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MARTINGALE HARDY SPACES WITH VARIABLE EXPONENTS

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Abstract. In this paper, we introduce Hardy spaces with variable exponents defined on a probability space and develop the martingale theory of variable Hardy spaces. We prove the weak-type and strong-type inequalities on Doob's maximal operator, and we get a $(1, p(\cdot), \infty)$-atomic decomposition for Hardy martingale spaces associated with conditional square functions. As applications, we obtain a dual theorem and the John–Nirenberg inequalities in the frame of variable exponents. The key ingredient is that we find a condition with a probabilistic characterization of $p(\cdot)$ to replace the so-called log-Hölder continuity condition in $\mathbb{R}^n$.

1. Introduction

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function such that $0 < \inf_{x\in\mathbb{R}^n} p(x) \leq \sup_{x\in\mathbb{R}^n} p(x) < \infty$. The space $L^{p(\cdot)}(\mathbb{R}^n)$, the Lebesgue space with variable exponent $p(\cdot)$, is defined as the set of all measurable functions f such that, for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx < \infty$$

with

$$\|f\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx \leq 1\Big\}.$$

Then $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ is a quasinormed space. Such Lebesgue spaces were introduced by Orlicz [24] in 1931 and studied by O. Kovàčik and J. Rákosník [17], X. Fan and

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D. Zhao [9], and others. We refer to two new monographs [3] and [7] for the recent progress on Lebesgue spaces with variable exponents and some applications in PDEs and variational integrals with nonstandard growth conditions. We also note that in recent years more attention has turned to the study of function spaces with variable exponent in harmonic analysis (see, e.g., [1], [4], [5], [8], [22], [26], [30]). Let $\Omega \subset \mathbb{R}^n$. We say that a function $p(\cdot) : \Omega \to \mathbb{R}$ is *locally log-Hölder continuous on* Ω if there exists $c_1 > 0$ such that

$$\big|p(x) - p(y)\big| \leq \frac{c_1}{\log(e + 1/|x-y|)} \tag{1.1}$$

for all $x, y \in \Omega$. Heavily relying on the so-called log-Hölder continuity conditions on the variable exponent functions, in the pioneering work [6], Diening proved that the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. An example in [25] showed that log-Hölder continuity of $p(x)$ is essentially the optimal condition when the maximal operator is bounded on variable exponent Lebesgue spaces defined on Euclidean spaces (even in the doubling metric measure spaces; see [12]). We refer to [18] for more questions related to the maximal operator in variable $L^{p(\cdot)}$.

Although variable exponent Lebesgue spaces on Euclidean space have attracted a steadily increasing interest over the last couple of years, the variable exponent framework has not yet been applied to the probability space setting. The purpose of the present paper is to introduce Hardy martingale spaces with variable exponent and to develop the martingale theory of variable Hardy spaces. To the best of our knowledge, our paper is the first treating Hardy martingale spaces from this perspective. For convenience, we first fix some notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $\mathcal{P} = \mathcal{P}(\Omega)$ denote the collection of all measurable functions $p(\cdot)$: $\Omega \longrightarrow (0, \infty)$, which is called a *variable exponent*. For a measurable set $A \subset \Omega$, we denote

$$p_+(A) = \sup_{x \in A} p(x), \qquad p_-(A) = \inf_{x \in A} p(x)$$

and

$$p_+ = p_+(\Omega), \qquad p_- = p_-(\Omega).$$

Compared with the Euclidean space $\mathbb{R}^n$, the probability space Ω has no natural metric structure. The main difficulty is how to overcome the log-Hölder continuity (1.1) when $p(x)$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The first aim of this paper is to discuss the weak-type and strong-type inequalities about Doob's maximal operator. Aoyama [2] proved that Doob's maximal inequality is true under some conditions. Namely, if $1 \leq p(\cdot) < \infty$ and there exists a constant C such that

$$\frac{1}{p(\cdot)} \leq C\mathbb{E}\Big(\frac{1}{p(\cdot)} \;\Big|\; \mathcal{F}_n\Big), \tag{1.2}$$

then

$$\mathbb{P}\big(\sup_n |f_n| > \lambda\big) \leq C_{p(\cdot)} \int_\Omega \Big(\frac{|f_\infty|}{\lambda}\Big)^{p(\cdot)} d\mathbb{P}, \quad \forall \lambda > 0. \tag{1.3}$$

And if $1 < p_- \le p_+ < \infty$ and $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \ge 0$, then

$$\big\|\sup_n |f_n|\big\|_{p(\cdot)} \le C_{p(\cdot)} \|f\|_{p(\cdot)}. \tag{1.4}$$

Obviously, the condition that $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \ge 0$ is quite strict. In 2013, Nakai and Sadasue [21] pointed out that there exists a variable exponent $p(\cdot)$ such that $p(\cdot)$ is not $\mathcal{F}_0$-measurable, but (1.4) still holds. In this paper, we obtain the weak-type inequality (1.3) without condition (1.2). Unfortunately, we cannot obtain (1.4) directly by using the weak-type inequality as the classical case. This is because the space $L^{p(\cdot)}$ is no longer a rearrangement invariant space, and the formula

$$\int_\Omega \big|f(x)\big|^p \, d\mathbb{P} = p \int_0^\infty t^{p-1} \mathbb{P}\big(x \in \Omega : \big|f(x)\big| > t\big)\, dt$$

has no variable exponent analogue (see [7]). In order to describe the strong-type Doob maximal inequality, we find the following condition without metric characterization of $p(x)$ to replace log-Hölder continuity in some sense; that is, there exists an absolute constant $K_{p(\cdot)} \ge 1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(A)^{p_-(A) - p_+(A)} \le K_{p(\cdot)}, \quad \forall A \in \mathcal{F}. \tag{1.5}$$

We often denote $K_{p(\cdot)}$ simply by K if there is no confusion. Under the condition of (1.5), we prove that (1.4) is true for any martingale with respect to the atom σ-algebra filtration. It should be mentioned that the condition (1.5) is not usually true (even in Euclidean space); however, if the exponent $p(x)$ has a nice uniform continuity with respect to Euclidean distance, then (1.5) holds. We refer to Lemma 3.2 in [6] for this fact.

The second aim of this paper is the atomic characterization of variable Hardy martingale spaces. Our result can be regarded as the probability version of [5] and [22]; we do not use the log-Hölder continuity of $p(x)$, and it seems that our proofs are simpler due to the stopping-time techniques used for them. Let $\mathcal{T}$ be the set of all stopping times with respect to $\{\mathcal{F}_n\}_{n\ge 0}$. For a martingale $f = (f_n)_{n\ge 0}$ and $\tau \in \mathcal{T}$, we denote the stopped martingale by $f^\tau = (f_n^\tau)_{n\ge 0} = (f_{n\wedge\tau})_{n\ge 0}$.

Definition 1.1. Given $p(\cdot) \in \mathcal{P}$, a measurable function a is called a $(1, p(\cdot), \infty)$-*atom* if there exists a stopping time $\tau \in \mathcal{T}$ such that

(1) $\mathbb{E}(a \mid \mathcal{F}_n) = 0$, $\forall n \le \tau$,
(2) $\|s(a)\|_\infty \le \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}$.

Denote by $A(s, p(\cdot), \infty)$ the set of all sequences of pairs (μ_k, a^k, τ_k), where μ_k are nonnegative numbers and a^k are $(1, p(\cdot), \infty)$-atoms satisfying (1) and (2).

In the remainder of the paper, we always denote $\underline{p} = \min\{p_-, 1\}$.

Definition 1.2. Given $p(\cdot) \in \mathcal{P}$, let us denote by $H_{p(\cdot)}^{s,at}$ the space of those martingales for which there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1, p(\cdot), \infty)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k, \quad \text{a.e.}, \tag{1.6}$$

and

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}}\Big(\frac{\mu_k\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{\underline{p}}\Big\}^{\frac{1}{\underline{p}}}\Big\|_{p(\cdot)}<\infty.$$

Let

$$\mathcal{A}\big(\{\mu_k\},\{a^k\},\{\tau_k\}\big)\equiv\Big\|\Big\{\sum_{k\in\mathbb{Z}}\Big(\frac{\mu_k\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{\underline{p}}\Big\}^{\frac{1}{\underline{p}}}\Big\|_{p(\cdot)}.$$

We define

$$\|f\|_{H^{s,at}_{p(\cdot)}}=\inf\mathcal{A}\big(\{\mu_k\},\{a^k\},\{\tau_k\}\big),\quad (\mu_k,a^k,\tau_k)\in A\big(s,p(\cdot),\infty\big),$$

where the infimum is taken over all decompositions of the form (1.6).

In Section 4, we prove that

$$H^s_{p(\cdot)}=H^{s,at}_{p(\cdot)},\quad p(\cdot)\in\mathcal{P},$$

with equivalent quasinorms (see Section 2 for the notation $H^s_{p(\cdot)}$). We give some applications of atomic decomposition in Section 5. Recall that the Lipschitz space $\Lambda_q(\alpha)(\alpha\geq 0,q\geq 1)$ is defined as the space of all functions $f\in L^q$ for which

$$\|f\|_{\Lambda_q(\alpha)}=\sup_{\tau}\big|\{\tau<\infty\}\big|^{-\frac{1}{q}-\alpha}\|f-f^\tau\|_q<\infty.$$

It was proved by Weisz in [27] that the dual space of $H^s_p(0<p\leq 1)$ is equivalent to $\Lambda_2(\alpha)(\alpha=1/p-1)$. The new Lipschitz space $\Lambda_q(\alpha(\cdot))$ is introduced in Section 5. Let $p(\cdot)$ satisfy (1.5). We obtain

$$(H^s_{p(\cdot)})^*=\Lambda_2\big(\alpha(\cdot)\big),\quad 0<p_-\leq p_+\leq 1,$$

where $\alpha(\cdot)=1/p(\cdot)-1$.

Finally, we get the John–Nirenberg inequality in the frame of variable exponents. If $p(\cdot)$ satisfies (1.5), then

$$\|f\|_{\mathrm{BMO}_1}\lesssim\|f\|_{\mathrm{BMO}_{p(\cdot)}}\lesssim\|f\|_{\mathrm{BMO}_1},\quad 1\leq p_-\leq p_+<\infty,$$

which can be regarded as the probability versions of Theorem 1.2 or Theorem 5.1 in [16] (see Section 5 for the definition of $\mathrm{BMO}_{p(\cdot)}$). Furthermore, we also obtain the exponential integrability form of the John–Nirenberg inequality for $\mathrm{BMO}_{p(\cdot)}$, which is the probability analogue of Theorem 3.2 in [13]. We note that the generalized John–Nirenberg inequalities were proved in the frame of rearrangement invariant spaces in [31], but the variable $L^{p(\cdot)}$ spaces are not rearrangement invariant spaces when $p(\cdot)$ is not a constant. Again, condition (1.5) plays an important role in the above results that lead to our estimating the $p(\cdot)$-norm of characterization function and to the availability of inverse Hölder inequalities.

Throughout this paper, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ denote the integer set, nonnegative integer set, and set of complex numbers, respectively. We denote by C the absolute positive constant, which can vary from line to line, and we denote by $C_{p(\cdot)}$ the constant depending only on $p(\cdot)$. The symbol $A\lesssim B$ stands for the inequality $A\leq CB$ or $A\leq C_{p(\cdot)}B$. If we write $A\approx B$, then it stands for $A\lesssim B\lesssim A$.

2. Preliminaries and lemmas

In this section, we give some preliminaries necessary to the whole paper. Given $p(\cdot) \in \mathcal{P}$, we always assume that $0 < p_- \leq p_+ < \infty$ if there is no special statement. The space $L^{p(\cdot)} = L^{p(\cdot)}(\Omega)$ is the collection of all measurable functions f defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for some $\lambda > 0$,

$$\rho(f/\lambda) = \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} d\mathbb{P} < \infty.$$

This becomes a quasi-Banach function space when it is equipped with the quasi-norm

$$\|f\|_{p(\cdot)} \equiv \inf\big\{\lambda > 0 : \rho(f/\lambda) \leq 1\big\}.$$

The following facts are well known (see, e.g., [22]):

(1) (Positivity) $\|f\|_{p(\cdot)} \geq 0$; $\|f\|_{p(\cdot)} = 0 \Leftrightarrow f \equiv 0$.
(2) (Homogeneity) $\|cf\|_{p(\cdot)} = |c| \cdot \|f\|_{p(\cdot)}$ for $c \in \mathbb{C}$.
(3) (The $\underline{p}$-triangle inequality) $\|f+g\|_{p(\cdot)}^{\underline{p}} \leq \|f\|_{p(\cdot)}^{\underline{p}} + \|g\|_{p(\cdot)}^{\underline{p}}$.

For $p(\cdot) \in \mathcal{P}$ and $p_- > 1$, we define the conjugate exponent $p'(\cdot)$ by the equation

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

We collect the following useful lemmas, which will be used in the paper.

Lemma 2.1 (see [5, p. 5]). *Let $p(\cdot) \in \mathcal{P}$, and let $p_- \geq 1$. Then, for all $r > 0$, we have*

$$\big\||f|^r\big\|_{p(\cdot)} = \|f\|_{rp(\cdot)}^r.$$

Lemma 2.2 (see [3, p. 24]). *Given $p(\cdot) \in \mathcal{P}$, we have, for all $f \in L^{p(\cdot)}$ and $\|f\|_{p(\cdot)} \neq 0$,*

$$\int_\Omega \Big|\frac{f(x)}{\|f\|_{p(\cdot)}}\Big|^{p(x)} d\mathbb{P} = 1.$$

Lemma 2.3 (see [9, Theorem 1.3] or [3, p. 22]). *Given $p(\cdot) \in \mathcal{P}$ and $f \in L^{p(\cdot)}$, we have*

(1) *$\|f\|_{p(\cdot)} < 1$ $(= 1, > 1)$ if and only if $\rho(f) < 1$ $(= 1, > 1)$;*
(2) *If $\|f\|_{p(\cdot)} > 1$, then $\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}$;*
(3) *If $0 < \|f\|_{p(\cdot)} \leq 1$, then $\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}$.*

Lemma 2.4 (Hölder's inequality; see [3]). *Given $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}$ such that*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)},$$

there exists a constant $C_{p(\cdot)}$ such that, for all $f \in L^{q(\cdot)}$, $g \in L^{r(\cdot)}$, we have $fg \in L^{p(\cdot)}$,

$$\|fg\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{q(\cdot)} \|g\|_{r(\cdot)}.$$

Now we introduce some standard notation from martingale theory. We refer to [10], [20], and [28] for the classical martingale space theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Recall that the conditional expectation operator relative to $\mathcal{F}_n$ is denoted by $\mathbb{E}_{\mathcal{F}_n}$; that is, $\mathbb{E}(f \mid \mathcal{F}_n) = \mathbb{E}_{\mathcal{F}_n}(f)$. A sequence of measurable functions $f = (f_n)_{n\geq 0} \subset L^1(\Omega)$ is called a *martingale* with respect to $(\mathcal{F}_n)$ if $\mathbb{E}_{\mathcal{F}_n}(f_{n+1}) = f_n$ for every $n \geq 0$. If in addition $f_n \in L^{p(\cdot)}$, f is called an $L^{p(\cdot)}$-*martingale* with respect to $(\mathcal{F}_n)$. In this case we set

$$\|f\|_{p(\cdot)} = \sup_{n\geq 0} \|f_n\|_{p(\cdot)}.$$

If $\|f\|_{p(\cdot)} < \infty$, f is called a *bounded* $L^{p(\cdot)}$-*martingale* and denoted by $f \in L^{p(\cdot)}$. For a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n\geq 0})$, define the maximal function and the conditional square function of f, respectively, as follows ($f_{-1} = f_0$):

$$M_m f = \sup_{n\leq m} |f_n|, \qquad Mf = \sup_{n\geq 1} |f_n|,$$
$$s_m(f) = \Big(\sum_{n=0}^{m} \mathbb{E}_{\mathcal{F}_{n-1}} |df_n|^2\Big)^{\frac{1}{2}}, \qquad s(f) = \Big(\sum_{n=0}^{\infty} \mathbb{E}_{\mathcal{F}_{n-1}} |df_n|^2\Big)^{\frac{1}{2}}.$$

Then we define the variable exponent martingale Hardy spaces analogous to classical martingale Hardy spaces as follows:

$$H^*_{p(\cdot)} = \big\{f = (f_n)_{n\geq 0} : Mf \in L^{p(\cdot)}\big\}, \qquad \|f\|_{H^*_{p(\cdot)}} = \|Mf\|_{p(\cdot)};$$
$$H^s_{p(\cdot)} = \big\{f = (f_n)_{n\geq 0} : s(f) \in L^{p(\cdot)}\big\}, \qquad \|f\|_{H^s_{p(\cdot)}} = \big\|s(f)\big\|_{p(\cdot)}.$$

3. Doob's maximal inequalities

In this section, we first prove the weak-type inequality (1.3) without condition (1.2). We begin with the following lemma.

Lemma 3.1. *Given $p(\cdot) \in \mathcal{P}$ and $1 \leq p_- \leq p_+ < \infty$, let $f = (f_n)_{0\leq n\leq\infty}$ be an $L^{p(\cdot)}$-martingale. Suppose that, for any stopping time τ,*

$$\mathbb{P}(\tau < \infty) < \int_{\{\tau<\infty\}} \frac{|f_\infty|}{\lambda}\, d\mathbb{P}, \quad \forall \lambda > 0.$$

Then there exists a constant $C_{p(\cdot)}$ such that

$$\mathbb{P}(\tau < \infty) \leq C_{p(\cdot)} \int_{\{\tau<\infty\}} \Big(\frac{|f_\infty|}{\lambda}\Big)^{p(x)} d\mathbb{P}, \quad \forall \lambda > 0.$$

Proof. We choose a sequence of simple functions $\{s_n\}_{n\geq 1}$ such that $p_+(\{\tau < \infty\}) \geq s_n \geq p_-(\{\tau < \infty\})$ for any n, and the sequence $\{s_n\}_{n\geq 1}$ increases monotonically to $p(x)$ on $\{\tau < \infty\}$. Then, for each n,

$$s_n(x) = \sum_{j=1}^{k_n} \alpha_{n,j} \chi_{A_{n,j}}(x),$$

where the sets $\{A_{n,j}\}$ are disjoint and $\bigcup_j A_{n,j} = \{\tau < \infty\}$.

By Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\int_{A_{n,j}} \frac{|f_\infty(x)|}{\lambda}\,d\mathbb{P} &\le \Big(\int_{A_{n,j}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{\alpha_{n,j}}\,d\mathbb{P}\Big)^{\frac{1}{\alpha_{n,j}}} \mathbb{P}(A_{n,j})^{\frac{1}{\alpha'_{n,j}}}\\ &\le \frac{1}{\alpha_{n,j}} \int_{A_{n,j}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{\alpha_{n,j}}\,d\mathbb{P} + \frac{\mathbb{P}(A_{n,j})}{\alpha'_{n,j}}\\ &\le \frac{1}{p_-(\{\tau<\infty\})} \int_{A_{n,j}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{s_n(x)}\,d\mathbb{P} + \frac{\mathbb{P}(A_{n,j})}{(p_+(\{\tau<\infty\}))'}.\end{aligned}$$

Adding the above inequalities with j from 1 to k_n, we have

$$\int_{\{\tau<\infty\}} \frac{|f_\infty(x)|}{\lambda}\,d\mathbb{P} \le \frac{1}{p_-(\{\tau<\infty\})} \int_{\{\tau<\infty\}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{s_n(x)}\,d\mathbb{P} + \frac{\mathbb{P}(\tau<\infty)}{(p_+(\{\tau<\infty\}))'}.$$

This inequality holds for all n, and hence the monotone convergence theorem implies that

$$\begin{aligned}\mathbb{P}(\tau<\infty) &< \int_{\{\tau<\infty\}} \frac{|f_\infty|}{\lambda}\,d\mathbb{P}\\ &\le \frac{1}{p_-(\{\tau<\infty\})} \int_{\{\tau<\infty\}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{p(x)}\,d\mathbb{P} + \frac{\mathbb{P}(\tau<\infty)}{(p_+(\{\tau<\infty\}))'}.\end{aligned} \tag{3.1}$$

Since $p_+<\infty$, we have $(p_+(\{\tau<\infty\}))'>1$. It follows that

$$\mathbb{P}(\tau<\infty)\Big(1-\frac{1}{(p_+(\{\tau<\infty\}))'}\Big) \le \frac{1}{p_-(\{\tau<\infty\})} \int_{\{\tau<\infty\}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{p(x)}\,d\mathbb{P}.$$

Therefore, by a simple calculation, we have

$$\mathbb{P}(\tau<\infty) \le C_{p(\cdot)} \int_{\{\tau<\infty\}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{p(x)}\,d\mathbb{P}.$$

□

The following theorem corresponds to Proposition 4 in [2].

Theorem 3.2. *Given $p(\cdot)\in\mathcal{P}$ and $1\le p_-\le p_+<\infty$, suppose that $f=(f_n)_{0\le n\le\infty}$ is a bounded $L^{p(\cdot)}$-martingale. Then*

$$\mathbb{P}(Mf>\lambda) \le C_{p(\cdot)} \int_\Omega \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{p(x)}\,d\mathbb{P}, \quad \forall\lambda>0.$$

Proof. For any $\lambda>0$, we define a stopping time $\tau=\inf\{n>0:|f_n|>\lambda\}$ (with the convention that $\inf\varnothing=\infty$). It is obvious that

$$\{Mf>\lambda\}=\{\tau<\infty\}$$

and

$$\{\tau<\infty\}\subset\big\{|f_\tau|>\lambda\big\}.$$

Note that $\mathbb{E}_{\mathcal{F}_\tau}(\frac{|f_\infty|}{\lambda})>1$ a.e. on the set $\{\tau<\infty\}$. We get that

$$\begin{aligned}\mathbb{P}(\tau<\infty) &= \int_{\{\tau<\infty\}} 1\, d\mathbb{P} \le \int_{\{\tau<\infty\}} \mathbb{E}_{\mathcal{F}_\tau}\Big(\frac{|f_\infty|}{\lambda}\Big)\, d\mathbb{P} \\ &= \int_{\{\tau<\infty\}} \frac{|f_\infty(x)|}{\lambda}\, d\mathbb{P}.\end{aligned}$$

It follows immediately from Lemma 3.1 that

$$\begin{aligned}\mathbb{P}(Mf>\lambda) = \mathbb{P}(\tau<\infty) &\le C_{p(\cdot)} \int_{\{\tau<\infty\}} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{p(x)}\, d\mathbb{P} \\ &\le C_{p(\cdot)} \int_{\Omega} \Big(\frac{|f_\infty(x)|}{\lambda}\Big)^{p(x)}\, d\mathbb{P}.\end{aligned}$$

The proof is complete. □

Lemma 3.3. *Given $p(\cdot)\in\mathcal{P}$, we have*

$$\big(\sup_{n\ge 0}|f_n|\big)^{p(\cdot)} = \sup_{n\ge 0}\big(|f_n|^{p(\cdot)}\big).$$

Although this lemma is very obvious, we will refer to it frequently below.

We now turn to consider the strong-type inequality (1.4). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let

$$\mathcal{D}_n = \{A_j^n\}_{j\ge 1}, \quad \text{for each } n\ge 0,$$

be decompositions of Ω such that $(\mathcal{B}_n)_{n\ge 0} = (\sigma(\mathcal{D}_n))_{n\ge 0}$ is increasing and $\mathcal{F} = \sigma(\bigcup_{n\ge 0}\mathcal{B}_n)$. It is clear that

$$\mathbb{E}_{\mathcal{B}_n}(f) = \sum_{j=1}^{\infty}\Big(\frac{1}{\mathbb{P}(A_j^n)}\int_{A_j^n} f(x)\, d\mathbb{P}\Big)\chi_{A_j^n}.$$

Then

$$\begin{aligned}\int_\Omega (Mf)^{p(x)}\, d\mathbb{P} &\le \int_\Omega \sup_n\Big\{\sum_{j=1}^{\infty}\Big(\frac{1}{\mathbb{P}(A_j^n)}\int_{A_j^n}|f(x)|\, d\mathbb{P}\Big)\chi_{A_j^n}\Big\}^{p(x)} d\mathbb{P} \\ &= \int_\Omega \Big\{\sup_n \sum_{j=1}^{\infty}\Big(\frac{1}{\mathbb{P}(A_j^n)}\int_{A_j^n}|f(x)|\, d\mathbb{P}\Big)^{\frac{p(x)}{p_-}}\chi_{A_j^n}\Big\}^{p_-} d\mathbb{P}.\end{aligned} \tag{3.2}$$

Lemma 3.4. *Let $p(\cdot)\in\mathcal{P}$, $1<p_-\le p_+<\infty$, and satisfy (1.5). Suppose that $f\in L^{p(\cdot)}$ and $\|f\|_{p(\cdot)}\le 1/2$. Then, for all measurable sets B,*

$$\Big(\frac{1}{\mathbb{P}(B)}\int_B |f(x)|\, d\mathbb{P}\Big)^{\frac{p(x)}{p_-}} \le K\Big(\frac{1}{\mathbb{P}(B)}\int_B |f(x)|^{\frac{p(x)}{p_-}}\, d\mathbb{P} + 1\Big).$$

Proof. Let $q(x)=p(x)/p_-$. Then, for any $x\in B$,

$$q(x)\le p(x) \qquad \text{and} \qquad 1\le q_-(B)\le p(x).$$

Let $f|_B(x)=f(x), x\in B$. Then $\|f|_B\|_{p(\cdot)}\le 1/2$. Let $g=f/\|f|_B\|_{p(\cdot)}$. It follows from Lemma 2.2 that

$$\int_B \Big|\frac{f(x)}{\|f|_B\|_{p(\cdot)}}\Big|^{q_-(B)}\,d\mathbb{P} = \int_{B\cap\{|g|\ge 1\}} \big|g(x)\big|^{q_-(B)}\,d\mathbb{P} + \int_{B\cap\{|g|<1\}} \big|g(x)\big|^{q_-(B)}\,d\mathbb{P}$$
$$\le 1+\mathbb{P}(\Omega).$$

Then

$$\|f|_B\|_{q_-(B)} \le \big(1+\mathbb{P}(\Omega)\big)^{\frac{1}{q_-(B)}}\|f|_B\|_{p(\cdot)} \le \big(1+\mathbb{P}(\Omega)\big)\|f\|_{p(\cdot)} \le 1.$$

Using Hölder's inequality and (1.5), we find that

$$\begin{aligned}
\Big(\frac{1}{\mathbb{P}(B)}\int_B \big|f(y)\big|\,d\mathbb{P}\Big)^{q(x)} &\le \Big(\frac{1}{\mathbb{P}(B)}\int_B \big|f(y)\big|^{q_-(B)}\,d\mathbb{P}\Big)^{\frac{q(x)}{q_-(B)}}\\
&= \mathbb{P}(B)^{-\frac{q(x)}{q_-(B)}}\|f|_B\|_{q_-(B)}^{q(x)}\\
&\le \mathbb{P}(B)^{-\frac{q(x)}{q_-(B)}}\|f|_B\|_{q_-(B)}^{q_-(B)}\\
&= \mathbb{P}(B)^{-\frac{q(x)-q_-(B)}{q_-(B)}}\frac{1}{\mathbb{P}(B)}\int_B \big|f(x)\big|^{q_-(B)}\,d\mathbb{P}\\
&\le \mathbb{P}(B)^{\frac{q_-(B)-q_+(B)}{q_-(B)}}\frac{1}{\mathbb{P}(B)}\int_B \big|f(y)\big|^{q_-(B)}\,d\mathbb{P}\\
&= \mathbb{P}(B)^{\frac{p_-(B)-p_+(B)}{p_-(B)}}\frac{1}{\mathbb{P}(B)}\int_B \big|f(y)\big|^{q_-(B)}\,d\mathbb{P}\\
&\le K^{\frac{1}{p_-(B)}}\Big(\frac{1}{\mathbb{P}(B)}\int_B \big(\big|f(y)\big|^{q(y)}+1\big)\,d\mathbb{P}\Big)\\
&\le K\Big(\frac{1}{\mathbb{P}(B)}\int_B \big(\big|f(y)\big|^{q(y)}+1\big)\,d\mathbb{P}\Big).
\end{aligned}$$
□

Theorem 3.5. *Let $\mathcal{D}_n = \{A_j^n\}_{j\ge 1}$ for each $n \ge 0$ be decompositions of Ω such that $(\mathcal{B}_n)_{n\ge 0} = (\sigma(\mathcal{D}_n))_{n\ge 0}$ is increasing and $\mathcal{F} = \sigma(\bigcup_{n\ge 0}\mathcal{B}_n)$. Let $p(\cdot)$ satisfy (1.5) and $1 < p_- \le p_+ < \infty$. Then, for any martingale $f \in L^{p(\cdot)}$ with respect to $(\mathcal{B}_n)_{n\ge 0}$,*

$$\big\|\sup_n |f_n|\big\|_{p(\cdot)} \le C_{p(\cdot)}\|f\|_{p(\cdot)}.$$

Proof. We assume that $\|f\|_{p(\cdot)} \le 1/2$ by homogeneity, and we let $q(x) = p(x)/p_-$. Then, by Lemma 3.4 and the classical Doob maximal inequality,

$$\begin{aligned}
&\int_\Omega \Big\{\sup_n \sum_{j=1}^\infty \Big(\frac{1}{\mathbb{P}(A_j^n)}\int_{A_j^n}\big|f(x)\big|\,d\mathbb{P}\Big)^{\frac{p(x)}{p_-}}\chi_{A_j^n}\Big\}^{p_-}\,d\mathbb{P}\\
&\le \int_\Omega \Big\{\sup_n \sum_{j=1}^\infty K\Big(\frac{1}{\mathbb{P}(A_j^n)}\int_{A_j^n}\big(\big|f(x)\big|^{\frac{p(x)}{p_-}}+1\big)\,d\mathbb{P}\Big)\chi_{A_j^n}\Big\}^{p_-}\,d\mathbb{P}\\
&= K^{p_-}\big\|\sup_n \mathbb{E}_{\mathcal{B}_n}\big(|f|^{q(\cdot)}+1\big)\big\|_{p_-}^{p_-}\\
&\le C_{p_-}K^{p_-}\big\||f|^{q(\cdot)}+1\big\|_{p_-}^{p_-} \le C.
\end{aligned}$$

By (3.2), we have $\int_\Omega (Mf)^{p(x)}\,d\mathbb{P} \le C$. Now the proof is complete. □

Remark 3.6. (1) We point out that there is a nonlog-Hölder continuous function $p(\cdot)$ for which the maximal operator is bounded on the corresponding Lebesgue spaces $L_{p(\cdot)}(\mathbb{R}^n)$ (see [23]).

(2) Note that condition (1.5) could not cover the example given by Nakai and Sadasue [21, p. 2169]. Indeed, we can verify a special case of their example. Let $((0,1],\Sigma,\mu)$ be a probability space such that μ is the Lebesgue measure and subalgebras $\{\Sigma_n\}_{n\geq 0}$ generated as follows:

$$\Sigma_n = \sigma\text{-algebra generated by atoms } \Big(\frac{j}{2^n},\frac{j+1}{2^n}\Big], \quad j=0,\dots,2^n-1.$$

For $n\geq 0$ we set $B_n=(0,\frac{1}{2^n}]$. Then

$$(0,1]=B_0\supset B_1\supset\cdots\supset B_n\cdots,$$

and we let

$$g(x)=\sin\big(h(x)\big),\quad h(x)=\sum_{n=1}^{\infty}\frac{1}{\ln(2^n e)}(2\chi_{B_n}-\chi_{B_{n-1}}).$$

Denote $h_m:=\sum_{n=1}^m\frac{1}{\ln(2^ne)}-\frac{1}{\ln(2^{m+1}e)}$, $m\geq 1$. It is easy to check that

$$h_m\to\infty \quad \text{as } m\to\infty. \tag{3.3}$$

Also, we have

$$0<h_{m+1}-h_m\leq\frac{2}{(m+1)\ln 2}<\frac{2\pi}{3},\quad m\geq 1. \tag{3.4}$$

Given N, we shall show that there exists $y\in B_N$ such that $1\geq g(y)\geq 1/2$. Choose the smallest integer k so that $h_N<2k\pi+\frac{\pi}{6}$. Then, from (3.3) and (3.4), it follows that there exists $j>N$ satisfying $h_j\in(2k\pi+\frac{\pi}{6},2k\pi+\frac{5\pi}{6})$. This means that, for any $y\in B_j\setminus B_{j+1}\subset B_N$, we have $1\geq g(y)\geq 1/2$. Similarly, there exists $z\in B_N$ such that $-1\leq g(z)\leq 0$. Now we obtain

$$\mu(B_N)^{g_-(B_N)-g_+(B_N)}=(2^N)^{g_+(B_N)-g_-(B_N)}\geq(2^N)^{g(y)-g(z)}\geq(2^N)^{1/2},$$

which implies that $g(\cdot)$ does not satisfy condition (1.5).

At the time of this writing, we do not know if the condition (1.5) is sufficient for Doob's maximal inequality in general probability spaces.

Problem 3.7. Let $p(\cdot)$ satisfy (1.5) with $1<p_-\leq p_+<\infty$. Then, for any martingale $f\in L^{p(\cdot)}$ with respect to $(\mathcal{F}_n)_{n\geq 0}$, do we have

$$\big\|\sup_n|f_n|\big\|_{p(\cdot)}\leq C_{p(\cdot)}\|f\|_{p(\cdot)}?$$

Remark 3.8. It is well known that $|\mathbb{E}_{\mathcal{F}_n}(f)|^p\leq\mathbb{E}_{\mathcal{F}_n}(|f|^p)$ for $1\leq p<\infty$; however, it is easy to give inverse examples to show that one can never expect a variable exponent version, namely,

$$\big|\mathbb{E}_{\mathcal{F}_n}(f)\big|^{p(\cdot)}\leq C_{p(\cdot)}\mathbb{E}_{\mathcal{F}_n}\big(|f|^{p(\cdot)}\big),\quad 1\leq p(\cdot)<\infty. \tag{3.5}$$

Hence the main difficulty in dealing with Problem 3.7 is how to overcome or avoid the use of inequality (3.5).

4. Atomic characterization of the variable Hardy martingale space

In this section we construct the atomic decomposition of the martingale Hardy space with variable exponents. Here we use Definitions 1.1 and 1.2.

Proposition 4.1. *Given $p(\cdot) \in \mathcal{P}$, let $f \in H^{s,at}_{p(\cdot)}$; that is, $f = \sum \mu_k a^k$.*

(1) *We have*

$$\Big(\sum_{k\in\mathbb{Z}} \mu_k^{p_+}\Big)^{\frac{1}{p_+}} \le \mathcal{A}\big(\{\mu_k\}, \{a^k\}, \{\tau_k\}\big).$$

(2) *If $p_+ \le 1$, then*

$$\sum_{k\in\mathbb{Z}} \mu_k \le \mathcal{A}\big(\{\mu_k\}, \{a^k\}, \{\tau_k\}\big).$$

(3) *For any $k \in \mathbb{Z}$, we have*

$$\|a^k\|_{H^{s,at}_{p(\cdot)}} \le 1.$$

Proof. (1) The convexity implies that

$$\begin{aligned}\int_\Omega \Big(\sum_{k\in\mathbb{Z}} \Big(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\lambda \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{\underline{p}}\Big)^{\frac{p(x)}{\underline{p}}} d\mathbb{P} &\ge \int_\Omega \sum_{k\in\mathbb{Z}} \Big(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\lambda \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{p(x)} d\mathbb{P} \\ &= \sum_{k\in\mathbb{Z}} \int_{\{\tau_k<\infty\}} \Big(\frac{\mu_k}{\lambda \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{p(x)} d\mathbb{P}.\end{aligned}$$

Now, if we set $\lambda = (\sum_{k\in\mathbb{Z}} \mu_k^{p_+})^{\frac{1}{p_+}}$, then we obtain

$$\int_\Omega \Big(\sum_{k\in\mathbb{Z}} \Big(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\lambda \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{\underline{p}}\Big)^{\frac{p(x)}{\underline{p}}} d\mathbb{P} \ge \sum_{k\in\mathbb{Z}} \Big(\frac{\mu_k}{\lambda}\Big)^{p_+} \int_\Omega \Big(\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{p(x)} d\mathbb{P} = 1.$$

By the definition of $\mathcal{A}(\{\mu_k\}, \{a^k\}, \{\tau_k\})$, we get the desired result.

(2) and (3) are obvious. □

Theorem 4.2. *Let $p(\cdot) \in \mathcal{P}$. If the martingale $f \in H^s_{p(\cdot)}$, then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1, p(\cdot), \infty)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that, for all $n \ge 0$,*

$$\sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_{\mathcal{F}_n} a^k = f_n \quad a.e. \tag{4.1}$$

and

$$\mathcal{A}\big(\{\mu_k\}, \{a^k\}, \{\tau_k\}\big) \lesssim H^s_{p(\cdot)}.$$

Moreover, the sum $\sum_{k\in\mathbb{Z}} \mu_k a^k$ converges to f in $H^s_{p(\cdot)}$. Conversely, if the martingale f has a decomposition of (4.1), then

$$\|f\|_{H^s_{p(\cdot)}} \lesssim \inf \mathcal{A}\big(\{\mu_k\}, \{a^k\}, \{\tau_k\}\big),$$

where the infimum is taken over all the decompositions of the form (4.1).

Proof. Assume that $f \in H^s_{p(\cdot)}$. Let us consider the following stopping times for all $k \in \mathbb{Z}$:

$$\tau_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}.$$

The sequence of these stopping times is obviously nondecreasing. For each stopping time τ, denote $f_n^\tau = f_{n\wedge\tau}$. It is easy to see that

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}).$$

Let

$$\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}, \quad \text{and let } a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

If $\mu_k = 0$, then let $a_n^k = 0$ for all $k \in \mathbb{Z}, n \in \mathbb{N}$. Then $(a_n^k)_{n\geq 0}$ is a martingale for each fixed $k \in \mathbb{Z}$. Since $s(f^{\tau_k}) = s_{\tau_k}(f) \leq 2^k$, we get

$$s\big((a_n^k)_{n\geq 0}\big) \leq \frac{s(f^{\tau_{k+1}}) + s(f^{\tau_k})}{\mu_k} \leq \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1}.$$

Hence it is easy to check that $(a_n^k)_{n\geq 0}$ is a bounded L_2-martingale. Consequently, there exists an element $a^k \in L_2$ such that $\mathbb{E}_{\mathcal{F}_n} a^k = a_n^k$. If $n \leq \tau_k$, then $a_n^k = 0$, and $s(a^k) \leq \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1}$. Thus we conclude that a^k is really a $(1, p(\cdot), \infty)$-atom.

Denote $\mathcal{O}_k = \{\tau_k < \infty\} = \{s(f) > 2^k\}$. Recalling that τ_k is nondecreasing for each $k \in \mathbb{Z}$, we have $\mathcal{O}_k \supset \mathcal{O}_{k+1}$. Then

$$\sum_{k\in\mathbb{Z}} \big(3 \cdot 2^k \chi_{\mathcal{O}_k}(x)\big)^{\underline{p}}$$

is the sum of the geometric sequence $\{(3 \cdot 2^k \chi_{\mathcal{O}_k}(x))^{\underline{p}}\}_{k\in\mathbb{Z}}$; thus, we can claim that

$$\sum_{k\in\mathbb{Z}} \big(3 \cdot 2^k \chi_{\mathcal{O}_k}(x)\big)^{\underline{p}} \approx \Big(\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{O}_k}(x)\Big)^{\underline{p}} \approx \Big(\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}}(x)\Big)^{\underline{p}}.$$

Indeed, for each fixed $x_0 \in \Omega$, there is $k_0 \in \mathbb{Z}$ such that $x_0 \in \mathcal{O}_{k_0}$ but $\notin \mathcal{O}_{k_0+1}$. Then

$$\begin{aligned}
\sum_{k=-\infty}^{k_0} \big(3 \cdot 2^k \chi_{\mathcal{O}_k}(x_0)\big)^{\underline{p}} &= \sum_{k=-\infty}^{k_0} (3 \cdot 2^k)^{\underline{p}} = (3 \cdot 2^{k_0})^{\underline{p}} \frac{1}{1 - 2^{-\underline{p}}} \\
&\lesssim (3 \cdot 2^{k_0})^{\underline{p}} \Big(\frac{1}{1 - \frac{1}{2}}\Big)^{\underline{p}} \\
&= \Big(\sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\mathcal{O}_k}(x_0)\Big)^{\underline{p}} \lesssim \Big(\sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}}(x_0)\Big)^{\underline{p}}.
\end{aligned}$$

Thus

$$\mathcal{A}\big(\{\mu_k\}, \{a^k\}, \{\tau_k\}\big) = \Big\| \Big\{ \sum_{k\in\mathbb{Z}} \Big(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\Big)^{\underline{p}} \Big\}^{\frac{1}{\underline{p}}} \Big\|_{p(\cdot)}$$

$$
\begin{aligned}
&= \Big\| \Big\{ \sum_{k\in\mathbb{Z}} (3\cdot 2^k \chi_{\{\tau_k<\infty\}})^{\underline{p}} \Big\}^{\frac{1}{\underline{p}}} \Big\|_{p(\cdot)} \\
&\lesssim \Big\| \sum_{k\in\mathbb{Z}} 3\cdot 2^k \chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}} \Big\|_{p(\cdot)} \\
&= \inf\Big\{ \lambda>0 : \int_\Omega \Big(\sum_{k\in\mathbb{Z}} \frac{3\cdot 2^k \chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}}(x)}{\lambda} \Big)^{p(x)} d\mathbb{P} \le 1 \Big\} \\
&= \inf\Big\{ \lambda>0 : \sum_{k\in\mathbb{Z}} \int_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}} \Big(\frac{3\cdot 2^k}{\lambda} \Big)^{p(x)} d\mathbb{P} \le 1 \Big\} \\
&\approx \inf\Big\{ \lambda>0 : \int_\Omega \Big(\frac{s(f)}{\lambda} \Big)^{p(x)} d\mathbb{P} \le 1 \Big\}.
\end{aligned}
$$

Therefore, we obtain

$$
\mathcal{A}\big(\{\mu_k\},\{a^k\},\{\tau_k\}\big) \lesssim \big\|s(f)\big\|_{p(\cdot)} = \|f\|_{H^s_{p(\cdot)}}.
$$

We now verify the sum $\sum_{k\in\mathbb{Z}} \mu_k a^k$ converges in $H^s_{p(\cdot)}$. By the equality $s(f-f^{\tau_k})^2 = s(f)^2 - s(f^{\tau_k})^2$, we have

$$
s(f-f^{\tau_k}), s(f^{\tau_k}) \le s(f) \qquad \text{and} \qquad s(f-f^{\tau_k}), s(f^{\tau_{-k}}) \to 0 \quad \text{a.e. as } k\to\infty.
$$

Consequently, by the dominated convergence theorem in variable $L^{p(\cdot)}$ (see [3, Theorem 2.62]),

$$
\Big\| f - \sum_{k=-M}^{N} \mu_k a^k \Big\|^{\underline{p}}_{H^s_{p(\cdot)}} \le \|f - f^{\tau_{N+1}}\|^{\underline{p}}_{H^s_{p(\cdot)}} + \|f^{\tau_{-M}}\|^{\underline{p}}_{H^s_{p(\cdot)}}
$$

converges to 0 a.e. as $M,N\to\infty$.

Conversely, by the definition of $(1,p(\cdot),\infty)$-atom, we have almost everywhere

$$
s(a) = s(a)\chi_{\{\tau<\infty\}} \le \big\|s(a)\big\|_\infty \chi_{\{\tau<\infty\}} \le \|\chi_{\{\tau<\infty\}}\|^{-1}_{p(\cdot)} \chi_{\{\tau<\infty\}},
$$

where a is a $(1,p(\cdot),\infty)$-atom. By the subadditivity of the conditional quadratic variation operator, we obtain

$$
s(f) \le \sum_{k\in\mathbb{Z}} \mu_k s(a^k) \le \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}.
$$

Thus

$$
\begin{aligned}
\|f\|_{H^s_{p(\cdot)}} = \big\|s(f)\big\|_{p(\cdot)} &\le \Big\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \Big\|_{p(\cdot)} \\
&\le \Big\| \Big\{ \sum_{k\in\mathbb{Z}} \Big(\mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \Big)^{\underline{p}} \Big\}^{\frac{1}{\underline{p}}} \Big\|_{p(\cdot)} \\
&= \mathcal{A}\big(\{\mu_k\},\{a^k\},\{\tau_k\}\big).
\end{aligned}
$$

Hence we can conclude that $\|f\|_{H^s_{p(\cdot)}} \approx \|f\|_{H^{s,at}_{p(\cdot)}}$ and the proof is complete. □

Remark 4.3. It is shown in Theorem 5.1 in [14] that, for the atomic decomposition of Hardy–Morrey spaces with variable exponents $p(\cdot)$ on $\mathbb{R}^n$, the exponent function $p(\cdot)$ is not necessary to be log-Hölder continuous.

5. Duality and the John–Nirenberg theorem

In this section, we establish the dual space of $H^s_{p(\cdot)}$ by the atomic decomposition established in Section 4 and prove the John–Nirenberg inequalities in the setting of variable exponents.

Proposition 5.1. *Let $p(\cdot) \in \mathcal{P}$ satisfy (1.5) with $0 < p_- \le p_+ < \infty$.*

(1) *If $q(\cdot) \in \mathcal{P}$ satisfies (1.5), then $p(\cdot) + q(\cdot)$ also satisfies (1.5).*
(2) *The term $\frac{1}{p(\cdot)}$ satisfies (1.5).*
(3) *If $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then $q(\cdot)$ satisfies (1.5).*
(4) *If $q(\cdot) \in \mathcal{P}$ satisfies (1.5) and $\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)}$, then $r(\cdot)$ satisfies (1.5).*

Proof. (1) Set $h(\cdot) = p(\cdot) + q(\cdot)$. Then

$$h_-(A) - h_+(A) \ge p_-(A) + q_-(A) - p_+(A) - q_+(A).$$

Hence

$$\mathbb{P}(A)^{h_-(A)-h_+(A)} \le \mathbb{P}(A)^{p_-(A)-p_+(A)+q_-(A)-q_+(A)} \le K_{p(\cdot)}K_{q(\cdot)} \triangleq K.$$

(2) We have

$$\mathbb{P}(A)^{1/p_+(A)-1/p_-(A)} = \mathbb{P}(A)^{\frac{p_-(A)-p_+(A)}{p_+(A)p_-(A)}} \le K_{p(\cdot)}^{\frac{1}{p_+(A)p_-(A)}}.$$

If $p_-(\Omega) \ge 1$, then $K_{p(\cdot)}^{\frac{1}{p_+(A)p_-(A)}} \le K_{p(\cdot)}$. If $0 < p_-(\Omega) < 1$, then

$$K_{p(\cdot)}^{\frac{1}{p_+(A)p_-(A)}} \le K_{p(\cdot)}^{1/p_-^2(\Omega)} \triangleq K.$$

(3) Set $h(\cdot) = 1 - \frac{1}{p(\cdot)}$. We get

$$\mathbb{P}(A)^{h_-(A)-h_+(A)} = \mathbb{P}(A)^{1-1/p_-(A)-1+1/p_+(A)} \le K_{p(\cdot)}^{\frac{1}{p_+(A)p_-(A)}} \le K_{p(\cdot)}^{1/p_-^2(\Omega)} \triangleq K.$$

Hence we have that $1 - \frac{1}{p(\cdot)}$ satisfies (1.5). Using (2), we get the desired result.

(4) It follows from (1) and (2). The proof is complete. □

It is easy to prove that, for all $B \in \mathcal{F}$,

$$\mathbb{P}(B)^{p_-(B)-p(x)}\big(\text{and } \mathbb{P}(B)^{p(x)-p_+(B)}\big) \le K, \quad \forall x \in B,$$

if $p(\cdot)$ satisfies (1.5). Using this result, we have the following lemma.

Lemma 5.2. *Let $p(\cdot) \in \mathcal{P}$ satisfy (1.5) with $0 < p_- \le p_+ < \infty$. Then, for all set $B \in \mathcal{F}$, we have*

$$\mathbb{P}(B)^{1/p_-(B)} \approx \mathbb{P}(B)^{1/p(x)} \approx \mathbb{P}(B)^{1/p_+(B)} \approx \|\chi_B\|_{p(\cdot)}, \quad \forall x \in B.$$

Proof. Obviously, we have $\mathbb{P}(B)^{1/p_-(B)} \le \mathbb{P}(B)^{1/p(x)} \le \mathbb{P}(B)^{1/p_+(B)}$ for all $x \in B$. Since (1.5), we have

$$\frac{\mathbb{P}(B)^{1/p(x)}}{\mathbb{P}(B)^{1/p_-(B)}} \le \mathbb{P}(B)^{\frac{p_-(B)-p(x)}{p_-(B)p(x)}} \le K_{p(\cdot)}^{\frac{1}{p_-^2(\Omega)}} \triangleq K.$$

This implies $\mathbb{P}(B)^{1/p(x)} \le K\mathbb{P}(B)^{1/p_-(B)}$.

Then it is easy to check that $\mathbb{P}(B)^{1/p_-(B)} \approx \mathbb{P}(B)^{1/p(x)} \approx \mathbb{P}(B)^{1/p_+(B)}$. And we have

$$\frac{\chi_B(x)}{\mathbb{P}(B)^{1/p_-(B)}} \approx \frac{\chi_B(x)}{\mathbb{P}(B)^{1/p(x)}};$$

that is,

$$\Big(\frac{\chi_B(x)}{\mathbb{P}(B)^{1/p_-(B)}}\Big)^{p(x)} \ge \frac{\chi_B(x)}{\mathbb{P}(B)} \ge \Big(\frac{\chi_B(x)}{K\mathbb{P}(B)^{1/p_-(B)}}\Big)^{p(x)},$$

and so

$$\int_\Omega \Big(\frac{\chi_B(x)}{\mathbb{P}(B)^{1/p_-(B)}}\Big)^{p(x)} d\mathbb{P} \approx \int_\Omega \frac{\chi_B(x)}{\mathbb{P}(B)}\, d\mathbb{P} = 1.$$

Consequently, $\|\chi_B\|_{p(\cdot)} \approx \mathbb{P}(B)^{1/p_-(B)}$, and we get the desired result. □

Remark 5.3. Lemma 5.2 is also true for $p_+ = \infty$. In this case, we need to employ a slightly different definition of $\|\cdot\|_{p(\cdot)}$ (see [3, Definition 2.16]).

Corollary 5.4. *Let $p(\cdot) \in \mathcal{P}$ satisfy (1.5) with $0 < p_- \le p_+ < \infty$.*

(1) *Then, for all set $B \in \mathcal{F}$, we have*

$$\|\chi_B\|_1 \approx \|\chi_B\|_{p(\cdot)}\|\chi_B\|_{q(\cdot)},$$

where

$$1 = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

(2) *Let $q(\cdot) \in \mathcal{P}$ satisfy (1.5). Then, for all set $B \in \mathcal{F}$, we have*

$$\|\chi_B\|_{r(\cdot)} \approx \|\chi_B\|_{p(\cdot)}\|\chi_B\|_{q(\cdot)},$$

where

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

Proof. It follows from Proposition 5.1 and Lemma 5.2 that

$$\|\chi_B\|_{r(\cdot)} \approx \mathbb{P}(B)^{\frac{1}{r(x)}} = \mathbb{P}(B)^{\frac{1}{p(x)}+\frac{1}{q(x)}} \approx \|\chi_B\|_{p(\cdot)}\|\chi_B\|_{q(\cdot)}, \quad \forall x \in B. \qquad \square$$

As an application of atomic decomposition, we now prove a duality theorem. First let us introduce the new Lipschitz spaces with variable exponents.

Definition 5.5. Given that $1/\alpha(\cdot)$ is a variable exponent ($1/\alpha(\cdot) = \infty$ is allowed) and a constant $1 \le q < \infty$, define $\Lambda_q(\alpha(\cdot))$ as the space of functions $f \in L^q$ for which

$$\|f\|_{\Lambda_q(\alpha(\cdot))} = \sup_{\tau\in\mathcal{T}} \|\chi_{\{\tau<\infty\}}\|_{\frac{1}{\alpha(\cdot)}}^{-1} \|\chi_{\{\tau<\infty\}}\|_q^{-1} \|f - f^\tau\|_q$$

is finite.

Theorem 5.6. *Given $p(\cdot)\in\mathcal{P}, 0<p_-\le p_+\le 1$ and $p(\cdot)$ satisfies (1.5). Then*

$$(H^s_{p(\cdot)})^*=\Lambda_2\big(\alpha(\cdot)\big),\qquad \alpha(x)=1/p(x)-1.$$

Proof. We first claim that $\alpha(\cdot)$ satisfies (1.5) by Proposition 5.1(1). Let $\varphi\in\Lambda_2(\alpha(\cdot))\subset L^2$, and for all $f\in L^2$ define

$$l_\varphi(f)=\mathbb{E}(f\varphi).$$

We shall show that l_φ is a bounded linear functional on $H^s_{p(\cdot)}$. By Theorem 4.2, we know that L^2 is dense in $H^s_{p(\cdot)}$. Take the same stopping times τ_k, atoms a^k, and nonnegative numbers μ_k as we do in Theorem 4.2. It follows from Theorem 4.2 that $f=\sum_{k\in\mathbb{Z}}\mu_k a^k$ $(\forall f\in L_2)$. Hence

$$l_\varphi(f)=\mathbb{E}(f\varphi)=\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}(a^k\varphi).$$

By the definition of the atom a^k, $\mathbb{E}(a^k\varphi)=\mathbb{E}(a^k(\varphi-\varphi^{\tau_k}))$ always holds. It follows from Corollary 5.4 that

$$\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}\approx\|\chi_{\{\tau_k<\infty\}}\|_{\frac{1}{\alpha(\cdot)}}\|\chi_{\{\tau_k<\infty\}}\|_2\|\chi_{\{\tau_k<\infty\}}\|_2.$$

Thus, using Hölder's inequality, we can conclude that

$$\begin{aligned}\big|l_\varphi(f)\big|&\le\sum_{k\in\mathbb{Z}}\mu_k\int_\Omega|a^k||\varphi-\varphi^{\tau_k}|\,d\mathbb{P}\\&\le\sum_{k\in\mathbb{Z}}\mu_k\|a^k\|_2\|\varphi-\varphi^{\tau_k}\|_2\\&\le\sum_{k\in\mathbb{Z}}\mu_k\frac{|\{\tau_k<\infty\}|^{\frac12}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\|\varphi-\varphi^{\tau_k}\|_2\\&\lesssim\sum_{k\in\mathbb{Z}}\mu_k\|\varphi\|_{\Lambda_2(\alpha(\cdot))}.\end{aligned}$$

Then we obtain from Proposition 4.1 and Theorem 4.2 that

$$\big|l_\varphi(f)\big|\lesssim\|f\|_{H^s_{p(\cdot)}}\|\varphi\|_{\Lambda_2(\alpha(\cdot))}.$$

Consequently, l_φ can be extended to $H^s_{p(\cdot)}$ uniquely.

On the other hand, let l be an arbitrary bounded linear functional on $H^s_{p(\cdot)}$. We shall show that there exists $\varphi\in\Lambda_2(\alpha(\cdot))$ such that $l=l_\varphi$ and

$$\|\varphi\|_{\Lambda_2(\alpha(\cdot))}\lesssim\|l\|.$$

Since $0<p_-\le p_+\le 1$, it follows from Lemma 2.1 and Theorem 2.8 in [17] that

$$\begin{aligned}\|f\|_{H^s_{p(\cdot)}}&=\big\|s(f)\big\|_{p(\cdot)}=\big\|s(f)^{p_-}\big\|^{\frac{1}{p_-}}_{\frac{p(\cdot)}{p_-}}\\&\le\big(2\big\|s(f)^{p_-}\big\|_{\frac{2}{p_-}}\big)^{\frac{1}{p_-}}=2^{\frac{1}{p_-}}\big\|s(f)\big\|_2=2^{\frac{1}{p_-}}\|f\|_2,\quad\forall f\in L^2.\end{aligned}$$

Then the space L^2 can be embedded continuously in $H^s_{p(\cdot)}$. Consequently, there exists $\varphi \in L^2$ such that

$$l(f) = \mathbb{E}(f\varphi), \quad \forall f \in L^2.$$

Let τ be an arbitrary stopping time, and let

$$g = \frac{\varphi - \varphi^\tau}{\|\varphi - \varphi^\tau\|_2 \|\chi_{\{\tau<\infty\}}\|_{\frac{1}{\alpha(\cdot)}} \|\chi_{\{\tau<\infty\}}\|_2}.$$

Then g is not necessarily a $(1, p(\cdot), \infty)$-atom, but it satisfies (1) in Definition 1.1, and thus we have

$$s(g) = s(g)\chi_{\{\tau<\infty\}}.$$

Since

$$\frac{1}{p(x)} = \frac{1}{2} + \frac{1}{1/\alpha(x)} + \frac{1}{2},$$

we have, by Hölder's inequality,

$$\begin{aligned}\|g\|_{H^s_{p(\cdot)}} &= \frac{\|s(\varphi - \varphi^\tau)\|_{p(\cdot)}}{\|\varphi - \varphi^\tau\|_2 \|\chi_{\{\tau<\infty\}}\|_{\frac{1}{\alpha(\cdot)}} \|\chi_{\{\tau<\infty\}}\|_2} \\ &\lesssim \frac{\|s(\varphi - \varphi^\tau)\|_2 \|\chi_{\{\tau<\infty\}}\|_{\frac{1}{\alpha(\cdot)}} \|\chi_{\{\tau<\infty\}}\|_2}{\|\varphi - \varphi^\tau\|_2 \|\chi_{\{\tau<\infty\}}\|_{\frac{1}{\alpha(\cdot)}} \|\chi_{\{\tau<\infty\}}\|_2} \\ &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\|l\| \gtrsim l(g) &= \mathbb{E}\big(g(\varphi - \varphi^\tau)\big) \\ &= \|\chi_{\{\tau<\infty\}}\|^{-1}_{\frac{1}{\alpha(\cdot)}} \|\chi_{\{\tau<\infty\}}\|_2^{-1} \|\varphi - \varphi^\tau\|_2,\end{aligned}$$

and we get that $\|\varphi\|_{\Lambda_2(\alpha(\cdot))} \lesssim \|l\|$ and the proof is complete. □

We now turn to the John–Nirenberg theorem with variable exponents. Recall that $\mathrm{BMO}_p (1 \leq p < \infty)$ is the space of those functions f for which

$$\|f\|_{\mathrm{BMO}_p} = \sup_{\tau \in \mathcal{T}} \|\chi_{\{\tau<\infty\}}\|_p^{-1} \|f - f^{\tau-1}\|_p < \infty.$$

Definition 5.7. Given that $p(\cdot) \in \mathcal{P}$ and $\mathcal{T}$ are the sets of all stopping times relative to $\{\mathcal{F}_n\}_{n\geq 0}$, define

$$\mathrm{BMO}_{p(\cdot)} = \big\{f = (f_n)_{n\geq 0} : \|f\|_{\mathrm{BMO}_{p(\cdot)}} < \infty\big\},$$

where

$$\|f\|_{\mathrm{BMO}_{p(\cdot)}} = \sup_{\tau \in \mathcal{T}} \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} \|f - f^{\tau-1}\|_{p(\cdot)}.$$

Lemma 5.8 (see [28]). *If* $1 \leq p < \infty$, *then*

$$\|f\|_{\mathrm{BMO}_1} \approx \|f\|_{\mathrm{BMO}_p}.$$

Proposition 5.9. *If $p(\cdot) \in \mathcal{P}$ satisfies (1.5) and $1 \le p_- \le p_+ < \infty$, then we have that, for all $f \in \mathrm{BMO}_1$*

$$\|f\|_{\mathrm{BMO}_1} \lesssim \|f\|_{\mathrm{BMO}_{p(\cdot)}} \lesssim \|f\|_{\mathrm{BMO}_1}.$$

Proof. By Hölder's inequality and Corollary 5.4, we have that

$$\begin{aligned}\frac{\|f - f^{\tau-1}\|_1}{\|\chi_{\{\tau<\infty\}}\|_1} &\lesssim \frac{\|f - f^{\tau-1}\|_{p(\cdot)}\|\chi_{\{\tau<\infty\}}\|_{p'(\cdot)}}{\|\chi_{\{\tau<\infty\}}\|_1} \\ &= \frac{\|f - f^{\tau-1}\|_{p(\cdot)}}{\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}} \cdot \frac{\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}\|\chi_{\{\tau<\infty\}}\|_{p'(\cdot)}}{\|\chi_{\{\tau<\infty\}}\|_1} \\ &\le C_{p(\cdot)}\|f\|_{\mathrm{BMO}_{p(\cdot)}},\end{aligned}$$

where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Hence $\|f\|_{\mathrm{BMO}_1} \lesssim \|f\|_{\mathrm{BMO}_{p(\cdot)}}$.

Since

$$\begin{aligned}\|f - f^{\tau-1}\|_{p(\cdot)} &\lesssim \|f - f^{\tau-1}\|_{p_+}\|\chi_{\{\tau<\infty\}}\|_{\frac{p_+p(\cdot)}{p_+-p(\cdot)}} \\ &= \frac{\|f - f^{\tau-1}\|_{p_+}}{\|\chi_{\{\tau<\infty\}}\|_{p_+}}\|\chi_{\{\tau<\infty\}}\|_{\frac{p_+p(\cdot)}{p_+-p(\cdot)}}\|\chi_{\{\tau<\infty\}}\|_{p_+},\end{aligned}$$

we have, by Lemma 5.8,

$$\|f - f^{\tau-1}\|_{p(\cdot)} \lesssim \|f\|_{\mathrm{BMO}_1}\|\chi_{\{\tau<\infty\}}\|_{\frac{p_+p(\cdot)}{p_+-p(\cdot)}}\|\chi_{\{\tau<\infty\}}\|_{p_+}.$$

Thus, by Corollary 5.4,

$$\begin{aligned}\frac{\|f - f^{\tau-1}\|_{p(\cdot)}}{\|\chi_{\tau<\infty}\|_{p(\cdot)}} &\lesssim \|f\|_{\mathrm{BMO}_1}\|\chi_{\{\tau<\infty\}}\|_{\frac{p_+p(\cdot)}{p_+-p(\cdot)}}\|\chi_{\{\tau<\infty\}}\|_{p_+}\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} \\ &\lesssim \|f\|_{\mathrm{BMO}_1}.\end{aligned}$$

This means that

$$\|f\|_{\mathrm{BMO}_{p(\cdot)}} \lesssim \|f\|_{\mathrm{BMO}_1}.$$

□

By applying Proposition 5.9, we prove the following exponential integrability form of the John–Nirenberg theorem, which should be compared with the very recent result of Theorem 3.2 in [13].

Theorem 5.10. *Let $p(\cdot) \in \mathcal{P}$ satisfy (1.5) and $1 \le p_- \le p_+ < \infty$. Then there exist constants $C_1, C_2 > 0$ such that, for every $f \in \mathrm{BMO}_1$ and $\tau \in \mathcal{T}$,*

$$\|\chi_{\{\tau<\infty\}\cap\{f-f_{\tau-1}\ge t\}}\|_{p(\cdot)} \le C_1 e^{-\frac{C_2 t}{\|f\|_{\mathrm{BMO}_1}}}\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}, \quad t > 0.$$

Proof. Using Lemma 2.1 and Theorem 5.9, we point out that, for $r \ge 1$,

$$\sup_\tau \frac{\||f - f^{\tau-1}|^r\|_{p(\cdot)}^{1/r}}{\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{1/r}} = \|f\|_{\mathrm{BMO}_{rp(\cdot)}} \le C\|f\|_{\mathrm{BMO}_1} \triangleq C_0.$$

This implies that

$$\big\||f - f^{\tau-1}|^r\big\|_{p(\cdot)} \le C_0^r \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}.$$

Then we get that

$$\|\chi_{\{\tau<\infty\}\cap\{f-f_{\tau-1}\ge t\}}\|_{p(\cdot)} \le \frac{1}{t^r}\big\||f - f^{\tau-1}|^r \chi_{\{\tau<\infty\}}\big\|_{p(\cdot)} \le \frac{C_0^r}{t^r}\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}.$$

If $t \ge 2C_0$, we take $r = \frac{t}{2C_0} \ge 1$. Then

$$\Big(\frac{C_0}{t}\Big)^r \le \frac{1}{2^r} = e^{-r\ln 2} = e^{-\frac{t}{2C_0}\ln 2} = e^{-\frac{t}{2C\|f\|_{\mathrm{BMO}_1}}\ln 2} = e^{-\frac{C_2 t}{\|f\|_{\mathrm{BMO}_1}}},$$

where $C_2 = \frac{1}{2C}\ln 2$.

If $t < 2C_0$, take $C_2 = \frac{1}{2C}\ln 2$. Then $e^{-\frac{C_2 t}{\|f\|_{\mathrm{BMO}_1}}} = (\frac{1}{2})^{\frac{t}{2C_0}} > 1/4$. Since

$$\{\tau < \infty\} \cap \{f - f_{\tau-1} \ge t\} \subset \{\tau < \infty\},$$

it follows that

$$\|\chi_{\{\tau<\infty\}\cap\{f-f_{\tau-1}\ge t\}}\|_{p(\cdot)} \le \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)} \le 4e^{-\frac{C_2 t}{\|f\|_{\mathrm{BMO}_1}}}\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}.$$

We conclude this proof. □

Remark 5.11. The result above depends on condition (1.5), and we refer to Corollary 3.5 in [15] for another John–Nirenberg theorem with a nonlog-Hölder exponent function $p(\cdot)$ on $\mathbb{R}^n$.

Remark 5.12. Recently, new results concerning martingale Hardy spaces with variable exponents have emerged (see [11], [19], [29]).

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