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SEMI-FREDHOLM THEORY ON HILBERT C^*-MODULES

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Abstract. We establish the semi-Fredholm theory on Hilbert C^*-modules as a continuation of Fredholm theory on Hilbert C^*-modules established by Mishchenko and Fomenko. We give a definition of a semi-Fredholm operator on Hilbert C^*-module, and we prove that these semi-Fredholm operators are those that are one-sided invertible modulo compact operators, that the set of proper semi-Fredholm operators is open, and many other results that generalize their classical counterparts.

1. Introduction

The Fredholm and semi-Fredholm theory on Hilbert and Banach spaces started with a focus on certain integral equations introduced in the pioneering work by Fredholm [2] in 1903. After that, the abstract theory of Fredholm and semi-Fredholm operators on Banach spaces was further developed in numerous papers; a representative recent result in the classical semi-Fredholm theory can be found in [17], for example. Fredholm theory on Hilbert C^*-modules as a generalization of Fredholm theory on Hilbert spaces was initiated by Mishchenko and Fomenko in [8], where they elaborated on the notion of a Fredholm operator on the standard module $H_{\mathcal{A}}$ and proved the generalization of the Atkinson theorem. Our aim in the present article is to study more general operators than Fredholm ones—namely, a generalization of semi-Fredholm operators, which we define and for which we establish several properties as analogues or generalized versions of properties of the classical semi-Fredholm operators on Hilbert and Banach spaces.

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Recall that if H is a Hilbert space, then F is a semi-Fredholm operator on H, denoted by $F \in \Phi_{\pm}(H)$ if $F \in B(H)$ and $\operatorname{ran} F$ is closed; that is, if there exists a decomposition

$$H = (\ker F)^{\perp} \oplus \ker F \xrightarrow{F} \operatorname{ran} F \oplus (\operatorname{ran} F)^{\perp} = H$$

with respect to which F has the matrix $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where F_1 is an isomorphism, and either $\dim \ker F < \infty$ or $\dim(\operatorname{ran} F)^{\perp} < \infty$. If $\dim \ker F < \infty$, then F is called an *upper semi-Fredholm operator* on H, denoted by $F \in \Phi_{+}(H)$, whereas if $\dim(\operatorname{ran} F)^{\perp} < \infty$, then F is called a *lower semi-Fredholm operator* on H, denoted by $F \in \Phi_{-}(H)$. If F is both an upper and lower semi-Fredholm operator on H, then F is said to be a *Fredholm operator* on H, denoted by $F \in \Phi(H)$. In the case when $F \in \Phi(H)$, the index of F is defined as $\operatorname{index} F = \dim \ker F - \dim(\operatorname{ran} F)^{\perp}$. Now, Hilbert C^*-modules are a natural generalization of Hilbert spaces when the field of scalars is replaced by an arbitrary C^*-algebra. (Some recent results in the theory of Hilbert C^*-modules can be found in [3], [5], [6], [9], [14].) In [8], a standard Hilbert C^*-modul over a unital C^*-algebra $\mathcal{A}$, denoted by $H_{\mathcal{A}}$, is considered and a definition of an $\mathcal{A}$-Fredholm operator F on $H_{\mathcal{A}}$ as a generalization of a Fredholm operator on Hilbert space H is achieved in the following way (see [8, Definition]): A (bounded $\mathcal{A}$ linear) operator $F : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called $\mathcal{A}$-*Fredholm* if

(1) it is adjointable;
(2) there exists a decomposition of the domain $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1$, and the range, $H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2$, where M_1, M_2, N_1, N_2 are closed $\mathcal{A}$-modules and N_1, N_2 have a finite number of generators, such that F has the matrix from $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$ with respect to these decompositions and $F_1 : M_1 \to M_2$ is an isomorphism.

It is then proved in [8] that some of the main results from the classical Fredholm theory on Hilbert spaces also hold when one considers this generalization of Fredholm operator on a Hilbert C^*-module. The idea in the present paper was to go further in this direction and to give, in a similar way, a definition of semi-Fredholm operators on $H_{\mathcal{A}}$, to investigate and prove generalized version in this setting of significantly many results from the classical semi-Fredholm theory on Hilbert and Banach spaces.

In Section 2, inspired by [8, Definition], we define upper and lower semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$. We consider F an upper semi-$\mathcal{A}$-Fredholm operator on $H_{\mathcal{A}}$ if all the conditions in the definition above hold except that N_2 does not need to be finitely generated, and we consider F a lower semi-$\mathcal{A}$-Fredholm operator on $H_{\mathcal{A}}$ if all the conditions in the definition above hold except that N_1 in this case does not need to be finitely generated. Then we show that the classes of upper and lower semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$ (denoted, respectively, by $\mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}(H_{\mathcal{A}})$) coincide with the inverse images of the sets of the left and right invertible elements, respectively, in the C^*-algebra $B^a(H_{\mathcal{A}})/K(H_{\mathcal{A}})$ under the quotient map, where $B^a(H_{\mathcal{A}})$ denotes the C^*-algebra of all bounded, adjointable operators on $H_{\mathcal{A}}$. Semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$ have been considered in [1] and [4]. In [1], semi-$\mathcal{A}$-Fredholm operators are defined to be those that are one-sided invertible modulo compact operators. However, in the following we give another definition of semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$ and then prove that these operators are exactly those that are one-sided invertible modulo compact operators. Moreover, we prove an analogue or generalized version of main results in [18, Section 1.2] and [18, Section 1.3], as well as some additional new results. We wish to remark that if one considers the classes of operators $\mathcal{M}\Phi(H_{\mathcal{A}})$, $\mathcal{M}\Phi_+(H_{\mathcal{A}})$, $\mathcal{M}\Phi_-(H_{\mathcal{A}})$ in the sense of our Definition 2.1, which are given in terms of the decompositions for given F, it is not obvious that $\mathcal{M}\Phi(H_{\mathcal{A}}) = \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}})$, as in the classical semi-Fredholm theory on Hilbert and Banach spaces. This is due to the fact that these decompositions may not be unique for $F \in B^a(H_{\mathcal{A}})$, whereas in the classical case one always considers the decomposition

$$H = (\ker F)^\perp \oplus \ker F \xrightarrow{F} \operatorname{ran} F \oplus (\operatorname{ran} F)^\perp = H,$$

where H is a Hilbert space and $F \in B(H)$. Key arguments in proving that $\mathcal{M}\Phi(H_{\mathcal{A}}) = \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}})$ is the equivalence between $\mathcal{M}\Phi_+$ property and the left invertibility in the C^*-algebra $B^a(H_{\mathcal{A}})/K(H_{\mathcal{A}})$ and the equivalence between $\mathcal{M}\Phi_-$-property and the right invertibility in $B^a(H_{\mathcal{A}})/K(H_{\mathcal{A}})$. This is also the main argument in proving that

$$\begin{aligned}&\mathcal{M}\Phi_+(H_{\mathcal{A}}), \qquad \mathcal{M}\Phi_-(H_{\mathcal{A}}),\\ &\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}), \qquad \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\end{aligned}$$

are semigroups, as we find in the classical semi-Fredholm theory on Banach spaces. On the other hand, Lemmas 2.16 and 2.17 are very important as well, and they are also used in proving several other fundamental results here, where we meet the challenge of nonuniqueness of the decomposition.

In Section 3, we prove analogous results of [18, Lemma 1.4.1] and [18, Lemma 1.4.2]. More precisely, we give generalizations on Hilbert C^*-modules of the results from the classical semi-Fredholm theory on Banach spaces connected with the Schechter's characterization of Φ_+ operators on Banach spaces given in [12]. In Section 4, we prove that $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ are open as analogues of the well-known result in the classical semi-Fredholm theory, which states that the sets $\Phi_+(X) \setminus \Phi(X)$ and $\Phi_+(X) \setminus \Phi(X)$ are open where X is a Banach space, which result was proved in [13]. Here again, Lemmas 2.16 and 2.17 are one of the main arguments in the proof. Also, we prove an analogue version on $H_{\mathcal{A}}$ of [18, Corollary 1.6.10] and [18, Lemma 1.6.11].

In Section 5, we give first a generalization on Hilbert C^*-modules of Φ_+^- and Φ_-^+ operators on Hilbert spaces. A natural generalization on $H_{\mathcal{A}}$ of the class $\Phi_+^-(X)$ as defined in [18, Definition 1.2.1], where X is a Banach space, would be the following: let $F \in B^a(H_{\mathcal{A}})$; then $F \in \mathcal{M}\Phi_+^-(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1 is finitely generated, and $N_1 \preceq N_2$ (i.e., N_1 is isomorphic to a closed submodule of N_2). If $\mathcal{A} = \mathbb{C}$ (i.e., if $H_{\mathcal{A}}$ is an ordinary Hilbert space), then this definition would coincide with [18, Definition 1.2.1] of the class of Φ_+^- operators on Hilbert spaces. However, if X is a Hilbert or a Banach space, then $\Phi_+(X) \setminus \Phi(X) \subseteq \Phi_+^-(X)$ when we consider [18, Definition 1.2.1] of classes $\Phi_+(X)$, $\Phi(X)$ and $\Phi_+^-(X)$, whereas if we consider this "generalized" definition of the class $\mathcal{M}\Phi_+^-(H_{\mathcal{A}})$, it is not true in general that

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+^-(H_{\mathcal{A}}).$$

This is due to the fact that, given a finitely generated closed submodule N_1 of $H_{\mathcal{A}}$ and a countably but not finitely generated closed submodule N_2 of $H_{\mathcal{A}}$, it is not true in general that N_1 is isomorphic to a closed submodule of N_2. Therefore, we define the class $\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ to be the class of operators in $B^a(H_{\mathcal{A}})$ which satisfy the conditions of the suggested "generalized" definition above of $\mathcal{M}\Phi_+^-(H_{\mathcal{A}})$. Moreover, we define $\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ be the subclass of $\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, where in the decomposition above we have in addition that N_2 is finitely generated. Then we set

$$\mathcal{M}\Phi_+^-(H_{\mathcal{A}}) = \tilde{\mathcal{M}}\Phi_+^{-\prime}(H_{\mathcal{A}}) \cup \big(\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\big).$$

In a similar way, we define the classes $\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$, $\tilde{\mathcal{M}}\Phi_+^{-\prime}(H_{\mathcal{A}})$ and we set

$$\mathcal{M}\Phi_-^+(H_{\mathcal{A}}) = \tilde{\mathcal{M}}\Phi_-^{+\prime}(H_{\mathcal{A}}) \cup \big(\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\big).$$

We also then prove that

$$\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}}) = \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}),$$
$$\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}}) = \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}).$$

In addition we show that if $F \in \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

is any decomposition with respect to which F has the matrix $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$, where F_1 is an isomorphism, and N_1, N_2 are finitely generated, then, if $K(A)$ satisfies the cancellation property, we will have $N_1 \preceq N_2$. A similar conclusion yields for operators in the class $\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$ only in this case $N_2 \preceq N_1$. In addition, we show that the classes

$$\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}}), \qquad \tilde{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}}), \qquad \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}), \qquad \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$$

are open. In the rest of this section, we work with these classes of operators and prove analogous or generalized versions of almost all results in [18, Section 1.9].

The generalized versions on $H_{\mathcal{A}}$ of the results from the classical semi-Fredholm theory on Banach and Hilbert spaces, which are presented here, demand different proofs from the proofs in the classical case. However, the techniques used in these proofs are to a certain extent inspired by the techniques used in the proofs of some of the results in [8]. Moreover, Section 1 and especially Section 4, in which we introduce new additional classes of operators as $\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$, $\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$, $\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, and $\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$, contain some new results and not just generalizations on $H_{\mathcal{A}}$ of results from the classical semi-Fredholm theory on Banach spaces.

2. Semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$

In this section, we define semi-$\mathcal{A}$-Fredholm operators on the standard module $H_{\mathcal{A}}$ and we prove some of the main properties and results concerning these operators. Throughout this article, let $\mathcal{A}$ be a unital C^*-algebra, let $H_{\mathcal{A}}$ be a standard module over $\mathcal{A}$, and let $B^a(H_{\mathcal{A}})$ denote the set of all bounded, adjointable operators on $H_{\mathcal{A}}$. In accord with [10, Definition 1.4.1], we say that a Hilbert C^*-module M over $\mathcal{A}$ is *finitely generated* if there exists a finite set $\{x_i\} \subseteq M$ such that M equals the linear span (over $\mathbf{C}$ and $\mathcal{A}$) of this set.

Definition 2.1. Let $F \in B^a(H_{\mathcal{A}})$. We consider F an *upper semi-$\mathcal{A}$-Fredholm operator* if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$, and N_1 is finitely generated. Similarly, we say that F is a *lower semi-$\mathcal{A}$-Fredholm operator* if all the above conditions hold (except that in this case we assume that N_2 (and not N_1) is finitely generated).

Set

$$\begin{aligned} \mathcal{M}\Phi_+(H_{\mathcal{A}}) &= \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm}\}, \\ \mathcal{M}\Phi_-(H_{\mathcal{A}}) &= \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm}\}, \\ \mathcal{M}\Phi(H_{\mathcal{A}}) &= \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_{\mathcal{A}}\}. \end{aligned}$$

Then obviously $\mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}})$. We are going to show later in this section that actually "=" holds. (Note that if M, N are two arbitrary Hilbert modules C^*-modules, then the definition above could be generalized to the classes $\mathcal{M}\Phi_+(M,N)$ and $\mathcal{M}\Phi_-(M,N)$.)

Recall that by [10, Definition 2.7.8] (originally given in [8]), when $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi$ decomposition for F, it holds that the index of F is defined by index $F = [N_1]-[N_2] \in K(\mathcal{A})$, where $[N_1]$ and $[N_2]$ denote the isomorphism classes of N_1 and N_2, respectively. By [10, Definition 2.7.9], the index is well defined and does not depend on the choice of $\mathcal{M}\Phi$ decomposition for F. As regards the K-group $K(\mathcal{A})$, it is worth mentioning that it is not true in general that $[M] = [N]$ implies that $M \cong N$ for two finitely generated submodules M, N of $H_{\mathcal{A}}$. If $K(\mathcal{A})$ satisfies the property that $[N] = [M]$ implies that $N \cong M$ for any two finitely generated, closed submodules M, N of $H_{\mathcal{A}}$, then $K(\mathcal{A})$ is said to satisfy the "cancellation property" mentioned in [16, Section 6.2].

Theorem 2.2. *Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent.*

(1) $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.

(2) *There exists* $D \in B^a(H_{\mathcal{A}})$ *such that* $DF = I + K$ *for some* $K \in K(H_{\mathcal{A}})$.

Proof. The proof consists of two parts.

(2) $\Rightarrow$ (1): If (2) holds, then $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$ by [10, Lemma 2.7.12]. Let $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ be a decomposition modulo DF with the matrix

$$\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix},$$

where $(DF)_1$ is an isomorphism and N_1, N_2 are finitely generated. We wish to show that $F(M_1)$ is closed and we will do it by showing that $F_{|_{M_1}}$ is bounded below. Suppose that this is not the case. Then there exists a sequence $\{x_n\} \subseteq M_1$ such that $\|x_n\| = 1$ for all n and $Fx_n \to 0$ as $n \to \infty$. Since D is bounded, we must have that $DFx_n \to 0$ as $n \to +\infty$. But this would mean that DF is not bounded below on M_1 as $\|x_n\| = 1$ for all n. This is a contradiction since $DF_{|_{M_1}}$ is an isomorphism. Hence we must have that F is bounded below on M_1, which means that $F(M_1)$ is closed.

Now, by [10, Theorem 2.7.6], the result which was originally proved in [15], we may assume that M_1 is orthogonally complementable in $H_{\mathcal{A}}$. Hence $F_{|_{M_1}}$ is adjointable, so by [10, Theorem 2.3.3], which was originally proved in [7], ran $F_{|_{M_1}}$ is orthogonally complementable in $H_{\mathcal{A}}$. Hence $H_{\mathcal{A}} = F(M_1) \oplus F(M_1)^{\perp}$. With respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \oplus F(M_1)^{\perp} = H_{\mathcal{A}},$$

F has the matrix $\left[\begin{smallmatrix} F_1 & F_2 \\ 0 & F_4 \end{smallmatrix}\right]$, where F_1 is an isomorphism. If we let

$$U = \begin{bmatrix} 1 & -F_1{}^{-1}F_2 \\ 0 & 1 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{U} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

then U is an isomorphism and with respect to the decomposition

$$H_{\mathcal{A}} = U(M_1) \tilde{\oplus} U(N_1) \xrightarrow{F} F(M_1) \tilde{\oplus} F(M_1)^{\perp} = H_{\mathcal{A}},$$

F has the matrix $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & \tilde{F}_4 \end{smallmatrix}\right]$. Since N_1 is finitely generated, $U(N_1)$ is finitely generated also, and hence $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.

(1) $\Rightarrow$ (2): Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which F has the matrix $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$, where F_1 is an isomorphism and N_1 is finitely generated. Since N_1 is finitely generated, it is orthogonally complementable in $H_{\mathcal{A}}$ by [10, Lemma 2.3.7], which was originally proved in [7]. Then, by the proof of [10, Theorem 2.7.6], we can deduce that $F_{|_{N_1^{\perp}}}$ is an isomorphism onto $F(N_1{}^{\perp})$. Now, $F(N_1{}^{\perp}) = \operatorname{ran} FP_{N_1{}^{\perp}}$, where $P_{N_1{}^{\perp}}$ denotes the orthogonal projection onto $N_1{}^{\perp}$. Since $FP_{N_1{}^{\perp}} \in B^a(H_{\mathcal{A}})$ and $F(N_1{}^{\perp})$

is closed, being isomorphic to ${N_1}^{\perp}$, by [10, Theorem 2.3.3], it follows that $F({N_1}^{\perp})$ is orthogonally complementable. With respect to the decomposition

$$H_{\mathcal{A}} = {N_1}^{\perp} \oplus N_1 \xrightarrow{F} F({N_1}^{\perp}) \oplus F({N_1}^{\perp})^{\perp} = H_{\mathcal{A}},$$

F has the matrix $\left[\begin{smallmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{F}_4 \end{smallmatrix}\right]$, where $\tilde{F}_1$ is an isomorphism. Clearly $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_4$ are then adjointable.

Let D be the operator which has the matrix $\left[\begin{smallmatrix} \tilde{F}_1^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = F({N_1}^{\perp}) \oplus F({N_1}^{\perp})^{\perp} \xrightarrow{D} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}}.$$

Then $D \in B^a(H_{\mathcal{A}})$ and $DF = \left[\begin{smallmatrix} 1 & \tilde{F}_1^{-1}\tilde{F}_2 \\ 0 & 0 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = {N_1}^{\perp} \oplus N_1 \xrightarrow{DF} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}}.$$

Let $K = \left[\begin{smallmatrix} 0 & \tilde{F}_1^{-1}\tilde{F}_2 \\ 0 & -1 \end{smallmatrix}\right]$ with respect to the same decomposition. Since N_1 is finitely generated, we have $K \in K(H_{\mathcal{A}})$. Moreover, $DF = I + K$. □

Theorem 2.3. *Let $D \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent:*

(1) *$D \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$,*

(2) *there exist $F \in B^a(H_{\mathcal{A}})$, $K \in K(H_{\mathcal{A}})$ such that $DF = I + K$.*

Proof. The proof consists of two parts.

(2) ⇒ (1): Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \overset{I+K}{\longrightarrow} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}.$$

be an $\mathcal{M}\Phi$ decomposition for $I + K$. As in the proof of Theorem 2.2, we deduce that $F(M_1)$ is closed and orthogonally complementable in $H_{\mathcal{A}}$.

With respect to the decomposition

$$H_{\mathcal{A}} = F(M_1) \tilde{\oplus} F(M_1)^{\perp} \xrightarrow{D} M_2 \oplus N_2 = H_{\mathcal{A}},$$

D has the matrix $\left[\begin{smallmatrix} D_1 & D_2 \\ 0 & D_4 \end{smallmatrix}\right]$, where D_1 is an isomorphism, as in the proof of Theorem 2.2, part (2) ⇒ (1), and we deduce that D has the matrix $\left[\begin{smallmatrix} D_1 & 0 \\ 0 & \tilde{D}_4 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = \tilde{U}\big(F(M_1)\big) \tilde{\oplus} \tilde{U}\big(F(M_1)^{\perp}\big) \xrightarrow{D} M_2 \oplus N_2 = H_{\mathcal{A}},$$

where $\tilde{U}$ is an isomorphism. Since N_2 is finitely generated, it follows that $D \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$.

(1) ⇒ (2): Let

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D'} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

be an $\mathcal{M}\Phi_{-}$ decomposition for D (so that N_2' is finitely generated). Since N_2' is finitely generated, it is orthogonally complementable by [10, Lemma 2.3.7]. Now, since

$$H_{\mathcal{A}} = M_2' \tilde{\oplus} N_2' = {N_2'}^{\perp} \tilde{\oplus} N_2',$$

we have that $P_{N_2'^{\perp}}{}_{|_{M_2'}}$ is an isomorphism from M_2' onto $N_2'^{\perp}$, where $P_{N_2'^{\perp}}$ denotes the orthogonal projection onto $N_2'^{\perp}$. Since D has the matrix $\left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_4 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

where D_1 is an isomorphism, it follows that $D^{-1}(N_2') = N_1'$, $\ker P_{N_2'^{\perp}} D = N_1'$, and $\operatorname{ran} P_{N_2'^{\perp}} D = P_{N_2'^{\perp}}(M_2') = N_2'^{\perp}$, which is closed. By [10, Theorem 2.3.3], $\ker P_{N_2'^{\perp}} D = N_1'$ is orthogonally complementable, so $H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1'$. Hence $\sqcap_{M_1'}{}_{|_{N_1'^{\perp}}}$ is an isomorphism from $N_1'^{\perp}$ onto M_1', where $\sqcap_{M_1'}$ denotes the projection onto M_1' along N_1'. Therefore, $P_{N_2'^{\perp}} D \sqcap_{M_1'}{}_{|_{N_1'^{\perp}}}$ is an isomorphism from $N_1'^{\perp}$ onto $N_2'^{\perp}$. But since $D^{-1}(N_2') = N_1'$ and $H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1'$, it follows that

$$P_{N_2'^{\perp}} D_{|_{N_1'^{\perp}}} = P_{N_2'^{\perp}} D \sqcap_{M_1'}{}_{|_{N_1'^{\perp}}} .$$

Hence, $P_{N_2'^{\perp}} D_{|_{N_1'^{\perp}}}$ is an isomorphism from $N_1'^{\perp}$ onto $N_2'^{\perp}$, being a composition of isomorphisms, so with respect to the decomposition

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D} N_2'^{\perp} \oplus N_2' = H_{\mathcal{A}},$$

D has the matrix $\left[\begin{smallmatrix} \tilde{D}_1 & 0 \\ \tilde{D}_3 & \tilde{D}_4 \end{smallmatrix}\right]$, where $\tilde{D}_1$ is an isomorphism.

Let $F = \left[\begin{smallmatrix} (\tilde{D}_1)^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = N_2'^{\perp} \oplus N_2' \xrightarrow{F} N_1'^{\perp} \oplus N_1' = H_{\mathcal{A}}.$$

Then $F \in B^a(H_{\mathcal{A}})$ and $DF = \left[\begin{smallmatrix} 1 & 0 \\ \tilde{D}_3 \tilde{D}_1^{-1} & 0 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = N_2'^{\perp} \oplus N_2' \xrightarrow{DF} N_2'^{\perp} \oplus N_2' = H_{\mathcal{A}}.$$

Since N_2' is finitely generated, it follows that, if we let the operator $K = \left[\begin{smallmatrix} 0 & 0 \\ D_3 D_1^{-1} & -1 \end{smallmatrix}\right]$ modulo the decomposition above, then $K \in (H_{\mathcal{A}})$. Moreover, $DF = I + K$. □

Recall that $B^a(H_{\mathcal{A}})$ is a C^*-algebra and that $K(H_{\mathcal{A}})$ is a closed two-sided ideal in $B^a(H_{\mathcal{A}})$. Hence, $B^a(H_{\mathcal{A}})/K(H_{\mathcal{A}})$ is also C^*-algebra, equipped with the quotient norm. We will call this algebra the *Calkin algebra*.

Corollary 2.4. *We have the following.*

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}}).$$

Proof. It suffices to show $\supseteq$. By Theorem 2.2, $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ consists of all elements that are left invertible in the Calkin algebra, whereas $\mathcal{M}\Phi_-(H_{\mathcal{A}})$ consists of all elements that are right invertible in the Calkin algebra by Theorem 2.3. Now by [10, Theorem 2.7.14] and also by the proof of [10, Lemma 2.7.15], we have that $\mathcal{M}\Phi(H_{\mathcal{A}})$ consists of all elements that are invertible in the Calkin algebra. The corollary follows. □

Corollary 2.5. *It holds that $\mathcal{M}\Phi_+(H_\mathcal{A})$ and $\mathcal{M}\Phi_-(H_\mathcal{A})$ are semigroups under multiplication.*

Proof. This follows directly from Theorem 2.2 and Theorem 2.3, as $\mathcal{M}\Phi_+(H_\mathcal{A})$ consists of all elements that are left invertible in the Calkin algebra, whereas $\mathcal{M}\Phi_+(H_\mathcal{A})$ consists of all elements that are right invertible in the Calkin algebra. □

Corollary 2.6. *Let $F, D \in B^a(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_-(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$.*

Proof. Suppose that $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$. By Theorem 2.2 there exists some $C \in B^a(H_\mathcal{A})$, $K \in K(H)$ such that $CDF = I + K$. Again, by Theorem 2.2 it follows that $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. The proof of the second statement of Corollary 2.6 is similar. □

Corollary 2.7. *Let $F, D \in B^a(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $F \in \mathcal{M}\Phi(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_-(H_\mathcal{A})$ and $D \in \mathcal{M}\Phi(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi_-(H_\mathcal{A})$.*

Proof. Suppose that $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and that $F \in \mathcal{M}\Phi(H_\mathcal{A})$. By Theorem 2.2 there exist some $C \in B^a(H_\mathcal{A})$, $K \in K(H_\mathcal{A})$ such that $CDF = I + K$, as $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$ by assumption. Moreover, since $F \in \mathcal{M}\Phi(H_\mathcal{A})$, by the proof of [10, Lemma 2.7.15] there exist some $F' \in B^a(H_\mathcal{A})$, $K' \in K(H_\mathcal{A})$ such that $FF' = I + K'$. Hence, $CDFF' = (CDF)F' = (I + K)F' = F' + KF'$ and $CDFF' = CD(FF') = CD(I + K') = CD + CDK'$. Therefore, $FF' + FKF' = FCD + FCDK'$. So $FCD = FF' + FKF' - FCDK' = I + K' + FKF' - FCDK'$. Since $K' + FKF' - FCDK' \in K(H_\mathcal{A})$, by Theorem 2.2 it follows that $D \in \mathcal{M}\Phi(H_\mathcal{A})$. The proof of the second statement of Corollary 2.7 is similar. □

Corollary 2.8. *Let $F, D \in B^a(H_\mathcal{A})$. If $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi(H_\mathcal{A})$. If $F \in \mathcal{M}\Phi_-(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi(H_\mathcal{A})$.*

Proof. Let $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$. Since $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, by the proof of [10, Lemma 2.7.15] there exists some $C \in B^a(H_\mathcal{A})$, $K \in K(H)$ such that $DFC = I + K$. By Theorem 2.3, we have then that $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$. So $D \in \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A})$. But $\mathcal{M}\Phi(H_\mathcal{A}) = \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A})$ by Corollary 2.4, so $D \in \mathcal{M}\Phi(H_\mathcal{A})$. The proof of the second statement of Corollary 2.8 is similar. □

Corollary 2.9. *If $D \in \mathcal{M}\Phi(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi(H_\mathcal{A})$. If $F \in \mathcal{M}\Phi(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi(H_\mathcal{A})$.*

Proof. Suppose that $D \in \mathcal{M}\Phi(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$. Since $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, by the proof of [10, Lemma 2.7.15] there exist some $C \in B^a(H_\mathcal{A})$, $K \in K(H)$ such that $DFC = I + K$.

Moreover, since $D \in \mathcal{M}\Phi(H_\mathcal{A})$, by the proof of [10, Lemma 2.7.15] there exist some $D' \in B^a(H_\mathcal{A})$, $K' \in K(H)$ such that $D'D = I + K'$. Hence $D'DFC = D'(DFC) = D'(I + K) = D' + D'K$ and $D'DFC = (D'D)FC = (I + K')FC = FC + K'FC$. Thus $D' + D'K = FC + K'FC$. Hence $D'D + D'KD = FCD + K'FCD$. But $D'D = I + K'$, so we obtain $I + K' + D'KD = FCD + K'FCD$. So

$FCD = I + K' + D'KD - K'FCD$. Since $(K' + D'KD - K'FCD) \in K(H_{\mathcal{A}})$, by Theorem 2.3 we have $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Now, since $DF \in \mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}})$, by Corollary 2.6 it follows that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ also. Hence

$$F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}}) = \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Corollary 2.5. The proof of the second statement of Corollary 2.9 is similar. □

Corollary 2.10. *We have that $\mathcal{M}\Phi_+(H_{\mathcal{A}})\backslash\mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}})\backslash\mathcal{M}\Phi(H_{\mathcal{A}})$ are two-sided ideals in $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}})$, respectively. In particular, they are semigroups under multiplication.*

Proof. Let $F, D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and suppose first that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. Since $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ is a semigroup by Corollary 2.5, $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Now, if $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$, by Corollary 2.8 we have $D \in \mathcal{M}\Phi(H_{\mathcal{A}})$. Then, by Corollary 2.9, it would follow that $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, which is a contradiction. Thus we must have that $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}})\backslash\mathcal{M}\Phi(H_{\mathcal{A}})$. Suppose next that $D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})\backslash\mathcal{M}\Phi(H_{\mathcal{A}})$. Again, if $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then, since $D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, by Corollary 2.8 we would have that $D \in \mathcal{M}\Phi(H_{\mathcal{A}})$, which is impossible. So, also in this case, we must have that $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. Similarly one can prove the statement for $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ □

Corollary 2.11. *Let $F \in B^a(M, N)$. Then $F \in \mathcal{M}\Phi_+(M, N)$ if and only if $F^* \in \mathcal{M}\Phi_-(N, M)$. Moreover, if $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\operatorname{index} F = -\operatorname{index} F^*$.*

Proof. Observe that it follows from the proofs of Theorems 2.2 and 2.3, part (1) ⇒ (2), which could be generalized to the case when $F \in B^a(M, N)$ (and not only when $F \in B^a(H_{\mathcal{A}})$), that if $F \in \mathcal{M}\Phi_+(M, N)$, then for F and consequently for F^* there exist decompositions

$$M = M_1 \oplus M_1^{\perp} \xrightarrow{F} M_2 \oplus M_2^{\perp} = N,$$

$$N = M_2 \oplus M_2^{\perp} \xrightarrow{F^*} M_1 \oplus M_1^{\perp} = M$$

with respect to which F and F^* have matrices

$$\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix},$$

$$\begin{bmatrix} F_1^* & 0 \\ F_2^* & F_4^* \end{bmatrix},$$

respectively, where F_1, F_1^* are isomorphisms and $M_1^{\perp}$ is finitely generated. Using the technique of diagonalization as in the proof of [10, Lemma 2.7.10], we deduce that $F^* \in \mathcal{M}\Phi_-(N, M)$ as $M_1^{\perp}$ is finitely generated. The proof is analogous when $F \in \mathcal{M}\Phi_-(N, M)$, only in this case $M_2^{\perp}$ is finitely generated. If in addition F is in $\mathcal{M}\Phi(H_{\mathcal{A}})$, then both $M_1^{\perp}$ and $M_2^{\perp}$ will be finitely generated. Again using the technique of diagonalization, one deduces easily that $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$ in this case and $\operatorname{index} F = [M_1^{\perp}] - [M_2^{\perp}]$, $\operatorname{index} F^* = [M_2^{\perp}] - [M_1^{\perp}]$, so $\operatorname{index} F = -\operatorname{index} F^*$. □

Lemma 2.12. *Let M be a closed submodule of $H_{\mathcal{A}}$ such that $H_{\mathcal{A}} = M \tilde{\oplus} N$ for some finitely generated submodule N. Let $F \in B^a(H_{\mathcal{A}})$, let J_M be the inclusion map from M into $H_{\mathcal{A}}$ and suppose that $FJ_M \in \mathcal{M}\Phi_+(M, H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.*

Proof. Consider a decomposition $M = M_1 \tilde{\oplus} M_2 \xrightarrow{FJ_M} \tilde{M}_1 \tilde{\oplus} \tilde{M}_2 = H_{\mathcal{A}}$ with respect to which

$$FJ_M = \begin{bmatrix} (FJ_M)_1 & 0 \\ 0 & (FJ_M)_4 \end{bmatrix},$$

where $(FJ_M)_1$ is an isomorphism and M_2 is finitely generated. Then F has the matrix

$$\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} (M_2 \tilde{\oplus} N) \xrightarrow{F} \tilde{M}_1 \tilde{\oplus} \tilde{M}_2 = H_{\mathcal{A}},$$

where F_1 is an isomorphism. Using the technique of diagonalization as in the proof of [10, Lemma 2.7.10] and the fact that $M_2 \tilde{\oplus} N$ is finitely generated as both M_2 and N are so, we deduce that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. □

Suppose now that $F \in B^a(H_{\mathcal{A}})$ and that $\operatorname{ran} F$ is closed. Then, again by [10, Theorem 2.3.3], $\operatorname{ran} F$ is orthogonally complementable in $H_{\mathcal{A}}$, so $J_{\operatorname{ran} F} \in B^a(\operatorname{ran} F, H_{\mathcal{A}})$.

Lemma 2.13. *Suppose that $D, F \in B^a(H_{\mathcal{A}})$, $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, and $\operatorname{ran} F$ is closed. Then $DJ_{\operatorname{ran} F} \in \mathcal{M}\Phi_+(\operatorname{ran} F, H_{\mathcal{A}})$.*

Proof. Let $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ be the decomposition with respect to which DF has the matrix

$$\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix},$$

where $(DF)_1$ is an isomorphism and N_1 is finitely generated.

Since $D(D^{-1}(M_2) \cap \operatorname{ran} F) = M_2$—that is, $DJ_{\operatorname{ran} F}(D^{-1}(M_2) \cap \operatorname{ran} F) = M_2$—we get that

$$\operatorname{ran} F = ((D^{-1}(M_2) \cap \operatorname{ran} F\big) \tilde{\oplus} (\big(D^{-1}(N_2) \cap \operatorname{ran} F\big).$$

To see this, let $y \in \operatorname{ran} F$, then $Dy = Dx_m + x_n$, for some $x_m \in D^{-1}(M_2) \cap \operatorname{ran} F$ and for some $x_n \in N_2$. Hence $x_n \in D(y - x_m)$, so $x_n \in D(\operatorname{ran} F)$. As $x_n \in N_2$ also, we have $x_n \in D(\operatorname{ran} F) \cap N_2$. Thus $(y - x_m) \in \operatorname{ran} F \cap D^{-1}(N_2)$. Since $y = x_m + (y - x_m)$, $x_m \in D^{-1}(M_2) \cap \operatorname{ran} F$ and $(y - x_m) \in D^{-1}(N_2) \cap \operatorname{ran} F$, we deduce that

$$\operatorname{ran} F = \big(D^{-1}(M_2) \cap \operatorname{ran} F\big) \tilde{\oplus} (\big(D^{-1}(N_2) \cap \operatorname{ran} F\big)$$

as $y \in \operatorname{ran} F$ was arbitrary. With respect to the decomposition

$$\operatorname{ran} F = \big(D^{-1}(M_2) \cap \operatorname{ran} F\big) \tilde{\oplus} (\big(D^{-1}(N_2) \cap \operatorname{ran} F\big) \xrightarrow{DJ_{\operatorname{ran} F}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

$DJ_{\operatorname{ran} F}$ has the matrix

$$\begin{bmatrix} (DJ_{\operatorname{ran} F})_1 & 0 \\ 0 & (DJ_{\operatorname{ran} F})_4 \end{bmatrix},$$

where $(DJ_{\operatorname{ran} F})_1$ is an isomorphism. Now, since DF has the matrix

$$\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

it is easily seen that $D^{-1}(N_2) \cap \operatorname{ran} F = F(N_1)$ which is finitely generated. We are done. □

Corollary 2.14. *Let V be a finitely generated Hilbert submodule of $H_{\mathcal{A}}$, $F \in B^a(H_{\mathcal{A}})$ and suppose that $P_{V^\perp} F \in \mathcal{M}\Phi_-(H_{\mathcal{A}}, V^\perp)$, where $P_{V^\perp}$ is the orthogonal projection onto $V^\perp$ along V. Then $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.*

Proof. Passing to the adjoints and using Lemma 2.12 together with Corollary 2.11, one obtains the result. □

Corollary 2.15. *Let $D, F \in B^a(H_{\mathcal{A}})$ and suppose that* $\operatorname{ran} D^*$ *is closed. If $DF \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, then $P_{\ker(D)^\perp} F \in \mathcal{M}\Phi_-(H_{\mathcal{A}}, \operatorname{ran}(D^*))$*

Proof. Observe that since $\operatorname{ran}(D^*)$ is closed, then by the proof of [10, Theorem 2.3.3], we have that $H_{\mathcal{A}} = \ker(D) \oplus \operatorname{ran}(D^*)$. Hence $\ker(D)^\perp = \operatorname{ran}(D^*)$. Passing to the adjoints and using Lemma 2.13 together with Corollary 2.11, one deduces the corollary. □

Lemma 2.16. *Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and suppose that there are two decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which F has matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \qquad \begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix},$$

respectively, where F_1, ${F_1}'$ are isomorphisms, N_1, ${N_1}'$, N_2 are closed and finitely generated, and ${N_2}'$ is just closed. Then ${N_2}'$ is finitely generated also.

Proof. Since N_1, ${N_1}'$ are finitely generated, by [10, Theorem 2.7.5] there exist an n such that

$$L_n = P \tilde{\oplus} p_n(N_1), \qquad P = M_1 \cap L_n, \qquad p_n(N_1) \cong N_1, \qquad \text{and}$$
$$L_n = P' \tilde{\oplus} p_n(N_1'), \qquad P' = M_1' \cap L_n, \qquad p_n(N_1') \cong N_1'.$$

Then

$$H_{\mathcal{A}} = L_n^\perp \tilde{\oplus} P \tilde{\oplus} N_1 = L_n^\perp \tilde{\oplus} P' \tilde{\oplus} N_1',$$

and consequently $\sqcap_{{M_1}_{|_{(L_n^\perp \tilde{\oplus} P)}}}$, $\sqcap_{{M_1'}_{|_{(L_n^\perp \tilde{\oplus} P')}}}$ are isomorphisms from $L_n^\perp \tilde{\oplus} P$ onto M_1 and from $L_n^\perp \tilde{\oplus} P'$ onto M_1', respectively, where $\sqcap_{{M_1}_{|_{(L_n^\perp \tilde{\oplus} P)}}}$, $\sqcap_{{M_1'}_{|_{(L_n^\perp \tilde{\oplus} P')}}}$ denote the restrictions of projections onto M_1 and M_1' along N_1 and N_1' restricted to $L_n^\perp \tilde{\oplus} P$ and $L_n^\perp \tilde{\oplus} P'$, respectively. Since $F(M_1) = M_2$ and $F(N_1) \in N_2$ and $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1$, it follows that

$$\sqcap_{M_2} F_{|_{(L_n^\perp \tilde{\oplus} P)}} = F \sqcap_{{M_1}_{|_{(L_n^\perp \tilde{\oplus} P)}}} = F_1 \sqcap_{{M_1'}_{|_{(L_n^\perp \tilde{\oplus} P')}}}$$

where $\sqcap_{M_2}$ denotes the projection onto M_2 along N_2. Hence $\sqcap_{M_2} F_{|_{(L_n^\perp \tilde{\oplus} P)}}$ is an isomorphism as $\sqcap_{{M_1}_{|_{(L_n^\perp \tilde{\oplus} P)}}}$ and F_1 are so. Similarly, $\sqcap_{M_2'} F_{|_{(L_n^\perp \tilde{\oplus} P')}}$ is an isomorphism, where $\sqcap_{M_2'}$ denotes the projection onto M_2' along N_2'.

We get then that F has the matrices

$$\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_3 & F_4 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{F}_1' & 0 \\ \tilde{F}_3' & F_4' \end{bmatrix},$$

with respect to the decompositions

$$H_{\mathcal{A}} = (L_n^\perp \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = (L_n^\perp \tilde{\oplus} P') \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

respectively, where $\tilde{F}_1 = \sqcap_{M_2} F_{|_{(L_n^\perp \tilde{\oplus} P)}}$, $\tilde{F}_1' = \sqcap_{M_2'} F_{|_{(L_n^\perp \tilde{\oplus} P')}}$ are isomorphisms. As in the proof of [10, Lemma 2.7.11], we let

$$V = \begin{bmatrix} 1 & 0 \\ -\tilde{F}_3 \tilde{F}_1^{-1} & 1 \end{bmatrix}, \qquad V' = \begin{bmatrix} 1 & 0 \\ -\tilde{F}_3' \tilde{F}_1'^{-1} & 1 \end{bmatrix},$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \xrightarrow{V} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = M_2' \tilde{\oplus} N_2' \xrightarrow{V} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}.$$

Then F has the matrices

$$\begin{bmatrix} \tilde{\tilde{F}}_1 & 0 \\ 0 & \tilde{\tilde{F}}_4 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\tilde{F}}_1' & 0 \\ 0 & \tilde{\tilde{F}}_4' \end{bmatrix},$$

with respect to the decompositions

$$H_{\mathcal{A}} = (L_n^\perp \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} V^{-1}(M_2) \tilde{\oplus} V^{-1}(N_2) = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = (L_n^\perp \tilde{\oplus} P') \tilde{\oplus} N_1' \xrightarrow{F} V'^{-1}(M_2') \tilde{\oplus} V'^{-1}(N_2') = H_{\mathcal{A}},$$

respectively, where $\tilde{\tilde{F}}_1$, $\tilde{\tilde{F}}_1'$ are isomorphism. Again, as in the proof of [10, Lemma 2.7.11], we change these decompositions into

$$H_{\mathcal{A}} = L_n^\perp \tilde{\oplus} (P \tilde{\oplus} N_1) \xrightarrow{F} F(L_n^\perp) \tilde{\oplus} \big(F(P) \tilde{\oplus} V^{-1}(N_2)\big) = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = L_n^\perp \tilde{\oplus} (P' \tilde{\oplus} N_1') \xrightarrow{F} F(L_n^\perp) \tilde{\oplus} \big(F(P') \tilde{\oplus} V'^{-1}(N_2')\big) = H_{\mathcal{A}},$$

and with respect to these decompositions, F has matrices

$$\begin{bmatrix} \tilde{\tilde{F}}_1 & 0 \\ 0 & \tilde{\tilde{F}}_4 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\tilde{F}}_1' & 0 \\ 0 & \tilde{\tilde{F}}_4' \end{bmatrix},$$

respectively, where $\tilde{\tilde{F}}_1$, $\tilde{\tilde{F}}_1'$ are isomorphisms.

As

$$H_{\mathcal{A}} = F(L_n^{\perp}) \tilde{\oplus} \big(F(P) \tilde{\oplus} V^{-1}(N_2)\big) = F(L_n^{\perp}) \tilde{\oplus} \big(F(P') \tilde{\oplus} V'^{-1}(N_2')\big) = H_{\mathcal{A}},$$

clearly we have

$$\big(F(P) \tilde{\oplus} V^{-1}(N_2)\big) \cong \big(F(P') \tilde{\oplus} V'^{-1}(N_2')\big).$$

Now, $F(P)$ and $V^{-1}(N_2)$ are finitely generated since $F_{|_P}$, V^{-1} are isomorphisms and P, N_2 are finitely generated. Hence, $(F(P) \tilde{\oplus} V^{-1}(N_2))$ is finitely generated, and consequently $(F(P') \tilde{\oplus} V'^{-1}(N_2'))$ is finitely generated, being isomorphic to a finitely generated submodule $(F(P) \tilde{\oplus} V^{-1}(N_2))$. Therefore, $V'^{-1}(N_2')$ is finitely generated, since it is generated by the images of the generators of $F(P') \tilde{\oplus} V'^{-1}(N_2')$ under the projection onto $V'^{-1}(N_2')$ along $F(P')$. But V' is an isomorphism, hence N_2' must be finitely generated. □

Lemma 2.17. *Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and let*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_2 is finitely generated, and N_1 is just closed. Then N_1 is finitely generated.

Proof. By Corollary 2.11, we have $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$ since $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, and moreover, by the proof of Theorem 2.3, N_1, N_2 are orthogonally complementable. With respect to the decomposition

$$H_{\mathcal{A}} = N_1{}^{\perp} \tilde{\oplus} N_1 \xrightarrow{F} N_2{}^{\perp} \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

F has the matrix

$$\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_3 & F_4 \end{bmatrix},$$

where $\tilde{F}_1$ is an isomorphism and hence, with respect to the decomposition

$$H_{\mathcal{A}} = N_2{}^{\perp} \tilde{\oplus} N_2 \xrightarrow{F^*} N_1{}^{\perp} \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

F^* has the matrix

$$\begin{bmatrix} \tilde{F}_1^* & \tilde{F}_3^* \\ 0 & \tilde{F}_4^* \end{bmatrix}.$$

Clearly, $\tilde{F}_1^*$ is an isomorphism since $\tilde{F}_1$ is an isomorphism. Then F^* has the matrix

$$\begin{bmatrix} \tilde{F}_1^* & 0 \\ 0 & F_4^* \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = U(N_2{}^{\perp}) \tilde{\oplus} U(N_2) \xrightarrow{F^*} N_1{}^{\perp} \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

where U is an isomorphism. But since $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$, $\tilde{F}_1^*$ is an isomorphism and $U(N_2)$ is finitely generated (as N_2 is finitely generated by assumption), we can use the preceding lemma to deduce that N_1 is finitely generated. □

Corollary 2.18. *Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and let*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = \tilde{M}_1 \tilde{\oplus} \tilde{N}_1 \xrightarrow{F} \tilde{M}_2 \tilde{\oplus} \tilde{N}_2 = H_{\mathcal{A}}$$

be two $\mathcal{M}\Phi_+$ decompositions for F. Then there exists some finitely generated submodules P and $\tilde{P}$ such that $(N_2 \tilde{\oplus} P) \cong (\tilde{N}_2 \tilde{\oplus} \tilde{P})$.

Proof. This proof follows from the proof of Lemma 2.16. □

Corollary 2.19. *Let $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and let*

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = \tilde{M}_1' \tilde{\oplus} \tilde{N}_1 \xrightarrow{D} \tilde{M}_2' \tilde{\oplus} \tilde{N}_2' = H_{\mathcal{A}}$$

be two $\mathcal{M}\Phi_-$ decompositions for D. Then there exists some finitely generated,closed submodules P' and $\tilde{P}'$ such that $(N_1' \tilde{\oplus} P') \cong (\tilde{N}_1{}' \tilde{\oplus} \tilde{P}')$.

Proof. This proof follows from the proof of Lemma 2.17. □

Lemma 2.20. *Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and suppose that* $\operatorname{ran} F$ *is closed. If*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$
$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

are two $\mathcal{M}\Phi_+$ decomposition for F, then $F(N_1)$, $F(N_1')$ are closed finitely generated projective modules and

$$[N_1] - \big[F(N_1)\big] = [N_1'] - \big[F(N_1')\big]$$

in $K(A)$.

Proof. First of all, it is obvious that

$$F(N_1) = \operatorname{ran} F \cap N_2, \qquad F(N_1') = \operatorname{ran} F \cap N_2'.$$

Now, since $\operatorname{ran} F$ is closed (by assumption), we have that $F(N_1)$, $F(N_1')$ are closed and also finitely generated as N_1, N_1' are also closed and also finitely generated. Then, by [10, Lemma 2.3.7],

$$F(N_1) \tilde{\oplus} \tilde{N}_2 = N_2, \qquad F(N_1') \tilde{\oplus} \tilde{N}_1' = N_2'$$

for some closed submodules $\tilde{N}_1'$, $\tilde{N}_2'$ of N_1, N_2, respectively.

Thus $H_{\mathcal{A}} = M_2 \tilde{\oplus} \tilde{N}_2 \tilde{\oplus} F(N_1)$ and $H_{\mathcal{A}} = M_2' \tilde{\oplus} \tilde{N}_2' \tilde{\oplus} F(N_1')$. Since $F(N_1)$, $F(N_1')$ are finitely generated, from [10, Theorem 2.7.5] it follows that $F(N_1)$, $F(N_1')$ are projective. Moreover, again by [10, Theorem 2.7.5], we may assume that there exists some m such that

$$N_1 \subseteq L_m, \qquad L_m = N_1 \tilde{\oplus} \tilde{P}_1, \qquad M_1 = \tilde{P}_1 \tilde{\oplus} L_m$$

and

$$L_m = \tilde{P}_1' \tilde{\oplus} p_m(N_1'), \qquad p_m(N_1') \tilde{=} N_1',$$

where p_m is the projection onto L_m along $L_m^{\perp}$ and P_1', $\tilde{P}_1$ are projective, finitely generated A-modules.

Set $L_m' = F(L_m) + F(N_1)$ and $L_m'' = F(L_m^{\perp})$. Note that $\operatorname{ran} F = L_m' \tilde{\oplus} L_m''$. By the arguments similar to the proof of [10, Theorem 2.7.9], we deduce that

$$[N_1] + [\tilde{P}_1] = [N_1'] + [P_1'] = [L_m],$$
$$\big[F(N_1)\big] + \big[F(\tilde{P}_1)\big] = \big[F(N_1')\big] + \big[F(P_1')\big] = [L_m'],$$
$$\big[F(\tilde{P}_1)\big] \cong [\tilde{P}_1], \big[F(P_1')\big] \cong [P_1'].$$

Hence,

$$[N_1] - \big[F(N_1)\big] = [N_1'] - \big[F(N_1')\big]. \qquad \square$$

3. Generalized Schechter characterization of $\mathcal{M}\Phi_+$ operators on $H_{\mathcal{A}}$

In this section we investigate the classes $\mathcal{M}\Phi_+(H_{\mathcal{A}})$, $B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and prove an analogue of some results concerning the classes $\Phi_+(X)$, $B(X) \setminus \Phi_+(X)$ (where X is a Banach space) in [12].

Lemma 3.1. *Let $F \in B^a(M,N)$ Then $F \in \mathcal{M}\Phi_+(M,N)$ if and only if there exists a closed, orthogonally complementable submodule $M' \subseteq M$ such that $F_{|_{M'}}$ is bounded below and $M'^{\perp}$ is finitely generated.*

Proof. If such M' exists, then $F(M')$ is closed in N. Moreover, as M' is orthogonally complementable, $F_{|_{M'}}$ is adjointable. By [10, Theorem 2.3.3], $F(M')$ is orthogonally complementable in N. Then with respect to the decomposition

$$M = M' \oplus M'^{\perp} \xrightarrow{F} F(M') \oplus F(M')^{\perp} = N,$$

F has the matrix

$$\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism.Using the technique of diagonalization as in the proof of [10, Lemma 2.7.10] and the fact that $M'^{\perp}$ is finitely generated, we deduce that $F \in \mathcal{M}\Phi_+(M,N)$. On the other hand, if $F \in \mathcal{M}\Phi_+(M,N)$, by the similar arguments as in the proof of [10, Theorem 2.7.6] we may assume that there exists a decomposition

$$M = M' \oplus M'^{\perp} \xrightarrow{F} N' \oplus N'' = N,$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism and $M'^{\perp}$ is finitely generated. □

Lemma 3.2. *Let $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there exists a sequence $\{x_k\} \subseteq H_{\mathcal{A}}$ and an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that*

$$x_k \in L_{n_k} \setminus L_{n_{k-1}} \quad \text{for all } k \in \mathbb{N}, \qquad \|x_k\| \leq 1 \quad \text{for all } k \in \mathbb{N},$$

and

$$\|Fx_k\| \leq 2^{1-2k} \quad \text{for all } k \in \mathbb{N}.$$

Proof. Since $F \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})$, there exists an

$$\tilde{x}_1 \subseteq H_{\mathcal{A}}, \qquad \|\tilde{x}_1\| \leq 1, \quad \text{such that } \|F\tilde{x}_1\| \leq \frac{1}{4}$$

because F is then not bounded below by the preceding lemma. As

$$\|P_{L_{n_1}^{\perp}} \tilde{x}_1\| \longrightarrow 0 \quad \text{when } n \to \infty,$$

there exists an $n_1 \in \mathbb{N}$ such that $\|P_{L_{n_1}^{\perp}} \tilde{x}_1\| \leq \frac{1}{\|F\|}\frac{1}{4}$ (here again $P_{L_{n_1}\perp}$ denotes the orthogonal projection onto $L_n^{\perp}$ along L_n). Hence,

$$\|FP_{L_{n_1}} \tilde{x}_1\| \leq \|F\tilde{x}_1\| + \|FP_{L_{n_1}^{\perp}} \tilde{x}_1\| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Set $x_1 = P_{L_{n_1}} \tilde{x}_1$, then

$$\|x_1\| \leq \|\tilde{x}_1\| \leq 1, \qquad \|Fx_1\| \leq \frac{1}{2}$$

and $x_1 \in L_{n_1}$. Suppose that there exists

$$x_1, \ldots, x_k \in H_{\mathcal{A}}, \qquad n_1 \leq n_2 \leq \cdots \leq n_k$$

such that the hypothesis of the lemma holds. By the preceding lemma, F is not bounded below on $L_{n_k}^{\perp}$, and hence we can find an $\tilde{x}_{k+1} \in L_{n_k}^{\perp}$ such that $\|\tilde{x}_{k+1}\| = 1$ and

$$\|F\tilde{x}_{k+1}\| \leq 2^{-2(k+1)}.$$

Again, since

$$\lim_{n \to 0} \|P_{L_n^{\perp}} \tilde{x}_{k+1}\| = 0,$$

there exits an $n_{k+1} \geq n_k$ such that

$$\|P_{L_{n_{k+1}}^{\perp}} \tilde{x}_{k+1}\| \leq \frac{1}{\|F\|} 2^{-2(k+1)}.$$

Then

$$\|FP_{L_{n_{k+1}}} \tilde{x}_{k+1}\| \leq \|F\tilde{x}_{k+1}\| + \|FP_{L_{n_{k+1}}^{\perp}} \tilde{x}_{k+1}\| \leq 2^{-2(k+1)} + 2^{-2(k+1)} = 2^{1-2(k+1)}.$$

Set $x_{k+1} = P_{L_{n_{k+1}}} \tilde{x}_{k+1}$. We then have $x_{k+1} \in L_{n_{k+1}} \setminus L_{n_k}$ (because $\tilde{x}_{k+1} \in L_{n_k}^{\perp}$),

$$\|x_{k+1}\| \leq \|\tilde{x}_{k+1}\| = 1 \qquad \text{and} \qquad \|Fx_{k+1}\| \leq 2^{1-2(k+1)}.$$

By induction, the lemma follows. □

4. Openness of the set of semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$

In this section we prove that the sets $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ are open in the norm topology, as an analogue of the result in [13]. Also, we derive some consequences. Recall that $\mathcal{M}\Phi(H_{\mathcal{A}})$ is open in the norm topology by [10, Lemma 2.7.10].

Theorem 4.1. *The sets $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ are open in $B^a(H_{\mathcal{A}})$, where $B^a(H_{\mathcal{A}})$ is equipped with the norm topology.*

Proof. Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. Then there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed and finitely generated, and N_2 is closed but *not* finitely generated. If $D \in B^a(H_{\mathcal{A}})$ such that $\|D\| < \epsilon$, then for ϵ small enough we may (by the same arguments as in the proof of [10, Lemma 2.7.10]) find isomorphisms U_1, U_2 such that $F+D$ has the matrix

$$\begin{bmatrix} (F+D)_1 & 0 \\ 0 & (F+D)_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = U_1(M_1) \tilde{\oplus} U_1(N_1) \xrightarrow{F+D} U_2^{-1}(M_2) \tilde{\oplus} U_2^{-1}(N_2) = H_{\mathcal{A}},$$

where $(F+D)_1$ is an isomorphism. Since U_2 is an isomorphism and N_2 is not finitely generated, it follows that $U_2^{-1}(N_2)$ is not finitely generated. Now, as $F+D$ has the matrix

$$\begin{bmatrix} (F+D)_1 & 0 \\ 0 & (F+D)_4 \end{bmatrix}$$

with respect to the decomposition above, where $(F+D)_1$ is an isomorphism and $U_1(N_1)$ is finitely generated whereas $U_2^{-1}(N_2)$ is *not* finitely generated, it follows by Lemma 2.16 that

$$(F+D) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

(because, by that lemma, if $F+D$ were $\mathcal{A}$-Fredholm, then $U_2^{-1}(N_2)$ would be finitely generated, which is a contradiction). The first part of the theorem follows, whereas the second part can be proved analogously or can be deduced directly from the first part by passing to the adjoints and using Corollary 2.11. ☐

Corollary 4.2. *If $F \in B^a(H_{\mathcal{A}})$ belongs to the boundary of $\mathcal{M}\Phi(H_{\mathcal{A}})$ in $B^a(H_{\mathcal{A}})$, then $F \notin \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$.*

Proof. The proof follows by the same arguments as in the proof of [18, Corollary 1.6.10] since

$$\mathcal{M}\Phi_{\pm}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) = \big(\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\big) \cup \big(\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\big)$$

is open in $B^a(H_{\mathcal{A}})$. □

Corollary 4.3. *Let $f : [0,1] \to B^a(H_{\mathcal{A}})$ be continuous and assume that $f([0,1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Then the following statements hold.*

(1) *If $f(0) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.*
(2) *If $f(0) \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.*
(3) *If $f(0) \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and* $\operatorname{index} f(0) = \operatorname{index} f(1)$.

Proof. We have that $\mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$ is a disjoint union of $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, $\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, and $\mathcal{M}\Phi(H_{\mathcal{A}})$. The first two sets are open by preceding theorem whereas $\mathcal{M}\Phi(H_{\mathcal{A}})$ is open by [10, Lemma 2.7.10]. Moreover, by assumption in the corollary, we have that $f([0,1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Since f is continuous by assumption, $f([0,1])$ must be connected in $B^a(H_{\mathcal{A}})$, and hence $f([0,1])$ must be completely contained in one of these three sets

$$\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}), \quad \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}), \quad \mathcal{M}\Phi(H_{\mathcal{A}})$$

(otherwise we would get a separation of $f([0,1])$ which is impossible). Thus (1), (2) and the first part of (3) folows. For the second part of (3), use [10, Lemma 2.7.10] together with the proof of [18, Lemma 1.6.1]. □

5. $\mathcal{M}\Phi_{+}^{-}$ and $\mathcal{M}\Phi_{-}^{+}$ operators on $H_{\mathcal{A}}$

In this section we construct certain classes of operators on $H_{\mathcal{A}}$ as a generalizations of classes $\Phi_{+}^{-}(H)$, $\Phi_{-}^{+}(H)$ (where H is a Hilbert space). Then we investigate and prove several properties concerning these new classes of operators on $H_{\mathcal{A}}$.

Definition 5.1. Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$. We say that $F \in \tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism and where N_1, N_2 are closed, finitely generated, and $N_1 \preceq N_2$ (i.e., N_1 is isomorphic to a closed submodule of N_2). We define similarly the class $\tilde{\mathcal{M}}\Phi_{-}^{+}(H_{\mathcal{A}})$, where the only difference in this case is that $N_2 \preceq N_1$. Then we set

$$\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) = \big(\tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})\big) \cup \big(\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\big)$$

and

$$\mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}}) = \big(\tilde{\mathcal{M}}\Phi_{-}^{+}(H_{\mathcal{A}})\big) \cup \big(\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})\big).$$

Lemma 5.2. *Suppose that $K(\mathcal{A})$ satisfies "the cancellation property." If $F \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, then for any decomposition*

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix},$$

where F_1' is an isomorphism, N_1', N_2' are finitely generated, and we have $N_1' \preceq N_2'$. Similarly, $N_1' \preceq N_2'$ if $F \in \tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$.

Proof. Given $F \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, choose a decomposition for F

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

as described in the definition above. Then $N_1 \cong N_{2,1} \preceq N_2$ for some closed submodule $N_{2,1}$ of N_2. Since N_1 is finitely generated, so is $N_{2,1}$; therefore, $N_{2,1}$, is orthogonally complementable in N_2. So $N_2 = N_{2,1} \oplus N_{2,2}$ for some closed submodule $N_{2,2}$ of N_2. Hence,

$$\operatorname{index} F = [N_1] - [N_2] = [N_{2,1}] - [N_{2,1}] - [N_{2,2}] = -[N_{2,2}].$$

Thus

$$\operatorname{index} F = [N_1'] - [N_2'] = -[N_{2,2}].$$

Taking the inverses on both sides of the equality in $K(A)$, we get

$$[N_2'] - [N_1'] = [N_{2,2}],$$

so

$$[N_2'] = [N_1'] + [N_{2,2}].$$

Since

$$[N_1'] + [N_{2,2}] = [N_1' \oplus N_{2,2}] = [N_2'],$$

it follows that

$$(N_1' \oplus N_{2,2}) \cong N_2'$$

as $K(\mathcal{A})$ satisfies the cancellation property.

Let $\tilde{\iota} : N_1' \oplus N_{2,2}' \longrightarrow N_2'$ be the isomorphism; then since $N_1' \oplus \{0\}$ is a closed submodule of $N_1' \oplus N_{2,2}$, it follows that $\tilde{\iota}(N_1' \oplus \{0\})$ is a closed submodule of N_2'. Thus $(N_1' \oplus \{0\}) \preceq N_2'$. But $N_1' \oplus \{0\} \cong N_1'$, so $N_1' \preceq N_2'$. One treats analogously the case when $F \in \tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$. $\square$

Lemma 5.3. *It holds that $\tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$ and $\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$ are semigroups under multiplication.*

Proof. Let $F, D \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$. Then there exist decompositions

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A},$$
$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

with respect to which F, D have matrices $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_4 \end{smallmatrix}\right]$, respectively, where F_1, D_1 are isomorphisms, N_1, N_2, N_1', N_2' are finitely generated, and moreover $N_1 \preceq N_2$, $N_1' \preceq N_2'$. By the proof of [10, Lemma 2.7.11], with respect to the decomposition

$$H_\mathcal{A} = \overline{\overline{M_1}} \tilde{\oplus} \overline{\overline{N_1}} \xrightarrow{DF} \overline{\overline{M_2'}} \tilde{\oplus} \overline{\overline{N_2'}} = H_\mathcal{A}$$

DF has the matrix $\left[\begin{smallmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{smallmatrix}\right]$, where $(DF)_1$ is an isomorphism,

$$\overline{\overline{N_1}} = U\big(F_1^{-1}(P) \tilde{\oplus} N_1\big), \qquad \overline{\overline{N_2'}} = D(P') \tilde{\oplus} N_2', \qquad (P \tilde{\oplus} N_2) \cong (P \tilde{\oplus} N_1') \cong L_n$$

for some n, $D_{|_P}$, $F_{|_{P'}}$ and U are isomorphisms. Since N_1 is isomorphic to a closed submodule of N_2 and $F_1^{-1}(P) \cong P$, it follows that $F_1^{-1}(P) \oplus N_1$ is isomorphic to a closed submodule of $P \oplus N_2$ (here we consider the direct sums of modules in the sense of [10, Example 1.3.4]). But since there are natural isomorphisms between $((F_1^{-1}(P) \tilde{\oplus} N_1))$ and $((F_1^{-1}(P) \oplus N_1))$, between $(P \tilde{\oplus} N_2)$ and $(P \oplus N_2)$, it follows that $F_1^{-1}(P) \tilde{\oplus} N_1$ is isomorphic to a closed submodule of $(P \tilde{\oplus} N_2)$. As U is an isomorphism, it follows that $\overline{\overline{N_1}} = U(F_1^{-1}(P) \tilde{\oplus} N_1)$ is isomorphic to a closed submodule of $P \tilde{\oplus} N_2$. Now, $P \tilde{\oplus} N_2$ is isomorphic to $P' \tilde{\oplus} N_1'$, so $\overline{\overline{N_1}}$ is isomorphic to a closed submodule of $P' \tilde{\oplus} N_1'$. Next, using that $P' \cong D(P')$ and that N_1' is isomorphic to a closed submodule of N_2', by the same arguments as above (considering direct sums of modules), we can deduce that $(P' \tilde{\oplus} N_1')$ is isomorphic to a closed submodule of $(D(P') \tilde{\oplus} N_2') = \overline{\overline{N_2'}}$, so $\overline{\overline{N_1}} \preceq (P' \tilde{\oplus} N_1') \preceq \overline{\overline{N_2'}}$. Thus $DF \in \tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$. Similarly one can show that $\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$ is a semigroup. $\square$

Lemma 5.4. *It holds that $\mathcal{M}\Phi_+^-(H_\mathcal{A})$ and $\mathcal{M}\Phi_-^+(H_\mathcal{A})$ are semigroups under multiplication.*

Proof. Let $F, D \in \mathcal{M}\Phi_+^-(H_\mathcal{A})$. By definition, $\mathcal{M}\Phi_+^-(H_\mathcal{A}) \subset \mathcal{M}\Phi_+(H_\mathcal{A})$, so $F, D \in \mathcal{M}\Phi_+(H_\mathcal{A})$. By Corollary 2.5, $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$. Now if $F, D \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, by the preceding lemma it follows that $DF \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$. If $D, F \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$, then $DF \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$ as $\mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$ is a semigroup by Corollary 2.10. If $F \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$ and $D \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, then in particular $D \in \mathcal{M}\Phi(H_\mathcal{A})$ as $\tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A}) \subseteq \mathcal{M}\Phi(H_\mathcal{A})$ by definition. By Corollary 2.9, it follows that DF can *not* be in $\mathcal{M}\Phi(H_\mathcal{A})$ as $F \notin \mathcal{M}\Phi(H_\mathcal{A})$. Since $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$, we get that $DF \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$. If $D \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$, it is clear that DF can not be an element of $\mathcal{M}\Phi(H_\mathcal{A})$. Indeed, if $DF \in \mathcal{M}\Phi(H_\mathcal{A})$ then by Corollary 2.6 we would get that $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$. Hence, $D \in \mathcal{M}\Phi_-(H_\mathcal{A}) \cap \mathcal{M}\Phi_+(H_\mathcal{A}))$ which is a contradiction as $\mathcal{M}\Phi_-(H_\mathcal{A}) \cap \mathcal{M}\Phi_+(H_\mathcal{A}) = \mathcal{M}\Phi(H_\mathcal{A})$ by Corollary 2.4. Collecting all these arguments together, we deduce that $\mathcal{M}\Phi_+^-(H_\mathcal{A})$ is a semigroup. Similarly one can show that $\mathcal{M}\Phi_-^+(H_\mathcal{A})$ is a semigroup. $\square$

Lemma 5.5. *It holds that* $\tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$ *and* $\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$ *are open.*

Proof. Given $F \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, let

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

be a decomposition, with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1, N_2 are finitely generated, and $N_1 \preceq N_2$. By the proof of [10, Lemma 2.7.10], there exists an $\epsilon > 0$ such that if $\|F - D\| < \epsilon$, then there exists a decomposition

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

with respect to which

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

where D_1 is an isomorphism, and moreover

$$M_1 \cong M_1', \qquad N_1 \cong N_1', \qquad M_2 \cong M_2', \qquad N_2 \cong N_2'.$$

Let

$$U_1 : N_1' \to N_1, \qquad U_2 : N_2 \to N_2'$$

be these isomorphisms. Since $N_1 \preceq N_2$, there exists an isomorphism $\tilde{\iota}$ from N_1 onto some closed submodule $\tilde{\iota}(N_1) \subseteq N_2$. Then $U_2\tilde{\iota}U_1$ is an isomorphism from N_1' onto $(U_2\tilde{\iota}U_1)(N_1)$ which is a closed submodule of N_2'. Thus $N_1' \preceq N_2'$ (and also N_1', N_2' are finitely generated as N_1, N_2 are so). Therefore, $D \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$. Similarly we can show that $\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$ is open. □

Definition 5.6. Let $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. We say that $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ if there exists a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed and finitely generated, and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$, only in this case $F \in \mathcal{M}\Phi_-(H_\mathcal{A})$, N_2 is finitely generated and $N_2 \preceq N_1$.

Proposition 5.7. *We have the following:*

$$\tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A}) = \mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) \cap \mathcal{M}\Phi(H_\mathcal{A}),$$
$$\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A}) = \mathcal{M}\Phi_-^{+'}(H_\mathcal{A}) \cap \mathcal{M}\Phi(H_\mathcal{A}).$$

Proof. By definition of $\tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, the inclusion $\subseteq$ is obvious. Let us show the other inclusion. To this end, choose some $D \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) \cap \mathcal{M}\Phi(H_\mathcal{A})$. Since $D \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$, there exists a decomposition

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

with respect to which D has the matrix $\left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_4 \end{smallmatrix}\right]$, where D_1 is an isomorphism, N_1' is finitely generated, and $N_1' \preceq N_2'$. On the other hand, since $D \in \mathcal{M}\Phi(H_\mathcal{A})$, there exists a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{D} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

with respect to which $D = \left[\begin{smallmatrix} \tilde{D}_1 & 0 \\ 0 & \tilde{D}_4 \end{smallmatrix}\right]$, where $\tilde{D}_1$ is an isomorphism and N_1, N_2 are finitely generated. By Lemma 2.16, N_2' must be then finitely generated. Hence, $D \in \tilde{\mathcal{M}}\Phi_+^{-'}(H_\mathcal{A})$. Similarly, using Lemma 2.17, one can show that

$$\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A}) = \mathcal{M}\Phi_-^{+'}(H_\mathcal{A}) \cap \mathcal{M}\Phi(H_\mathcal{A}). \qquad \square$$

Remark 5.8. We have $\mathcal{M}\Phi_+^{-'} \subseteq \mathcal{M}\Phi_+^-$ and $\mathcal{M}\Phi_-^{+'} \subseteq \mathcal{M}\Phi_-^+$, and on Hilbert spaces "=" holds due to that given any finite-dimensional subspace N_1 and infinite-dimensional subspace N_2, then N_1 is isomorphic to a closed subspace of N_2.

Lemma 5.9. *The sets $\mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$ and $\mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ are open. Moreover, if $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ and $K \in K(H_\mathcal{A})$, then*

$$(F + K) \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A}).$$

If $F \in \mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$ and $K \in K(H_\mathcal{A})$, then

$$(F + K) \in \mathcal{M}\Phi_-^{+'}(H_\mathcal{A}).$$

Proof. Suppose that $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ and choose a decomposition.

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

such that $N_1 \preceq N_2$ as described in the Definition 5.6. Then, again by the proof of [10, Lemma 2.7.10], we have that there exists an $\epsilon > 0$ such that if $\|F - D\| < \epsilon$, then there exists a decomposition

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

with respect to which D has the matrix

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

where D_1 is an isomorphism, and $N_1' \cong N_1$, $N_2' \cong N_2$. Therefore, by the same arguments as in the proof of Lemma 5.5, we have $N_1' \preceq N_2'$ as $N_1 \preceq N_2$. Thus D is in $\mathcal{M}\Phi_+^-(H_\mathcal{A})'$ also, so $\mathcal{M}\Phi_+^-(H_\mathcal{A})'$ is open. Next, let $K \in K(H_\mathcal{A})$. By the proof of [10, Lemma 2.7.13] there exists an L_n such that $F + K$ has the matrix

$$\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = U_1'(L_n^{\perp}) \tilde{\oplus} U_1'(L_n) \xrightarrow{F+K} U_2'^{-1}(FL_n^{\perp}) \tilde{\oplus} U_2'^{-1}\big(F(P) \tilde{\oplus} N_2\big) = H_{\mathcal{A}},$$

where $(F+K)_1$ is an isomorphism, $L_n = N_1 \tilde{\oplus} P$, $P = M_1 \cap L_n$, and $P \cong F(P)$ for some closed, finitely generated submodule P (here F, N_1, N_2 are as given above). Now, since N_1 is isomorphic to a closed submodule of N_2, then clearly $P \tilde{\oplus} N_1$ is isomorphic to a closed submodule of $F(P) \tilde{\oplus} N_2$ as $P \cong F(P)$. Therefore,

$$(P \tilde{\oplus} N_1) \preceq \big(F(P) \tilde{\oplus} N_2\big).$$

Since U_1', U_2' are isomorphisms, then

$$U_1'(L_n) = U_1'(P \tilde{\oplus} N_1) \preceq U_2'^{-1}\big(F(P) \tilde{\oplus} N_2\big),$$

so $(F+K) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. Similarly one proves the statements for $\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$. □

Theorem 5.10. *Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent:*

(1) *$F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$;*
(2) *there exist $D \in B^a(H_{\mathcal{A}})$, $K \in K(H_{\mathcal{A}})$ such that D is bounded below and $F = D + K$*

Proof. The proof consists of two parts.

(1) ⇒ (2): Let $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ and let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition as given in the Definition 5.6, so that N_1 is finitely generated, $N_1 \preceq N_2$, and $F_{|_{M_1}} \to M_2$ is an isomorphism. Since N_1 is finitely generated, by the proof of [10, Theorem 2.7.6] we may assume that $M_1 = N_1^{\perp}$. Let ι be the isomorphism from N_1 onto a closed submodule $\iota(N_1) \subseteq N_2$. Set $D = F + (\iota - F)P_{N_1}$, where P_{N_1} is the orthogonal projection onto N_1. Then $(\iota - F)P_{N_1}$ is in $K(H_{\mathcal{A}})$ and in addition $D = F + (\iota - F)P_{N_1} = FP_{M_1} + \iota P_{N_1}$. Since $F_{|_{M_1}}$ is an isomorphism from M_1 onto M_2, ι is an isomorphism from N_1 onto $\iota(N_1) \subseteq N_2$, and $H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2$, it follows that D is bounded below being an isomorphism of $H_{\mathcal{A}}$ onto $M_2 \tilde{\oplus} \iota(N_1)$ which is a closed submodule of $H_{\mathcal{A}}$. Moreover $F = D + (F - \iota)P_{N_1}$ and $(F - \iota)P_{N_1}$ is compact. Note that ιP_{N_1} is indeed adjointable: Since $\iota : N_1 \to \iota(N_1) \subseteq N_2$ and N_1 is self-dual being finitely generated, then by [10, Proposition 2.5.2], the result which was originally proved in [11], ι is adjointable. Moreover, since $\iota(N_1)$ is finitely generated being isomorphic to N_1, it follows that $\iota(N_1)$ is an orthogonal direct summand in $H_{\mathcal{A}}$ by [10, Lemma 2.3.7]. Hence, the inclusion $J_{\iota(N_1)} : \iota(N_1) \to H_{\mathcal{A}}$ is adjointable. Also P_{N_1} is adjointable, so $\iota P_{N_1} = J_{\iota(N_1)}\iota P_{N_1} \in B^a(H_{\mathcal{A}})$.

(2) ⇒ (1): If $D \in B^a(H_{\mathcal{A}})$ is bounded below, then obviously $D \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. As $K \in K(\mathcal{A})$, by the preceding lemma we get that $(D + K) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. □

Proposition 5.11. *We have the following.*

(1) $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}) \Leftrightarrow F^* \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$.
(2) $F \in \widetilde{\mathcal{M}\Phi}_+^{-}(H_{\mathcal{A}}) \Leftrightarrow F^* \in \widetilde{\mathcal{M}\Phi}_-^{+}(H_{\mathcal{A}})$.
(3) $F \in \mathcal{M}\Phi_+^{-}(H_{\mathcal{A}}) \Leftrightarrow F^* \in \mathcal{M}\Phi_-^{+}(H_{\mathcal{A}})$.

Proof. The proofs consists of three parts.

(1) Let $F \in \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$; choose a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

modulo F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, $N_1 \preceq N_2$, and N_1 is finitely generated. Again, by the proof of [10, Theorem 2.7.6], we may assume that $M_1 = N_1^{\perp}$ modulo the decomposition

$$H_{\mathcal{A}} = {N_1}^{\perp} \tilde{\oplus} N_1 \xrightarrow{F} F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} = H_{\mathcal{A}},$$

F has the matrix

$$\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{F}_4 \end{bmatrix},$$

where $\tilde{F}_1$ is an isomorphism, and $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_4$ are adjointable, so

$$F^* = \begin{bmatrix} \tilde{F}_1^* & 0 \\ \tilde{F}_2^* & \tilde{F}_4^* \end{bmatrix}$$

modulo the decomposition

$$H_{\mathcal{A}} = F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} \xrightarrow{F^*} {N_1}^{\perp} \tilde{\oplus} N_1 = H_{\mathcal{A}}.$$

Moreover, since $\tilde{F}_1^*$ is an isomorphism, F^* has the matrix

$$\begin{bmatrix} \tilde{\tilde{F}}_1^* & 0 \\ 0 & \tilde{F}_4^* \end{bmatrix}$$

modulo the decomposition

$$H_{\mathcal{A}} = {N_2}^{\perp} \tilde{\oplus} N_2 \xrightarrow{F^*} V^{-1}({N_1}^{\perp}) \tilde{\oplus} V^{-1}(N_1) = H_{\mathcal{A}},$$

where

$$V = \begin{bmatrix} 1 & 0 \\ -\tilde{F}_2^*(\tilde{F}_1^*)^{-1} & 1 \end{bmatrix}$$

modulo the decomposition

$$H_{\mathcal{A}} = {N_1}^{\perp} \oplus N_1 \xrightarrow{V} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}},$$

so that V is an isomorphism and also $\tilde{\tilde{F}}_1^*$ is an isomorphism. Now, since V is an isomorphism and there exists an isomorphism $\iota : N_1 \to \iota(N_1) \subseteq N_2$ (as $N_1 \preceq N_2$), we get that $\iota V : V^{-1}(N_1) \to \iota(N_1) \subseteq N_2$ is an isomorphism, so $V^{-1}(N_1) \preceq N_2$. Moreover, $V^{-1}(N_1)$ is finitely generated since N_1 is also finitely generated. Therefore, $F^* \in \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$. Conversely, if $F \in \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$, let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be an $\mathcal{M}\Phi_{-}^{+'}$ decomposition for F; then $N_2 \preceq N_1$, and N_2 is finitely generated. By the proof of Theorem 2.3 part (1) $\Rightarrow$ (2), F has the matrix

$$\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_3 & \tilde{F}_4 \end{bmatrix}$$

modulo the decomposition

$$H_{\mathcal{A}} = N_1{}^{\perp} \tilde{\oplus} N_1 \xrightarrow{F} N_2{}^{\perp} \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where $\tilde{F}_1$, $\tilde{F}_3$, $\tilde{F}_4$ are adjointable and $\tilde{F}_1$ is an isomorphism. Then F^* has the matrix

$$\begin{bmatrix} \tilde{F}_1^* & \tilde{F}_2^* \\ 0 & \tilde{F}_4^* \end{bmatrix}$$

modulo the decomposition

$$H_{\mathcal{A}} = N_2{}^{\perp} \tilde{\oplus} N_2 \xrightarrow{F^*} N_1{}^{\perp} \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

and $\tilde{F}_1^*$ is an isomorphism. Hence,

$$F^* = \begin{bmatrix} \tilde{F}_1^* & 0 \\ 0 & \tilde{\tilde{F}}_4^* \end{bmatrix}$$

modulo the decomposition

$$H_{\mathcal{A}} = U(N_2{}^{\perp}) \tilde{\oplus} U(N_2) \xrightarrow{F^*} N_1{}^{\perp} \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

where

$$U = \begin{bmatrix} 1 & -\tilde{F}_1^{*^{-1}}(\tilde{F}_3^*) \\ 0 & 1 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = N_2{}^{\perp} \oplus N_2 \xrightarrow{V} N_2{}^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

so that U is an isomorphism. Since $\iota : N_2 \Rightarrow \iota(N_2) \subseteq N_1$ is an isomorphism, then $\iota U^{-1} : U(N_2) \to \iota(N_2) \subseteq N_1$ is also an isomorphism, so $U(N_2) \preceq N_1$. Thus $F^* \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$.

(2) Use (1) together with the fact that

$$F \in \mathcal{M}\Phi(H_{\mathcal{A}}) \quad \Leftrightarrow \quad F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Corollary 2.11 and the fact that

$$\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}}) = \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}),$$
$$\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}}) = \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Proposition 5.7.

(3) Use (2) together with the fact that

$$F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) \quad \Leftrightarrow \quad F^* \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Corollary 2.11 and the fact that

$$\mathcal{M}\Phi_+^-(H_\mathcal{A}) = \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A}) \cup \big(\mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})\big),$$
$$\mathcal{M}\Phi_-^+(H_\mathcal{A}) = \tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A}) \cup \big(\mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})\big)$$

by Definition 5.1. □

Definition 5.12. We set $M^a(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is bounded below}\}$.

Lemma 5.13. *Let $Q^a(H_\mathcal{A}) = \{D \in B^a(H_\mathcal{A}) \mid D$ is surjective $\}$. Then $F \in M^a(H_\mathcal{A})$ if and only if $F^* \in Q^a(H_\mathcal{A})$.*

Proof. Let $F \in M^a(H_\mathcal{A})$. By the proof of [10, Theorem 2.3.3], as $\operatorname{ran} F$ is closed in this case, we have that $\operatorname{ran} F^*$ is also closed. Moreover, by the proof of [10, Theorem 2.3.3] since $\operatorname{ran} F^*$ is closed, we also have $H_\mathcal{A} = \ker F \oplus \operatorname{ran} F^*$. Since $\ker F = \{0\}$, it follows that $H_\mathcal{A} = \operatorname{ran} F^*$.

Conversely, if $F^* \in Q^a(H_\mathcal{A})$, then $\ker F = \operatorname{ran} F^{*\perp} = \{0\}$, so F is injective. Moreover, since $\operatorname{ran} F^* = H_\mathcal{A}$ is closed, then $\operatorname{ran} F$ is closed also (again by the proof of [10, Theorem 2.3.3]). By the Banach open mapping theorem, it follows that F is an isomorphism from $H_\mathcal{A}$ onto its image. Thus F is bounded below. □

Corollary 5.14. *Let $D \in B^a(H_\mathcal{A})$. The following statements are equivalent:*

(1) $D \in \mathcal{M}\Phi_-^{+\prime}(H_\mathcal{A})$,
(2) *there exist $Q \in Q^a(H_\mathcal{A})$, $K \in K(H_\mathcal{A})$ such that $D = Q + K$.*

Proof. The proof follows from Theorem 5.10, Proposition 5.11 part (1), and Lemma 5.13 by passing to the adjoints. □

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