Banach J. Math. Anal. 10 (2016), no. 4, 671–685
http://dx.doi.org/10.1215/17358787-3649260
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

AN EXTENSION OF A THEOREM OF SCHOENBERG TO PRODUCTS OF SPHERES

J. C. GUELLA, V. A. MENEGATTO, and A. P. PERON*

Communicated by D. Han

Abstract. We present a characterization for the continuous, isotropic, and positive definite kernels on a product of spheres along the lines of a classical result of Schoenberg on positive definiteness on a single sphere. We also discuss a few issues regarding the characterization, including topics for future investigation.

1. Introduction

We consider the problem of characterizing positive definite kernels on a product of spheres. Our focus will be on continuous and isotropic kernels, keeping the setting originally adopted by Schoenberg in his influential work published in 1942 (see [20]).

As usual, let S^m denote the unit sphere in the $(m+1)$-dimensional space $\mathbb{R}^{m+1}$, and let S^∞ denote the unit sphere in the usual real ℓ^2 space (here denoted by $\mathbb{R}^\infty$ for convenience). Throughout the present article, we will be dealing with real, continuous, and isotropic kernels on the product $S^m \times S^M$, $m, M = 1, 2, \ldots, \infty$. When speaking of continuity, we will assume that each sphere is endowed with its usual geodesic distance. The *isotropy* (zonality) of a kernel K on $S^m \times S^M$ refers to the fact that

$$K\big((x,z),(y,w)\big) = f(x \cdot y, z \cdot w), \quad x, y \in S^m, z, w \in S^M,$$

Received Jul. 1, 2015; Accepted Dec. 19, 2015.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 43A35; Secondary 33C50, 33C55, 42A10, 42A82.
Keywords. positive definiteness, spherical harmonics, isotropy, Gegenbauer polynomials, addition formula.

for some real function f on $[-1,1]^2$, where "$\cdot$" stands for the inner product of both $\mathbb{R}^{m+1}$ and $\mathbb{R}^{M+1}$. In particular, the concept introduced above demands the usual notion of isotropy on each sphere involved. In many places, we will refer to f as the *isotropic part* of K.

Recall that if X is a nonempty set, a kernel K is *positive definite* on X if

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu K(x_\mu, x_\nu) \geq 0,$$

for $n \geq 1$, distinct points $x_1, x_2, \ldots, x_n$ on X, and real scalars $c_1, c_2, \ldots, c_n$. In other words, for any $n \geq 1$ and any distinct points $x_1, x_2, \ldots, x_n$ on X, the $n \times n$ matrix with entries $K(x_\mu, x_\nu)$ is nonnegative definite. In this paper, we will present a characterization for the positive definiteness of a continuous and isotropic kernel on $X = S^m \times S^M$ based upon Fourier expansions.

Isotropy and positive definiteness for kernels on a single sphere were first considered by Schoenberg in [20]. He showed that a continuous and isotropic kernel K on S^m is positive definite if and only if $K(x,y) = g(x \cdot y)$, $x, y \in S^m$, in which the isotropic part g of K has a series representation in the form

$$g(t) = \sum_{k=0}^{\infty} a_k^m P_k^m(t), \quad t \in [-1,1],$$

where $a_k^m \geq 0$, $k \in \mathbb{Z}_+$, and $\sum_{k=0}^{\infty} a_k P_k^m(1) < \infty$. The symbol P_k^m stands for the usual Gegenbauer polynomial of degree k associated with the rational $(m-1)/2$, as discussed in [21]. This outstanding result of Schoenberg is far-reaching and has ramifications in distance geometry, statistics, spherical designs, approximation theory, and so on. In approximation theory, positive definite kernels are used in interpolation of scattered data over the sphere (see [8]). The importance of this problem in many areas of science and engineering is reflected in the literature, where different methods to solve such a problem have been proposed. Given n distinct data points $x_1, x_2, \ldots, x_n$ on S^m and a target function $h : S^m \to \mathbb{R}$, the interpolation problem itself requires the finding of a continuous function $s : S^m \to \mathbb{R}$ of the form

$$s(x) = \sum_{j=1}^{n} \lambda_j g(x \cdot x_j), \quad x \in S^m, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R},$$

so that $s(x_i) = h(x_i)$, $i = 1, 2, \ldots, n$. If we choose the prescribed function g to be the isotropic part of a convenient positive definite kernel K, then the interpolation problem has a unique solution for any n and any n data points.

Potential applications of an extension of Schoenberg's result to a product of spheres is still pending at this time. So far, we were unable to find any practical problem where such a characterization could enter in a decisive manner. However, we have to point out that characterizations similar to the one presented here (e.g., positive definiteness on a product of a sphere and a compact group) are useful in probability theory and stochastic processes, namely, in the understanding of certain space-time covariance functions (see [5], [12]). The bounding of codes in products of manifolds may be a branch of mathematics where positive definiteness—in the way we consider here—could have some applications (see [3] and references therein).

This paper is outlined as follows. In Section 2, we present several technical results that culminate with a characterization for the continuous, isotropic, and positive definite kernels on $S^m \times S^M$, $m, M < \infty$. In Section 3, we complete this circle of ideas by reaching a similar characterization in the cases in which at least one of the spheres involved is the real Hilbert sphere S^∞. Finally, Section 4 contains a few relevant remarks along with the description of future lines of investigation on the subject.

2. Positive definiteness on $S^m \times S^M$, $m, M < \infty$

The results in this section will converge to an extension of Schoenberg's theorem to a product of spheres $S^m \times S^M$ in the case when both m and M are finite. During the completion of this paper, we learned that such a characterization can be obtained from extensions of Bochner's work on positive definiteness, for instance, like that described in [2] (see Theorem 4.11). The proof we offer here is quite more down-to-earth and extrapolates some of Schoenberg's original arguments to the product $S^m \times S^M$. In particular, it exposes the real difficulties one has going from positive definiteness on a single sphere to positive definiteness on a product of spheres.

We will need a series of well-known results involving Gegenbauer polynomials and also a few facts from the analysis on the sphere. We suggest the classical reference [21] for the first topic and [9], [18] for the other.

The orthogonality relation for Gegenbauer polynomials reads as follows (see [9, p. 10]):

$$\int_{-1}^{1} P_n^m(t) P_k^m(t)(1-t^2)^{(m-2)/2}\, dt = \frac{\tau_{m+1}}{\tau_m} \frac{m-1}{2n+m-1} P_n^m(1)\delta_{n,k},$$

in which τ_{m+1} is the surface area of S^m; that is,

$$\tau_{m+1} := \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)}.$$

Since Schoenberg's characterization for positive definiteness on S^m is based upon Fourier expansions with respect to the orthogonal family $\{P_n^m : n = 0, 1, \ldots\}$, it is quite natural to expect that a similar characterization for positive definiteness on $S^m \times S^M$ will require expansions with respect to the tensor family

$$\left\{(t,s) \in [-1,1]^2 \to P_k^m(t) P_l^M(s) : k, l = 0, 1, \ldots\right\}.$$

The first important fact to be noted about the functions in the family above is this.

Lemma 2.1. *If $k, l \in \mathbb{Z}_+$, then $(t,s) \in [-1,1]^2 \to P_k^m(t)P_l^M(s)$ is the isotropic part of a positive definite kernel on $S^m \times S^M$.*

Proof. This follows from the definition of positive definiteness, Schoenberg's original characterization for positive definite kernels, and the Schur product theorem (see [13, p. 458]). The latter asserts that the entrywise product of two nonnegative definite matrices of the same order is itself a nonnegative definite matrix. □

The tensor family is orthogonal on $[-1,1]^2$ with respect to the weight function

$$w_{m,M}(t,s) = (1-t^2)^{(m-2)/2}(1-s^2)^{(M-2)/2}, \quad t,s \in [-1,1].$$

The (k,l)-Fourier coefficient of a function $f : [-1,1]^2 \to \mathbb{R}$ from the usual space $L^1([-1,1]^2, w_{m,M})$ of integrable functions in $[-1,1]^2$ with respect to $w_{m,M}$, is

$$\hat{f}_{k,l} := \frac{1}{\tau_k^m \tau_l^M} \int_{[-1,1]^2} f(t,s) P_k^m(t) P_l^M(s)\, dw_{m,M}(t,s), \quad k,l \in \mathbb{Z}_+,$$

in which

$$\tau_k^m := \frac{\tau_{m+1}}{\tau_m} \frac{m-1}{2k+m-1} P_k^m(1), \quad k \in \mathbb{Z}_+.$$

The next lemma describes an alternative way for computing these Fourier coefficients. The symbol σ_m will denote the surface measure on S^m.

Lemma 2.2. *If f belongs to $L^1([-1,1]^2, w_{m,M})$, then the Fourier coefficient $\hat{f}_{k,l}$ is a positive multiple of*

$$\int_{S^m \times S^M} \int_{S^m \times S^M} f(x\cdot y, z\cdot w) P_k^m(x\cdot y) P_l^M(z\cdot w)\, d\sigma_m(y)\, d\sigma_M(w)\, d\sigma_m(x)\, d\sigma_M(z).$$

Proof. If $m, M \geq 2$, it suffices to employ the Funk–Hecke formula (see [9, p. 11]) in the expression defining the Fourier coefficient. The Funk–Hecke formula states that

$$\begin{aligned}\int_{S^M} & g(z\cdot w) P_l^M(z\cdot w)\, d\sigma_M(w) \\ &= \tau_{M-1} \int_{-1}^{1} g(s) P_l^M(s)(1-s^2)^{(M-2)/2}\, ds, \quad z \in S^M,\end{aligned}$$

whenever $l \in \mathbb{Z}_+$ and $g \in L^2([-1,1], w_M)$. Using the formula with

$$g(s) = \int_{-1}^{1} f(t,s) P_k^m(t)(1-t^2)^{(m-2)/2}\, dt, \quad s \in [-1,1],$$

it is promptly seen that $\hat{f}_{k,l}$ is a positive multiple of

$$\int_{S^M} \int_{-1}^{1} f(t, z\cdot w) P_k^m(t)(1-t^2)^{(m-2)/2}\, dt P_l^M(z\cdot w)\, d\sigma_M(w).$$

Applying a similar argument in the internal integral reveals that $\hat{f}_{k,l}$ is a positive multiple of

$$\int_{S^M} \int_{S^m} f(x\cdot y, z\cdot w) P_k^m(x\cdot y) P_l^M(z\cdot w)\, d\sigma_m(y)\, d\sigma_M(w).$$

Integration with respect to the remaining variables concludes the proof. In the cases in which either $m = 1$ or $M = 1$, the arguments demand the replacement of the Funk–Hecke formula with direct computation. □

An interesting point to be observed is that the multiple constant in the statement of the preceding lemma does not depend upon k and l.

Lemma 2.3. *If f is the continuous and isotropic part of a positive definite kernel on $S^m \times S^M$, then*

$$\int_{S^m\times S^M}\left[\int_{S^m\times S^M} f(x\cdot y, z\cdot w)\,d\sigma_m(x)\,d\sigma_M(z)\right]d\sigma_m(y)\,d\sigma_M(w) \geq 0.$$

Proof. It suffices to write the double integral I in the statement of the theorem as a double limit of Riemann sums. Indeed, we can select a sequence $\{\mathcal{P}_n : n = 0, 1, \ldots\}$ of partitions of $S^m \times S^M$ in such a way that $\mathcal{P}_n = \{Q_1^n, Q_2^n, \ldots, Q_{\alpha(n)}^n\}$, the sequence $\{\alpha(n)\}$ increases to ∞, and the sequences of diameters $\{\mathrm{diam}(Q_j^n)\}$ satisfy $\lim_{n\to\infty}\mathrm{diam}(Q_j^n) = 0$. Picking points $(x_j^n, z_j^n) \in Q_j^n$, we can write

$$I = \lim_{N\to\infty}\sum_{J=1}^{\alpha(N)}\left[\int_{S^m\times S^M} f(x\cdot x_J^N, z\cdot z_J^N)\,d\sigma_m(x)\,d\sigma_M(z)\right]\mathrm{vol}(Q_J^N).$$

Repeating the procedure with the resulting integral leads to

$$\begin{aligned} I &= \lim_{N\to\infty}\sum_{J=1}^{\alpha(N)}\left[\lim_{n\to\infty}\sum_{j=1}^{\alpha(n)} f(x_j^n\cdot x_J^N, z_j^n\cdot z_J^N)\,\mathrm{vol}(Q_j^n)\right]\mathrm{vol}(Q_J^N)\\ &= \lim_{N\to\infty}\lim_{n\to\infty}\sum_{J=1}^{\alpha(N)}\sum_{j=1}^{\alpha(n)}\mathrm{vol}(Q_j^n)\,\mathrm{vol}(Q_J^N) f(x_j^n\cdot x_J^N, z_j^n\cdot z_J^N). \end{aligned}$$

Since the double limit above exists, it follows that

$$I = \lim_{n\to\infty}\sum_{j,J=1}^{\alpha(n)}\mathrm{vol}(Q_j^n)\,\mathrm{vol}(Q_J^n) f(x_j^n\cdot x_J^n, z_j^n\cdot z_J^n).$$

If f is the isotropic part of a positive definite kernel on $S^m \times S^M$, then each double sum in the last expression above is clearly nonnegative. In particular, the limit itself is nonnegative as well. □

We now combine the three lemmas above in order to obtain the following result.

Lemma 2.4. *If f is the continuous and isotropic part of a positive definite kernel on $S^m \times S^M$, then $\hat{f}_{k,l} \geq 0$, $k, l \in \mathbb{Z}_+$.*

Proof. Let us fix k and l. Lemma 2.1 and the Schur product theorem guarantee that the function

$$(t,s) \in [-1,1]^2 \to f(t,s)P_k^m(t)P_l^M(s)$$

is the continuous and isotropic part of a positive definite kernel on $S^m \times S^M$. Taking into account this information and that provided by Lemma 2.2, an application of Lemma 2.3 concludes the proof. □

Next, we recall one of the several generating formulas for the Gegenbauer polynomials, the Poisson identity (see [9, p. 419]).

Lemma 2.5. *If $r \in [0,1)$, then*

$$\frac{1-r^2}{(1-2tr+r^2)^{(m+1)/2}} = \sum_{k=0}^{\infty} \frac{2k+m-1}{m-1} P_k^m(t) r^k, \quad t \in [-1,1].$$

If $r_0 \in [0,1)$ is fixed, then the convergence of the series is absolute and uniform for $(r,t) \in [0,r_0] \times [-1,1]$.

We are ready to prove the following auxiliary result.

Lemma 2.6. *Let f be the continuous and isotropic part of a kernel on $S^m \times S^M$. If $r, \rho \in [0,1)$, then the double series*

$$\sum_{k,l=0}^{\infty} \hat{f}_{k,l} P_k^m(1) P_l^M(1) r^k \rho^l$$

converges. As a matter of fact, there exists a positive constant C, that depends upon f only, so that

$$\Big| \sum_{k,l=0}^{\infty} \hat{f}_{k,l} P_k^m(1) P_l^M(1) r^k \rho^l \Big| \leq C, \quad r, \rho \in [0,1).$$

Proof. Let $a_{k,l}^{r,\rho}$ denote the general term of the series in the statement of the lemma. It is promptly seen that

$$a_{k,l}^{r,\rho} = C_1 \int_{-1}^{1} \int_{-1}^{1} f(t,s) \frac{2k+m-1}{m-1} P_k^m(t) r^k \frac{2l+M-1}{M-1} P_l^M(s) \rho^l \, dw_{m,M}(t,s),$$

in which

$$C_1 = \frac{\tau_m \tau_M}{\tau_{m+1}\tau_{M+1}}.$$

On the other hand, Lemma 2.5 implies that

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}^{r,\rho} \leq C \int_{-1}^{1} \int_{-1}^{1} \frac{(1-r^2)(1-\rho^2)}{(1-2rt+r^2)^{(m+1)/2}(1-2\rho s+s^2)^{(M+1)/2}} \, dw_{m,M}(t,s),$$

whenever $r, \rho \in [0,1)$, in which $C := C_1 \max\{|f(t,s)| : -1 \leq t, s \leq 1\}$. It remains to verify that the double integral

$$\int_{[-1,1]^2} \frac{1-r^2}{(1-2rt+r^2)^{(m+1)/2}} \frac{1-\rho^2}{(1-2\rho s+s^2)^{(M+1)/2}} \times (1-t^2)^{(m-2)/2}(1-s^2)^{(M-2)/2} \, dt \, ds$$

is finite. But this follows from the well-known property of the Poisson kernels (see [18, p. 47])

$$\int_{-1}^{1} \frac{1-r^2}{(1-2rt+r^2)^{(m+1)/2}} (1-t^2)^{(m-2)/2} \, dt = \frac{\tau_{m+1}}{\tau_m}.$$

The proof is complete. □

Proposition 2.7. *If f is the continuous and isotropic part of a positive definite kernel on $S^m \times S^M$, then the double series*

$$\sum_{k,l=0}^{\infty} \hat{f}_{k,l} P_k^m(t) P_l^M(s)$$

converges absolutely and uniformly for $(t,s) \in [-1,1]^2$.

Proof. Let f be the continuous and isotropic part of a positive definite kernel on $S^m \times S^M$. Due to Lemma 2.4, we know already that all the Fourier coefficients $\hat{f}_{k,l}$ are nonnegative. In particular, the double sequence $\{s_{p,q}\}$ of partial sums of the double series

$$\sum_{k,l=0}^{\infty} \hat{f}_{k,l} P_k^m(1) P_l^M(1)$$

is monotonically increasing; that is, $s_{p,q} \leq s_{\mu,\nu}$ when $p \leq \mu$ and $q \leq \nu$. On the other hand, the preceding lemma produces the inequality

$$\sum_{k=0}^{p} \sum_{l=0}^{q} \hat{f}_{k,l} P_k^m(1) P_l^M(1) r^k \rho^l \leq C, \quad p,q \in \mathbb{Z}_+, r,\rho \in [0,1),$$

for some $C > 0$. By taking a double limit when $r,\rho \to 1^+$, we deduce that the double sequence of partial sums $\{s_{p,q}\}$ is bounded above. A classical result from the theory of double sequences (see [11, p. 373]) implies that $\{s_{p,q}\}$ converges. An application of the Weierstrass M-test adapted to double series of functions leads to the convergence quoted in the statement of the proposition. □

The main result in this section is the following.

Theorem 2.8. *Let K be a continuous and isotropic kernel on $S^m \times S^M$. It is positive definite on $S^m \times S^M$ if and only if its isotropic part f has a representation in the form*

$$f(t,s) = \sum_{k,l=0}^{\infty} a_{k,l} P_k^m(t) P_l^M(s), \quad t,s \in [-1,1],$$

in which $a_{k,l} \geq 0$, $k,l \in \mathbb{Z}_+$, and $\sum_{k,l=0}^{\infty} a_{k,l} P_k^m(1) P_l^M(1) < \infty$.

Proof. If the isotropic part f of K has the representation announced in the theorem, the series appearing there is uniformly and absolutely convergent. In particular, Lemma 2.1 implies that f is a pointwise double limit of functions which are isotropic parts of positive definite kernels on $S^m \times S^M$. Consequently, K itself is positive definite on $S^m \times S^M$. Conversely, assume that K is positive definite on $S^m \times S^M$, and write f to denote its isotropic part. Lemma 2.4, along with Proposition 2.7 and its proof, supplies a function

$$g(s,t) = \sum_{k,l=0}^{\infty} \hat{f}_{k,l} P_k^m(t) P_l^M(s), \quad t,s \in [-1,1],$$

for which $\hat{f}_{k,l} \geq 0$, $k,l \in \mathbb{Z}_+$, and $\sum_{k,l=0}^{\infty} \hat{f}_{k,l} P_k^m(1) P_l^M(1) < \infty$. In particular, since the series representing g is uniformly convergent in $[-1,1]^2$, g is continuous in

$[-1,1]^2$. The orthogonality on $[-1,1]^2$ of the tensor family quoted in Lemma 2.1 with respect to $w_{m,M}$ implies that

$$\hat{f}_{k,l} - \hat{g}_{k,l} = 0, \quad k,l \in \mathbb{Z}_+.$$

Thus, $f = g$ and, consequently, f has the representation in the statement of the theorem with $a_{k,l} = \hat{f}_{k,l}$, $k,l \in \mathbb{Z}_+$. □

3. Positive definiteness on $S^\infty \times S^M$

In this section, we extend Theorem 2.8 to the cases in which either $m = \infty$ or $M = \infty$.

Every sphere S^m can be isometrically embedded in S^∞. In particular, a positive definite kernel on $S^\infty \times S^M$ is positive definite on $S^m \times S^M$, for $m = 1, 2, \ldots$. Likewise, if f is the continuous and isotropic part of a positive definite kernel on $S^\infty \times S^M$, then it is the continuous and isotropic part of a positive definite kernel on $S^m \times S^M$, for $m = 1, 2, \ldots$. Hence, for every $m \geq 1$, we have a representation for f in the form

$$f(t,s) = \sum_{k,l=0}^{\infty} \hat{f}_{k,l}^{m,M} P_k^m(t) P_l^M(s), \quad t,s \in [-1,1],$$

in which

$$\hat{f}_{k,l}^{m,M} = \frac{1}{\tau_k^m \tau_l^M} \int_{[-1,1]^2} f(t,s) P_k^m(t) P_l^M(s)\, dw_{m,M}(t,s) \geq 0, \quad k,l \in \mathbb{Z}_+,$$

and $\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hat{f}_{k,l}^{m,M} P_k^{m-1}(1) P_l^{M-1}(1) < \infty$. Below, we will normalize the above expressions by writing

$$R_k^m = \frac{P_k^m}{P_k^m(1)}, \quad k \in \mathbb{Z}_+,$$

and

$$f(t,s) = \sum_{k,l=0}^{\infty} \check{f}_{k,l}^{m,M} R_k^m(t) R_l^M(s), \quad t,s \in [-1,1],$$

where

$$\check{f}_{k,l}^{m,M} := P_k^m(1) P_l^M(1) \hat{f}_{k,l}^{m,M}, \quad k,l \in \mathbb{Z}_+.$$

The notation and remarks above enter in the statement and proof of the next lemma.

Lemma 3.1. *Let f be the continuous and isotropic part of a positive kernel on $S^\infty \times S^M$. If k and l are fixed, then the sequence $\{\check{f}_{k,l}^{2m,M} : m = 1, 2, \ldots\}$ is convergent.*

Proof. Using the following recurrence relation for Gegenbauer polynomials (see [21, p. 84]),

$$(1-t^2) P_k^{m+2}(t) = \frac{(k+m-1)(k+m)}{(m-1)(2k+m+1)} P_k^m(t) - \frac{(k+1)(k+2)}{(m-1)(2k+m+1)} P_{k+2}^m(t),$$

it is easy to deduce that

$$\check{f}_{k,l}^{m+2,M} = \frac{(k+m-1)(k+m)}{m(2k+m-1)}\check{f}_{k,l}^{m,M} - \frac{(k+1)(k+2)}{m(2k+m+3)}\check{f}_{k+2,l}^{m,M}, \quad m \geq 1.$$

Consequently,

$$\begin{aligned}|\check{f}_{k,l}^{m+2,M} - \check{f}_{k,l}^{m,M}| &= \Big|\frac{k(k-1)}{m(2k+m-1)}\check{f}_{k,l}^{m,M} - \frac{(k+1)(k+2)}{m(2k+m+3)}\check{f}_{k+2,l}^{m,M}\Big| \\ &\leq \Big[\frac{k(k-1)}{m(2k+m-1)} + \frac{(k+1)(k+2)}{m(2k+m+3)}\Big] f(1,1), \quad m \geq 1.\end{aligned}$$

As an obvious consequence, $\{\check{f}_{k,l}^{2m,M}\}$ is a Cauchy sequence of real numbers and therefore convergent. □

Lemma 3.2. *If f is the continuous and isotropic part of a positive definite kernel on $S^\infty \times S^M$, then the double series*

$$\sum_{k,l=0}^{\infty} \check{f}_{k,l}^{m,M} t^k R_l^M(s)$$

converges for $(t,s) \in (-1,1)^2$, uniformly in m.

Proof. In order to see that the series converges in $(-1,1)^2$ for a fixed m, it suffices to show that $\sum_{k=0}^\infty \sum_{l=0}^\infty \check{f}_{k,l}^{m,M}|t|^k$ converges. Recalling Tonelli's theorem for convergence of double series (see [11, p. 384]), that will follow as long as $\sum_{l=0}^\infty \check{f}_{k,l}^{m,M}$ converges for all k and the iterated series $\sum_{k=0}^\infty(\sum_{l=0}^\infty \check{f}_{k,l}^{m,M})|t|^k$ converges. But both assertions follow from the inequalities

$$\sum_{l=0}^{\infty} \check{f}_{k,l}^{m,M} \leq \sum_{\mu,l=0}^{\infty} \check{f}_{\mu,l}^{m,M} = f(1,1), \quad k = 0,1,\ldots,$$

and

$$\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{\infty} \check{f}_{k,l}^{m,M}\Big)|t|^k \leq f(1,1)\sum_{k=0}^{\infty}|t|^k = \frac{f(1,1)}{1-|t|}, \quad t \in (-1,1).$$

As for the uniform convergence in m, it suffices to observe that

$$\sum_{k,l=0}^{\infty} \check{f}_{k,l}^{m,M} t^k R_l^M(s) \leq f(1,1)\sum_{k=0}^{\infty}|t|^k, \quad t,s \in (-1,1).$$

The proof is complete. □

The next lemma is a technical result that can be found proved in [20, (4.4)].

Lemma 3.3. *For $t \in (-1,1)$ fixed, the sequence $\{R_k^m(t)\}$ converges to t^k as $m \to \infty$, uniformly in k.*

Theorem 3.4. *Let K be a continuous and isotropic kernel on $S^\infty \times S^M$. It is positive definite on $S^\infty \times S^M$ if and only if its isotropic part f has a representation in the form*

$$f(t,s) = \sum_{k,l=0}^{\infty} a_{k,l}^M t^k R_l^M(s),$$

in which $a_{k,l}^M \geq 0$, $k,l \in \mathbb{Z}_+$, and $\sum_{k,l=0}^{\infty} a_{k,l}^M < \infty$.

Proof. For each k and l, the function $(t,s) \in [-1,1]^2 \to t^k R_l^M(s)$ is the isotropic part of a positive definite kernel on $S^\infty \times S^M$. Hence, if f has the representation described in the statement of the theorem, then K is a pointwise limit of positive definite kernels. In particular, it is positive definite itself. Conversely, assume that K is positive definite. Without loss of generality, we can assume that K is nonzero. Hence, we can also assume that its isotropic part f satisfies $f(1,1) > 0$. Since f is the isotropic part of a positive definite kernel on each product $S^m \times S^M$, then for each pair (k,l), we may consider the sequence of normalized Fourier coefficients $\{\check{f}_{k,l}^{m,M}\}$. Lemma 3.1 authenticates the definition

$$\check{f}_{k,l}^M := \lim_{m\to\infty} \check{f}_{k,l}^{2m,M}, \quad k,l = 0,1,\ldots,$$

while Lemma 3.2 guarantees that

$$\lim_{m\to\infty} \sum_{k,l=0}^{\infty} \check{f}_{k,l}^{2m,M} t^k R_l^M(s) = \sum_{k,l=0}^{\infty} \check{f}_{k,l}^M t^k R_l^M(s), \quad t,s \in (-1,1).$$

To proceed, we fix $(t,s) \in (-1,1)^2$ and $\epsilon > 0$. From the previous limit, we can select m_0 so that

$$\left| \sum_{k,l=0}^{\infty} \check{f}_{k,l}^{2m,M} t^k R_l^M(s) - \sum_{k,l=0}^{\infty} \check{f}_{k,l}^M t^k R_l^M(s) \right| < \frac{\epsilon}{2}, \quad m \geq m_0.$$

By Lemma 3.3, we can select m_1 so that

$$\left| R_k^{2m}(t) - t^k \right| < \frac{\epsilon}{2f(1,1)}, \quad k = 0,1,\ldots,m \geq m_1.$$

It is now clear that

$$\begin{aligned} \left| \sum_{k,l=0}^{\infty} \check{f}_{k,l}^{2m,M} R_k^{2m}(t) R_l^M(s) - \sum_{k,l=0}^{\infty} \check{f}_{k,l}^{2m,M} t^k R_l^M(s) \right| &< \frac{\epsilon}{2f(1,1)} \sum_{k,l=0}^{\infty} \check{f}_{k,l}^{2m,M} \\ &\leq \frac{\epsilon}{2}, \quad m \geq m_1. \end{aligned}$$

Thus, with the help of an arbitrarily large m, we can use both implications above to deduce that

$$0 \leq \left| f(t,s) - \sum_{k,l=0}^{\infty} \check{f}_{k,l}^M t^k R_l^M(s) \right| < \epsilon.$$

Hence,

$$f(t,s)=\sum_{k,l=0}^{\infty}\check{f}_{k,l}^{M}t^{k}R_{l}^{M}(s),\quad t,s\in(-1,1).$$

The coefficients $\check{f}_{k,l}^{M}$ are obviously nonnegative. If $\sum_{k,l=0}^{\infty}\check{f}_{k,l}^{M}$ were not convergent, we could select a positive integer N so that

$$\sum_{k,l=0}^{N}\check{f}_{k,l}^{M}\geq 2f(1,1).$$

Picking a $\tau\in(0,1)$ so that $\tau^{N}>1/2$, we would reach

$$f(\tau,1)=\sum_{k,l=0}^{\infty}\check{f}_{k,l}^{M}\tau^{k}\geq\sum_{k,l=0}^{N}\check{f}_{k,l}^{M}\tau^{k}>f(1,1),$$

a contradiction with the positive definiteness of f. Having guaranteed the uniform convergence of the series in the representation for f above and invoking the continuity of f in $[-1,1]^{2}$, we now can let $t,s\to 1^{-}$ and $t,s\to -1^{+}$ in the representation formula, in order to conclude that it holds in $[-1,1]^{2}$. Thus, f has the representation announced in the statement of the theorem, in which $a_{k,l}^{M}:=\check{f}_{k,l}^{M}$, $k,l\in\mathbb{Z}_{+}$. □

An adaptation of the arguments used in the proof of the preceding theorem is all that is needed in order to deduce the following complement.

Theorem 3.5. *Let K be a continuous and isotropic kernel on $S^{\infty}\times S^{\infty}$. It is positive definite on $S^{\infty}\times S^{\infty}$ if and only if its isotropic part f has a representation in the form*

$$f(t,s)=\sum_{k,l=0}^{\infty}a_{k,l}t^{k}s^{l},$$

in which $a_{k,l}\geq 0$, $k,l\in\mathbb{Z}_{+}$, and $\sum_{k,l=0}^{\infty}a_{k,l}<\infty$.

4. Final remarks

In view of the characterization for the continuous, isotropic, and positive definite kernels on a product of the form $S^{m}\times S^{M}$ obtained in the previous sections, one may ask what are the other relevant questions regarding that class of kernels. We will mention a few of them in this final section of the paper along with some additional results.

Let us begin with the strictly positive definite kernels. A continuous, isotropic, and positive definite kernel K on $S^{m}\times S^{M}$ is *strictly positive definite of order n* on $S^{m}\times S^{M}$ if its isotropic part f satisfies

$$\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}c_{\mu}c_{\nu}f(x_{\mu}\cdot x_{\nu},w_{\mu}\cdot w_{\nu})>0$$

whenever the n points $(x_{1},w_{1}),(x_{2},w_{2}),\ldots,(x_{n},w_{n})$ of $S^{m}\times S^{M}$ are distinct and the scalars c_{μ} are not all zero. So, for a fixed n, an interesting question would be to characterize, via the main theorems proved here, the continuous, isotropic, and strictly positive definite kernels of order n on $S^m \times S^M$. This is a challenging problem even on a single sphere, as one can see in [15]. To characterize the continuous, isotropic, and strictly positive definite kernels of all orders on $S^m \times S^M$ seems quite more promising, since a similar characterization on single spheres is found in [7], [14], [16], and [19]. Since the achievement of such a characterization on a product of spheres would demand additional techniques, these characterization problems will be considered elsewhere.

Positive definiteness on a product of spheres allows an intermediate notion of strict positive definiteness. A continuous, isotropic, and positive definite kernel K on $S^m \times S^M$ is DC-*strictly positive definite of order* n on $S^m \times S^M$ (DC stands for "distinct components") if its isotropic part f satisfies

$$\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} c_\mu c_\nu f(x_\mu \cdot x_\nu, w_\mu \cdot w_\nu) > 0$$

whenever the n points $x_1, x_2, \ldots, x_n$ of S^m are distinct, the n points $w_1, w_2, \ldots, w_n$ of S^M are distinct, and the scalars c_μ are not all zero. Obviously, a strictly positive definite kernel of order n on $S^m \times S^M$ is DC-strictly positive definite of order n on $S^m \times S^M$, but not conversely (unless $n = 1$). Thus, to characterize the continuous, isotropic, and positive definite kernels on $S^m \times S^M$ which are DC-strictly positive definite of order n on $S^m \times S^M$ would be an interesting problem as well. The same remark applies to DC-strict positive definiteness of all orders. Likewise, we do not intend to consider this problem here.

As for consistent methods to construct continuous, isotropic, and (strictly) positive definite kernels on $S^m \times S^M$, one may think of methods based on the use of known classes of continuous, isotropic, and (strictly) positive definite kernels on a single sphere. Here is a simple one, a generalization of Lemma 2.1.

Proposition 4.1. *If f is the continuous and isotropic part of a positive definite kernel on S^m and g is the continuous and isotropic part of a positive definite kernel on S^M, then the function h given by the formula*

$$h(t,s) = f(t)g(s), \quad t, s \in [-1,1],$$

is the isotropic part of a positive definite kernel on $S^m \times S^M$. Further, if f is the isotropic part of a strictly positive definite kernel of order n on S^m (resp., g is the isotropic part of a strictly positive definite kernel of order n on S^M) and $g(1) > 0$ (resp., $f(1) > 0$), then h is the isotropic part of a strictly positive definite kernel of order n on $S^m \times S^M$.

Proof. The first assertion of the theorem is a consequence of the Schur product theorem. As for the second one, it follows from Oppenheim's inequality (see [13, p. 480]). □

If the intention is to obtain a deeper example, one may employ completely monotonic functions in two variables. A continuous function $g : [0,\infty)^2 \to \mathbb{R}$ is *completely monotonic* on $(0,\infty)^2$ if it is C^∞ in $(0,\infty)^2$ and

$$(-1)^{n_1+n_2}\frac{\partial^{n_1+n_2} g}{\partial u^{n_1}\,\partial v^{n_2}}(u,v)\geq 0, \quad u,v>0, n_1,n_2\in\mathbb{Z}_+.$$

It is known that a function g as above can be represented in the form

$$g(u,v)=\int_{[0,\infty)^2} e^{-tu-sv}\,d\rho(t,s), \quad u,v>0,$$

in which ρ is a σ-additive and nonnegative measure on $[0,\infty)^2$ satisfying $0<\rho((0,\infty)^2)\leq\rho([0,\infty)^2)\leq\infty$ (see [6, p. 87]).

A positive scalar multiple of a completely monotonic function on $(0,\infty)^2$ is itself completely monotonic on $(0,\infty)^2$. Likewise, the sum and product of two completely monotonic functions on $(0,\infty)^2$ are completely monotonic on $(0,\infty)^2$. If $g,h:[0,\infty)\to\mathbb{R}$ are usual completely monotonic functions on $(0,\infty)$, then $F(u,v)=g(u)h(v)$ is completely monotonic on $(0,\infty)^2$. In particular, $(u,v)\in[0,\infty)^2\to\exp(-u)\exp(-v)$ and $(u,v)\in[0,\infty)^2\to 1/(1+u)^\alpha(1+v)^\beta$, $\alpha,\beta\geq 0$, are completely monotonic on $(0,\infty)^2$. Additional examples can be found in [17].

For actual examples of positive definite kernels on $S^m\times S^M$ based on completely monotonic functions, here is a concise method to produce them.

Proposition 4.2. *If g is completely monotonic on $(0,\infty)^2$, then*

$$f(t,s):=g(\arccos t,\arccos s)$$

is the isotropic part of a positive definite kernel on $S^m\times S^M$. Further, if g is nonconstant, then f is the isotropic part of a strictly positive definite kernel of all orders on $S^m\times S^M$.

Proof. If $x_1,\ldots,x_n\in S^m$, $w_1,\ldots,w_n\in S^M$, and $c_1,\ldots,c_n$ are real scalars, the integral representation for g previously described implies that

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu f(x_\mu\cdot x_\nu, w_\mu\cdot w_\nu)=\int_{[0,\infty)^2}\sum_{\mu,\nu=1}^{n} c_\mu c_\nu e^{-t\arccos(x_\mu\cdot x_\nu)-s\arccos(w_\mu\cdot w_\nu)}\,d\rho(t,s),$$

that is,

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu f(x_\mu\cdot x_\nu, w_\mu\cdot w_\nu)=\int_{[0,\infty)^2}\sum_{\mu,\nu=1}^{n} c_\mu c_\nu e^{-td_m(x_\mu\cdot x_\nu)-sd_M(w_\mu\cdot w_\nu)}\,d\rho(t,s),$$

in which d_m and d_M are the usual geodesic distances on S^m and S^M, respectively. A result proved in [1, pp. 9–10] reveals that d_m and d_M are kernels of negative type. Consequently, the matrices with entries $-td_m(x_\mu,x_\nu)-sd_M(w_\mu,w_\nu)$ are almost nonnegative definite (see [10, p. 135]). A classical result from the theory of positive definite kernels (see [4, p. 74]) now implies that $\exp(-td_m-sd_M)$ is a positive definite kernel on $S^m\times S^M$. Thus, the initial quadratic form is nonnegative and the first assertion of the proposition is proved. As for the second one, it suffices to observe that if the points (x_μ,w_μ) are distinct, then the matrix with entries $-td_m(x_\mu,x_\nu)-sd_M(w_\mu,w_\nu)$ has no pair of identical rows when $t,s>0$. In that case, the kernel $(x,z,y,w)\in(S^m\times S^M)^2\to\exp[-td_m(x,y)-sd_M(z,w)]$ is, in fact, strictly positive definite on $S^m \times S^M$. If g is nonconstant, then the original quadratic form is always positive, unless all the c_μ are zero. □

If one decides to go the other way around in the search for positive definiteness on a single sphere from positive definiteness on a product of spheres, two simple results are as follows.

Proposition 4.3. *If f is the continuous and isotropic part of a (strictly) positive definite kernel on $S^m \times S^M$, then $t \to f(t,1)$ and $s \to f(1,s)$ are the isotropic parts of (strictly) positive definite kernels on S^m and S^M, respectively.*

Proposition 4.4. *If f is the continuous and isotropic part of a* DC*-strictly positive definite kernel on $S^m \times S^M$, then $t \to f(t,t)$ is the isotropic part of a strictly positive definite kernel on $S^{m\wedge M}$, in which $m \wedge M = \min\{m, M\}$.*

We found it hard to think of additional examples.

Acknowledgments. The authors were partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) grants #2012/22161-3, #2014/00277-5, and #2014/25796-5, respectively.

References

1. R. Alexander and K. B. Stolarsky, *Extremal problems of distance geometry related to energy integrals*, Trans. Amer. Math. Soc. **193** (1974), 1–31. Zbl 0293.52005. MR0350629. 683
2. C. Bachoc, *Semidefinite programming, harmonic analysis and coding theory*, preprint, arXiv:0909.4767v2 [cs.IT]. 673
3. C. Bachoc, Y. Ben-Haim, and S. Litsyn, *Bounds for codes in products of spaces, Grassmann, and Stiefel manifolds*, IEEE Trans. Inform. Theory **54** (2008), no. 3, 1024–1035. Zbl pre05516852. MR2445048. DOI 10.1109/TIT.2007.915916. 673
4. C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions*, Grad. Texts in Math. **100**, Springer, New York, 1984. Zbl 0619.43001. MR0747302. DOI 10.1007/978-1-4612-1128-0. 683
5. C. Berg and E. Porcu, *From Schoenberg coefficients to Schoenberg functions*, preprint, to appear in Constr. Approx., arXiv:1505.05682v2 [math.CA]. 672
6. S. Bochner, *Harmonic Analysis and the Theory of Probability*, Univ. of California Press, Berkeley and Los Angeles, 1955. Zbl 0068.11702. MR0072370. 683
7. D. Chen, V. A. Menegatto, and X. Sun, *A necessary and sufficient condition for strictly positive definite functions on spheres*, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2733–2740. Zbl 1125.43300. MR1974330. DOI 10.1090/S0002-9939-03-06730-3. 682
8. E. W. Cheney, "Approximation using positive definite functions" in *Approximation Theory VIII, Vol. 1 (College Station, TX, 1995)*, Ser. Approx. Decompos. **6**, World Sci. Publ., River Edge, NJ, 1995, 145–168. Zbl 1137.41338. MR1471725. 672
9. F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer Monogr. Math., Springer, New York, 2013. Zbl 1275.42001. MR3060033. DOI 10.1007/978-1-4614-6660-4. 673, 674, 675
10. W. F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Grundlehren Math. Wiss. **207**, Springer, New York, 1974. Zbl 0278.30004. MR0486556. 683
11. S. R. Ghorpade and B. V. Limaye, *A Course in Multivariable Calculus and Analysis*, Undergrad. Texts Math., Springer, New York, 2010. Zbl 1186.26001. MR2583676. DOI 10.1007/978-1-4419-1621-1. 677, 679
12. T. Gneiting, *Strictly and non-strictly positive definite functions on spheres*, Bernoulli **19** (2013), no. 4, 1327–1349. Zbl 1283.62200. MR3102554. DOI 10.3150/12-BEJSP06. 672

13. R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge Univ. Press, Cambridge, 2013. Zbl 1267.15001. MR2978290. 674, 682
14. V. A. Menegatto, *Strictly positive definite kernels on the Hilbert sphere*, Appl. Anal. **55** (1994), no. 1–2, 91–101. Zbl 0873.41005. MR1379646. DOI 10.1080/00036819408840292. 682
15. V. A. Menegatto, *Strict positive definiteness on spheres*, Analysis (Munich) **19** (1999), no. 3, 217–233. Zbl 0978.42013. MR1715171. DOI 10.1524/anly.1999.19.3.217. 682
16. V. A. Menegatto, C. P. Oliveira, and A. P. Peron, *Strictly positive definite kernels on subsets of the complex plane*, Comput. Math. Appl. **51** (2006), no. 8, 1233–1250. Zbl 1153.41307. MR2235825. DOI 10.1016/j.camwa.2006.04.006. 682
17. K. S. Miller and S. G. Samko, *Completely monotonic functions*, Integral Transforms Spec. Funct. **12** (2001), no. 4, 389–402. Zbl 1035.26012. MR1872377. DOI 10.1080/10652460108819360. 683
18. C. Müller, *Analysis of Spherical Symmetries in Euclidean Spaces*, Appl. Math. Sci. **129**, Springer, New York, 1998. Zbl 0884.33001. MR1483320. DOI 10.1007/978-1-4612-0581-4. 673, 676
19. A. Ron and X. Sun, *Strictly positive definite functions on spheres in Euclidean spaces*, Math. Comp. **65** (1996), no. 216, 1513–1530. Zbl 0853.42018. MR1370856. DOI 10.1090/S0025-5718-96-00780-6. 682
20. I. J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. **9** (1942), 96–108. Zbl 0063.06808. MR0005922. 671, 672, 679
21. G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. **XXIII**, Amer. Math. Soc., Providence, 1975. Zbl 0305.42011. MR0372517. 672, 673, 678

Departamento de Matemática, ICMC-USP – São Carlos, Caixa Postal 668, 13560-970 – São Carlos SP, Brasil.

E-mail address: jeanguella@gmail.com; menegatt@icmc.usp.br; apperon@icmc.usp.br