \;\Rightarrow\; fx \perp fy, \quad \forall x, y \in X. \tag{1.3}$$

For two linear mappings $f, g\colon X \to Y$, an analogous property

$$x \perp y \;\Rightarrow\; fx \perp gy, \quad \forall x, y \in X \tag{1.4}$$

was introduced in [6], and the following characterization has been proved.

Theorem 1.2 ([6, Theorem 3.9])**.** *Let X, Y be inner product spaces, and let $f, g\colon X \to Y$ be linear mappings. The following conditions are equivalent with some $\gamma \in \mathbb{K}$:*

(1) $x \perp y \Rightarrow fx \perp gy$, $\forall x, y \in X$,
(2) $\langle fx \mid gy\rangle = \gamma\langle x \mid y\rangle$, $\forall x, y \in X$.

Moreover, each of the above conditions implies (and in complex spaces is also equivalent to)

(3) $\langle fx \mid gx\rangle = \gamma\|x\|^2$, $\forall x \in X$.

In subsequent parts of the paper, we describe approximate solutions of (1.2), as well as the class of the mappings which approximately preserves orthogonality. We also deal with stability problems for (1.2) and (1.4).

2. Stability of the orthogonality equation with two unknown functions

2.1. **Stability.** As before, X and Y are inner product spaces and $f, g\colon X \to Y$. Assume that the pair (f,g) is in some sense an approximate solution of (1.2). The question is how much (f,g) differs from an exact solution of (1.2). This is a standard problem in the theory of stability of functional equations (see monographs [8], [10], and numerous papers). Sometimes it happens that each approximate solution of a given equation is in fact an exact solution. We will call such a phenomenon a *superstability*. For the orthogonality equation (1.1), various types of stability have been considered.

Let us consider the classical Ulam–Hyers approach (see [17], [7]). Namely, assume that, with some $\varepsilon \geq 0$, we have

$$\big|\big\langle f(x) \,\big|\, g(y)\big\rangle - \langle x \mid y\rangle\big| \leq \varepsilon, \quad x,y \in X. \tag{2.1}$$

Observe that both f and g must be injective. Indeed, assuming $f(x_1) = f(x_2)$, we have, for an arbitrary $y \in X$,

$$\begin{aligned}\big|\langle x_1 - x_2 \mid y\rangle\big| &\leq \big|\langle x_1 \mid y\rangle - \big\langle f(x_1) \,\big|\, g(y)\big\rangle\big| + \big|\big\langle f(x_2) \,\big|\, g(y)\big\rangle - \langle x_2 \mid y\rangle\big| \\ &\leq 2\varepsilon.\end{aligned}$$

Taking $y := n(x_1 - x_2)$, we get $\|x_1 - x_2\| \leq \sqrt{\frac{2\varepsilon}{n}}$ for $n \in \mathbb{N}$; hence, $x_1 = x_2$.

We say that $f\colon X \to Y$ is *δ-surjective* if and only if for each $y \in Y$ there exists an x in X such that $\|f(x) - y\| \leq \delta$. A mapping $g\colon X \to Y$ is called *ε-additive* whenever $\|g(x+y) - g(x) - g(y)\| \leq \varepsilon$ for all $x, y \in X$. Let us note that approximate surjectivity of one of the mappings f, g satisfying (2.1) implies the approximate additivity of the other one.

Proposition 2.1. *Let X, Y be inner product spaces. If $f, g\colon X \to Y$ satisfy (2.1) and f is δ-surjective ($\delta \geq 0$), then*

$$\big\|g(x+y) - g(x) - g(y)\big\| \leq \delta + \sqrt{3\varepsilon}, \quad x, y \in X.$$

Proof. For arbitrary $x, y, z \in X$, we have

$$\begin{aligned}\big|\big\langle f(z) \,\big|\, g(x+y) - g(x) - g(y)\big\rangle\big| &\leq \big|\big\langle f(z) \,\big|\, g(x+y)\big\rangle - \langle z \mid x+y\rangle\big| \\ &\quad + \big|\big\langle f(z) \,\big|\, g(x)\big\rangle - \langle z \mid x\rangle\big| \\ &\quad + \big|\big\langle f(z) \,\big|\, g(y)\big\rangle - \langle z \mid y\rangle\big| \\ &\leq 3\varepsilon.\end{aligned} \tag{2.2}$$

Fix arbitrarily $x, y \in X$, and let $v := g(x+y) - g(x) - g(y)$. Since f is δ-surjective, there exists $z \in X$ such that $\|f(z) - v\| \leq \delta$. Let $u := f(z) - v$. Then $f(z) = u + v$ and $\|u\| \leq \delta$. Moreover, from (2.2), $|\langle u + v \mid v\rangle| \leq 3\varepsilon$. Thus

$$\|v\|^2 - \big|\langle u \mid v\rangle\big| \leq \big|\|v\|^2 + \langle u \mid v\rangle\big| = \big|\langle u + v \mid v\rangle\big| \leq 3\varepsilon,$$

and hence

$$\|v\|^2 \le 3\varepsilon + \big|\langle u \mid v\rangle\big| \le 3\varepsilon + \|u\|\|v\| \le 3\varepsilon + \delta\|v\|.$$

Solving the inequality $\|v\|^2 - \delta\|v\| - 3\varepsilon \le 0$, one gets

$$0 \le \|v\| \le \frac{\delta + \sqrt{\delta^2 + 12\varepsilon}}{2} \le \delta + \sqrt{3\varepsilon}.$$ □

In particular, if f is surjective ($\delta = 0$), then we have

$$\big\|g(x+y) - g(x) - g(y)\big\| \le \sqrt{3\varepsilon}, \quad x, y \in X.$$

Under the assumption of approximate surjectivity of one of the mappings f, g, we obtain the first stability result for equation (1.2).

Proposition 2.2. *Let X be an inner product space, and let Y be a Hilbert space. Assume that $f, g\colon X \to Y$ satisfy (2.1) (with some $\varepsilon \ge 0$). If f is δ-surjective, then there exists $g_0\colon X \to Y$ such that (f, g_0) satisfies (1.2) and $\|g(x) - g_0(x)\| \le \delta + \sqrt{3\varepsilon}$.*

Proof. According to Proposition 2.1, g is $(\delta + \sqrt{3\varepsilon})$-additive. By the classical Hyers theorem (see [7, Theorem 1]), the mapping

$$g_0(x) := \lim_{n\to\infty} 2^{-n} g(2^n x), \quad x \in X$$

is well defined and additive. Moreover,

$$\big\|g(x) - g_0(x)\big\| \le \delta + \sqrt{3\varepsilon}.$$

Using (2.1), putting $2^n y$ in place of y, and dividing by 2^n, we obtain

$$\big|\big\langle f(x) \bigm| 2^{-n} g(2^n y)\big\rangle - \langle x \mid 2^{-n} 2^n y\rangle\big| \le \frac{\varepsilon}{2^n}, \quad x, y \in X;$$

hence, letting $n \to \infty$,

$$\big\langle f(x) \bigm| g_0(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X.$$ □

Assuming surjectivity of one mapping, one gets superstability of (1.2).

Theorem 2.3. *Let X be an inner product space, let Y be a Hilbert space, and let $f, g\colon X \to Y$ satisfy (2.1) (with some $\varepsilon \ge 0$). If f is surjective, then (1.2) holds true.*

Proof. Due to Proposition 2.2, for some $g_0\colon X \to Y$, the pair (f, g_0) satisfies (1.2). Since f is surjective, it follows from Theorem 1.1 that f and g_0 are linear. Using the linearity of f, we get from (2.1), for arbitrary $x, y \in X$,

$$\big|\big\langle f(x) \bigm| g(y)\big\rangle - \langle x \mid y\rangle\big| = 2^{-n}\big|\big\langle f(2^n x) \bigm| g(y)\big\rangle - \langle 2^n x \mid y\rangle\big| \le \frac{\varepsilon}{2^n},$$

and, finally, letting $n \to \infty$, $\langle f(x) \mid g(y)\rangle = \langle x \mid y\rangle$. □

Notice that δ-surjectivity of f implies that $f(X)^{\perp} = \{0\}$ (the reverse is not true—see the example below). Indeed, suppose that $0 \neq y_0 \in f(X)^{\perp}$; then also $ny_0 \perp f(X)$ for $n \in \mathbb{N}$. On the other hand, for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that $\|f(x_n) - ny_0\| \leq \delta$. Thus $n^2\|y_0\|^2 \leq \|f(x_n)\|^2 + n^2\|y_0\|^2 = \|f(x_n) - ny_0\|^2 \leq \delta^2$—a contradiction.

The condition $f(X)^{\perp} = \{0\}$ is not sufficient for superstability.

Example 2.4. Given $X = Y = l^2$, we define $f = g : l^2 \to l^2$ by

$$f(x) = (\sqrt{\varepsilon}, x_1, x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in l^2.$$

One has $f(X)^{\perp} = \{0\}$, but there is no superstability. However, there exists $g_0(x) := (0, x_1, x_2, \ldots)$ such that $\langle f(x) \mid g_0(y)\rangle = \langle x \mid y\rangle$ for all x, y and $\|g(x) - g_0(x)\| = \sqrt{\varepsilon}$.

Assuming linearity of f or g and using similar elementary techniques, one may prove superstability for the more general class of approximate solutions (see [2] for one unknown mapping). Let us start with a simple observation.

Proposition 2.5. *Let X, Y be inner product spaces, and let $f, g\colon X \to Y$ be linear mappings. Suppose that, with some $p, q \in \mathbb{R}$ such that $p \neq 1$ or $q \neq 1$,*

$$\big|\big\langle f(x) \bigm| g(y)\big\rangle - \langle x \mid y\rangle\big| \leq \varepsilon\|x\|^p\|y\|^q, \quad x, y \in X \setminus \{0\}.$$

Then (1.2) holds.

Proof. Assume that $p > 1$. For $x, y \in X \setminus \{0\}$, we have

$$\big|\big\langle f(x) \bigm| g(y)\big\rangle - \langle x \mid y\rangle\big| = 2^n\big|\big\langle f(2^{-n}x) \bigm| g(y)\big\rangle - \langle 2^{-n}x \mid y\rangle\big| \leq 2^{n(1-p)}\varepsilon\|x\|^p\|y\|^q.$$

The right-hand side tends to 0 as $n \to \infty$; hence $\langle f(x) \mid g(y)\rangle = \langle x \mid y\rangle$ (for $x = 0$ or $y = 0$ it is obvious). If $p < 1$, then we replace 2 by $1/2$, and for $q \neq 1$ the proof is analogous. $\square$

Apparently, the exceptional case $p = q = 1$ is much more difficult to handle and also more interesting. It is treated in the following theorem but only under some additional assumptions.

Theorem 2.6. *Let X, Y be Hilbert spaces, and let $f, g\colon X \to Y$ be linear and bounded. Assume that g is invertible. If*

$$\big|\langle fx \mid gy\rangle - \langle x \mid y\rangle\big| \leq \varepsilon\|x\|\|y\|, \quad x, y \in X, \tag{2.3}$$

then there exists a linear and bounded mapping $f_0\colon X \to Y$ such that

$$\big\langle f_0(x) \bigm| g(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X,$$

and

$$\|f_0 - f\| \leq \varepsilon\|g^{-1}\|.$$

Proof. Let $f_0 := (g^{-1})^*$. Then, for arbitrary $x, y \in X$,

$$\langle f_0x \mid gy\rangle = \big\langle (g^{-1})^*x \bigm| gy\big\rangle = \langle x \mid g^{-1}gy\rangle = \langle x \mid y\rangle.$$

We have also, for $x, y \in X$,

$$\big|\big\langle (f - f_0)x \bigm| gy\big\rangle\big| = \big|\langle fx \mid gy\rangle - \langle x \mid y\rangle\big| \leq \varepsilon\|x\|\|y\|.$$

Take an arbitrary element $z \in Y$ and $y = g^{-1}z$. From the above we have

$$\big|\big\langle (f - f_0)x \bigm| z \big\rangle\big| \le \varepsilon \|g^{-1}\| \|x\| \|z\|,$$

and hence

$$\big\|(f - f_0)x\big\| = \sup_{\|z\|=1} \big|\big\langle (f - f_0)x \bigm| z \big\rangle\big| \le \varepsilon \|g^{-1}\| \|x\|, \quad x \in X,$$

and, finally,

$$\|f - f_0\| \le \varepsilon \|g^{-1}\|. \qquad \square$$

2.2. **Decomposition of approximate solutions.** In this section, we follow [2], where a decomposition of approximate solutions of the orthogonality equation with a single unknown function was studied. A similar decomposition of exact solutions of (1.2) was shown in [13].

We start with two auxiliary results.

Lemma 2.7. *Let $X \ne \emptyset$ be a set, let Y be a Hilbert space, and let $f, g\colon X \to Y$ be arbitrary mappings. Then there exist a subspace Y_0 of $\overline{\operatorname{Lin} g(X)}$ and mappings $f_1, g_1\colon X \to Y_0$, $f_2\colon X \to g(X)^\perp$, $g_2\colon X \to Y_0^\perp \cap \overline{\operatorname{Lin} g(X)}$ such that*

$$\big\langle f_1(x) \bigm| g_1(y) \big\rangle = \big\langle f(x) \bigm| g(y) \big\rangle, \quad x, y \in X, \tag{2.4}$$

$$f = f_1 + f_2, \qquad g = g_1 + g_2, \tag{2.5}$$

$$\overline{\operatorname{Lin} f_1(X)} = \overline{\operatorname{Lin} g_1(X)} = Y_0. \tag{2.6}$$

Proof. We will use several times the projection theorem for Hilbert spaces. Let $f_1, f_2\colon X \to Y$ be functions such that

$$f = f_1 + f_2,$$
$$f_1(x) \in \overline{\operatorname{Lin} g(X)}, \qquad f_2(x) \in g(X)^\perp, \quad x \in X.$$

Further, let $g_1, g_2\colon X \to Y$ be functions such that

$$g = g_1 + g_2,$$
$$g_1(x) \in \overline{\operatorname{Lin} f_1(X)}, \qquad g_2(x) \in f_1(X)^\perp, \quad x \in X.$$

Let $Y_0 := \overline{\operatorname{Lin} f_1(X)}$. We observe that

$$\begin{aligned}\big\langle f(x) \bigm| g(y) \big\rangle &= \big\langle f_1(x) + f_2(x) \bigm| g(y) \big\rangle = \big\langle f_1(x) \bigm| g(y) \big\rangle + \big\langle f_2(x) \bigm| g(y) \big\rangle \\ &= \big\langle f_1(x) \bigm| g(y) \big\rangle = \big\langle f_1(x) \bigm| g_1(y) \big\rangle + \big\langle f_1(x) \bigm| g_2(y) \big\rangle \\ &= \big\langle f_1(x) \bigm| g_1(y) \big\rangle, \quad x, y \in X.\end{aligned}$$

Since $g_1(X) \subset \overline{\operatorname{Lin} f_1(X)} \subset \overline{\operatorname{Lin} g(X)}$, we have

$$g_2(y) = g(y) - g_1(y) \in \overline{\operatorname{Lin} g(X)}, \quad y \in X.$$

Fix $x \in X$. Then $f_1(x) = u + v$, where $u \in \overline{\operatorname{Lin} g_1(X)} \subset \overline{\operatorname{Lin} g(X)}$, $v \in g_1(X)^\perp \cap \overline{\operatorname{Lin} g(X)}$.

Hence we have

$$\begin{aligned}
0 &= \big\langle v \bigm| g_1(y)\big\rangle = \big\langle f_1(x) \bigm| g_1(y)\big\rangle - \big\langle u \bigm| g_1(y)\big\rangle \\
&= \big\langle f_1(x) \bigm| g(y)\big\rangle - \big\langle u \bigm| g(y)\big\rangle = \big\langle f_1(x) - u \bigm| g(y)\big\rangle \\
&= \big\langle v \bigm| g(y)\big\rangle, \quad y \in X.
\end{aligned}$$

Then we get

$$v \in g(X)^{\perp} \cap \overline{\operatorname{Lin} g(X)} = \{0\},$$

which means that $f_1(x) = u \in \overline{\operatorname{Lin} g_1(X)}$, and so we have $f_1(X) \subset \overline{\operatorname{Lin} g_1(X)}$. Since $g_1(X) \subset \overline{\operatorname{Lin} f_1(X)}$, then $\overline{\operatorname{Lin} g_1(X)} = \overline{\operatorname{Lin} f_1(X)}$. □

Lemma 2.8. *Let X be a set, and let Y be an inner product space. Assume that mappings $f_1, g_1 \colon X \to Y$ satisfy*

$$\big\|f_1(x)\big\| \le \alpha(x), \quad x \in X,$$

or

$$\big\|g_1(x)\big\| \le \beta(x), \quad x \in X,$$

where $\alpha, \beta \colon X \to [0, \infty)$. Then there exist mappings $f_2 \colon X \to \overline{\operatorname{Lin} f_1(X)}$, $g_2 \colon X \to \overline{\operatorname{Lin} g_1(X)}$ such that

$$\big\langle f_2(x) \bigm| g_2(y)\big\rangle = 0, \quad x, y \in X, \tag{2.7}$$

$$\big\|f_1(x) - f_2(x)\big\| \le \alpha(x), \quad x \in X, \tag{2.8}$$

$$\big\|g_1(x) - g_2(x)\big\| \le \beta(x), \quad x \in X. \tag{2.9}$$

Proof. Assume that $\|f_1(x)\| \le \alpha(x)$ for all $x \in X$. Define $f_2(x) := 0$, and define $g_2(x) := g_1(x)$, $x \in X$. Conditions (2.7)–(2.9) are satisfied. If $\|g_1(x)\| \le \beta(x)$, $x \in X$, then we take $f_2 = f_1$ and $g_2 \equiv 0$. □

The main result from [2] reads as follows.

Theorem 2.9 ([2, Proposition 1, Theorem 1])**.** *Let X be an inner product space, let Y be a Hilbert space, and let $f \colon X \to Y$ satisfy*

$$\big|\big\langle f(x) \bigm| f(y)\big\rangle - \langle x \mid y\rangle\big| \le \Phi(x, y), \quad x, y \in X,$$

with $\Phi \colon X \times X \to [0, \infty)$ satisfying for some $c > 0$ the condition

$$\lim_{m+n\to\infty, m,n\in\mathbb{N}} c^{m+n}\Phi(c^{-m}x, c^{-n}y) = 0, \quad x, y \in X.$$

Then there exist a linear isometry $I : X \to Y$ (a solution of (1.1)) and a mapping $b : X \to Y$ such that

$$\big\|b(x)\big\| \le \sqrt{\Phi(x, x)}, \quad x \in X,$$

$$\big\langle I(x) \bigm| b(y)\big\rangle = 0, \quad x, y \in X,$$

and

$$f(x) = I(x) + b(x), \quad x \in X.$$

Moreover, such a decomposition is unique.

That is to say that each approximate solution of (1.1) can be decomposed into an exact solution of the equation and some disturbance. Moreover, one can notice that, under some additional assumptions (e.g., $\dim X = \dim Y < \infty$), b must vanish, which explains superstability in such a case. Some counterparts of the above result for two unknown mappings are given in the following theorems.

Theorem 2.10. *Let X be an inner product space, and let Y be a Hilbert space. Suppose that mappings $f, g\colon X \to Y$ with a control function $\Phi\colon X \times X \to [0, \infty)$ satisfy the following assumptions:*

$$\big|\big\langle f(x) \;\big|\; g(y)\big\rangle - \langle x \mid y\rangle\big| \le \Phi(x, y), \quad x, y \in X \setminus \{0\}, \tag{2.10}$$

$$\overline{\operatorname{Lin} f(X)} = \overline{\operatorname{Lin} g(X)} = Y, \tag{2.11}$$

and, with some $c, d \in \mathbb{K} \setminus \{0\}$,

$$\forall x, y \in X \quad \lim_{n\to\infty} |c|^{-n}\Phi(c^n x, y) = \lim_{n\to\infty} |d|^{-n}\Phi(x, d^n y) = 0, \tag{2.12}$$

$$\forall x \in X \quad \liminf_{n\to\infty}\big\|c^{-n} f(c^n x)\big\| < \infty \qquad \text{or} \qquad \operatorname{Lin} g(X) = Y, \tag{2.13}$$

$$\forall x \in X \quad \liminf_{n\to\infty}\big\|d^{-n} g(d^n x)\big\| < \infty \qquad \text{or} \qquad \operatorname{Lin} f(X) = Y. \tag{2.14}$$

Then there exist linear mappings $f_0, g_0\colon X \to Y$ and mappings $f_1, g_1\colon X \to Y$ such that $f = f_0 + f_1$, $g = g_0 + g_1$, and

$$\big\langle f_0(x) \;\big|\; g_0(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X, \tag{2.15}$$

$$\big|\big\langle f_1(x) \;\big|\; g_1(y)\big\rangle\big| \le \Phi(x, y), \quad x, y \in X \setminus \{0\}, \tag{2.16}$$

$$f_1(X) \subset g_0(X)^\perp, \qquad g_1(X) \subset f_0(X)^\perp. \tag{2.17}$$

Proof. Fix $x \in X$, fix $z \in Y$, and fix $\varepsilon > 0$. Let $F_n(x) = c^{-n} f(c^n x)$. First we assume that, for all $x \in X$, $\liminf_{n\to\infty} \|F_n(x)\| = M < \infty$. If $M = 0$, then there exists a subsequence $(F_{n_k}(x))_{k\in\mathbb{N}}$ of $(F_n(x))_{n\in\mathbb{N}}$ which is convergent (to zero). Now assume that $M > 0$. Then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\big\|F_{n_k}(x)\big\| \le 2M, \quad k \in \mathbb{N}.$$

Due to (2.11), there exist $m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in \mathbb{K} \setminus \{0\}$ and $y_1, \dots, y_m \in X$ such that

$$\Big\|z - \sum_{i=1}^m \alpha_i g(y_i)\Big\| < \frac{\varepsilon}{8M}.$$

Let $N \in \mathbb{N}$ be such that, for all $i \in \{1, \dots, m\}$,

$$|c|^{-n_k}\Phi(c^{n_k} x, y_i) < \frac{\varepsilon}{4m|\alpha_i|}, \quad k \ge N.$$

Then for $k, l \ge N$ we have

$$\begin{aligned} \big|\big\langle F_{n_k}(x) - F_{n_l}(x) \;\big|\; z\big\rangle\big| &\le \Big|\Big\langle F_{n_k}(x) - F_{n_l}(x) \;\Big|\; \sum_{i=1}^m \alpha_i g(y_i)\Big\rangle\Big| \\ &\quad + \Big|\Big\langle F_{n_k}(x) - F_{n_l}(x) \;\Big|\; z - \sum_{i=1}^m \alpha_i g(y_i)\Big\rangle\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} |\alpha_i| \Big(\big| \big\langle F_{n_k}(x) \mid g(y_i) \big\rangle - \langle x \mid y_i \rangle \big| \\
&\quad + \big| \langle x \mid y_i \rangle - \big\langle F_{n_l}(x) \mid g(y_i) \big\rangle \big| \Big) \\
&\quad + \big\| F_{n_k}(x) - F_{n_l}(x) \big\| \Big\| z - \sum_{i=1}^{m} \alpha_i g(y_i) \Big\| \\
&\leq \sum_{i=1}^{m} |\alpha_i| \big(|c^{-n_k}| \Phi(c^{n_k} x, y_i) + |c^{-n_l}| \Phi(c^{n_l} x, y_i) \big) \\
&\quad + \big(\|F_{n_k}(x)\| + \|F_{n_l}(x)\| \big) \Big\| z - \sum_{i=1}^{m} \alpha_i g(y_i) \Big\| \\
&< \sum_{i=1}^{m} |\alpha_i| \frac{2\varepsilon}{4m|\alpha_i|} + 4M \frac{\varepsilon}{8M} = \varepsilon.
\end{aligned}
$$

Therefore, $(F_{n_k}(x))_{k\in\mathbb{N}}$ is weakly Cauchy and hence a weakly convergent sequence.

Now, assume that $\operatorname{Lin} g(X) = Y$. Then there exist $m \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{K}\setminus\{0\}$, $y_1, \ldots, y_m \in X$ such that $z = \sum_{i=1}^{m} \alpha_i g(y_i)$. Let $N \in \mathbb{N}$ be such that, for all $i \in \{1, \ldots, m\}$,

$$
|c|^{-n} \Phi(c^n x, y_i) < \frac{\varepsilon}{2m|\alpha_i|}, \quad n \geq N.
$$

Then for $k, l \geq N$ we have

$$
\begin{aligned}
\big| \big\langle F_k(x) - F_l(x) \mid z \big\rangle \big| &\leq \Big| \Big\langle F_k(x) - F_l(x) \,\Big|\, \sum_{i=1}^{m} \alpha_i g(y_i) \Big\rangle \Big| \\
&\leq \sum_{i=1}^{m} |\alpha_i| \Big(\big| \big\langle F_k(x) \mid g(y_i) \big\rangle - \langle x \mid y_i \rangle \big| \\
&\quad + \big| \langle x \mid y_i \rangle - \big\langle F_l(x) \mid g(y_i) \big\rangle \big| \Big) \\
&\leq \sum_{i=1}^{m} |\alpha_i| \big(|c|^{-k} \Phi(c^k x, y_i) + |c|^{-l} \Phi(c^l x, y_i) \big) \\
&< \sum_{i=1}^{m} |\alpha_i| \frac{2\varepsilon}{2m|\alpha_i|} = \varepsilon,
\end{aligned}
$$

which, as above, shows weak convergence of $(F_n(x))_{n\in\mathbb{N}}$.

Thus, in both cases of (2.13), there exists a weakly convergent subsequence $(F_{n_k}(x))_{k\in\mathbb{N}}$ of $(F_n(x))_{n\in\mathbb{N}}$. We denote its weak limit by $f_0(x)$.

Observe that, by (2.12), for an arbitrary $y \in X$,

$$
\big| \big\langle F_{n_k}(x) \mid g(y) \big\rangle - \langle x \mid y \rangle \big| \leq |c|^{-n_k} \Phi(c^{n_k} x, y) \to 0, \quad \text{for } k \to \infty,
$$

and hence

$$
\big\langle f_0(x) \mid g(y) \big\rangle = \lim_{k\to\infty} \big\langle F_{n_k}(x) \mid g(y) \big\rangle = \langle x \mid y \rangle, \quad x, y \in X.
$$

Property (2.11) yields $g(X)^{\perp} = \{0\}$, and hence by Theorem 1.1, f_0 is linear.

Similarly, for $G_n(x) = d^{-n}g(d^n x)$, $x \in X$, there exists a linear mapping $g_0\colon X \to Y$ such that, for every $x \in X$, $g_0(x)$ is a weak limit of some subsequence of the sequence $(G_n(x))_{n\in\mathbb{N}}$ and

$$\big\langle f(x) \bigm| g_0(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X.$$

The above property also gives

$$\big\langle F_{n_k}(x) \bigm| g_0(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X,$$

and, finally,

$$\big\langle f_0(x) \bigm| g_0(y)\big\rangle = \lim_{k\to\infty}\big\langle F_{n_k}(x) \bigm| g_0(y)\big\rangle = \langle x \mid y\rangle, \quad x, y \in X.$$

It is easy to observe that, for $f_1 := f - f_0$, $g_1 := g - g_0$, we have

$$\begin{aligned}\big\langle f_1(x) \bigm| g_0(y)\big\rangle &= \big\langle f(x) - f_0(x) \bigm| g_0(y)\big\rangle = \langle x \mid y\rangle - \langle x \mid y\rangle = 0, \quad x, y \in X,\\ \big\langle f_0(x) \bigm| g_1(y)\big\rangle &= \big\langle f_0(x) \bigm| g(y) - g_0(y)\big\rangle = \langle x \mid y\rangle - \langle x \mid y\rangle = 0, \quad x, y \in X,\end{aligned}$$

which shows (2.17). To show (2.16), notice that

$$\begin{aligned}\big|\big\langle f_1(x) \bigm| g_1(y)\big\rangle\big| &= \big|\big\langle f(x) - f_0(x) \bigm| g(y) - g_0(y)\big\rangle\big|\\ &= \big|\big\langle f(x) \bigm| g(y)\big\rangle - \big\langle f(x) \bigm| g_0(y)\big\rangle\\ &\quad - \big\langle f_0(x) \bigm| g(y)\big\rangle + \big\langle f_0(x) \bigm| g_0(y)\big\rangle\big|\\ &= \big|\big\langle f(x) \bigm| g(y)\big\rangle - \langle x \mid y\rangle\big| \le \Phi(x, y).\end{aligned}$$

□

Remark 2.11. Notice that, in view of Lemma 2.7, the assumption (2.11) can be omitted without loss of generality. The same concerns the subsequent results.

Theorem 2.12. *Let X be an inner product space, and let Y be a Hilbert space. Let $f, g\colon X \to Y$ be functions such that (2.11) holds and*

$$\big|\big\langle f(x) \bigm| g(y)\big\rangle - \langle x \mid y\rangle\big| \le \varphi(x)\psi(y), \quad x, y \in X \setminus \{0\}, \tag{2.18}$$

given $\varphi, \psi\colon X \to [0,\infty)$. Assume that there exist $c, d \in \mathbb{K} \setminus \{0\}$ such that

$$\begin{aligned}&\lim_{n\to\infty} |c|^{-n}\varphi(c^n x) = \lim_{n\to\infty} |d|^{-n}\psi(d^n x) = 0, \quad \forall x \in X,\\ &\liminf_{n\to\infty}\big\|c^{-n}f(c^n x)\big\| < \infty, \quad \forall x \in X \qquad \textit{or} \qquad \operatorname{Lin} g(X) = Y,\\ &\liminf_{n\to\infty}\big\|d^{-n}g(d^n x)\big\| < \infty, \quad \forall x \in X \qquad \textit{or} \qquad \operatorname{Lin} f(X) = Y.\end{aligned}$$

Moreover, assume that there exists $M > 0$ such that

$$\big\|f(x) - c^{-n}f(c^n x)\big\| \le M\varphi(x), \quad x \in X, n \in \mathbb{N},$$

or

$$\big\|g(x) - d^{-n}g(d^n x)\big\| \le M\psi(x), \quad x \in X, n \in \mathbb{N}.$$

Then there exist mappings $F, G\colon X \to Y$ such that

$$\begin{aligned}\big\langle F(x) \bigm| G(y)\big\rangle &= \langle x \mid y\rangle, \quad x, y \in X,\\ \big\|f(x) - F(x)\big\| &\le M\varphi(x), \quad x \in X,\\ \big\|g(x) - G(x)\big\| &\le M\psi(x), \quad x \in X.\end{aligned}$$

Proof. The proof relies on Theorem 2.10 and Lemma 2.8. Notice that with $\Phi(x, y) := \varphi(x)\psi(y)$, $x, y \in X$, all the assumptions of Theorem 2.10 are satisfied, and hence there exist suitable mappings f_0, g_0, f_1, g_1. Notice that f_1, g_1 satisfy the assumptions of Lemma 2.8 with $\alpha(x) = M\varphi(x)$ and $\beta(x) = M\psi(x)$, $x \in X$. (It follows, in particular, from the definition of f_0, g_0.) Therefore, there exist mappings f_2, g_2 such that conditions (2.7)–(2.9) are satisfied. Now, taking $F = f_0 + f_2$, $G = g_0 + g_2$, and using, in particular, the fact that $f_0(X) \perp g_2(X)$, $f_2(X) \perp g_0(X)$, $f_2(X) \perp g_2(X)$, and $\langle f_0(x) \mid g_0(y)\rangle = \langle x \mid y\rangle$, $x, y \in X$, we get the assertion. $\square$

The following result has a simple, direct proof. However, it can also be immediately derived from Theorem 2.12.

Corollary 2.13. *Let X be an inner product space, and let Y be a Hilbert space. Let $f, g\colon X \to Y$ be functions such that (2.11) holds and*

$$\left|\big\langle f(x) \;\big|\; g(y)\big\rangle - \langle x \mid y\rangle\right| \le \varphi(x)\psi(y), \quad x, y \in X \setminus \{0\},$$

with $\varphi, \psi\colon X \to [0, \infty)$. Assume that there exist $c, d \in \mathbb{K} \setminus \{0\}$ such that f is c-homogeneous and g is d-homogeneous ($f(cx) = cf(x)$, $g(dx) = dg(x)$, $x \in X$). Moreover, assume that

$$\lim_{n\to\infty} |c|^{-n}\varphi(c^n x) = \lim_{n\to\infty} |d|^{-n}\psi(d^n x) = 0, \quad \forall x \in X.$$

Then (f, g) satisfies the orthogonality equation (1.2).

Proof. The assumptions of Theorem 2.12 are satisfied with an arbitrary $M > 0$. Hence it follows that $f = F$, $g = G$. $\square$

3. Stability of the orthogonality-preserving property with two unknown functions

3.1. **Approximate orthogonality-preserving property.** For inner product spaces X, Y and one linear mapping $f\colon X \to Y$ the *approximate orthogonality-preserving property*

$$x \perp y \;\Rightarrow\; fx \perp^{\varepsilon} fy, \quad \forall x, y \in X, \tag{3.1}$$

was introduced and examined in [4]. Here the approximate orthogonality relation is defined (for $\varepsilon \in [0, 1)$) by

$$u \perp^{\varepsilon} v \;\Leftrightarrow\; \big|\langle u \mid v\rangle\big| \le \varepsilon\|u\|\|v\|.$$

The stability of the orthogonality-preserving property was proved in [5] for the finite-dimensional case and in [16] for the general case. The results have been generalized in various ways, in particular, in [1], [3], [9], [11], [14], [18], [19], and [20].

Now, we will concentrate our investigations on the following condition, which we call the *approximate orthogonality-preserving property for two linear mappings* $f, g\colon X \to Y$. We assume that, for some $\varepsilon \in [0, 1)$,

$$x \perp y \;\Rightarrow\; fx \perp^{\varepsilon} gy, \quad \forall x, y \in X. \tag{3.2}$$

Lemma 3.1. *Suppose that X, Y are inner product spaces and that $f, g\colon X \to Y$ are linear mappings. Then the property (3.2) is equivalent to*

$$\left|\langle fx \mid gy\rangle - \frac{\langle fy \mid gy\rangle}{\|y\|^2}\langle x \mid y\rangle\right| \le \varepsilon\left\|fx - \frac{\langle x \mid y\rangle}{\|y\|^2}fy\right\|\|gy\|, \quad x, y \in X, y \neq 0. \tag{3.3}$$

Proof. Assume (3.2), and fix two vectors $x, y \in X$, $y \neq 0$. Notice that $x - \frac{\langle x|y\rangle}{\|y\|^2}y \perp y$, and hence $f(x - \frac{\langle x|y\rangle}{\|y\|^2}y) \perp^\varepsilon gy$ and (3.3) follows. The reverse is clear. □

Proposition 3.2. *Suppose that X, Y are Hilbert spaces and that $f, g\colon X \to Y$ are linear mappings satisfying (3.2). If $\overline{g(X)} = Y$, then f is continuous and* $\ker g \subset \ker f$.

Proof. We apply the closed graph theorem. Let $x_n \to 0$, and let $fx_n \to z$. It follows from (3.3) that

$$\left|\langle fx_n \mid gy\rangle - \frac{\langle fy \mid gy\rangle}{\|y\|^2}\langle x_n \mid y\rangle\right| \le \varepsilon\left\|fx_n - \frac{\langle x_n \mid y\rangle}{\|y\|^2}fy\right\|\|gy\|, \quad y \in X \setminus \{0\}.$$

Letting $n \to \infty$, the above inequality becomes

$$\left|\langle z \mid gy\rangle\right| \le \varepsilon\|z\|\|gy\|, \quad y \in X.$$

Since $\overline{g(X)} = Y$, it means that $z \perp^\varepsilon Y$, and hence $z = 0$. Thus the graph of f is closed and f must be continuous. We also have $x \perp y \Rightarrow gx \perp^\varepsilon fy$; hence, applying again (3.3), we get

$$\left|\langle gx \mid fy\rangle - \frac{\langle gy \mid fy\rangle}{\|y\|^2}\langle x \mid y\rangle\right| \le \varepsilon\left\|gx - \frac{\langle x \mid y\rangle}{\|y\|^2}gy\right\|\|fy\|, \quad x, y \in X, y \neq 0.$$

This for $y_0 \in \ker g$ becomes

$$\left|\langle gx \mid fy_0\rangle\right| \le \varepsilon\|gx\|\|fy_0\|, \quad x \in X,$$

which yields $fy_0 \perp^\varepsilon Y$, and hence $fy_0 = 0$. Thus $\ker g \subset \ker f$. □

Corollary 3.3. *Suppose that X, Y are Hilbert spaces and that $f, g\colon X \to Y$ are linear mappings satisfying (3.2). If $\overline{f(X)} = \overline{g(X)} = Y$, then f and g are continuous and* $\ker f = \ker g$.

3.2. **Stability of the orthogonality-preserving property.** Let us start with the following observation.

Proposition 3.4. *Let X, Y be inner product spaces, and let $f, g, f_0, g_0\colon X \to Y$ be linear mappings. Assume that f_0, g_0 satisfy (1.4); that is, for each $x, y \in X$,*

$$x \perp y \quad \Rightarrow \quad f_0x \perp g_0y.$$

Suppose that f, g are sufficiently close to f_0, g_0, respectively; namely, that, for an $\varepsilon \in [0, 1]$ and all $x, y \in X$,

$$\|fx - f_0x\| \le \frac{\varepsilon}{3}\|fx\| \qquad \text{and} \qquad \|gy - g_0y\| \le \frac{\varepsilon}{3}\|gy\|. \tag{3.4}$$

Then the pair (f, g) satisfies (3.2).

Proof. According to Theorem 1.2, for some $\gamma \in \mathbb{K}$, we have

$$\langle f_0 x \mid g_0 y \rangle = \gamma \langle x \mid y \rangle, \quad x, y \in X.$$

From inequalities (3.4) we get

$$\|f_0 x\| \le \Big(1 + \frac{\varepsilon}{3}\Big)\|fx\|, \quad x \in X, \qquad \text{and} \qquad \|g_0 y\| \le \Big(1 + \frac{\varepsilon}{3}\Big)\|gy\|, \quad y \in X.$$

For $x, y \in X$, we then have

$$\begin{aligned}
\big|\langle fx \mid gy \rangle - \gamma\langle x \mid y \rangle\big| &= \big|\langle fx \mid gy \rangle - \langle f_0 x \mid g_0 y \rangle\big| \\
&= \big|\langle fx - f_0 x \mid gy - g_0 y \rangle + \langle fx - f_0 x \mid g_0 y \rangle \\
&\quad + \langle f_0 x \mid gy - g_0 y \rangle\big| \\
&\le \|fx - f_0 x\|\|gy - g_0 y\| + \|fx - f_0 x\|\|g_0 y\| \\
&\quad + \|f_0 x\|\|gy - g_0 y\| \\
&\le \frac{\varepsilon}{3}(2 + \varepsilon)\|fx\|\|gy\| \\
&\le \varepsilon\|fx\|\|gy\|,
\end{aligned}$$

and (3.2) follows. □

Thus, roughly speaking, if f, g are close to mappings f_0, g_0 satisfying the orthogonality-preserving property, then the pair (f, g) approximately preserves orthogonality. It is our goal to answer a question of whether the reverse is true; that is, whether for each pair (f, g) approximately preserving orthogonality there exists a pair (f_0, g_0) which satisfies exactly the orthogonality-preserving property and is close to (f, g).

We are going to present a counterpart to the characterization in Theorem 1.2. We will need a simple lemma.

Lemma 3.5. *Let X be an inner product space, $x, y \in X$, $\|x\| = \|y\| = 1$, $\lambda \in \mathbb{K}$, $|\lambda| = 1$. Then*

$$\|x + \lambda y\|\|x - \lambda y\| \le 2.$$

Proof. We have

$$\begin{aligned}
\|x + \lambda y\|^2\|x - \lambda y\|^2 &= \big(2 + 2\operatorname{Re}\langle x \mid \lambda y \rangle\big)\big(2 - 2\operatorname{Re}\langle x \mid \lambda y \rangle\big) \\
&= 4\Big(1 - \big(\operatorname{Re}\langle x \mid \lambda y \rangle\big)^2\Big) \le 4.
\end{aligned}$$ □

For linear mappings $f, g \colon X \to Y$, we will consider the following assumption concerning their joint boundedness:

$$x \perp y \;\Rightarrow\; \|fx\|\|gy\| \le M\|x\|\|y\|, \quad \forall x, y \in X, \tag{3.5}$$

with some positive number M.

Obviously, if f and g are linear and bounded mappings, then the above condition is satisfied with $M = \|f\|\|g\|$. On the other hand, if X is a Hilbert space and f, g are nonzero linear mappings, and (3.5) holds true, then f and g have to be bounded. Indeed, let $x_0 \notin \ker f$, and let $X_0 := \operatorname{Lin} x_0$. For an arbitrary $y \in X_0^{\perp}$, we have $\|g(y)\| \le \frac{M\|x_0\|}{\|fx_0\|}\|y\|$; that is, g is bounded on $X_0^{\perp}$ and obviously also on X_0. Thus g is bounded on $X = X_0 \oplus X_0^\perp$. Similarly, one can show that f is bounded. The reason for considering (3.5) is that for some f, g, the constant M appearing in (3.5) may be less than $\|f\|\|g\|$.

Theorem 3.6. *Suppose that X, Y are inner product spaces and that $f, g\colon X \to Y$ are linear mappings satisfying (3.2) and (3.5). Then, for an arbitrary $y_0 \in X$ such that $\|y_0\| = 1$ and $\gamma := \langle fy_0 \mid gy_0\rangle$,*

$$\big|\langle fx \mid gy\rangle - \gamma\langle x \mid y\rangle\big| \le 4M\varepsilon\|x\|\|y\|, \quad x, y \in X. \tag{3.6}$$

Moreover,

$$\big|\langle fx \mid gx\rangle - \gamma\|x\|^2\big| \le 2M\varepsilon\|x\|^2, \quad x \in X. \tag{3.7}$$

Proof. Fix arbitrary $x, y \in X$ such that $\|x\| = \|y\| = 1$. Let $\lambda = 1$ if $x \perp y$ and $\lambda = \frac{\langle x|y\rangle}{|\langle x|y\rangle|}$, otherwise. Then $|\lambda| = 1$, $x+\lambda y \perp x - \lambda y$. Thus $f(x+\lambda y) \perp^\varepsilon g(x-\lambda y)$ and $f(x-\lambda y) \perp^\varepsilon g(x+\lambda y)$. It follows from this, (3.5), and Lemma 3.5 that

$$\begin{aligned}\big|\big\langle f(x+\lambda y) \bigm| g(x-\lambda y)\big\rangle\big| &\le \varepsilon\big\|f(x+\lambda y)\big\|\big\|g(x-\lambda y)\big\| \\ &\le M\varepsilon\|x+\lambda y\|\|x-\lambda y\| \\ &\le 2M\varepsilon.\end{aligned}$$

Similarly, we have

$$\big|\big\langle f(x-\lambda y) \bigm| g(x+\lambda y)\big\rangle\big| \le 2M\varepsilon.$$

Using the above estimations, it follows that

$$\begin{aligned}\big|\langle fx \mid gx\rangle - \langle fy \mid gy\rangle\big| &= \big|\langle fx \mid gx\rangle - \langle \lambda fy \mid \lambda gy\rangle\big| \\ &= \frac{|\langle f(x+\lambda y) \mid g(x-\lambda y)\rangle + \langle f(x-\lambda y) \mid g(x+\lambda y)\rangle|}{2} \\ &\le \frac{2M\varepsilon + 2M\varepsilon}{2} = 2M\varepsilon.\end{aligned}$$

Thus we have obtained

$$\|x\| = \|y\| = 1 \quad \Rightarrow \quad \big|\langle fx \mid gx\rangle - \langle fy \mid gy\rangle\big| \le 2M\varepsilon.$$

Now, take $x, y_0 \in X$ such that $x \neq 0$ and $\|y_0\| = 1$. We then have

$$\left|\left\langle f\Big(\frac{x}{\|x\|}\Big) \Bigm| g\Big(\frac{x}{\|x\|}\Big)\right\rangle - \langle fy_0 \mid gy_0\rangle\right| \le 2M\varepsilon.$$

With $\gamma := \langle fy_0 \mid gy_0\rangle$, it yields

$$\big|\langle fx \mid gx\rangle - \gamma\|x\|^2\big| \le 2M\varepsilon\|x\|^2,$$

and hence (3.7) is proved. Now fix $x, y \in X$, $y \neq 0$. Define $\alpha := -\frac{\langle x|y\rangle}{\|y\|^2}$. Then $x + \alpha y \perp y$ and, consequently,

$$\begin{aligned}\big|\langle fx \mid gy\rangle - \gamma\langle x \mid y\rangle\big| &= \big|\big\langle f(x+\alpha y - \alpha y) \bigm| gy\big\rangle - \gamma\langle x + \alpha y - \alpha y \mid y\rangle\big| \\ &= \big|\big\langle f(x+\alpha y) \bigm| gy\big\rangle - \big\langle f(\alpha y) \bigm| gy\big\rangle + \gamma\langle \alpha y \mid y\rangle\big| \\ &\le \varepsilon\big\|f(x+\alpha y)\big\|\|gy\| + |\alpha| \cdot 2M\varepsilon\|y\|^2 \\ &\le M\varepsilon\|x+\alpha y\|\|y\| + \big|\langle x \mid y\rangle\big| \cdot 2M\varepsilon\end{aligned}$$

$$\begin{aligned}&\le M\varepsilon\big(|\alpha|\|y\| + \|x\|\big)\|y\| + \|x\|\|y\| \cdot 2M\varepsilon\\&= 4M\varepsilon\|x\|\|y\|.\end{aligned}$$

If $y = 0$, then the desired inequality holds trivially. □

Remark 3.7. The definition of γ admits that its value can be equal to 0. However, without loss of generality one may assume that $\gamma \neq 0$. Suppose that $\langle fy_0 \mid gy_0\rangle = 0$ for all y_0 such that $\|y_0\| = 1$. This would imply that $\langle fx \mid gx\rangle = 0$ for all $x \in X$. But in such a case, inequality (3.6) is satisfied with any $\gamma' \in [0, 2M\varepsilon]$. Indeed, for any $x, y \in X$, $y \neq 0$, and $\alpha := -\frac{\langle x|y\rangle}{\|y\|^2}$, we have $x + \alpha y \perp y$ and, similarly as above,

$$\begin{aligned}\big|\langle fx \mid gy\rangle - \gamma'\langle x \mid y\rangle\big| &= \big|\big\langle f(x+\alpha y) \bigm| gy\big\rangle - \alpha\langle fy \mid gy\rangle + \gamma'\langle \alpha y \mid y\rangle\big|\\&= \big|\big\langle f(x+\alpha y) \bigm| gy\big\rangle + \gamma'\langle \alpha y \mid y\rangle\big|\\&\le \big\|f(x+\alpha y)\big\|\|gy\| + \gamma'|\alpha|\big|\langle y \mid y\rangle\big|\\&\le M\varepsilon\|x+\alpha y\|\|y\| + \gamma'\big|\langle x \mid y\rangle\big|\\&\le 4M\varepsilon\|x\|\|y\|.\end{aligned}$$

Now, let us consider a particular case where f, g are linear and bounded mappings on a complex Hilbert space. Then the constant 4 in (3.6) can be replaced by 1 (which turns out to be the best approximation).

Theorem 3.8. *Suppose that X is a complex Hilbert space and that $f, g\colon X \to X$ are linear mappings satisfying (3.2) and (3.5). Then there exists a constant $\gamma \in \mathbb{C}$ such that*

$$\big|\langle fx \mid gy\rangle - \gamma\langle x \mid y\rangle\big| \le M\varepsilon\|x\|\|y\|, \quad x, y \in X, \tag{3.8}$$

and

$$\|g^*f - \gamma \mathrm{Id}\| = \min\big\{\|g^*f - \lambda\mathrm{Id}\| : \lambda \in \mathbb{C}\big\}.$$

Proof. Define $\varphi\colon \mathbb{C} \to \mathbb{R}$, $\varphi(\lambda) := \|g^*f - \lambda\mathrm{Id}\|$. Since φ is a convex mapping and $\lim_{|\lambda|\to\infty} \varphi(\lambda) = \infty$, then φ attains its minimum; that is, there exists $\gamma \in \mathbb{C}$ such that

$$\|g^*f - \gamma\mathrm{Id}\| = \min\big\{\|g^*f + \lambda\mathrm{Id}\| : \lambda \in \mathbb{C}\big\}.$$

It is known (see [1], [12]) that, for an arbitrary linear and bounded operator $A\colon X \to X$,

$$\min\big\{\|A + \lambda\mathrm{Id}\| : \lambda \in \mathbb{C}\big\} = \sup\big\{\big|\langle Ax \mid y\rangle\big| : \|x\| = \|y\| = 1, x \perp y\big\}.$$

Thus we have

$$\begin{aligned}\|g^*f - \gamma\mathrm{Id}\| &= \sup\big\{\big|\langle g^*fx \mid y\rangle\big| : \|x\| = \|y\| = 1, x \perp y\big\}\\&= \sup\big\{\big|\langle fx \mid gy\rangle\big| : \|x\| = \|y\| = 1, x \perp y\big\}\\&\le \varepsilon \sup\big\{\|fx\|\|gy\| : \|x\| = \|y\| = 1, x \perp y\big\}\\&\le M\varepsilon \sup\big\{\|x\|\|y\| : \|x\| = \|y\| = 1, x \perp y\big\}\\&= M\varepsilon.\end{aligned}$$

Now, for arbitrary $x, y \in X$, we get from the above estimation

$$\begin{aligned}\big|\langle fx \mid gy\rangle - \gamma\langle x \mid y\rangle\big| &= \big|\langle g^*fx - \gamma x \mid y\rangle\big| \le \|g^*f - \gamma \mathrm{Id}\|\|x\|\|y\| \\ &\le M\varepsilon\|x\|\|y\|. \qquad \square\end{aligned}$$

Below we give a direct application of Theorem 3.8 to the case $f = g$.

Corollary 3.9. *Let X be a Hilbert space over $\mathbb{C}$, and let $f\colon X \to X$ be a nonzero linear mapping satisfying (3.1). Then f is continuous and there exists $\gamma \in \mathbb{C}$ such that*

$$\big|\langle fx \mid fy\rangle - \gamma\langle x \mid y\rangle\big| \le \varepsilon\|f\|^2\|x\|\|y\|, \quad x, y \in X. \tag{3.9}$$

Moreover, $(1-\varepsilon)\|f\|^2 \le |\gamma|$.

Proof. It follows from [4, Theorem 2] that f is continuous. Applying Theorem 3.8 for $f = g$, one gets (3.9). For $y = x$, we also get $|\|fx\|^2 - \gamma\|x\|^2| \le \varepsilon\|f\|^2\|x\|^2$, which yields $\|fx\|^2 \le (|\gamma| + \varepsilon\|f\|^2)\|x\|^2$. Passing to the supremum over $\|x\| = 1$, we get $\|f\|^2 \le |\gamma| + \varepsilon\|f\|^2$. $\square$

The inequality (3.9) improves the respective ones given in [4] and [16].

Finally, we present a result concerning the stability of the orthogonality-preserving property for two linear mappings.

Theorem 3.10. *Let X, Y be Hilbert spaces, and let $f, g\colon X \to Y$ be linear mappings satisfying (3.2) and (3.5). Moreover, assume that g is invertible. Then there exists a linear mapping $f_0\colon X \to Y$ such that*

$$x \perp y \quad \Rightarrow \quad f_0x \perp gy, \quad x, y \in X, \tag{3.10}$$

and $\|f - f_0\| \le 4M\varepsilon\|g^{-1}\|$.

Proof. It follows from Theorem 3.6 that, with some $\gamma \in \mathbb{K}$,

$$\big|\langle fx \mid gy\rangle - \gamma\langle x \mid y\rangle\big| \le 4M\varepsilon\|x\|\|y\|,$$

and we may assume that $\gamma \neq 0$ (see Remark 3.7). Let $f_1 := \frac{1}{\gamma}f$. Then we have

$$\big|\langle f_1x \mid gy\rangle - \langle x \mid y\rangle\big| \le \frac{4M\varepsilon}{|\gamma|}\|x\|\|y\|,$$

and from Theorem 2.6 there exists $f_2\colon X \to Y$ such that

$$\langle f_2x \mid gy\rangle = \langle x \mid y\rangle, \quad x, y \in X,$$

and

$$\|f_2 - f_1\| \le \frac{4M\varepsilon}{|\gamma|}\|g^{-1}\|.$$

Now, take $f_0 := \gamma f_2$ to get the assertion. $\square$

3.3. **Applications.** Theorems 3.6 and 3.8 yield the following result.

Theorem 3.11. *Let X be an inner product space, and let $f\colon X \to X$ be a linear and bounded mapping satisfying*

$$x \perp y \;\Rightarrow\; fx \perp^{\varepsilon} y, \forall x, y \in X.$$

Then there exists $\gamma \in \mathbb{K}$ such that

$$\|fx - \gamma x\| \leq 4\varepsilon \|f\| \|x\|, \quad x \in X.$$

If X is a complex Hilbert space, then the above estimation can be strengthened to

$$\|fx - \gamma x\| \leq \varepsilon \|f\| \|x\|, \quad x \in X.$$

Proof. For the general case, we apply Theorem 3.6 with $g = \mathrm{Id}$ and $M = \|f\|$. For some $\gamma \in \mathbb{K}$ it follows that

$$\left|\langle fx - \gamma x \mid y\rangle\right| \leq 4\varepsilon \|f\| \|x\| \|y\|, \quad x, y \in X.$$

For $y = fx - \gamma x$, we get

$$\|fx - \gamma x\|^2 \leq 4\varepsilon \|f\| \|x\| \|fx - \gamma x\|,$$

and hence either $\|fx - \gamma x\| = 0$ or $\|fx - \gamma x\| \leq 4\varepsilon \|f\| \|x\|$ and the assertion follows. If X is a complex Hilbert space, then we use Theorem 3.8 and replace the constant 4 by 1. □

In particular, for $\varepsilon = 0$, we get that a linear and bounded mapping $f\colon X \to X$ satisfies

$$x \perp y \;\Rightarrow\; fx \perp y, \quad \forall x, y \in X,$$

if and only if $fx = \gamma x$, $x \in X$ for some $\gamma \in \mathbb{K}$. This assertion, however, can be obtained without the assumption of boundedness of f (see [6, Corollary 3.6]).

The following result can be considered as a generalization of Theorem 3.11. We assume here that X is a Hilbert space.

Theorem 3.12. *Let X be a Hilbert space, and let $T, U\colon X \to X$ be linear and bounded operators on X. Suppose that U is a surjective isometry and that (3.2) holds true; that is,*

$$x \perp y \;\Rightarrow\; Tx \perp^{\varepsilon} Uy, \quad \forall x, y \in X. \tag{3.11}$$

Then there exists $\gamma \in \mathbb{K}$ such that

$$\|T - \gamma U\| \leq 4\varepsilon \|T\|. \tag{3.12}$$

Moreover, if $\varepsilon < \frac{1}{4}$, then $\gamma \neq 0$.

In the case when X is a complex Hilbert space, there exists $\gamma \in \mathbb{C}$ such that

$$\|T - \gamma U\| \leq \varepsilon \|T\|, \tag{3.13}$$

and if $\varepsilon < 1$, then $\gamma \neq 0$.

Proof. The condition (3.5) is satisfied for $f = T$ and $g = U$ with $M = \|T\|$, and hence by Theorem 3.6 we get

$$\big|\langle Tx \mid Uy\rangle - \gamma\langle x \mid y\rangle\big| \leq 4\varepsilon\|T\|\|x\|\|y\|, \quad x, y \in X.$$

Next we get, for $x, y \in X$,

$$\big|\langle U^*Tx - \gamma x \mid y\rangle\big| \leq 4\varepsilon\|T\|\|x\|\|y\|.$$

Putting $U^*Tx - \gamma x$ in place of y, we get, for an arbitrary $x \in X$,

$$\|U^*Tx - \gamma x\|^2 \leq 4\varepsilon\|T\|\|x\|\|U^*Tx - \gamma x\|.$$

Therefore, $\|U^*Tx - \gamma x\| \leq 4\varepsilon\|T\|\|x\|$ for $x \in X$, and hence $\|U^*T - \gamma \mathrm{Id}\| \leq 4\varepsilon\|T\|$. Finally,

$$\|T - \gamma U\| = \|U^*T - \gamma \mathrm{Id}\| \leq 4\varepsilon\|T\|.$$

In a similar way, by Theorem 3.8, we obtain $\|T - \gamma U\| \leq \varepsilon\|T\|$ for $\mathbb{K} = \mathbb{C}$. □

In some sense, the reverse result is also true.

Theorem 3.13. *Let X be a Hilbert space, and let T, U be linear and bounded operators on X. Assume that U is an isometry (not necessarily surjective) and that there exists $\gamma \neq 0$ such that*

$$\|Tx - \gamma Ux\| \leq \varepsilon\|Tx\|, \quad x \in X.$$

Then the operators T, U satisfy (3.11).

Proof. For $x, z \in X$, we have

$$\big|\langle Tx \mid z\rangle - \langle \gamma Ux \mid z\rangle\big| = \big|\langle Tx - \gamma Ux \mid z\rangle\big| \leq \|Tx - \gamma Ux\|\|z\| \leq \varepsilon\|Tx\|\|z\|.$$

Putting Uy ($y \in X$) in place of z, we obtain

$$\big|\langle Tx \mid Uy\rangle - \gamma\langle x \mid y\rangle\big| = \big|\langle Tx \mid Uy\rangle - \gamma\langle Ux \mid Uy\rangle\big| \leq \varepsilon\|Tx\|\|Uy\|, \quad x, y \in X,$$

and the assertion follows. □

Let X be a Hilbert space over $\mathbb{C}$. Let $U, T \in \mathcal{B}(X)$ be isometries. Suppose that the first one is surjective, whereas the second one is not. It is clear that both mappings preserve orthogonality; that is,

$$x \perp y \;\Rightarrow\; Ux \perp Uy, \quad \forall x, y \in X, \qquad \text{and} \qquad x \perp y \;\Rightarrow\; Tx \perp Ty, \quad \forall x, y \in X.$$

However, the pair (T, U) cannot, even approximately, preserve orthogonality (for any $\varepsilon \in [0, 1)$).

Theorem 3.14. *Let X be a complex Hilbert space. Let $U, T \in \mathcal{B}(X)$ be isometries. Suppose that U is unitary, and assume that T is not surjective. Then there is no $\varepsilon \in [0, 1)$ such that the condition*

$$x \perp y \;\Rightarrow\; Tx \perp^{\varepsilon} Uy, \quad \forall x, y \in X$$

holds true.

Proof. Assume, contrary to our claims, that for some $\varepsilon \in [0,1)$ there is $x \perp y \Rightarrow Tx \perp^{\varepsilon} Uy$. Applying Theorem 3.8, we get

$$\big|\langle Tx \mid Uy\rangle - \gamma\langle x \mid y\rangle\big| \le \varepsilon\|x\|\|y\|, \quad x, y \in X.$$

Putting U^*z in place of y, we get

$$\big|\langle Tx \mid UU^*z\rangle - \gamma\langle x \mid U^*z\rangle\big| \le \varepsilon\|x\|\|U^*z\|, \quad x, z \in X,$$

and hence (since $UU^* = \mathrm{Id} = U^*U$ and $\|U^*z\| = \|z\|$) we have

$$\big|\langle Tx \mid z\rangle - \gamma\langle Ux \mid z\rangle\big| \le \varepsilon\|x\|\|z\|$$

and

$$\big|\langle Tx - \gamma Ux \mid z\rangle\big| \le \varepsilon\|x\|\|z\|.$$

Passing to the supremum over $\|x\| \le 1$, $\|z\| \le 1$, we obtain $\|T - \gamma U\| \le \varepsilon$. It is easy to notice that $T^*T = \mathrm{Id}$ (but $TT^* \ne \mathrm{Id}$) and that $\|T^*\| = 1$. Therefore, we have

$$\|\mathrm{Id} - \gamma T^*U\| = \|T^*T - \gamma T^*U\| = \big\|T^*(T - \gamma U)\big\| \le \|T^*\|\|T - \gamma U\| \le \varepsilon < 1.$$

It follows that $\|\mathrm{Id} - \gamma T^*U\| < 1$, and hence γT^*U is invertible; hence, T^* is also invertible, and finally T is invertible, which is a contradiction. □

Corollary 3.15. *If $T, U \in \mathcal{B}(X)$ are isometries such that $T(X) \subsetneqq U(X)$, then there is no $\varepsilon \in [0,1)$ such that the condition*

$$x \perp y \;\Rightarrow\; Tx \perp^{\varepsilon} Uy, \quad x, y \in X$$

holds.

Proof. There is a linear surjective isometry $A\colon U(X) \to X$. It is enough to consider AT and AU. □

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