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WEAK FRAMES IN HILBERT C^* -MODULES WITH APPLICATION IN GABOR ANALYSIS

DAMIR BAKIĆ

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ABSTRACT. In the first part, we describe the dual $\ell^2(\mathbf{A})'$ of the standard Hilbert C^* -module $\ell^2(\mathbf{A})$ over an arbitrary (not necessarily unital) C^* -algebra \mathbf{A} . When \mathbf{A} is a von Neumann algebra, this enables us to construct explicitly a self-dual Hilbert \mathbf{A} -module $\ell^2_{\text{strong}}(\mathbf{A})$ that is isometrically isomorphic to $\ell^2(\mathbf{A})'$, which contains $\ell^2(\mathbf{A})$, and whose \mathbf{A} -valued inner product extends the original inner product on $\ell^2(\mathbf{A})$. This serves as a concrete realization of a general construction for Hilbert C^* -modules over von Neumann algebras introduced by Paschke. Then we introduce a concept of a weak Bessel sequence and a weak frame in Hilbert C^* -modules over von Neumann algebras. The dual $\ell^2(\mathbf{A})'$ is recognized as a suitable target space for the analysis operator. We describe fundamental properties of weak frames such as the correspondence with surjective adjointable operators, the canonical dual, the reconstruction formula, and so on, first for self-dual modules and then, working in the dual, for general modules. The last part describes a class of Hilbert C^* -modules over $L^\infty(I)$, where I is a bounded interval on the real line, that appears naturally in connection with Gabor (i.e., Weyl–Heisenberg) systems. We then demonstrate that Gabor Bessel systems and Gabor frames in $L^2(\mathbb{R})$ are in a bijective correspondence with weak Bessel systems and weak frames of translates by a in these modules over $L^\infty[0, \frac{1}{b}]$, where $a, b > 0$ are the lattice parameters. In this setting new proofs of several classical results on Gabor frames are demonstrated and some new ones are obtained.

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1. Introduction

Frame theory for Hilbert C^* -modules is now about two decades old. Introduced by Frank and Larson in the late 1990s, frame theory has since served both as a useful tool and as a subject of research interest in its own right. It turns out that frames in Hilbert C^* -modules share many properties with classical frames for Hilbert spaces. However, this is limited only to the class of—in Frank–Larson terminology—standard frames. Those are the frames for which the corresponding analysis operator takes values in the standard Hilbert module $\ell^2(\mathbf{A})$. Frames in some weaker sense have not been studied up to now. There are some generalizations such as modular g -frames (a number of recent articles have featured g -frames for Hilbert C^* -modules) and outer frames (see, e.g., [2]). The latter class is well suited for Hilbert C^* -modules over nonunital C^* -algebras, but in cases where the underlying C^* -algebra possesses a unit, the outer frames simply become the usual standard frames.

So, the question of an appropriate concept of a weak frame (in any sense weaker than with respect to the norm) is still open. On the other hand, researchers have encountered situations in which such kind of modular frames could play a role (see [8, p. 97], [10, pp. 11–12]).

The reason why no theory of weak modular frames has been developed to date, even for some special classes of C^* -algebras, lies in the fact that in order to study such frames one has to introduce a suitable target space (a Hilbert C^* -module) for the analysis operator. And since no such target module was available, the whole concept remained unfounded.

Here, in the first part of the paper, we describe the dual $\ell^2(\mathbf{A})'$ of the standard Hilbert C^* -module $\ell^2(\mathbf{A})$ over an arbitrary C^* -algebra \mathbf{A} . It is to some extent surprising that there is no complete description of this dual available from the existing literature, bearing in mind the importance of the standard module $\ell^2(\mathbf{A})$ (which goes back to Kasparov [18]). It turns out that this description is particularly nice when the underlying algebra is a von Neumann algebra. When this is the case, the dual $\ell^2(\mathbf{A})'$ is realized in a concrete way precisely as we know (from the theoretical viewpoint; see [25]) it should be: as a self-dual Hilbert \mathbf{A} -module $\ell^2_{\text{strong}}(\mathbf{A})$ that is isometrically isomorphic to $\ell^2(\mathbf{A})'$, which contains $\ell^2(\mathbf{A})$, and whose \mathbf{A} -valued inner product extends the original inner product on $\ell^2(\mathbf{A})$; see Theorem 2.3 and Corollary 2.5 below.

Once the dual module was identified and described, it was natural to try to introduce a concept of a weak frame for Hilbert C^* -modules over von Neumann algebras (see Definition 3.2 below). In fact, the convergence of the series in the frame definition is dictated by the inner product in $\ell^2(\mathbf{A})'$; it was of substantial importance to ensure that the analysis operators for such weak frames take values in the dual module $\ell^2(\mathbf{A})'$. Another necessary property of the analysis (as well as the synthesis) operator is adjointability. This is the reason why we restrict ourselves in the first step to the class of self-dual modules over von Neumann algebras. In the second step we extend the theory to general Hilbert modules over von Neumann algebras by working in the dual. It turns out that weak Bessel

sequences and weak modular frames have properties, with respect to some topology weaker than the norm topology, similar to those of standard Bessel sequences and frames.

In the last part of the paper we apply this new concept. It turns out that our weak frames are well suited for application in Gabor analysis. This was already indicated in [8] and [10]. Weak frames (resp., weak Bessel sequences) of translates in the Lebesgue–Bochner module $L_{\frac{1}{b}}^\infty(\ell^2)$ of the form $(T_{na} \frac{1}{\sqrt{b}}g)_{n \in \mathbb{Z}}$ are in a bijective correspondence with Gabor frames (resp., Bessel sequences) $G(g, a, b)$ in $L^2(\mathbb{R})$, where a and b are the translation and modulation parameters. Thus, weak frames are nonstandard modular frames for a Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$ in the sense that perfectly fits into Gabor analysis. We remark that in [10, pp. 11–12], Coco and Lammers had already observed the correspondence of Gabor frames and certain nonstandard modular frames in $L_{\frac{1}{b}}^\infty(\ell^2)$, while noting that a concept of nonstandard modular frames had not been developed at the time, so the idea of using Hilbert C^* -modules in that context could not be exploited. Here, after developing the theory of weak modular frames, we explore this connection by demonstrating in this new (modular) light simple proofs of some well-known results concerning Gabor Bessel sequences and Gabor frames (see Theorem 6.5 and Remarks 6.8 and 6.12 below). In addition, a modular form of the Walnut representation is proved (Proposition 6.15) and, as a consequence, a result of Walnut type for classical Gabor Bessel sequences is obtained (Theorem 6.16).

This shows that the Hilbert C^* -module technique might serve as a promising tool in future investigations of Gabor systems. It should be noted that Hilbert modules are already recognized as an important tool in harmonic analysis: in [17] (see also [21], [22]) Rieffel’s Heisenberg modules are naturally related with Gabor (multi-window) frames. We refer the reader to [17] where it is demonstrated that Heisenberg modules provide the natural setting for the duality theory of Gabor systems.

The paper is organized as follows. In the rest of this [Introduction](#), we fix our terminology and notation regarding the modular part of the paper. All necessary preliminaries and notation concerned with Gabor systems will be introduced at the beginning of Section 5. Section 2 is completely devoted to the identification of the dual $\ell^2(\mathbf{A})'$ of the standard Hilbert C^* -module $\ell^2(\mathbf{A})$. We have included some of the arguments already known in the literature in an attempt to make the exposition as self-contained as possible. The main results are Theorem 2.3 and its Corollary 2.5.

In Section 3 weak Bessel sequences and weak frames are introduced and their fundamental properties are derived. As mentioned above, it was necessary to restrict our discussion to the class of Hilbert C^* -modules over von Neumann algebras in order to ensure that the target space for the analysis operators is again a Hilbert C^* -module over the same algebra. In this section we develop the theory of weak frames (resp., weak Bessel sequences) by obtaining all fundamental results such as unconditional convergence of the series representing synthesis operators (with respect to a certain weak topology), invertibility of the frame operator, the reconstruction formula, and so on.

In Section 4 we discuss the correspondence of adjointable operators on the standard Hilbert C^* -module $\ell^2(\mathbf{A})$ over a von Neumann algebra \mathbf{A} with infinite matrices with coefficients from \mathbf{A} . Among other results, a generalization of the Schur test is proved. It should be pointed out that the results in this section are obtained under the additional assumption that the underlying von Neumann algebra is commutative. Section 5 introduces Hilbert C^* -modules and spaces relevant for Gabor analysis and discusses some of their properties. The central space in our study is $L^\infty[0, \frac{1}{b}]$ -module $L^\infty_{\frac{1}{b}}(\ell^2)$, which is a Hilbert C^* -module unitarily equivalent to the dual $\ell^2(L^\infty[0, \frac{1}{b}])'$ of $\ell^2(L^\infty[0, \frac{1}{b}])$. Here b is an arbitrary positive number that will in fact play the role of the modulation parameter for a given Gabor system. The main result of the section is Theorem 5.9 in which we establish a bijective correspondence of Gabor frames/Bessel sequences and weak frames/weak Bessel sequences of translates in $L^\infty_{\frac{1}{b}}(\ell^2)$.

Finally, in Section 6 we discuss some consequences. Various results on Gabor systems are (re)obtained. In particular, a discussion on Walnut representation is included.

Readers who are primarily interested in the Gabor part of the paper may want to start with Section 5 and refer back to previous sections (i.e., Sections 3 and 4) as needed for theoretical background concerning weak modular frames.

Throughout the present paper we work with left Hilbert C^* -modules. Recall that a Hilbert C^* -module over a C^* -algebra \mathbf{A} is a complex vector space X that is also a left \mathbf{A} -module equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbf{A}$ that is linear in the first variable, antilinear in the second variable, and satisfies $\langle x, x \rangle \geq 0$ for all $x \in X$, $\langle x, x \rangle = 0$ if and only if $x = 0$, $\langle ax, y \rangle = a\langle x, y \rangle$ and $\langle y, x \rangle = \langle x, y \rangle^*$ for all $x, y \in X$, $a \in \mathbf{A}$, and such that X is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

We opted (between two equivalent choices) to work with left Hilbert C^* -modules despite some technical difficulties that arise in establishing correspondence of operators with infinite matrices. This was motivated by possible applications in Gabor analysis where passing back and forth between Gabor systems in $L^2(\mathbb{R})$ and modular Bessel sequences is needed, and hence it is more convenient to work with inner products that are linear in the same (i.e., the first) argument in both structures. If X and Y are Hilbert C^* -modules over the same C^* -algebra, then we denote by $\mathbb{B}(X, Y)$ the Banach space of all adjointable operators from X into Y .

The key role in our considerations in Section 2 will be played by the standard Hilbert C^* -module $\ell^2(\mathbf{A})$ that is defined as

$$\ell^2(\mathbf{A}) = \left\{ (a_n)_n : a_n \in \mathbf{A}, \sum_{n=1}^{\infty} a_n a_n^* \text{ converges in norm in } \mathbf{A} \right\}$$

and equipped with the inner product

$$\langle (a_n)_n, (b_n)_n \rangle = \sum_{n=1}^{\infty} a_n b_n^*.$$

It turns out that this series converges in \mathbf{A} . In fact, it converges unconditionally since by polarization it can be written as a linear combination of four series of the form $\sum_{n=1}^{\infty} c_n c_n^*$, with $(c_n)_n \in \ell^2(\mathbf{A})$ and it is well known that when a series of positive elements converges in a C^* -algebra, it necessarily converges unconditionally.

This implies that there is no loss of generality in working with countable systems indexed by natural numbers (i.e., sequences), although in the second part of the paper we will naturally use indexation over the integers when working with Gabor systems.

We will often assume (in particular, in Section 2) that our C^* -algebra \mathbf{A} acts nondegenerately on a Hilbert space H . This is not a restriction since any C^* -algebra can be faithfully and nondegenerately represented on a Hilbert space. We denote by $LM(\mathbf{A})$, $RM(\mathbf{A})$, and $M(\mathbf{A})$ the sets of left multipliers, right multipliers, and (two-sided) multipliers of \mathbf{A} , respectively. Recall that $LM(\mathbf{A})$ and $RM(\mathbf{A})$ are Banach algebras, while $M(\mathbf{A})$ is a C^* -algebra. Note also that all these algebras are contained in the closure $\overline{\mathbf{A}}^s$ of \mathbf{A} in the strong operator topology which in this situation coincides with the bicommutant \mathbf{A}'' of \mathbf{A} . (We refer the reader to [26] and [30] for general facts on C^* -algebras and Hilbert C^* -modules. Further notation will be explained in the course of exposition.)

2. The dual $\ell^2(\mathbf{A})'$

Given a Hilbert C^* -module X over a C^* -algebra \mathbf{A} , we denote by X^* its adjointable dual, that is, $X^* = \mathbb{B}(X, \mathbf{A})$, and we denote by X' its dual, that is, the Banach space of all bounded module maps from X into \mathbf{A} . By a “module map” we mean a linear operator $L : X \rightarrow \mathbf{A}$ which satisfies $L(ax) = aL(x)$ for all x in X and a in \mathbf{A} . The dual X' becomes a right Banach module over \mathbf{A} with the action of \mathbf{A} on X' defined by $(L, b) \mapsto Lb$, $(Lb)(x) = L(x)b$, x in X .

Clearly, X^* is a closed subspace (in fact, a submodule) of X' . We also know that X is embedded in $X^* \subseteq X'$ in a standard way: for x in X we define the map $L_x : X \rightarrow \mathbf{A}$ by $L_x(y) = \langle y, x \rangle$. It is evident that L_x belongs to $X^* = \mathbb{B}(X, \mathbf{A})$. Indeed, L_x^* is given by $L_x^*(a) = ax$, $a \in \mathbf{A}$. So, we have the map

$$\varphi : X \rightarrow X^* \subseteq X', \quad \varphi(x) = L_x, \quad L_x(y) = \langle y, x \rangle. \quad (1)$$

It is easy to show that φ is an antilinear isometry. What is more, it turns out with the help of the Cohen–Hewitt factorization (each x in X is of the form $x = b^*v$ for some v in X and b^* in \mathbf{A}) that every L_x is in fact a “rank-one” operator $\theta_{v,b}$, where $\theta_{v,b}(y) = \langle y, v \rangle b$, $y \in X$. It is also known that the image of φ coincides with the space of all “compact” operators $\mathbb{K}(X, \mathbf{A})$ (see Lemma 2.32 in [27]). A Hilbert C^* -module X is considered self-dual if each bounded module map (i.e., an element of X') arises by taking the inner product with some fixed element of X . Thus, X is self-dual if for each $L \in X'$ there exists $x \in X$ such that $L = L_x$.

It is always desirable to determine precisely both the dual and the adjointable dual of a Hilbert C^* -module under consideration. Our goal in this section is to identify $\ell^2(\mathbf{A})^*$ and $\ell^2(\mathbf{A})'$, where $\ell^2(\mathbf{A})$ is the standard Hilbert C^* -module over \mathbf{A}

(sometimes called the Hilbert space over \mathbf{A}). We do not impose any restrictions on the underlying C^* -algebra \mathbf{A} ; in particular, we do not assume that \mathbf{A} is unital.

Let us begin by recalling the definition of the multiplier module $M(\ell^2(\mathbf{A}))$ of $\ell^2(\mathbf{A})$ (cf. [3]):

$$M(\ell^2(\mathbf{A})) = \left\{ (c_n)_n : c_n \in M(\mathbf{A}), \sum_{n=1}^{\infty} c_n c_n^* \text{ converges strictly} \right\}. \quad (2)$$

It is known (see [3, Theorem 2.1]) that $M(\ell^2(\mathbf{A}))$ is a Hilbert C^* -module over $M(\mathbf{A})$ with the $M(\mathbf{A})$ -valued inner product on $M(\ell^2(\mathbf{A}))$ given by

$$\langle (c_n)_n, (d_n)_n \rangle = (\text{strict}) \sum_{n=1}^{\infty} c_n d_n^*.$$

Here the strict topology on $M(\mathbf{A})$ is the locally convex topology generated by the seminorms $x \mapsto \|ax\|$ and $x \mapsto \|xa\|$, $x \in M(\mathbf{A})$, for all $a \in \mathbf{A}$. Since we assumed that \mathbf{A} acts nondegenerately on H , each strictly convergent net converges also in the strong operator topology to the same limit. Note that $\ell^2(\mathbf{A})$ is contained in $M(\ell^2(\mathbf{A}))$ and that the norm on $\ell^2(\mathbf{A})$ inherited from $M(\ell^2(\mathbf{A}))$ coincides with the original Hilbert C^* -norm defined on $\ell^2(\mathbf{A})$. Clearly, when \mathbf{A} is unital, we have $M(\mathbf{A}) = \mathbf{A}$ and consequently $M(\ell^2(\mathbf{A})) = \ell^2(\mathbf{A})$.

We will work in another, even larger Hilbert C^* -module over $\overline{\mathbf{A}}^s = \mathbf{A}''$. Let

$$\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s) = \left\{ (c_n)_n : c_n \in \overline{\mathbf{A}}^s, \sup_N \left\| \sum_{n=1}^N c_n c_n^* \right\| < \infty \right\}. \quad (3)$$

Observe that the condition $\sup_N \left\| \sum_{n=1}^N c_n c_n^* \right\| < \infty$ implies that the series $\sum_{n=1}^{\infty} c_n c_n^*$ converges strongly. On the other hand, applying the uniform boundedness principle we conclude that the converse is also true. Thus, (3) can be rewritten as

$$\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s) = \left\{ (c_n)_n : c_n \in \overline{\mathbf{A}}^s, \sum_{n=1}^{\infty} c_n c_n^* \text{ converges strongly} \right\}. \quad (4)$$

Since $M(\mathbf{A}) \subseteq \overline{\mathbf{A}}^s$ and the strong operator topology is weaker than the strict topology on $M(\mathbf{A})$, by comparing (2) and (4), we see that $M(\ell^2(\mathbf{A}))$ is contained in $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$. There is another space of our interest in between:

$$\ell_{\text{strong}}^2(LM(\mathbf{A})) = \left\{ (c_n)_n : c_n \in LM(\mathbf{A}), \sup_N \left\| \sum_{n=1}^N c_n c_n^* \right\| < \infty \right\}. \quad (5)$$

Thus, we have the following chain of inclusions:

$$\ell^2(\mathbf{A}) \subseteq M(\ell^2(\mathbf{A})) \subseteq \ell_{\text{strong}}^2(LM(\mathbf{A})) \subseteq \ell_{\text{strong}}^2(\overline{\mathbf{A}}^s). \quad (6)$$

Proposition 2.1. *If $\mathbf{A} \subseteq \mathbb{B}(H)$ is a C^* -algebra that acts nondegenerately on H , then $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$ is a Hilbert C^* -module over $\overline{\mathbf{A}}^s$ with the inner product defined by*

$$\langle (x_n)_n, (y_n)_n \rangle = (\text{strong}) \sum_{n=1}^{\infty} x_n y_n^*,$$

where (strong) refers to the strong operator topology on $B(H)$. The norm on $M(\ell^2(\mathbf{A}))$ inherited from $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$ coincides with the original Hilbert C^* -norm on $M(\ell^2(\mathbf{A}))$. In addition, $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$ contains $\ell^2_{\text{strong}}(LM(\mathbf{A}))$ and $M(\ell^2(\mathbf{A}))$ as closed subspaces.

Proof. Let $(x_n)_n$ and $(y_n)_n$ be any two sequences in $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$.

We first claim that the series $\sum_{n=1}^{\infty} x_n y_n^*$ converges strongly. Let $C = \sup\{\|\sum_{n=1}^N x_n x_n^*\| : N \in \mathbb{N}\}$. Since the sequence $(\|\sum_{n=1}^N y_n y_n^*\|)_N$ is bounded, the series $\sum_{n=1}^{\infty} y_n y_n^*$ converges strongly. Take any ξ in H , and fix $\epsilon > 0$. Then there exists N_0 such that

$$N_2 > N_1 \geq N_0 \Rightarrow \left\| \left(\sum_{n=N_1+1}^{N_2} y_n y_n^* \right) \xi \right\| < \epsilon.$$

Recall the strong version of the Cauchy–Schwarz inequality that holds true in any Hilbert C^* -module X : $\langle x, y \rangle^* \langle x, y \rangle \leq \| \langle x, x \rangle \| \langle y, y \rangle$ (see [19, Proposition 1.1]). Applying this inequality in the Hilbert C^* -module $(\overline{\mathbf{A}}^s)^{N_2-N_1}$, we get

$$\left(\sum_{n=N_1+1}^{N_2} x_n y_n^* \right)^* \left(\sum_{n=N_1+1}^{N_2} x_n y_n^* \right) \leq \left\| \sum_{n=N_1+1}^{N_2} x_n x_n^* \right\| \left(\sum_{n=N_1+1}^{N_2} y_n y_n^* \right).$$

We now apply the operators from both sides of this inequality to ξ and take the inner product in H by ξ . In this way, we obtain

$$\left\| \left(\sum_{n=N_1+1}^{N_2} x_n y_n^* \right) \xi \right\|^2 \leq C \left\langle \left(\sum_{n=N_1+1}^{N_2} y_n y_n^* \right) \xi, \xi \right\rangle \leq C \epsilon \|\xi\|^2.$$

This proves our claim: the series $\sum_{n=1}^{\infty} x_n y_n^*$ converges strongly. Observe that this implies two things. First, we conclude that the series $\sum_{n=1}^{\infty} (x_n + y_n)^* (x_n + y_n)$ converges strongly and hence by the uniform boundedness principle the sequence $(\sum_{n=1}^N (x_n + y_n)(x_n + y_n)^*)_N$ is bounded; thus, $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$ is closed under addition. Second, we now have a well-defined $\overline{\mathbf{A}}^s$ -valued inner product on $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$ given by

$$\langle (x_n)_n, (y_n)_n \rangle = (\text{strong}) \sum_{n=1}^{\infty} x_n y_n^*.$$

Notice that the strong convergence of the series $\sum_{n=1}^{\infty} x_n y_n^*$ does not imply a priori the strong convergence of the series $\sum_{n=1}^{\infty} y_n x_n^*$. However, by the preceding discussion both series do converge strongly and hence weakly, which implies that $\langle (x_n)_n, (y_n)_n \rangle = \langle (y_n)_n, (x_n)_n \rangle^*$ for all sequences $(x_n)_n, (y_n)_n$ from $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$. So, $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$ has the structure of an inner product $\overline{\mathbf{A}}^s$ -module.

Let us now show that this module is complete. Take a Cauchy sequence $(c_n)_n$ in $\ell^2_{\text{strong}}(\overline{\mathbf{A}}^s)$, and put $c_n = (x_k^n)_k, n \in \mathbb{N}$. Fix $\epsilon > 0$. Then there exists n_0 with the property

$$m, n \geq n_0 \Rightarrow \|c_n - c_m\| < \epsilon, \tag{7}$$

which means that

$$m, n \geq n_0 \Rightarrow \left\| \left(\text{strong} \right) \sum_{k=1}^{\infty} (x_k^n - x_k^m)(x_k^n - x_k^m)^* \right\| < \epsilon^2. \tag{8}$$

Since the sequence $(\sum_{k=1}^N (x_k^n - x_k^m)(x_k^n - x_k^m)^*)_N$ is an increasing strongly convergent sequence of positive operators, we conclude that

$$\left\| (x_k^n - x_k^m)(x_k^n - x_k^m)^* \right\| < \epsilon^2, \quad \forall k \in \mathbb{N} \tag{9}$$

and hence

$$\|x_k^n - x_k^m\| < \epsilon, \quad \forall k \in \mathbb{N}.$$

This shows that $(x_k^n)_n$ is a *norm*-Cauchy sequence in \overline{A}^s . Put

$$x_k^0 = \lim_{n \rightarrow \infty} x_k^n \in \overline{A}^s, \quad k \in \mathbb{N} \tag{10}$$

and

$$c_0 = (x_1^0, x_2^0, x_3^0, \dots).$$

Consider again n_0 for which we have (7) and (8), and fix $n \geq n_0$. Then for each K we have

$$\begin{aligned} \left\| \sum_{k=1}^K (x_k^n - x_k^0)(x_k^n - x_k^0)^* \right\| &= \lim_{m \rightarrow \infty} \left\| \sum_{k=1}^K (x_k^n - x_k^m)(x_k^n - x_k^m)^* \right\| \\ &\leq \limsup_{m \rightarrow \infty} \left\| \left(\text{strong} \right) \sum_{k=1}^{\infty} (x_k^n - x_k^m)(x_k^n - x_k^m)^* \right\| \\ &= \limsup_{m \rightarrow \infty} \|c_n - c_m\|^2 \stackrel{(7)}{\leq} \epsilon^2. \end{aligned} \tag{11}$$

This tells us that the sequence $(\sum_{k=1}^K (x_k^n - x_k^0)(x_k^n - x_k^0)^*)_K$ is bounded, and hence $c_n - c_0$ belongs to $\ell_{\text{strong}}^2(\overline{A}^s)$. In particular, c_0 is in $\ell_{\text{strong}}^2(\overline{A}^s)$. Moreover,

$$\begin{aligned} \|c_n - c_0\|^2 &= \left\| \left(\text{strong} \right) \sum_{k=1}^{\infty} (x_k^n - x_k^0)(x_k^n - x_k^0)^* \right\| \\ &\leq \sup_K \left\| \sum_{k=1}^K (x_k^n - x_k^0)(x_k^n - x_k^0)^* \right\| \stackrel{(11)}{\leq} \epsilon^2. \end{aligned}$$

Thus, $\ell_{\text{strong}}^2(\overline{A}^s)$ is complete, so it is a Hilbert C^* -module over \overline{A}^s .

To prove the second assertion, take any $(t_n)_n \in M(\ell^2(A))$. Its original norm arising from the Hilbert $M(A)$ -module structure on $M(\ell^2(A))$ is equal to $\|(\text{strict}) \sum_{n=1}^{\infty} t_n t_n^*\|^{1/2}$. Since A acts nondegenerately on H , strict convergence in $M(\ell^2(A))$ induced by A implies convergence in the strong operator topology. So, $(\text{strict}) \sum_{n=1}^{\infty} t_n t_n^* = (\text{strong}) \sum_{n=1}^{\infty} t_n t_n^*$ and hence $\|(\text{strict}) \sum_{n=1}^{\infty} t_n t_n^*\|^{1/2} = \|(\text{strong}) \sum_{n=1}^{\infty} t_n t_n^*\|^{1/2}$.

Finally, recall our observation preceding (10). (See also (10).) If we have a sequence $(c_n)_n$, $c_n = (x_k^n)_k$ such that each x_k^n belongs to some norm-closed subalgebra \mathbf{B} of $\overline{\mathbf{A}}^s$ and if $(c_n)_n$ converges in our Hilbert C^* -module $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$ to $c_0 = (c_k^0)_k$, then all c_k^0 (i.e., the componentwise limits) also belong to \mathbf{B} . This proves that $\ell_{\text{strong}}^2(LM(\mathbf{A}))$ is closed in $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$. As for $M(\ell^2(\mathbf{A}))$, if $c_0 = \lim_{n \rightarrow \infty} c_n$ in $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$ with $c_n \in M(\ell^2(\mathbf{A}))$ for every n , then $(c_n)_n$ is a Cauchy sequence in $M(\ell^2(\mathbf{A}))$ and since it is a Hilbert C^* -module (hence, complete) and its original norm coincides with the norm inherited from $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$, we must have $c_0 \in M(\ell^2(\mathbf{A}))$. \square

Remark 2.2. Observe that, although $\ell_{\text{strong}}^2(LM(\mathbf{A}))$ has the structure of a Banach \mathbf{A} -module, it is only a closed subspace, and not a submodule of $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$.

We can now state our main theorem in this section.

Theorem 2.3. *Let $\mathbf{A} \subseteq \mathbb{B}(H)$ be a C^* -algebra that acts nondegenerately on H . Then the map*

$$\begin{aligned} \varphi : \ell_{\text{strong}}^2(LM(\mathbf{A})) &\rightarrow \ell^2(\mathbf{A})', & \varphi((t_n)_n) &= L_{(t_n)_n}, \\ L_{(t_n)_n}((a_n)_n) &= \sum_{n=1}^{\infty} a_n t_n^*, \end{aligned}$$

where this series converges in norm in \mathbf{A} , is an antilinear isometric isomorphism of Banach \mathbf{A} -modules. Moreover, its restriction to $M(\ell^2(\mathbf{A}))$

$$\varphi|_{M(\ell^2(\mathbf{A}))} : M(\ell^2(\mathbf{A})) \rightarrow \ell^2(\mathbf{A})^*$$

is an antilinear isometric isomorphism of Hilbert $M(\mathbf{A})$ -modules. In particular, $\varphi|_{\ell^2(\mathbf{A})}$ extends the embedding introduced in (1): $\varphi(x) = L_x$, $x \in \ell^2(\mathbf{A})$.

Observe that the definition of φ makes sense since $t \mapsto t^*$ is a bijection from $LM(\mathbf{A})$ to $RM(\mathbf{A})$ and hence $at^* \in \mathbf{A}$ for each $a \in \mathbf{A}$ and $t \in LM(\mathbf{A})$.

Note that in the nonunital case the adjointable dual $\ell^2(\mathbf{A})^*$ is much larger than $\ell^2(\mathbf{A})$ —a remarkable, but somewhat surprising fact. We will explain later in Remark 2.10 the reason why the idea of passing from $\ell^2(\mathbf{A})$ to $\ell^2(\tilde{\mathbf{A}})$, where $\tilde{\mathbf{A}}$ is the minimal unitization of \mathbf{A} , and trying to describe the dual of $\ell^2(\mathbf{A})$ in terms of $\tilde{\mathbf{A}}$ turns out to be rather naive and of no help.

To prove this theorem we need a couple of auxiliary results. But first, we proceed with some comments and consequences.

When \mathbf{A} is unital we have $LM(\mathbf{A}) = M(\mathbf{A}) = \mathbf{A}$, and the sequence of inclusions (6) becomes

$$\ell^2(\mathbf{A}) \subseteq \ell_{\text{strong}}^2(\mathbf{A}) \subseteq \ell_{\text{strong}}^2(\overline{\mathbf{A}}^s). \quad (12)$$

Also note that the assumed nondegenerate action of \mathbf{A} on H implies that the unit in \mathbf{A} is the identity operator on H . Thus, we have the following.

Corollary 2.4. *Let $\mathbf{A} \subseteq \mathbb{B}(H)$ be a C^* -algebra that contains the identity operator on H . Then the map*

$$\begin{aligned} \varphi : \ell_{\text{strong}}^2(\mathbf{A}) &\rightarrow \ell^2(\mathbf{A})', & \varphi((t_n)_n) &= L_{(t_n)_n}, \\ L_{(t_n)_n}((a_n)_n) &= \sum_{n=1}^{\infty} a_n t_n^*, \end{aligned}$$

where this series converges in norm in \mathbf{A} , is an antilinear isometric isomorphism of Banach \mathbf{A} -modules. Moreover, its restriction to $\ell^2(\mathbf{A})$

$$\varphi|_{\ell^2(\mathbf{A})} : \ell^2(\mathbf{A}) \rightarrow \ell^2(\mathbf{A})^*$$

is an antilinear isometric isomorphism of Hilbert \mathbf{A} -modules.

We note that, in the unital case, the isometric one-to-one correspondence of sequences in $\ell_{\text{strong}}^2(\mathbf{A})$ and bounded module maps from $\ell^2(\mathbf{A})'$ is proved in Proposition 2.5.5 in [24] (see also Lemma 4.1 and Corollary 4.2 in [12]). Observe also that one easily concludes from Theorem 2.3 that $\ell^2(\mathbf{A})$ is self-dual if and only if \mathbf{A} is finite-dimensional—a fact that is first proved in [12].

Finally, we can make a further specialization by assuming that \mathbf{A} is a von Neumann algebra. When this is the case, we have $\overline{\mathbf{A}}^s = \mathbf{A}$ and the above sequence of inclusions (12) reduces to just two Hilbert \mathbf{A} -modules:

$$\ell^2(\mathbf{A}) \subseteq \ell_{\text{strong}}^2(\mathbf{A}). \tag{13}$$

The preceding corollary applies. Here we see that $\ell^2(\mathbf{A})'$ is isometrically isomorphic to the Hilbert \mathbf{A} -module $\ell_{\text{strong}}^2(\mathbf{A})$. (Note that without the assumption that \mathbf{A} is strongly closed, $\ell_{\text{strong}}^2(\mathbf{A})$ is not a Hilbert \mathbf{A} -module.) In fact, even more is true.

Corollary 2.5. *Let \mathbf{A} be a von Neumann algebra. Then $\ell^2(\mathbf{A})'$ is antilinearly isometrically isomorphic to the Hilbert \mathbf{A} -module $\ell_{\text{strong}}^2(\mathbf{A})$. Moreover, $\ell_{\text{strong}}^2(\mathbf{A})$ is self-dual.*

Proof. We only need to prove self-duality. Let $L \in \ell_{\text{strong}}^2(\mathbf{A})'$. Then, by Corollary 2.4, the restriction of L to $\ell^2(\mathbf{A})$ is of the form $(a_n)_n \mapsto \sum_{n=1}^{\infty} a_n t_n^*$ for some sequence $(t_n)_n \in \ell_{\text{strong}}^2(\mathbf{A})$. Consider $L_{(t_n)_n} : \ell_{\text{strong}}^2(\mathbf{A}) \rightarrow \mathbf{A}$, $L_{(t_n)_n}((x_n)_n) = \langle (x_n)_n, (t_n)_n \rangle = (\text{strong}) \sum_{n=1}^{\infty} x_n t_n^*$. Clearly, L and $L_{(t_n)_n}$ coincide on $\ell^2(\mathbf{A})$. In other words, $L - L_{(t_n)_n} \in \ell_{\text{strong}}^2(\mathbf{A})'$ satisfies $(L - L_{(t_n)_n})|_{\ell^2(\mathbf{A})} = 0$. We claim that $L - L_{(t_n)_n} = 0$. This can be seen in the following way.

By Theorem 3.2 in [25] there is an \mathbf{A} -valued inner product $[\cdot, \cdot]$ on $\ell_{\text{strong}}^2(\mathbf{A})$ which extends the original inner product on $\ell^2(\mathbf{A})$ and such that $\ell_{\text{strong}}^2(\mathbf{A})$ is a self-dual Hilbert C^* -module. Moreover, the norm on $\ell_{\text{strong}}^2(\mathbf{A})$ arising from $[\cdot, \cdot]$ coincides with the operator norm $\ell^2(\mathbf{A})'$ and hence, by our Theorem 2.3, with the norm arising from our inner product introduced in Proposition 2.1. This is enough to conclude (see the last part of the proof of Theorem 3.2 in [25]) that $L - L_{(t_n)_n} = 0$. □

Notice that we now have a concrete realization of Paschke’s construction of a self-dual Hilbert C^* -module structure on $\ell^2(\mathbf{A})'$ extending that which is defined on $\ell^2(\mathbf{A})$.

Let us now turn to the proof of Theorem 2.3. We first need a lemma that is known (see, e.g., Lemma 1.4 in [2]). The proof is included here for completeness.

Lemma 2.6. *Let $\mathbf{A} \subseteq \mathbb{B}(H)$ be a C^* -algebra that acts nondegenerately on H . Let $(t_n)_n$ be a sequence of operators in $LM(\mathbf{A})$. Then the following two conditions are equivalent:*

- (a) *The series $\sum_{n=1}^{\infty} a_n t_n^*$ is norm-convergent for all $(a_n)_n$ from $\ell^2(\mathbf{A})$.*
- (b) *The sequence $(\sum_{n=1}^N t_n t_n^*)_N$ is bounded.*

If (a) and (b) are satisfied, then $L : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$, $L((a_n)_n) = \sum_{n=1}^{\infty} a_n t_n^*$ is a bounded module map for which we have $\|L\| = \lim_{N \rightarrow \infty} \|L^N\| = \lim_{N \rightarrow \infty} \|\sum_{n=1}^N t_n t_n^*\|^{\frac{1}{2}}$ where, for each natural number N , the operator $L^N : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$ is defined as the N th partial sum, $L^N((a_n)_n) = \sum_{n=1}^N a_n t_n^*$.

Proof. Assume (a). First note that each L^N is bounded and $\|L^N\| \leq \|\sum_{n=1}^N t_n t_n^*\|^{\frac{1}{2}}$. This is a direct consequence of the Cauchy–Schwarz inequality in the Hilbert $\mathbb{B}(H)$ -module $\mathbb{B}(H)^N$ (the direct sum of N copies of $\mathbb{B}(H)$). Since by the assumption the sequence $(L^N)_N$ strongly converges to L , the uniform boundedness principle tells us that L is bounded. It should also be observed that L and all L^N do take values in \mathbf{A} since each t_n is a left centralizer of \mathbf{A} and hence t_n^* is a right centralizer, so $a t_n^* \in \mathbf{A}$ for all a in \mathbf{A} .

Since each L^N is a restriction of L^{N+1} and at the same time a restriction of L , we have

$$\|L^1\| \leq \|L^2\| \leq \|L^3\| \leq \dots \leq \|L\|. \quad (14)$$

On the other hand, we have for each $x \in \ell^2(\mathbf{A})$

$$\|Lx\| = \lim_{N \rightarrow \infty} \|L^N x\| \leq \liminf_N \|L^N\| \|x\|,$$

which gives us

$$\|L\| \leq \liminf_N \|L^N\|. \quad (15)$$

From (14) and (15) we conclude that $\|L\| = \lim_{N \rightarrow \infty} \|L^N\|$.

Let us now prove that $\|L^N\| = \|\sum_{n=1}^N t_n t_n^*\|^{\frac{1}{2}}$. Denote for simplicity $\|\sum_{n=1}^N t_n t_n^*\|$ by c^N . We already know from the beginning of the proof that $\|L^N\| \leq \sqrt{c^N}$, so we only need to show that $\|L^N x\|$ can be made arbitrarily close to $\sqrt{c^N}$ by choosing a suitable x from the closed unit ball in $\ell^2(\mathbf{A})$.

Let $(e_\lambda)_\lambda$ be an approximate unit for \mathbf{A} . Observe that $e_\lambda \xi \rightarrow \xi$ for each ξ in H since \mathbf{A} acts nondegenerately on H .

Let us now take, for each λ , the sequence $x_\lambda = \frac{1}{\sqrt{c^N}}(t_1 e_\lambda, \dots, t_N e_\lambda, 0, 0, \dots) \in \ell^2(\mathbf{A})$ (recall that t_n 's are left centralizers which ensures that each $t_n e_\lambda$ belongs to \mathbf{A}).

We now recall a well-known inequality that holds true in every C^* -algebra: $baa^*b^* \leq \|a\|^2 bb^*$. Since $\|e_\lambda\| \leq 1$, this gives us

$$\sum_{n=1}^N t_n e_\lambda e_\lambda t_n^* \leq \sum_{n=1}^N t_n t_n^*$$

and hence

$$\|x_\lambda\| = \frac{1}{\sqrt{c^N}} \left\| \sum_{n=1}^N t_n e_\lambda e_\lambda t_n^* \right\|^{\frac{1}{2}} \leq \frac{1}{\sqrt{c^N}} \left\| \sum_{n=1}^N t_n t_n^* \right\|^{\frac{1}{2}} = 1.$$

Thus, all x_λ 's are in the closed unit ball. Now observe that

$$\|L^N(x_\lambda)\| = \frac{1}{\sqrt{c^N}} \left\| \sum_{n=1}^N t_n e_\lambda t_n^* \right\| = \frac{1}{\sqrt{c^N}} \sup \left\{ \left\| \sum_{n=1}^N t_n e_\lambda t_n^* \xi \right\| : \xi \in H, \|\xi\| \leq 1 \right\}.$$

In fact, this is enough to conclude the desired equality $\|L^N\| = \sqrt{c^N}$. One can argue as follows. Fix $\epsilon > 0$. There exists $\xi_0 \in H$ such that $\|\xi_0\| \leq 1$ and

$$\left| c^N - \left\| \sum_{n=1}^N t_n t_n^* \xi_0 \right\| \right| < \epsilon.$$

For this ξ_0 and t_1, \dots, t_n , there exists λ_0 such that

$$\left| \left\| \sum_{n=1}^N t_n e_{\lambda_0} t_n^* \xi_0 \right\| - \left\| \sum_{n=1}^N t_n t_n^* \xi_0 \right\| \right| < \epsilon.$$

This last approximation was possible since $e_\lambda t_n^* \xi_0 \rightarrow t_n^* \xi_0$ for all $n = 1, 2, \dots, N$.

After all, we conclude that $\|L^N(x_{\lambda_0})\|$ is sufficiently close to $\sqrt{c^N}$. In this way we have proved the implication (a) \Rightarrow (b) and the second assertion of the lemma. It remains to prove that (b) implies (a).

Assume (b), and choose a positive number c such that $\|\sum_{n=1}^N t_n t_n^*\| \leq c$ for every N in \mathbb{N} . Take any sequence $(a_n)_n$ from $\ell^2(\mathbf{A})$. If $\epsilon > 0$ is given, then we can find N_0 with the property

$$N_2 > N_1 \geq N_0 \Rightarrow \left\| \sum_{n=N_1+1}^{N_2} a_n a_n^* \right\| < \epsilon^2.$$

From this we conclude that

$$\left\| \sum_{n=1}^{N_2} a_n t_n^* - \sum_{n=1}^{N_1} a_n t_n^* \right\| = \left\| \sum_{n=N_1+1}^{N_2} a_n t_n^* \right\| \leq \left\| \sum_{n=N_1+1}^{N_2} a_n a_n^* \right\|^{\frac{1}{2}} \left\| \sum_{n=N_1+1}^{N_2} t_n t_n^* \right\|^{\frac{1}{2}} < \sqrt{c}\epsilon,$$

where the first inequality is obtained using the Cauchy–Schwarz inequality in the $\mathbb{B}(H)$ -Hilbert module $\mathbb{B}(H)^{N_2-N_1}$.

Thus, $(\sum_{n=1}^N a_n t_n^*)_N$ is a Cauchy sequence. □

Remark 2.7. It is much easier to prove the inequality $\|L^N\| \geq \|\sum_{n=1}^N t_n^* t_n\|^{\frac{1}{2}} = \sqrt{c^N}$ when all t_n 's belong not merely to $LM(\mathbf{A})$ but to \mathbf{A} . If so, one just observes that

$$\left\| L^N \left(\frac{1}{\sqrt{c^N}} (t_1, t_2, \dots, t_N, 0, 0, \dots) \right) \right\| = \frac{1}{\sqrt{c^N}} \left\| \sum_{n=1}^N t_n t_n^* \right\| = \left\| \sum_{n=1}^N t_n t_n^* \right\|^{\frac{1}{2}}.$$

Even if we take t_n from $M(\mathbf{A})$ the proof is easy. Namely, in that case we observe that all L^N are adjointable operators; one easily checks that $(L^N)^* : \mathbf{A} \rightarrow \ell^2(\mathbf{A})$ is then given by $(L^N)^* a = (at_1, at_2, \dots, at_N, 0, 0, \dots)$. Thus, we have $L^N(L^N)^* a =$

$(\sum_{n=1}^N t_n t_n^*)a^*$ for all a in \mathbf{A} and, since \mathbf{A} is an essential ideal in $M(\mathbf{A})$, this implies that $\|L^N(L^N)^*\| = \|\sum_{n=1}^N t_n t_n^*\|$.

However, in the remainder of this section we will need the full force of the preceding lemma with the sequence $(t_n)_n$ of elements of $LM(\mathbf{A})$.

Suppose now that either of two equivalent conditions from Lemma 2.6 is satisfied. Clearly, the operator L is a module map, so we have $L \in \ell^2(\mathbf{A})'$. It is now natural to ask whether L is adjointable. We provide the answer (or rather a reformulation of this question) in our next lemma.

Before stating the lemma, it is convenient to recall a few facts concerning the multiplier module $M(\ell^2(\mathbf{A}))$ of $\ell^2(\mathbf{A})$. It is known (see [3]) that $M(\ell^2(\mathbf{A}))$ is the completion of $\ell^2(\mathbf{A})$ with respect to the strict topology induced by \mathbf{A} . This is the topology on $M(\ell^2(\mathbf{A}))$ induced by the family of seminorms $x \mapsto \|ax\|$, $a \in \mathbf{A}$, and $x \mapsto \|\langle x, y \rangle\|$, $y \in \ell^2(\mathbf{A})$.

In general, each Hilbert \mathbf{A} -module X possesses the strict completion $M(X)$ which is a Hilbert C^* -module over $M(\mathbf{A})$. As one might expect, when a C^* -algebra \mathbf{A} is regarded as a Hilbert C^* -module over itself, its strict completion coincides with $M(\mathbf{A})$. Conveniently enough, each operator $T \in \mathbb{B}(X, Y)$ extends by strict continuity to a unique operator $T_M \in \mathbb{B}(M(X), M(Y))$. Moreover, we know that $(T_M)^* = (T^*)_M$.

Before we state our next result, we need to establish one more notational convention. Given an element a from a C^* -algebra \mathbf{A} and a natural number n , we denote by $a^{(n)}$ the sequence $(0, \dots, 0, a, 0, 0, \dots)$ whose only possibly nontrivial entry a is on the n th place. Clearly, $a \mapsto a^{(n)}$ is an embedding of \mathbf{A} into $\ell^2(\mathbf{A})$.

Lemma 2.8. *Let $\mathbf{A} \subseteq \mathbb{B}(H)$ be a C^* -algebra that acts nondegenerately on H . Let $(t_n)_n$ be a sequence of operators in $LM(\mathbf{A})$. Then the following two conditions are equivalent:*

- (a) *The series $\sum_{n=1}^\infty a_n t_n^*$ is norm-convergent for all $(a_n)_n$ from $\ell^2(\mathbf{A})$ and the operator $L : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$ defined by $L((a_n)_n) = \sum_{n=1}^\infty a_n t_n^*$ is adjointable.*
- (b) *We have $(t_n)_n \in M(\ell^2(\mathbf{A}))$.*

Proof. Assume (a). So, there exists $L^* \in \mathbb{B}(\mathbf{A}, \ell^2(\mathbf{A}))$. Consider the extended operators $L_M \in \mathbb{B}(M(\ell^2(\mathbf{A})), M(\mathbf{A}))$ and $(L^*)_M \in \mathbb{B}(M(\mathbf{A}), M(\ell^2(\mathbf{A})))$.

Denote by e the unit element in $M(\mathbf{A})$ (which is in fact the identity operator on H). We first claim that

$$L_M(e^{(n)}) = t_n^*, \quad \forall n \in \mathbb{N}. \tag{16}$$

Indeed, if $(e_\lambda)_\lambda$ is an approximate unit for \mathbf{A} , then we have

$$\begin{aligned} L_M(e^{(n)}) &= (\text{strict}) \lim_\lambda L_M(e_\lambda^{(n)}) = (\text{strict}) \lim_\lambda L(e_\lambda^{(n)}) \\ &= (\text{strict}) \lim_\lambda e_\lambda t_n^* = (\text{left strict}) \lim_\lambda e_\lambda t_n^* = t_n^*. \end{aligned}$$

Note that the first equality above comes from the strict continuity of L_M since $e_\lambda^{(n)}$ converges strictly in $M(\ell^2(\mathbf{A}))$ to $e^{(n)}$.

Since L_M takes values in $M(\mathbf{A})$, (16) in particular shows us that $t_n^* \in M(\mathbf{A})$, so our assumption (a) forces that the original sequence $(t_n)_n$ of left multipliers

consists in fact of two-sided multipliers. What is more, we can now obtain a simple formula for L^*a . Namely, we know from the discussion preceding this lemma that $L^*a = (L^*)_M a = (L_M)^*a$ and now we see that the n th component of L^*a is

$$\langle L^*a, e^{(n)} \rangle = \langle (L_M)^*a, e^{(n)} \rangle = \langle a, L_M e^{(n)} \rangle \stackrel{(16)}{=} \langle a, t_n^* \rangle = at_n.$$

Hence, the adjoint operator $L^* \in \mathbb{B}(\mathbf{A}, \ell^2(\mathbf{A}))$ is given by

$$L^*a = (at_1, at_2, at_3, \dots), \quad a \in \mathbf{A}. \quad (17)$$

In particular, since $L^*a \in \ell^2(\mathbf{A})$, (17) implies that

$$\sum_{n=1}^{\infty} at_n t_n^* a^* \text{ converges in norm in } \mathbf{A}, \quad \forall a \in \mathbf{A}. \quad (18)$$

Let $x_N = (t_1, t_2, \dots, t_N, 0, 0, \dots) \in M(\ell^2(\mathbf{A}))$, $N \in \mathbb{N}$. We now claim that the sequence $(x_N)_N$ is strictly Cauchy in $M(\ell^2(\mathbf{A}))$.

This is seen as follows. First, for each $(a_n)_n$ from $\ell^2(\mathbf{A})$ we have $\langle (a_n)_n, x_N \rangle = \sum_{n=1}^N a_n t_n^*$ which is by our assumption (a) norm-convergent and hence Cauchy. Second, for each a in \mathbf{A} the sequence $(ax_N)_N$ is also Cauchy since (assuming $N_2 > N_1$) $\|ax_{N_2} - ax_{N_1}\|^2 = \|\sum_{n=N_1+1}^{N_2} at_n t_n^* a^*\|^2$, which is small enough by (18).

Since $M(\ell^2(\mathbf{A}))$ is strictly complete, there exists $s = (s_1, s_2, s_3, \dots) \in M(\ell^2(\mathbf{A}))$ for which we have $s = \text{strict } \lim_{N \rightarrow \infty} x_N$. In particular, this implies that $\lim_{N \rightarrow \infty} \langle a^{(n)}, x_N \rangle = \langle a^{(n)}, s \rangle$ for all $a \in \mathbf{A}$ and n in \mathbb{N} . In other words, we have $at_n^* = as_n^*$ for all a and n . This is enough to conclude that $t_n^* = s_n^*$ for all n and this gives us $s = (t_1, t_2, t_3, \dots) \in M(\ell^2(\mathbf{A}))$. After all, we have obtained the implication (a) \Rightarrow (b).

Let us now assume (b). By the definition of $M(\ell^2(\mathbf{A}))$, this means that the series $\sum_{n=1}^{\infty} t_n t_n^*$ \mathbf{A} -strictly converges. In particular, this implies that $\sum_{n=1}^{\infty} t_n t_n^* a$ converges in norm in $\mathbb{B}(H)$ for all a in \mathbf{A} . As A acts nondegenerately on H , the Cohen–Hewitt factorization theorem tells us that each $\xi \in H$ is of the form $\xi = a\eta$ for some $a \in \mathbf{A}$ and $\eta \in H$. Hence, $\sum_{n=1}^{\infty} t_n t_n^*$ converges strongly in $\mathbb{B}(H)$. This in turn implies, via the uniform boundedness principle, that the sequence $(\sum_{n=1}^N t_n t_n^*)_N$ is bounded. By Lemma 2.6, the series $\sum_{n=1}^{\infty} a_n t_n^*$ converges for every $(a_n)_n$ in $\ell^2(\mathbf{A})$. Hence, we have the operator $L : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$ defined by $L((a_n)_n) = \sum_{n=1}^{\infty} a_n t_n^*$.

Let us now define $R : \mathbf{A} \rightarrow \ell^2(\mathbf{A})$ by $Ra = (at_1, at_2, at_3, \dots)$. Observe that R is well defined because, by our assumption (b), $(t_n)_n \in M(\ell^2(\mathbf{A}))$ and $\ell^2(\mathbf{A})$ is the \mathbf{A} -ideal submodule of $M(\ell^2(\mathbf{A}))$.

To end the proof it remains to show that $L^* = R$. But this is evidently true since

$$\langle L((a_n)_n), b \rangle = \sum_{n=1}^{\infty} a_n t_n^* b^* = \langle (a_n)_n, Rb \rangle$$

for all $(a_n)_n$ from $\ell^2(\mathbf{A})$ and b in \mathbf{A} . □

We are now in position to prove Theorem 2.3.

Proof of Theorem 2.3. We already know from Lemma 2.6 that

$$\begin{aligned} \varphi : \ell_{\text{strong}}^2(LM(\mathbf{A})) &\rightarrow \ell^2(\mathbf{A})', & \varphi((t_n)_n) &= L_{(t_n)_n}, \\ L_{(t_n)_n}((a_n)_n) &= \sum_{n=1}^{\infty} a_n t_n^*, \end{aligned}$$

is a well-defined isometric antilinear module map. Let us show that φ is a surjection.

Take any $L \in \ell^2(\mathbf{A})'$. For a fixed $n \in \mathbb{N}$, we have a bounded module map $\mathbf{A} \rightarrow \mathbf{A}$ defined by $a \mapsto L(a^{(n)})$. By Theorem 1.5 from [20] there exists a right multiplier $s_n \in RM(\mathbf{A})$ such that $L(a^{(n)}) = a s_n$. (In fact, Theorem 1.5 in [20] establishes a bijective correspondence between X' and $LM(\mathbb{K}(X, \mathbf{A}))$. Here we have $X = \mathbf{A}$ and $\mathbb{K}(\mathbf{A}) \simeq \mathbf{A}$ where “compact” operators on \mathbf{A} are described as maps $\mathbf{A} \rightarrow \mathbf{A}$ of the form $a \mapsto at$ for some fixed $t \in \mathbf{A}$. In addition, one should take into account that $RM(\mathbf{A}) = LM(\mathbf{A})^*$.)

Put $s_n = t_n^*$, where t_n is the corresponding left multiplier. In this way we have obtained a sequence $(t_n)_n$ in $LM(\mathbf{A})$ such that $L(a^{(n)}) = a t_n^*$ for each n in \mathbb{N} . In particular, we have for every N in \mathbb{N}

$$L((a_1, a_2, \dots, a_N, 0, 0, \dots)) = \sum_{n=1}^N a_n t_n^*, \quad \forall a_1, a_2, \dots, a_N \in \mathbf{A}. \quad (19)$$

The proof will be finished when we show that $(t_n)_n \in \ell_{\text{strong}}^2(LM(\mathbf{A}))$. But this is easy. We know from (19) by applying Lemma 2.6 that $\|\sum_{n=1}^N t_n t_n^*\| \leq L$, for each N in \mathbb{N} . This shows that the sequence $(t_n)_n$ does belong to $\ell_{\text{strong}}^2(LM(\mathbf{A}))$ and now (19) implies that $L = L_{(t_n)_n} = \varphi((t_n)_n)$.

It remains to show that the restriction

$$\begin{aligned} \varphi|_{M(\ell^2(\mathbf{A}))} : M(\ell^2(\mathbf{A})) &\rightarrow \ell^2(\mathbf{A})^*, & \varphi((t_n)_n) &= L_{(t_n)_n}, \\ L_{(t_n)_n}((a_n)_n) &= \sum_{n=1}^{\infty} a_n t_n^*, \end{aligned}$$

is an antilinear isometric isomorphism of Hilbert $M(\mathbf{A})$ -modules. By Lemma 2.8, φ maps $M(\ell^2(\mathbf{A}))$ into $\ell^2(\mathbf{A})^*$. By the first part of the proof and by the last assertion of Proposition 2.1, $\varphi|_{M(\ell^2(\mathbf{A}))} : M(\ell^2(\mathbf{A})) \rightarrow \ell^2(\mathbf{A})^*$ is an isometry. Basically, this is what we had to show because we know from [3] that $M(\ell^2(\mathbf{A})) = \mathbb{B}(\mathbf{A}, \ell^2(\mathbf{A}))$ and $\ell^2(\mathbf{A})^* = \mathbb{B}(\ell^2(\mathbf{A}), \mathbf{A})$. For the reader's convenience, we include the argument which shows that $\varphi|_{M(\ell^2(\mathbf{A}))}$ is surjective onto $\ell^2(\mathbf{A})^*$.

Take an arbitrary adjointable $L : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$. Since $L^* \in \mathbb{B}(\mathbf{A}, \ell^2(\mathbf{A}))$ exists, we can extend it to $(L^*)_M \in \mathbb{B}(M(\mathbf{A}), M(\ell^2(\mathbf{A})))$. Observe that $M(\mathbf{A})$ contains the identity operator on H which we again denote by e . Since $(L^*)_M$ is a module map, we have

$$(L^*)_M(m) = (L^*)_M(me) = m \cdot (L^*)_M(e), \quad \forall m \in M(\mathbf{A}).$$

Denote $(L^*)_M(e) = (t_n)_n \in M(\ell^2(\mathbf{A}))$. We now compute:

$$\begin{aligned} L_M((y_n)_n) &= L_M((y_n)_n)e = \langle L_M((y_n)_n), e \rangle = \langle (y_n)_n, (L^*)_M(e) \rangle \\ &= \langle (y_n)_n, (t_n)_n \rangle, \quad \forall (y_n)_n \in M(\ell^2(\mathbf{A})). \end{aligned}$$

In particular, this gives us

$$L((a_n)_n) = L_M((a_n)_n) = \langle (a_n)_n, (t_n)_n \rangle, \quad \forall (a_n)_n \in \ell^2(\mathbf{A}). \quad \square$$

We end this section with some examples and comments.

Example 2.9. Let H be a separable Hilbert space with an orthonormal basis $(\varepsilon_n)_n$. For each $n \in \mathbb{N}$, denote by e_n the orthogonal projection to $\text{span}\{\varepsilon_n\}$. Let $\mathbf{A} = \mathbb{K}(H)$ —the C^* -algebra of all compact operators on H .

Clearly, $e_n \in \mathbb{K}(H) \subset M(\mathbb{K}(H)) = \mathbb{B}(H)$. As the sequence $(\sum_{n=1}^N e_n e_n^*)_N = (\sum_{n=1}^N e_n)_N$ is bounded, Lemma 2.6 implies that the operator $L : \ell^2(\mathbb{K}(H)) \rightarrow \mathbb{K}(H)$ given by $L((a_n)_n) = \sum_{n=1}^\infty a_n e_n$ is a well-defined bounded module map. Since all bounded module maps of Hilbert C^* -modules over $\mathbb{K}(H)$ are adjointable (see [23, Theorem 1]), implication (b) \Rightarrow (a) from Lemma 2.8 shows that $(e_n)_n \in M(\ell^2(\mathbb{K}(H)))$.

Indeed, this last conclusion can also be obtained directly. In any case, L belongs to the adjointable dual $\ell^2(\mathbb{K}(H))^*$. It is useful to observe that $(e_n)_n$ does not belong to $\ell^2(\mathbb{K}(H))$.

Remark 2.10. Corollary 2.4 tells us that, when \mathbf{A} contains the unit, $\ell^2(\mathbf{A})^*$ is in a bijective correspondence with $\ell^2(\mathbf{A})$. When \mathbf{A} is nonunital, this is not the case since then $M(\ell^2(\mathbf{A}))$ contains $\ell^2(\mathbf{A})$ as a proper subset. (Usually $M(\ell^2(\mathbf{A}))$ is way bigger than $\ell^2(\mathbf{A})$.) This is demonstrated by the preceding example where we have concluded that $(e_n)_n \in M(\ell^2(\mathbb{K}(H)))$, but, clearly, $(e_n)_n \notin \ell^2(M(\mathbb{K}(H))) = \ell^2(\mathbb{B}(H)) \supseteq \ell^2(\mathbb{K}(H))$.

It is tempting to try to describe $\ell^2(\mathbf{A})^*$ in the nonunital case by passing to $\ell^2(\tilde{\mathbf{A}})$, where $\tilde{\mathbf{A}}$ is the minimal unitization of \mathbf{A} . Bearing in mind the unital case, one could try to establish a correspondence of $\ell^2(\mathbf{A})^*$ with $\ell^2(\tilde{\mathbf{A}})$. Let us take a closer look at that idea.

Take any $L \in \ell^2(\mathbf{A})^*$. Recall that this means that L is an adjointable operator from $\ell^2(\mathbf{A})$ to \mathbf{A} . But now $\ell^2(\mathbf{A})$ can be regarded as a Hilbert C^* -module over the unital C^* -algebra $\tilde{\mathbf{A}}$, so we can regard L as a map, call it \tilde{L} , from $\ell^2(\mathbf{A})$ into $\tilde{\mathbf{A}}$. Is \tilde{L} adjointable? If we assume that the answer is yes, it would follow that $Lx = \tilde{L}x = \tilde{L}x \cdot e = \langle \tilde{L}x, e \rangle = \langle x, (\tilde{L})^*e \rangle$ for all x in $\ell^2(\mathbf{A})$ and this leads to the conclusion that L is represented by the sequence $(\tilde{L})^*e \in \ell^2(\mathbf{A})$.

However, this cannot be true in general. Example 2.9 provides L that is represented by the sequence $(e_n)_n$ which does not belong to $\ell^2(\mathbf{A})$. So our innocent looking assumption that the map \tilde{L} is adjointable was wrong. Although \tilde{L} acts precisely as L , it need not be adjointable.

Alternatively, one can try to extend L to an adjointable map $L^{(e)} : \ell^2(\tilde{\mathbf{A}}) \rightarrow \tilde{\mathbf{A}}$. If this is possible, L will be represented by the sequence $(L^{(e)})^*e \in \ell^2(\tilde{\mathbf{A}})$. But again, this fails in general. Observe that the representing sequence $(e_n)_n$ in the preceding example does not belong to $\ell^2(\tilde{\mathbf{A}})$. This is simply because the series $\sum_{n=1}^\infty e_n e_n^* = \sum_{n=1}^\infty e_n$ does not converge in norm.

The point is that there is no easy way to extend L to an adjointable map of modules over unital C^* -algebras. In order to do so, one has to go all the way up to the maximal essential extension $M(\ell^2(\mathbf{A}))$ of $\ell^2(\mathbf{A})$. Only there, extensions of adjointable operators are available (as we mentioned in the discussion preceding Lemma 2.8) thanks to the strict continuity.

Example 2.11. Take a separable Hilbert space H . Let H_1 be a closed subspace of H such that both H_1 and H_1^\perp are infinite-dimensional. Denote by $p \in \mathbb{B}(H)$ the orthogonal projection to H_1 . Consider the C^* -algebra $\mathbf{A} = C^*(\mathbb{K}(H), p)$ generated by p and all compact operators. Clearly, \mathbf{A} acts nondegenerately on H .

Let us now take an orthonormal basis $(\varepsilon_n)_n$ for H_1 . Again for each $n \in \mathbb{N}$, we denote by e_n the orthogonal projection to $\text{span}\{\varepsilon_n\}$.

Consider the sequence (e_1, e_2, e_3, \dots) . We first observe that all the e_n 's belong to $LM(\mathbf{A})$ (in fact, they belong to \mathbf{A}) and that the sequence $(\sum_{n=1}^N e_n e_n^*)_N = (\sum_{n=1}^N e_n)_N$ is bounded. Applying Lemma 2.6 we conclude that $L : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$, defined by $L((a_n)_n) = \sum_{n=1}^\infty a_n e_n$, is bounded. Thus, $L \in \ell^2(\mathbf{A})'$.

We claim that L is not adjointable, that is, $L \in \ell^2(\mathbf{A})' \setminus \ell^2(\mathbf{A})^*$. To see this, it suffices by Lemma 2.8 to prove that $(e_n)_n \notin M(\ell^2(\mathbf{A}))$. Thus, we must show that the series $\sum_{n=1}^\infty e_n e_n^* = \sum_{n=1}^\infty e_n$ does not converge in the strict topology on $B(H)$ induced by our C^* -algebra \mathbf{A} . But this is clear. The strict convergence fails because of the presence of p in \mathbf{A} . Indeed, $(\sum_{n=1}^N e_n p)_N$ cannot converge in norm since $\sum_{n=1}^N e_n p = \sum_{n=1}^N e_n$ for all N and we know that $\sum_{n=1}^\infty e_n$ is not norm-convergent.

Remark 2.12. Let us also note the following observation. If bounded module maps L_1 and L_2 from $\ell^2(\mathbf{A})'$ coincide on the set $c_{00}(\mathbf{A})$ of all finite sequences (all but finitely many components are equal to zero), then we must have $L_1 = L_2$ simply because the set of finite sequences is norm-dense in $\ell^2(\mathbf{A})$.

On the other hand, the set of all finite sequences is not norm-dense in $\ell_{\text{strong}}^2(LM(\mathbf{A}))$ (and, in particular, it cannot be dense in $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$). Recall that $M(\ell^2(\mathbf{A}))$ is closed in $\ell_{\text{strong}}^2(\overline{\mathbf{A}}^s)$, and observe that all finite sequences obviously belong to $M(\ell^2(\mathbf{A}))$; in fact, they are already in $\ell^2(\mathbf{A})$.

The sequence $x = (e_1, e_2, e_3, \dots)$ of one-dimensional orthogonal projections from Example 2.11 may serve as a simple illustration. Recall that this was the representing sequence for a module map $L \in \ell^2(\mathbf{A})'$. So, we know that $x = (e_1, e_2, e_3, \dots)$ must be in $\ell_{\text{strong}}^2(LM(\mathbf{A}))$ (which is also easily seen by a direct verification). For N in \mathbb{N} , let $x_N = (e_1, e_2, \dots, e_N, 0, 0, \dots)$. Clearly, the sequence $(x_N)_N$ cannot converge to x in the Hilbert C^* -norm on $\ell_{\text{strong}}^2(LM(\mathbf{A}))$. In fact, if $N < M$, then we have $\|x_M - x_N\|^2 = \|\langle x_M - x_N, x_M - x_N \rangle\| = \|\sum_{n=N+1}^M e_n\| = 1$.

3. Weak frames

Throughout this section, X will denote a left Hilbert C^* -module over a von Neumann algebra \mathbf{A} acting on a Hilbert space H . We point out that the hypothesis that \mathbf{A} be a von Neumann algebra is essential.

Recall from the preceding section that, when \mathbf{A} is a von Neumann algebra, $\ell^2(\mathbf{A})$ —the standard Hilbert C^* -module over \mathbf{A} —is contained as a closed submodule in its dual

$$\ell_{\text{strong}}^2(\mathbf{A}) = \left\{ (a_n)_n : a_n \in \mathbf{A}, \sup_N \left\| \sum_{n=1}^N a_n a_n^* \right\| < \infty \right\},$$

which is a self-dual Hilbert \mathbf{A} -module with the inner product

$$\langle (a_n)_n, (b_n)_n \rangle = (\text{strong}) \sum_{n=1}^{\infty} a_n b_n^*$$

that extends the inner product defined on $\ell^2(\mathbf{A})$.

In the first part of this section we additionally assume that X is self-dual.

Definition 3.1. Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . The *weak-strong topology* on X is defined as the weak topology induced by the maps $L_y : X \rightarrow \mathbf{A}$, $L_y(x) = \langle x, y \rangle$, $y \in X$, where $\mathbf{A} \subseteq \mathbb{B}(H)$ is regarded with the respect to the strong operator topology.

A net $(x_\lambda)_\lambda$ in X converges weak-strong to $x \in X$, which we denote as $x = (\text{weak-strong}) \lim_\lambda x_\lambda$, if and only if

$$\langle x, y \rangle = (\text{strong}) \lim_\lambda \langle x_\lambda, y \rangle, \quad \forall y \in X.$$

Here again we use our assumption that \mathbf{A} is a von Neumann algebra: this guarantees that $(\text{strong}) \lim_\lambda \langle x_\lambda, y \rangle$, if it exists, belongs to \mathbf{A} , so it makes sense to require that this limit is equal to $\langle x, y \rangle$ which is, by the definition of a Hilbert \mathbf{A} -module, an element of \mathbf{A} .

It should also be observed that the weak-strong topology is Hausdorff. Namely, the strong operator topology on \mathbf{A} is Hausdorff and the family L_y , $y \in X$, obviously separates points of X .

Definition 3.2. Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . A sequence $(x_n)_n$ in X is called a *weak frame* for X if there exist positive constants A and B such that

$$A \langle x, x \rangle \leq (\text{strong}) \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x, x_n \rangle^* \leq B \langle x, x \rangle, \quad \forall x \in X. \quad (20)$$

A sequence for which the second inequality in (20) is satisfied for some constant B is said to be a *weak Bessel sequence*. The constants A and B are called *frame bounds*. If $A = B = 1$, that is, if

$$(\text{strong}) \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x, x_n \rangle^* = \langle x, x \rangle, \quad \forall x \in X, \quad (21)$$

then the sequence $(x_n)_n$ is called a *weak Parseval frame*.

Note that it is implicitly required in the above definition that the series in (20) converges strongly in \mathbf{A} . We know that this is equivalent to the condition $\sup_N \|\sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^*\| < \infty$ (this is already noted in establishing equivalence of (3) and (4)). Recall from [13] that in the definition of a standard frame one requires norm-convergence of the series $\sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x, x_n \rangle^*$. Hence, each standard frame in X is a weak frame. However, there are weak frames that are not standard.

Example 3.3. Let H be an infinite-dimensional separable Hilbert space with an orthonormal basis $(\varepsilon_n)_{n \in \mathbb{N}}$. For each n in \mathbb{N} , denote by e_n the one-dimensional projection to $\text{span}\{\varepsilon_n\}$. Clearly, we have (strong) $\sum_{n=1}^{\infty} e_n = e$, where e denotes the identity operator on H .

Consider $\mathbb{B}(H)$ —the algebra of all bounded linear operators on H —as a Hilbert C^* -module over itself. We know that $\mathbb{B}(H)$ is self-dual (cf. Theorem 1.5 from [20]). We claim that $(e_n)_n$ is a weak nonstandard Parseval frame for $\mathbb{B}(H)$. Indeed, for each $a \in \mathbb{B}(H)$ the series $\sum_{n=1}^{\infty} \langle a, e_n \rangle \langle a, e_n \rangle^* = \sum_{n=1}^{\infty} a e_n a^* = a(\sum_{n=1}^{\infty} e_n) a^*$ converges strongly to $aa^* = \langle a, a \rangle$.

On the other hand, this is not a standard frame; that is, the series $\sum_{n=1}^{\infty} \langle a, e_n \rangle \times \langle a, e_n \rangle^* = \sum_{n=1}^{\infty} a e_n a^*$ cannot converge in norm to aa^* for all $a \in \mathbb{B}(H)$. Indeed, $aa^* = (\text{norm}) \sum_{n=1}^{\infty} a e_n a^*$ forces aa^* (and hence a) to be a compact operator.

Our first goal is to introduce analysis and synthesis operators for weak frames. In order to ensure adjointability of these operators, the self-duality assumption is needed. On the other hand, all we need can be done even for weak Bessel sequences. Before stating the theorem, we recall our notational convention: for $a \in \mathbf{A}$ and $n \in \mathbb{N}$, we denote by $a^{(n)} \in \ell^2(\mathbf{A})$ the sequence with a on the n th place and zeros elsewhere. The unit element in \mathbf{A} (i.e., the identity operator on the underlying Hilbert space H) is denoted by e .

Theorem 3.4. *Let X be a self-dual Hilbert C^* -module X over a von Neumann algebra \mathbf{A} , and let $(x_n)_n$ be a sequence in X . Then the following two conditions are equivalent:*

- (a) $(x_n)_n$ is a weak Bessel sequence,
- (b) $\sup_N \|\sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^*\| < \infty$ for all x in X .

If $(x_n)_n$ is a weak Bessel sequence, then the map

$$U : X \rightarrow \ell_{\text{strong}}^2(\mathbf{A}), \quad Ux = (\langle x, x_n \rangle)_n$$

is adjointable. The adjoint operator $U^* : \ell_{\text{strong}}^2(\mathbf{A}) \rightarrow X$ is given by

$$U^*((a_n)_n) = (\text{weak-strong}) \sum_{n=1}^{\infty} a_n x_n, \quad \forall (a_n)_n \in \ell_{\text{strong}}^2(\mathbf{A}). \quad (22)$$

In particular, $U^*e^{(n)} = x_n$ for all $n \in \mathbb{N}$.

Proof. Clearly, (a) implies (b). To prove the reverse implication, suppose that (b) is satisfied. Then $U : X \rightarrow \ell_{\text{strong}}^2(\mathbf{A})$, $Ux = (\langle x, x_n \rangle)_n$, is a well-defined module map. One now shows, precisely as it is done in the proof of Theorem 2.1 in [2],

that U is bounded using the closed graph theorem. For the reader's convenience, we include the proof.

Let $(y, (a_n)_n) = \lim_{k \rightarrow \infty} (y_k, Uy_k)$, where $y_k, y \in X, (a_n)_n \in \ell^2_{\text{strong}}(\mathbf{A})$. For each $m \in \mathbb{N}$ and all $k \in \mathbb{N}$, we have

$$\begin{aligned} \langle (a_n)_n - Uy_k, (a_n)_n - Uy_k \rangle &= (\text{strong}) \sum_{n=1}^{\infty} (a_n - \langle y_k, x_n \rangle) (a_n - \langle y_k, x_n \rangle)^* \\ &\geq (a_m - \langle y_k, x_m \rangle) (a_m - \langle y_k, x_m \rangle)^*. \end{aligned}$$

It follows that

$$\|a_m - \langle y_k, x_m \rangle\| \leq \|(a_n)_n - Uy_k\|.$$

Since by the assumption we have $(a_n)_n = \lim_{k \rightarrow \infty} Uy_k$ and $y = \lim_{k \rightarrow \infty} y_k$, this implies that

$$a_m = \lim_{k \rightarrow \infty} \langle y_k, x_m \rangle = \langle y, x_m \rangle.$$

As m was arbitrary, this shows that $(a_n)_n = Uy$. So, the graph of U is closed.

Knowing that U is a bounded module map, we can apply Theorem 2.8 in [25] to conclude that $\langle Ux, Ux \rangle \leq \|U\|^2 \langle x, x \rangle$ for all $x \in X$; thus, $(x_n)_n$ is a weak Bessel sequence. In this way we have proved that (b) implies (a). Moreover, since X is self-dual, by Proposition 3.4 from [25] we conclude that U is in fact an adjointable operator.

It remains to obtain (22). We need to prove that for each $(a_n)_n \in \ell^2_{\text{strong}}(\mathbf{A})$, the sequence $(\sum_{n=1}^N a_n x_n)_N$ converges in the weak-strong topology on X to $U^*((a_n)_n)$. Thus, we must show that

$$(\text{strong}) \lim_{N \rightarrow \infty} \left\langle \sum_{n=1}^N a_n x_n, y \right\rangle = \langle U^*((a_n)_n), y \rangle, \quad \forall y \in X.$$

Take any $y \in X$. Then we have

$$\begin{aligned} (\text{strong}) \lim_{N \rightarrow \infty} \left\langle \sum_{n=1}^N a_n x_n, y \right\rangle &= (\text{strong}) \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n \langle x_n, y \rangle \\ &= (\text{strong}) \sum_{n=1}^{\infty} a_n \langle y, x_n \rangle^* \\ &= \langle (a_n)_n, Uy \rangle \\ &= \langle U^*((a_n)_n), y \rangle. \end{aligned}$$

Finally, the equality $U^*e^{(n)} = x_n$ for all n in \mathbb{N} now follows from

$$\langle x, U^*e^{(n)} \rangle = \langle Ux, e^{(n)} \rangle = \langle x, x_n \rangle, \quad \forall x \in X. \quad \square$$

Definition 3.5. The operators U and U^* are called the *analysis* and *synthesis* operators, respectively.

Remark 3.6. After concluding in the preceding proof that U is bounded one might try to prove the existence of the adjoint U^* directly, as it is known from literature for standard frames. In the first step, one could put $U^*e^{(n)} = x_n$ for every n in \mathbb{N} . And this works fine: for each x in X we would then have $\langle x, U^*e^{(n)} \rangle = \langle x, x_n \rangle = \langle Ux, e^{(n)} \rangle$. But the next step is precisely where the proof of Theorem 4.1 from [13] breaks down in this situation. The obstruction lies in the fact already observed in Remark 2.12 that the set of all finite sequences is not dense in $\ell^2_{\text{strong}}(\mathbf{A})$. And this is the reason why we needed self-duality of X to ensure adjointability of U .

Remark 3.7. Suppose that $(x_n)_n$ is a weak Bessel sequence in a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . Denote by U the analysis operator. We claim that

$$U^*((a_n)_n) = (\text{norm}) \sum_{n=1}^{\infty} a_n x_n, \quad \forall (a_n)_n \in \ell^2(\mathbf{A}). \quad (23)$$

To see this, we first recall that for each $(a_n)_n \in \ell^2(\mathbf{A})$ we have

$$(a_n)_n = (\text{norm}) \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n e^{(n)} = (\text{norm}) \lim_{N \rightarrow \infty} (a_1, a_2, \dots, a_N, 0, 0, \dots). \quad (24)$$

Since U^* is norm-continuous, this gives us

$$U^*(a_n)_n = (\text{norm}) \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n U^*e^{(n)} = (\text{norm}) \sum_{n=1}^{\infty} a_n x_n.$$

In general, we cannot conclude norm-convergence of the series $\sum_{n=1}^{\infty} a_n x_n$ for sequences from $\ell^2_{\text{strong}}(\mathbf{A}) \setminus \ell^2(\mathbf{A})$. This is clearly seen from the following example.

Take $X = \ell^2_{\text{strong}}(\mathbf{A})$, and consider the sequence $(e^{(n)})_n$. This sequence is weak Bessel (in fact, it is a weak Parseval frame) in X . Here norm-convergence of the series $\sum_{n=1}^{\infty} a_n x_n$, that is, $\sum_{n=1}^{\infty} a_n e^{(n)}$, is in fact equation (24) for which we know that it is satisfied only for those sequences $(a_n)_n \in \ell^2_{\text{strong}}(\mathbf{A})$ that belong to $\ell^2(\mathbf{A})$.

It is also useful to note that although the inner product in $\ell^2_{\text{strong}}(\mathbf{A})$ is defined in terms of strong convergence, we have $\langle (a_n)_n, Ux \rangle = (\text{norm}) \sum_{n=1}^{\infty} a_n \langle x, x_n \rangle^*$ for all $(a_n)_n \in \ell^2(\mathbf{A})$ and $x \in X$. This is because Ux , being an element of $\ell^2_{\text{strong}}(\mathbf{A})$, induces a bounded module map $L_{Ux} : \ell^2(\mathbf{A}) \rightarrow \mathbf{A}$ and, by Lemma 2.6, the series $\sum_{n=1}^{\infty} a_n \langle x, x_n \rangle^*$ is norm-convergent whenever $(a_n)_n$ is an element of $\ell^2(\mathbf{A})$.

Remark 3.8. In fact, the sequence $(e^{(n)})_n$ from Remark 3.7 is more than a weak Parseval frame for $\ell^2_{\text{strong}}(\mathbf{A})$. Note that we have $\langle e^{(n)}, e^{(m)} \rangle = \delta_{n,m}e$ for all $n, m \in \mathbb{N}$ which shows that this is an orthogonal sequence. Since it is a weak Parseval frame, we also have $U^*U = I$, that is,

$$(c_n)_n = (\text{weak-strong}) \sum_{n=1}^{\infty} c_n e^{(n)}, \quad \forall (c_n)_n \in \ell^2_{\text{strong}}(\mathbf{A}).$$

Thus, $(e^{(n)})_n$ is what can be called a *weak orthonormal basis* for $\ell^2_{\text{strong}}(\mathbf{A})$. Since its analysis operator coincides with the identity operator, we will refer in the remainder of this article to $(e^{(n)})_n$ as the *canonical weak basis* for $\ell^2_{\text{strong}}(\mathbf{A})$. (Observe that

$(e^{(n)})_n$ is at the same time a standard orthonormal basis for $\ell^2(\mathbf{A})$.) In general, one can define weak Riesz bases as those weak frames whose analysis operators are invertible. In the present paper we omit further discussion on weak Riesz bases.

Proposition 3.9. *Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . Let $T : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow X$ be a bounded module map. Then the sequence $(x_n)_n$, where $x_n = Te^{(n)}$, $n \in \mathbb{N}$, is a weak Bessel sequence in X whose analysis operator coincides with T^* .*

Proof. First observe that since $\ell^2_{\text{strong}}(\mathbf{A})$ is self-dual, T is adjointable by Proposition 3.4 from [25]. We now have for all x in X and n in \mathbb{N} ,

$$\langle x, x_n \rangle = \langle x, Te^{(n)} \rangle = \langle T^*x, e^{(n)} \rangle.$$

The last term is precisely the n th component of the sequence T^*x ; hence, $T^*x = (\langle x, x_n \rangle)_n$. Moreover, as T^*x belongs to $\ell^2_{\text{strong}}(\mathbf{A})$, we do have $\sup_N \left\| \sum_{n=1}^N \langle x, x_n \rangle \times \langle x, x_n \rangle^* \right\| < \infty$. By Theorem 3.4, this implies that $(x_n)_n$ is a weak Bessel sequence. \square

Remark 3.10. If X is a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} , then Theorem 3.4 and Proposition 3.9 show us that weak Bessel sequences in X are in a bijective correspondence with bounded module maps $T : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow X$.

The following theorem provides us with a sufficient frame condition which is considerably easier to check than the original condition from Definition 3.2.

Theorem 3.11. *Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . Then a sequence $(x_n)_n$ in X is a weak frame for X if and only if there exists a constant $A > 0$ such that*

$$A\|x\|^2 \leq \sup_N \left\| \sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^* \right\| < \infty, \quad \forall x \in X. \tag{25}$$

Proof. We have already observed that the series $\sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x, x_n \rangle^*$ converges strongly if and only if $\sup_N \left\| \sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^* \right\| < \infty$, in which case we have

$$\left\| \left(\text{strong} \right) \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x, x_n \rangle^* \right\| = \sup_N \left\| \sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^* \right\|.$$

The desired conclusion now follows from our Theorem 3.4 and Corollary 2.2 from [1]. \square

Corollary 3.12. *Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . A sequence $(x_n)_n$ in X is a weak frame for X if and only if there is a surjective bounded module map $T : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow X$ such that $Te^{(n)} = x_n$ for each n in \mathbb{N} . If $(x_n)_n$ is a weak frame in X and if $R : X \rightarrow Y$ is a surjective bounded module map, where Y is a self-dual Hilbert \mathbf{A} -module, then the sequence $(Rx_n)_n$ is a weak frame in Y .*

Proof. Suppose that $(x_n)_n$ is a weak frame in X . Then by (25) the analysis operator U is bounded from below and hence has a closed range. This implies that U^* is a surjection.

To prove the converse, suppose that we are given a surjective bounded module map $T : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow X$. First, T is adjointable by Proposition 3.4 from [25]. By Theorem 3.2 from [19], T^* is injective and has a closed range which implies that T^* is bounded from below. As in the proof of Proposition 3.9, we conclude that T^* is in fact the analysis operator of the sequence $(x_n)_n$. Now Theorem 3.11 applies. The second assertion of the corollary is an immediate consequence of the first one. \square

Corollary 3.13. *Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . If $(x_n)_n$ is a weak frame in X , then the sequence $(x_{\sigma(n)})_n$ is also a weak frame for every permutation σ of the set \mathbb{N} .*

Proof. Take any permutation σ , and fix $N_1 \in \mathbb{N}$. Then we have

$$\left\| \sum_{n=1}^{N_1} \langle x, x_{\sigma(n)} \rangle \langle x, x_{\sigma(n)} \rangle^* \right\| \leq \sup_N \left\| \sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^* \right\|$$

and

$$\left\| \sum_{n=1}^{N_1} \langle x, x_n \rangle \langle x, x_n \rangle^* \right\| \leq \sup_N \left\| \sum_{n=1}^N \langle x, x_{\sigma(n)} \rangle \langle x, x_{\sigma(n)} \rangle^* \right\|.$$

This is enough to conclude that

$$\sup_N \left\| \sum_{n=1}^N \langle x, x_{\sigma(n)} \rangle \langle x, x_{\sigma(n)} \rangle^* \right\| = \sup_N \left\| \sum_{n=1}^N \langle x, x_n \rangle \langle x, x_n \rangle^* \right\|.$$

The conclusion now follows from Theorem 3.11. \square

Remark 3.14. The conclusion of the preceding corollary applies also to Bessel sequences. This in turn implies that when working with the synthesis operator of a weak Bessel sequence (or a weak frame) $(x_n)_n$ the relevant series $\sum_{n=1}^{\infty} c_n x_n$ converges unconditionally in the weak-strong topology for every sequence $(c_n)_n \in \ell^2_{\text{strong}}(\mathbf{A})$.

This is essential in applications since we will often encounter countable systems naturally indexed by sets other than the set of natural numbers. In particular, when dealing with systems indexed by the set of the integers \mathbb{Z} , we will be allowed to restrict our considerations to symmetric partial sums $\sum_{n=-N}^N c_n x_n$, $N \in \mathbb{N}$.

Theorem 3.15. *Let X be a self-dual Hilbert C^* -module over a von Neumann algebra \mathbf{A} . Suppose that $(x_n)_n$ is a weak frame in X with the analysis operator U . Then the frame operator U^*U is invertible, the sequence $((U^*U)^{-1}x_n)_n$ is a weak frame in X , and the following reconstruction formula holds:*

$$x = (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, x_n \rangle (U^*U)^{-1}x_n, \quad \forall x \in X. \quad (26)$$

Proof. As U is bounded from below and has a closed range, one shows that U^*U is invertible by a standard argument. Corollary 3.12 tells us that $((U^*U)^{-1}x_n)_n$ is a weak frame in X . To prove the reconstruction formula, first observe that each adjointable operator $R : X \rightarrow X$ is continuous in the weak-strong topology. Indeed, suppose that $x = (\text{weak-strong}) \lim_{\lambda} x_{\lambda}$, and fix any $y \in X$. Then we have

$$\langle Rx, y \rangle = \langle x, R^*y \rangle = (\text{strong}) \lim_{\lambda} \langle x_{\lambda}, R^*y \rangle = (\text{strong}) \lim_{\lambda} \langle Rx_{\lambda}, y \rangle,$$

which shows that $Rx = (\text{weak-strong}) \lim_{\lambda} Rx_{\lambda}$. Now we have

$$U^*Ux \stackrel{(22)}{=} (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \quad \forall x \in X.$$

Applying $(U^*U)^{-1}$ to both sides and using its weak-strong continuity, we get

$$x = (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, x_n \rangle (U^*U)^{-1}x_n, \quad \forall x \in X. \quad \square$$

Definition 3.16. The weak frame $((U^*U)^{-1}x_n)_n$ from the preceding theorem is called the *canonical dual* of $(x_n)_n$.

Remark 3.17. Denote by V the analysis operator of the canonical dual $((U^*U)^{-1}x_n)_n$. Then the reconstruction formula (26) simply reads $V^*U = I$, where I is the identity operator on X . But this is obviously equivalent to $U^*V = I$, which means that we also have

$$x = (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, (U^*U)^{-1}x_n \rangle x_n, \quad \forall x \in X. \quad (27)$$

This tells us that $((U^*U)^{-1}x_n)_n$ and $(x_n)_n$ are dual to each other in a symmetric way.

By a standard argument one also shows that $((U^*U)^{-\frac{1}{2}}x_n)_n$ is a weak Parseval frame. This follows from the equality

$$x = (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, (U^*U)^{-\frac{1}{2}}x_n \rangle (U^*U)^{-\frac{1}{2}}x_n, \quad \forall x \in X \quad (28)$$

and the fact that a sequence $(y_n)_n$ in X is a weak Parseval frame if and only if it has the property

$$x = (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, y_n \rangle y_n, \quad \forall x \in X. \quad (29)$$

Our next proposition basically says that unitary operators of Hilbert C^* -modules map weak frames into weak frames. This is something that is certainly expected, but one should observe that in the proof that follows we again encounter a step for which we need the assumption that the underlying C^* -algebra is a von Neumann algebra (i.e., any isomorphism of von Neumann algebras is normal; see [26, Proposition 2.5.2]).

Suppose that X and Y are left Hilbert C^* -modules over C^* -algebras \mathbf{A} and \mathbf{B} , respectively, and suppose that $\varphi : \mathbf{A} \rightarrow \mathbf{B}$ is a morphism of C^* -algebras. Recall from [3] that a map $\Phi : X \rightarrow Y$ is said to be a φ -morphism of X and Y if it satisfies $\langle \Phi(x_1), \Phi(x_2) \rangle = \varphi(\langle x_1, x_2 \rangle)$ for all x_1, x_2 from X . It turns out that such a map necessarily satisfies $\Phi(ax) = \varphi(a)\phi(x)$ for all $x \in X$ and $a \in \mathbf{A}$. We say that Φ is a unitary operator of Hilbert C^* -modules if both Φ and φ are bijections (in fact, it suffices to require that Φ is a surjection and that φ is injective). When this is the case, both Φ and φ are isometries.

Proposition 3.18. *Suppose that X and Y are self-dual Hilbert C^* -modules over von Neumann algebras \mathbf{A} and \mathbf{B} , respectively, that $\varphi : \mathbf{A} \rightarrow \mathbf{B}$ is an isomorphism, and that $\Phi : X \rightarrow Y$ is a φ -unitary operator. Then a sequence $(x_n)_n$ is a weak frame (resp., Bessel sequence) in X with bounds A and B if and only if the sequence $(\Phi(x_n))_n$ is a weak frame (resp., Bessel sequence) in Y with the same frame (resp., Bessel) bounds.*

Proof. This essentially follows from two important properties of isomorphisms of von Neumann algebras. First, we know that $a \leq b$ implies that $\varphi(a) \leq \varphi(b)$ (this is already true when φ is a morphism of arbitrary C^* -algebras) and second, that φ is normal. This latter property means that when $(a_n)_n$ is an increasing strongly convergent net in \mathbf{A} such that $a = (\text{strong}) \lim_n a_n$, then $\varphi(a) = (\text{strong}) \lim_n \varphi(a_n)$.

Suppose now that we are given a weak frame $(x_n)_n$ in X with frame bounds A and B . Then we have

$$A\langle x, x \rangle \leq (\text{strong}) \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x, x_n \rangle^* \leq B\langle x, x \rangle, \quad \forall x \in X.$$

We now use all the properties of φ to obtain

$$A\varphi(\langle x, x \rangle) \leq (\text{strong}) \sum_{n=1}^{\infty} \varphi(\langle x, x_n \rangle) \varphi(\langle x, x_n \rangle)^* \leq B\varphi(\langle x, x \rangle), \quad \forall x \in X.$$

Since Φ is a φ -morphism, this gives us

$$\begin{aligned} A\langle \Phi(x), \Phi(x) \rangle &\leq (\text{strong}) \sum_{n=1}^{\infty} \langle \Phi(x), \Phi(x_n) \rangle \langle \Phi(x), \Phi(x_n) \rangle^* \\ &\leq B\langle \Phi(x), \Phi(x) \rangle, \quad \forall x \in X. \end{aligned}$$

Finally, since Φ is a surjection, this tells us that $(\Phi(x_n))_n$ is a weak frame in Y . The same reasoning applied to the maps Φ^{-1} and φ^{-1} proves the converse. \square

We end the section with a brief discussion on a more general situation in which we do not assume self-duality of X . Suppose that X is an arbitrary Hilbert C^* -module over a von Neumann algebra A . As it is well known (see [25]), each Hilbert \mathbf{A} -module X can be embedded into a self-dual \mathbf{A} -module X' in such a way that the inner product on X' extends the original inner product given on X . In fact, X' is—with a slight abuse of notation—a twisted copy of the dual of X . As X' is self-dual, all references made in this section to weak Bessel sequences and weak frames apply to X' .

However, sometimes our real interest lies in a Hilbert C^* -module X that is not self-dual. Then the question arises: Are there weak frames for X in the context of a broader ambient module X' ? Certainly, weak frames for X can be obtained by working in the dual module X' . In particular, for all weak frames in X' the reconstruction formula (26) is still valid and applies, in particular, to all elements from X . The difference now is that such frames may be outer from the X -perspective in the sense that some of the frame elements x_n (or even all of them) may belong to $X' \setminus X$.

On the other hand, it is natural to ask what can be said about standard frames for X in this more general context. To provide the answer we first need a couple of auxiliary results which are obtained in (or can be easily concluded from) [25].

Our first lemma is proved in [25]. Since it is not explicitly stated there, we include it here for future reference.

Lemma 3.19 (see [25]). *Let X be a Hilbert C^* -module over a von Neumann algebra \mathbf{A} . Suppose that $f \in X'$ is orthogonal to all x in X . Then $f = 0$. (In other words, the orthogonal complement of X in X' is trivial.)*

Proof. We have $\langle x, f \rangle = 0$ for all x in X . Then by Theorem 3.2 from [25] (when f is regarded as a bounded module map from X in \mathbf{A}), we have $f(x) = 0, \forall x \in X$. By the last paragraph of the proof of Theorem 3.2 in [25] this forces $f = 0$. \square

We once again recall Propositions 3.4 and 3.6 from [25]. By Proposition 3.6 in [25], if X and Y are Hilbert C^* -modules over a von Neumann algebra \mathbf{A} , then each bounded module map $T : X \rightarrow Y$ extends uniquely to a bounded module map $T' : X' \rightarrow Y'$. (Note that this provides us with an alternative argument for the proof of Lemma 3.19.) Let us call this extended map T' the *standard extension* of T .

Another fact that we need is Proposition 3.4 from [25]: if X and Y are self-dual Hilbert C^* -modules over a von Neumann algebra \mathbf{A} , each bounded module map $T : X \rightarrow Y$ is adjointable. (In fact, for this conclusion we do not need self-duality of Y .)

Lemma 3.20. *Suppose that X and Y are Hilbert C^* -modules over a von Neumann algebra \mathbf{A} and that $T : X \rightarrow Y$ is an adjointable operator. Then the standard extensions $T' : X' \rightarrow Y'$ and $(T^*)' : Y' \rightarrow X'$ are adjoint to each other. In other words, T' is an adjointable operator and $(T')^* = (T^*)'$.*

Proof. To show that $(T')^* = (T^*)'$ it suffices to prove that these operators coincide on Y (since the standard extension is the unique extension). Take any $y \in Y$. We want to prove that $(T')^*y = (T^*)'y$. As both sides of this equality belong to X' , we can apply Lemma 3.19. Hence, it is enough to show that $\langle (T')^*y - (T^*)'y, x \rangle = 0$ for all x in X . Fix $x \in X$. Then

$$\langle (T')^*y - (T^*)'y, x \rangle = \langle y, T'x \rangle - \langle T^*y, x \rangle = \langle y, Tx \rangle - \langle y, Tx \rangle = 0. \quad \square$$

Lemma 3.21. *Suppose that X is a Hilbert C^* -module over a von Neumann algebra \mathbf{A} and that $T : X \rightarrow X$ is an invertible bounded module map. Then its standard extension $T' : X' \rightarrow X'$ is also invertible.*

Proof. Observe that T^{-1} is also a bounded module map. Consider the standard extensions T' and $(T^{-1})'$. For each x in X we now have $(T^{-1})'T'x = (T^{-1})'Tx = T^{-1}Tx = x$. Thus, $(T^{-1})'T'$ extends the identity operator I_X on X . Since the standard extension is unique, and since the identity operator $I_{X'}$ on X' obviously extends I_X , this forces $(T^{-1})'T' = I_{X'}$.

In precisely the same way we conclude that $T'(T^{-1})' = I_{X'}$. Thus, T' is invertible. \square

Proposition 3.22. *Let $(x_n)_n$ be a standard frame in a Hilbert C^* -module X over a von Neumann algebra \mathbf{A} . Then $(x_n)_n$ is a weak frame for X' .*

Proof. Denote by $U : X \rightarrow \ell^2(\mathbf{A})$ the analysis operator of $(x_n)_n$. We know from [13] that U is adjointable, that we have $U^*e^{(n)} = x_n$ for all n , and that $U^*U : X \rightarrow X$ is an invertible operator. Recall that $\ell^2(\mathbf{A})' = \ell^2_{\text{strong}}(\mathbf{A})$. Consider the standard extensions $U' : X' \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$ and $(U^*)' : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow X'$.

By Lemma 3.20, we have $(U^*)' = (U')^*$. So, $(U')^*U' = (U^*)'U'$ is the standard extension of U^*U . Since U^*U is an invertible operator, we conclude from Lemma 3.21 that $(U')^*U'$ is invertible. In particular, $(U')^*$ must be surjective. So, $(U')^*$ is an adjointable surjection. Recall from Remark 3.7 that $(e^{(n)})_n$ is a weak frame for $\ell^2_{\text{strong}}(\mathbf{A})$. By applying the second assertion of Corollary 3.12, we conclude that $((U')^*e^{(n)})_n = ((U^*)'e^{(n)})_n = (U^*e^{(n)})_n = (x_n)_n$ is a weak frame for X' . \square

Remark 3.23. Note that $(x_n)_n$ will be a weak Parseval frame in X' if $(x_n)_n$ is a standard Parseval frame in X . The proof also shows: if $(x_n)_n$ is a standard Bessel sequence in X , then $(x_n)_n$ is a weak Bessel sequence in X' .

Proposition 3.24. *Let X be a Hilbert C^* -module over a von Neumann algebra \mathbf{A} . Let $(x_n)_n$ be a weak frame (resp., Bessel sequence) in X' such that $x_n \in X$ for all $n \in \mathbb{N}$. Let $Y = \{x \in X : (\langle x, x_n \rangle)_n \in \ell^2(\mathbf{A})\}$. If Y is dense in X , then $(x_n)_n$ is a standard frame (resp., Bessel sequence) in X .*

Proof. Suppose that $(x_n)_n$ is a weak Bessel sequence, and denote by $U : X' \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$ its analysis operator. We know from Remark 3.7 that

$$U^*((a_n)_n) = (\text{norm}) \sum_{n=1}^{\infty} a_n x_n, \quad \forall (a_n)_n \in \ell^2(\mathbf{A}).$$

Note that

$$Y = \{x \in X' : Ux \in \ell^2(\mathbf{A})\} \cap X.$$

Clearly, Y is a submodule of X . We also conclude that Y is closed because U is adjointable and hence a bounded operator. By the assumption, Y is dense in X and therefore $Y = X$. Thus, we have $Ux \in \ell^2(\mathbf{A})$ for all x in X and, since each x_n is in X , this is enough to conclude that $(x_n)_n$ is a standard Bessel sequence in X (see Theorem 2.1 in [2]).

If, in addition, we assume that $(x_n)_n$ is a weak frame (not merely a weak Bessel sequence), then it follows that $(x_n)_n$ is a standard frame in X . In fact, its analysis operator is $U_0 = U|_X$ and it is bounded from below since U is bounded from below. \square

In light of Propositions 3.22 and 3.24 and Remark 3.23, one may ask whether there are weak frames (resp., weak Bessel sequences) for X' that are not standard Bessel sequences or frames for X . Indeed, there are.

Example 3.25. Consider an infinite-dimensional separable Hilbert space H decomposed as $H = \bigoplus_{n=1}^{\infty} H_n$, where $\dim H_n = \infty$ for all n . Observe that the elements of H can be identified as sequences $(\xi_n)_n$, $\xi_n \in H_n$, satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

Let $X = \mathbb{K}(H)$ be the Hilbert $\mathbb{B}(H)$ -module consisting of all compact operators acting on H . It is known that X is not self-dual; by Theorem 1.5 from [20] we know that $X' = \mathbb{B}(H)$, where $\mathbb{B}(H)$ is regarded as a Hilbert C^* -module over itself.

Denote by q_n , $n \in \mathbb{N}$, the orthogonal projections to H_n 's. As in Example 3.3, one easily shows that the series $\sum_{n=1}^{\infty} q_n$ converges in the strong operator topology to the identity operator.

From this we conclude for each $x \in \mathbb{B}(H)$ that $x = (\text{strong}) \sum_{n=1}^{\infty} xq_n = (\text{strong}) \sum_{n=1}^{\infty} \langle x, q_n \rangle q_n$. In particular, this can be rewritten as in (29): $x = (\text{weak-strong}) \sum_{n=1}^{\infty} \langle x, q_n \rangle q_n$ which means that the sequence $(q_n)_n$ is a weak Parseval frame for $\mathbb{B}(H)$. On the other hand, since the range of each q_n is infinite-dimensional, $q_n \notin \mathbb{K}(H)$ for all n in \mathbb{N} . Therefore, our weak frame $(q_n)_n$ for the dual X' does not arise as a standard frame for the Hilbert C^* -module X we started from.

4. Bounded operators on $\ell^2(\mathbf{A})$ and infinite matrices

In this section we discuss conditions on a sequence $(x_n)_n$ sufficient to ensure the weak Bessel property. Recall from Lemma 3.5.1 in [9] that a sequence $(x_n)_n$ is Bessel in a Hilbert space H if and only if the Gram matrix associated with $(x_n)_n$ defines a bounded operator on ℓ^2 . In the setting of Hilbert C^* -modules there is no such result for standard frames since it is not enough to have a bounded operator on $\ell^2(\mathbf{A})$; one has additionally to know that this operator is adjointable and this does not follow automatically from boundedness. Here we prove a similar result for weak Bessel sequences in self-dual Hilbert C^* -modules over von Neumann algebras. For some technical reasons which will become clear from the context, we additionally assume that the underlying von Neumann algebra is commutative.

Suppose that we have a weak Bessel sequence $(x_n)_n$ with a Bessel bound B in a self-dual Hilbert C^* -module X over a von Neumann algebra \mathbf{A} . Consider the canonical weak basis $(e^{(n)})_n$ for $\ell^2_{\text{strong}}(\mathbf{A})$ from Remark 3.8 (which is also a standard basis for $\ell^2(\mathbf{A})$). We know that the corresponding analysis operator $U : X \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$, $Ux = (\langle x, x_n \rangle)_n$, is an adjointable operator. Hence, $UU^* \in \mathbb{B}(\ell^2_{\text{strong}}(\mathbf{A}))$.

Recall from Remark 3.7 that the n th component $UU^*((a_k)_k)_n$ of the sequence $UU^*((a_k)_k)$ is

$$UU^*((a_k)_k)_n = (\text{norm}) \sum_{k=1}^{\infty} a_k \langle x_k, x_n \rangle \in \mathbf{A}, \quad \forall (a_k)_k \in \ell^2(\mathbf{A}), \quad (30)$$

where norm-convergence is ensured by the fact that the sequence $(\langle x_n, x_k \rangle)_k$ belongs to $\ell^2_{\text{strong}}(\mathbf{A})$ and hence defines an element from $\ell^2(\mathbf{A})'$, so Lemma 2.6

applies. Moreover, since $UU^*((a_k)_k) \in \ell^2_{\text{strong}}(\mathbf{A})$, we have from (30)

$$\sup_N \left\| \sum_{n=1}^N \left(\sum_{k=1}^{\infty} a_k \langle x_k, x_n \rangle \right) \left(\sum_{k=1}^{\infty} a_k \langle x_k, x_n \rangle \right)^* \right\| \leq B^2 \|(a_k)_k\|^2. \quad (31)$$

Note that (30) implies that

$$\langle UU^*e^{(k)}, e^{(n)} \rangle = (UU^*e^{(k)})_n = \langle x_k, x_n \rangle, \quad \forall n, k. \quad (32)$$

Let us now introduce an infinite matrix Γ with entries Γ_{nk} in \mathbf{A} defined by

$$\Gamma_{nk} = \langle x_k, x_n \rangle, \quad n, k \in \mathbb{N}.$$

The matrix Γ is called the Gramian associated with the sequence $(x_n)_n$. We see from (32) that Γ is in fact the matrix representation of UU^* with respect to the canonical basis $(e^{(n)})_n$.

We now suppose that the underlying von Neumann algebra \mathbf{A} is commutative. Then (30) can be rewritten as

$$\begin{aligned} UU^*((a_k)_k)_n &= (\text{norm}) \sum_{k=1}^{\infty} \langle x_k, x_n \rangle a_k \\ &= (\text{norm}) \sum_{k=1}^{\infty} \Gamma_{nk} a_k, \quad \forall (a_k)_k \in \ell^2(\mathbf{A}), \end{aligned} \quad (33)$$

which shows us that UU^* acts on elements $(a_k)_k \in \ell^2(\mathbf{A})$ simply as matrix multiplication:

$$\begin{aligned} UU^*(a_k)_k &= \Gamma \cdot (a_k)_k = \begin{bmatrix} \langle x_1, x_1 \rangle & \langle x_2, x_1 \rangle & \langle x_3, x_1 \rangle & \dots \\ \langle x_1, x_2 \rangle & \langle x_2, x_2 \rangle & \langle x_3, x_2 \rangle & \dots \\ \langle x_1, x_3 \rangle & \langle x_2, x_3 \rangle & \langle x_3, x_3 \rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \\ &\in \ell^2_{\text{strong}}(\mathbf{A}). \end{aligned} \quad (34)$$

Now we can reverse the process and ask what can be said about the sequence $(x_n)_n$ if its Gramian defines by (34) a bounded operator on $\ell^2(\mathbf{A})$ with values in $\ell^2_{\text{strong}}(\mathbf{A})$.

Proposition 4.1. *Let $(x_n)_n$ be a sequence in a self-dual Hilbert C^* -module over a commutative von Neumann algebra \mathbf{A} . Suppose that its Gramian Γ defines by $(a_k)_k \mapsto \Gamma \cdot (a_k)_k$ as in (34) a bounded operator $\ell^2(\mathbf{A}) \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$ with bound B . Then $(x_n)_n$ is a weak Bessel sequence in X with a Bessel bound B .*

Proof. By the assumption, we have $\|\Gamma \cdot (a_k)_k\| \leq B\|(a_k)_k\|$ for all $(a_k)_k \in \ell^2(\mathbf{A})$, that is,

$$\begin{aligned} \sup_N \left\| \sum_{n=1}^N \left(\sum_{k=1}^{\infty} \langle x_k, x_n \rangle a_k \right) \left(\sum_{k=1}^{\infty} \langle x_k, x_n \rangle a_k \right)^* \right\| \\ \leq B^2 \|(a_k)_k\|^2, \quad \forall (a_k)_k \in \ell^2(\mathbf{A}). \end{aligned} \quad (35)$$

We follow the proof of Lemma 3.5.1 in [9]. Take any $(a_k)_k \in \ell^2(\mathbf{A})$ and natural numbers N, M such that $N < M$. Then we have

$$\begin{aligned}
& \left\| \sum_{n=1}^M a_n x_n - \sum_{n=1}^N a_n x_n \right\|^4 \\
&= \left\| \sum_{n=N+1}^M a_n x_n \right\|^4 \\
&= \left\| \left\langle \sum_{n=N+1}^M a_n x_n, \sum_{m=N+1}^M a_m x_m \right\rangle \right\|^2 \\
&= \left\| \sum_{m=N+1}^M \left\langle \sum_{n=N+1}^M a_n x_n, x_m \right\rangle a_m^* \right\|^2 \\
&\leq (\text{applying the Cauchy–Schwarz inequality in } \mathbf{A}^{M-N}) \\
&\leq \left\| \sum_{m=N+1}^M \left(\sum_{n=N+1}^M a_n \langle x_n, x_m \rangle \right) \left(\sum_{n=N+1}^M a_n \langle x_n, x_m \rangle \right)^* \right\| \cdot \left\| \sum_{m=N+1}^M a_m a_m^* \right\| \\
&\leq (\text{passing in the first term to the sequence } (0, \dots, 0, a_{N+1}, \dots, a_M, 0, \dots)) \\
&= \left\| \sum_{m=N+1}^M \left(\sum_{n=1}^{\infty} a_n \langle x_n, x_m \rangle \right) \left(\sum_{n=1}^{\infty} a_n \langle x_n, x_m \rangle \right)^* \right\| \cdot \left\| \sum_{m=N+1}^M a_m a_m^* \right\| \\
&\stackrel{(35)}{\leq} B^2 \left\| \sum_{m=N+1}^M a_m a_m^* \right\| \cdot \left\| \sum_{m=N+1}^M a_m a_m^* \right\|.
\end{aligned}$$

So, we have obtained

$$\left\| \sum_{n=1}^M a_n x_n - \sum_{n=1}^N a_n x_n \right\|^2 \leq B \left\| \sum_{n=N+1}^M a_n a_n^* \right\|.$$

This shows us that $\sum_{n=1}^{\infty} a_n x_n$ converges in norm in X for each sequence $(a_k)_k$ from $\ell^2(\mathbf{A})$. In other words, we have a well-defined map $T : \ell^2(\mathbf{A}) \rightarrow X$ by the formula $T((a_k)_k) = \sum_{n=1}^{\infty} a_n x_n$. Clearly, T is a module map.

Repeating the above computation, we conclude that T is bounded by B . By Proposition 3.6 from [25], T extends to a bounded module map $T : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow X$. Since we obviously have $T e^{(n)} = x_n$ for every $n \in \mathbb{N}$, Proposition 3.9 above implies that $(x_n)_n$ is a weak Bessel sequence. \square

In light of the preceding proposition, it is of interest to find practical sufficient conditions on an infinite matrix $M = (m_{ij})$ with entries in \mathbf{A} which will ensure that the map defined on $\ell^2(\mathbf{A})$ by

$$L_M((a_k)_k) = M \cdot (a_k)_k = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots \\ m_{21} & m_{22} & m_{23} & \dots \\ m_{31} & m_{32} & m_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \quad (36)$$

defines a bounded module map with values in $\ell_{\text{strong}}^2(\mathbf{A})$ (or, possibly, even in $\ell^2(\mathbf{A})$). Note that here again we need commutativity of the underlying algebra to ensure that the resulting map is a module map.

For this purpose, we provide a generalization of the Schur test for infinite matrices. Suppose that $M = (m_{ij})$ is an infinite matrix with entries in a commutative von Neumann algebra \mathbf{A} . Let

$$p_{jk} = (m_{jk}^* m_{jk})^{\frac{1}{2}} \geq 0, \quad \forall j, k \in \mathbb{N}.$$

Observe that, since \mathbf{A} is a von Neumann algebra, p_{jk} and $p_{jk}^{\frac{1}{2}}$ belong to \mathbf{A} for all j and k . Consider the following condition: there exist positive real constants B_c and B_r such that

$$\sum_{j=1}^{\infty} \|m_{jk}\| \leq B_c, \quad \forall k \in \mathbb{N} \quad \text{and} \quad \sum_{k=1}^{\infty} \|m_{jk}\| \leq B_r, \quad \forall j \in \mathbb{N}. \quad (37)$$

There are two more conditions related to the preceding one:

$$\begin{aligned} \left\| (\text{norm}) \sum_{j=1}^{\infty} p_{jk} \right\| &\leq B_c, \quad \forall k \in \mathbb{N} \quad \text{and} \\ \left\| (\text{norm}) \sum_{k=1}^{\infty} p_{jk} \right\| &\leq B_r, \quad \forall j \in \mathbb{N}, \end{aligned} \quad (38)$$

and

$$\begin{aligned} \left\| (\text{strong}) \sum_{j=1}^{\infty} p_{jk} \right\| &\leq B_c, \quad \forall k \in \mathbb{N} \quad \text{and} \\ \left\| (\text{strong}) \sum_{k=1}^{\infty} p_{jk} \right\| &\leq B_r, \quad \forall j \in \mathbb{N}. \end{aligned} \quad (39)$$

Note that we implicitly assume that the series in (38) and (39) converge in indicated topologies in \mathbf{A} . Observe now that

$$\|p_{jk}\|^2 = \|p_{jk}^2\| = \|m_{jk}^* m_{jk}\| = \|m_{jk}\|^2, \quad \forall j, k \in \mathbb{N}.$$

Thus, we require in (38) that the series $\sum_{j=1}^{\infty} p_{jk}$ and $\sum_{k=1}^{\infty} p_{jk}$ converge in norm in \mathbf{A} , while (37) means that the same series converge absolutely in \mathbf{A} . Hence, (37) \Rightarrow (38) \Rightarrow (39).

In the proofs of the following two propositions we repeatedly use the following facts from general theory of C^* -algebras. First, if $(a_k)_k$ and $(b_k)_k$ are sequences in a C^* -algebra \mathbf{A} such that $0 \leq a_k \leq b_k$ for all k 's and such that $\sum_{k=1}^{\infty} b_k$ converges

in \mathbf{A} , then $\sum_{k=1}^\infty a_k$ also converges in \mathbf{A} . (Reason: $\sum_{k=K_1}^{K_2} a_k \leq \sum_{k=K_1}^{K_2} b_k$ and hence $\|\sum_{k=K_1}^{K_2} a_k\| \leq \|\sum_{k=K_1}^{K_2} b_k\|$.)

Second, if $a, b \in \mathbf{A}$ are such that $0 \leq a \leq b$, then we have $c^*ac \leq c^*bc$ for all $c \in \mathbf{A}$. Finally, for all $a, c \in \mathbf{A}$ we have $a^*c^*ca \leq \|c^*c\|a^*a$.

Proposition 4.2. *Let $M = (m_{ij})$ be an infinite matrix with entries in a commutative von Neumann algebra \mathbf{A} that satisfies (38) (with $p_{jk} = (m_{jk}^*m_{jk})^{\frac{1}{2}}$ for all j, k). Then the operator L_M defined by (36), that is,*

$$(L_M(a_k)_k)_j = \sum_{k=1}^\infty m_{jk}a_k, \quad j \in \mathbb{N}, (a_k)_k \in \ell^2(\mathbf{A}), \tag{40}$$

defines a bounded module map $L_M : \ell^2(\mathbf{A}) \rightarrow \ell^2(\mathbf{A})$ that extends to an adjointable map $L_M : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$ such that $\|L_M\| \leq (B_r B_c)^{\frac{1}{2}}$.

Proof. Take any $(a_k)_k \in \ell^2(\mathbf{A})$, and write $m_{jk} = u_{jk}p_{jk}$, $a_k = v_kq_k$, $j, k \in \mathbb{N}$, where all u_{jk} and v_k are partial isometries and $q_k = (a_k a_k^*)^{\frac{1}{2}} \geq 0$. Observe that all these operators belong to \mathbf{A} since \mathbf{A} is a von Neumann algebra. In addition, let $z_{jk} = u_{jk}v_k$ for all $j, k \in \mathbb{N}$. Since \mathbf{A} is commutative, we can write

$$m_{jk}a_k = u_{jk}p_{jk}v_kq_k = (z_{jk}p_{jk}^{\frac{1}{2}})(p_{jk}^{\frac{1}{2}}q_k), \quad \forall j, k \in \mathbb{N}. \tag{41}$$

We first claim that

$$x = (p_{jk}^{\frac{1}{2}}z_{jk}^*)_k \in \ell^2(\mathbf{A}). \tag{42}$$

To see this, first observe that for each k we have

$$(p_{jk}^{\frac{1}{2}}z_{jk}^*)(p_{jk}^{\frac{1}{2}}z_{jk}^*)^* = z_{jk}^*p_{jk}^{\frac{1}{2}}p_{jk}^{\frac{1}{2}}z_{jk} = p_{jk}^{\frac{1}{2}}z_{jk}^*z_{jk}p_{jk}^{\frac{1}{2}} \leq \|z_{jk}^*z_{jk}\|p_{jk}^{\frac{1}{2}}p_{jk}^{\frac{1}{2}} \leq p_{jk},$$

which implies that the series

$$\sum_{k=1}^\infty (p_{jk}^{\frac{1}{2}}z_{jk}^*)(p_{jk}^{\frac{1}{2}}z_{jk}^*)^*$$

converges since by the assumption the series $\sum_{k=1}^\infty p_{jk}$ converges in norm in \mathbf{A} and we have

$$\sum_{k=1}^\infty (p_{jk}^{\frac{1}{2}}z_{jk}^*)(p_{jk}^{\frac{1}{2}}z_{jk}^*)^* \leq \sum_{k=1}^\infty p_{jk}. \tag{43}$$

Next we claim that

$$y = (p_{jk}^{\frac{1}{2}}q_k)_k \in \ell^2(\mathbf{A}). \tag{44}$$

Indeed, we have

$$\begin{aligned} (p_{jk}^{\frac{1}{2}}q_k)(p_{jk}^{\frac{1}{2}}q_k)^* &= q_k p_{jk}^{\frac{1}{2}} p_{jk}^{\frac{1}{2}} q_k = p_{jk}^{\frac{1}{2}} a_k a_k^* p_{jk}^{\frac{1}{2}} \leq p_{jk}^{\frac{1}{2}} \langle (a_k)_k, (a_k)_k \rangle p_{jk}^{\frac{1}{2}} \\ &\leq \| \langle (a_k)_k, (a_k)_k \rangle \| p_{jk}, \end{aligned}$$

which shows us that the series

$$\sum_{k=1}^\infty (p_{jk}^{\frac{1}{2}}q_k)(p_{jk}^{\frac{1}{2}}q_k)^*$$

converges since $\sum_{k=1}^{\infty} p_{jk}$ is norm-convergent and

$$\sum_{k=1}^{\infty} (p_{jk}^{\frac{1}{2}} q_k) (p_{jk}^{\frac{1}{2}} q_k)^* \leq \sum_{k=1}^{\infty} p_{jk}. \quad (45)$$

We now apply the Cauchy–Schwarz inequality $\langle x, y \rangle^* \langle x, y \rangle \leq \|\langle x, x \rangle\| \cdot \langle y, y \rangle$ in the Hilbert C^* -module $\ell^2(\mathbf{A})$ to the sequences x from (42) and y from (44) to obtain

$$\begin{aligned} \left(\sum_{k=1}^{\infty} m_{jk} a_k \right) \left(\sum_{k=1}^{\infty} m_{jk} a_k \right)^* &\stackrel{(41)}{=} \left(\sum_{k=1}^{\infty} z_{jk} p_{jk}^{\frac{1}{2}} p_{jk}^{\frac{1}{2}} q_k \right) \left(\sum_{k=1}^{\infty} z_{jk} p_{jk}^{\frac{1}{2}} p_{jk}^{\frac{1}{2}} q_k \right)^* \\ &= \langle x, y \rangle^* \langle x, y \rangle \\ &\leq \|\langle x, x \rangle\| \cdot \langle y, y \rangle \\ &\stackrel{(43)}{\leq} \left\| \sum_{k=1}^{\infty} p_{jk} \right\| \left\| \left(\sum_{k=1}^{\infty} p_{jk}^{\frac{1}{2}} q_k^2 p_{jk}^{\frac{1}{2}} \right) \right\| \\ &\stackrel{(38)}{\leq} B_r \sum_{k=1}^{\infty} p_{jk}^{\frac{1}{2}} q_k^2 p_{jk}^{\frac{1}{2}}. \end{aligned} \quad (46)$$

Finally, we now have

$$\begin{aligned} \sum_{j=1}^{\infty} (L_M((a_k)_k))_j (L_M((a_k)_k))_j^* &\stackrel{(40)}{=} \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} m_{jk} a_k \right) \left(\sum_{k=1}^{\infty} m_{jk} a_k \right)^* \\ &\stackrel{(46)}{\leq} \sum_{j=1}^{\infty} B_r \sum_{k=1}^{\infty} p_{jk}^{\frac{1}{2}} q_k^2 p_{jk}^{\frac{1}{2}} \\ &= \sum_{k=1}^{\infty} B_r \left(\sum_{j=1}^{\infty} p_{jk} \right) q_k^2 \\ &\leq B_r \left\| \sum_{j=1}^{\infty} p_{jk} \right\| \sum_{k=1}^{\infty} q_k^2 \\ &\stackrel{(38)}{\leq} B_r B_c \sum_{k=1}^{\infty} a_k a_k^* \\ &= B_r B_c \langle (a_k)_k, (a_k)_k \rangle. \end{aligned}$$

Obviously, this implies that the series $\sum_{j=1}^{\infty} (L_M((a_k)_k))_j (L_M((a_k)_k))_j^*$ converges, which means that $L_M((a_k)_k)$ is well defined and also $\|L_M((a_k)_k)\|^2 \leq B_r B_c \|(a_k)_k\|^2$. By Proposition 3.6 from [25], L_M extends to a bounded module map $L_M : \ell_{\text{strong}}^2(\mathbf{A}) \rightarrow \ell_{\text{strong}}^2(\mathbf{A})$. Since $\ell_{\text{strong}}^2(\mathbf{A})$ is self-dual, L_M is by Proposition 3.4 from [25] adjointable. \square

Remark 4.3. Proposition 4.2 guarantees that the extended operator $L_M : \ell_{\text{strong}}^2(\mathbf{A}) \rightarrow \ell_{\text{strong}}^2(\mathbf{A})$ is adjointable. However, one can ask whether the originally defined operator $L_M : \ell^2(\mathbf{A}) \rightarrow \ell^2(\mathbf{A})$ is adjointable.

This will be the case if our matrix M satisfies (37). We have already noted that (37) \Rightarrow (38); thus, Proposition 4.2 applies to infinite matrices $M = (m_{ij})$ that satisfy (37). Moreover, it is easy to check that in this situation the matrix $M^* = (m_{ji}^*)$ also satisfies (37) (with the roles of the constants B_r and B_c interchanged) and hence defines a bounded operator on $\ell^2(\mathbf{A})$. By an easy verification one proves that this defines the adjoint operator to the operator induced by the original matrix M .

It is a little bit subtler with those matrices M that satisfy (38). We now must additionally assume that $M = M^*$ in order to conclude that condition (38) is satisfied also for M^* . Then, we again have adjointability of the operator under consideration.

Conveniently enough, we will most often use Proposition 4.2 applied to the Gramian Γ associated to some sequence $(x_n)_n$ and then, since for all n, k we have $\langle x_n, x_k \rangle^* = \langle x_k, x_n \rangle^*$, our matrix Γ has the property $\Gamma = \Gamma^*$.

Proposition 4.4. *Let $M = (m_{ij})$ be an infinite matrix with entries from a commutative von Neumann algebra \mathbf{A} that satisfies (39) (with $p_{jk} = (m_{jk}^* m_{jk})^{\frac{1}{2}}$ for all j, k). Then the operator L_M defined by (36), that is,*

$$(L_M(a_k)_k)_j = \sum_{k=1}^{\infty} m_{jk} a_k, \quad j \in \mathbb{N}, (a_k)_k \in \ell^2(\mathbf{A}), \tag{47}$$

defines a bounded module map $L_M : \ell^2(\mathbf{A}) \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$ that extends to an adjointable map $L_M : \ell^2_{\text{strong}}(\mathbf{A}) \rightarrow \ell^2_{\text{strong}}(\mathbf{A})$ such that $\|L_M\| \leq (B_r B_c)^{\frac{1}{2}}$.

Proof. We must show, for each $x = (a_k)_k \in \ell^2(\mathbf{A})$, that

$$L_x((d_j)_j) = \left\langle (d_j)_j, \left(\sum_{k=1}^{\infty} m_{jk} a_k \right)_j \right\rangle, \quad (d_j)_j \in \ell^2(\mathbf{A}), \tag{48}$$

defines a bounded module map on $\ell^2(\mathbf{A})$. Let us first take an arbitrary finite sequence $x = (a_1, \dots, a_K, 0, 0, \dots)$. Consider also finite $d = (d_1, \dots, d_N, 0, 0, \dots)$. Then we have

$$\|L_x(d)\|^2 = \left\| \sum_{j=1}^N d_j \left(\sum_{k=1}^K m_{jk} a_k \right)^* \right\|^2 = \left\| \sum_{j=1}^N \sum_{k=1}^K d_j a_k^* m_{jk}^* \right\|^2.$$

Writing again $m_{jk} = u_{jk} p_{jk}$, we get

$$\|L_x(d)\|^2 = \left\| \sum_{j=1}^N \sum_{k=1}^K d_j a_k^* p_{jk} u_{jk}^* \right\|^2 = \left\| \sum_{j=1}^N \sum_{k=1}^K (p_{jk}^{\frac{1}{2}} d_j) (a_k^* p_{jk}^{\frac{1}{2}} u_{jk}^*) \right\|^2.$$

We now regard this last double sum as the inner product in the Hilbert C^* -module $(\mathbf{A}^K)^N = (\mathbf{A} \times \mathbf{A} \times \dots \times \mathbf{A})^N$. (In order to do so, we understand that $a_{jk} = a_k$ for all j and also $d_{jk} = d_j$ for all k .) Then we use the Cauchy–Schwarz inequality in $(\mathbf{A} \times \mathbf{A} \times \dots \times \mathbf{A})^N$. In this way, continuing the above computation we obtain

$$\begin{aligned}
\|L_x(d)\|^2 &= \left\| \sum_{j=1}^N \sum_{k=1}^K (p_{jk}^{\frac{1}{2}} d_j) (a_k^* p_{jk}^{\frac{1}{2}} u_{jk}^*) \right\|^2 \\
&\stackrel{\text{C-S}}{\leq} \left\| \sum_{j=1}^N \sum_{k=1}^K p_{jk}^{\frac{1}{2}} d_j d_j^* p_{jk}^{\frac{1}{2}} \right\| \cdot \left\| \sum_{j=1}^N \sum_{k=1}^K u_{jk} p_{jk}^{\frac{1}{2}} a_k a_k^* p_{jk}^{\frac{1}{2}} u_{jk}^* \right\| \\
&\leq \left\| \sum_{j=1}^N \sum_{k=1}^K p_{jk}^{\frac{1}{2}} d_j^* d_j p_{jk}^{\frac{1}{2}} \right\| \cdot \left\| \sum_{j=1}^N \sum_{k=1}^K p_{jk}^{\frac{1}{2}} a_k^* a_k p_{jk}^{\frac{1}{2}} \right\| \\
&= \left\| \sum_{j=1}^N d_j^* \left(\sum_{k=1}^K p_{jk} \right) d_j \right\| \cdot \left\| \sum_{k=1}^K a_k^* \left(\sum_{j=1}^N p_{jk} \right) a_k \right\| \\
&\leq \left\| \sum_{j=1}^N \left\| \sum_{k=1}^K p_{jk} \right\| d_j^* d_j \right\| \cdot \left\| \sum_{k=1}^K \left\| \sum_{j=1}^N p_{jk} \right\| a_k^* a_k \right\| \\
&\stackrel{(39)}{\leq} \left\| \sum_{j=1}^N B_r d_j^* d_j \right\| \cdot \left\| \sum_{k=1}^K B_c a_k^* a_k \right\| \\
&= B_c B_r \|x\|^2 \|d\|^2.
\end{aligned}$$

This proves that L_x is bounded on finite sequences; thus, we can extend it to a bounded module map L_x defined on $\ell^2(\mathbf{A})$ such that $\|L_x\| \leq (B_c B_r)^{\frac{1}{2}} \|x\|$. Keeping in mind that $\ell_{\text{strong}}^2(\mathbf{A})$ is the dual of $\ell^2(\mathbf{A})$, we conclude that there exists a unique $x' = (a'_k)_k \in \ell_{\text{strong}}^2(\mathbf{A})$ such that

$$L_x((d_j)_j) = \langle (d_j)_j, x' \rangle, \quad \forall (d_j)_j \in \ell^2(\mathbf{A}). \quad (49)$$

Comparing this to (48), we conclude that $(\sum_{k=1}^{\infty} m_{jk} a_k)_j = a'_j$ for every $j \in \mathbb{N}$ and therefore $(\sum_{k=1}^{\infty} m_{jk} a_k)_j \in \ell_{\text{strong}}^2(\mathbf{A})$. Moreover, we know that $\|(\sum_{k=1}^{\infty} m_{jk} a_k)_j\| = \|(a'_j)_j\| = \|L_x\| \leq (B_c B_r)^{\frac{1}{2}} \|x\|$. This proves that (47) defines a bounded operator L_M on the set of finite sequences which extends to $L_M : \ell^2(\mathbf{A}) \rightarrow \ell_{\text{strong}}^2(\mathbf{A})$ such that $\|L_M\| \leq (B_c B_r)^{\frac{1}{2}}$.

The proof is finished again by a combined application of Propositions 3.4 and 3.6 from [25]. \square

5. Gabor systems as modular sequences of translates

In this section we describe two families of Hilbert C^* -modules that are naturally connected with Gabor analysis. Our goal in this section is to establish a bijective correspondence between Gabor frames (resp., Bessel sequences) and modular weak frames (resp., Bessel sequences) of translates. As in the preceding section, an important role will be played by the standard Hilbert \mathbf{A} -module $\ell^2(\mathbf{A})$ and its dual $\ell_{\text{strong}}^2(\mathbf{A})$, where \mathbf{A} is a von Neumann algebra. All of what is said holds, mutatis mutandis, for ℓ^2 -modules indexed by the set of all integers. To indicate indexation over \mathbb{Z} we will write $\ell_{\mathbb{Z}}^2(\mathbf{A})$ and $\ell_{\mathbb{Z}, \text{strong}}^2(\mathbf{A})$. Since all relevant sums converge unconditionally, we will operate with symmetric partial sums of the form $\sum_{n=-N}^N a_n b_n^*$, (strong) $\sum_{n=-N}^N a_n b_n^*$, and so on. Also, here and in the

rest of the paper, in each norm or inner product under consideration the ambient space will be indicated in the subscript, for example, $\|\cdot\|_{L^2(\mathbb{R})}$, $\langle \cdot, \cdot \rangle_{\ell^2_{\mathbb{Z}, \text{strong}}(\mathbb{A})}$, and so forth.

Let $a > 0$. Recall that $L^\infty[0, a]$ is a von Neumann algebra with pointwise operations, complex conjugation playing the role of the involution and the norm $\|f\|_{L^\infty[0, a]} = \text{ess sup}_{x \in [0, a]} |f(x)|$. Observe that each function $f \in L^\infty[0, a]$ naturally extends to a function $f^a \in L^\infty(\mathbb{R})$, where $f^a : \mathbb{R} \rightarrow \mathbb{C}$ denotes the function which is defined by $f^a(x + ka) = f(x)$ for all $x \in [0, a]$ and $k \in \mathbb{Z}$. We also note that $L^\infty[0, a]$ is naturally represented on the Hilbert space $L^2[0, a]$ via $R : L^\infty[0, a] \rightarrow \mathbb{B}(L^2[0, a])$, $R(f) = R_f$, $R_f(g) = fg$, $g \in L^2[0, a]$. It is convenient to note the following observation for future reference.

Remark 5.1. Suppose that $(h_N)_N$ is a sequence in $L^\infty[0, a]$ such that $0 \leq h_1(x) \leq h_2(x) \leq \dots$ a.e. and $\sup_N \|h_N\|_{L^\infty[0, a]} < \infty$. Then, since $L^\infty[0, a]$ is a von Neumann algebra, there exists $h = (\text{strong}) \lim_{N \rightarrow \infty} h_N \in L^\infty[0, a]$. Moreover, we also have $h(x) = \lim_{N \rightarrow \infty} h_N(x)$ pointwise a.e.

To see this, first observe that, since $[0, a]$ has finite measure, strong convergence coincides with convergence in measure on bounded sets. Thus, here we conclude that h is the limit in measure of the sequence $(h_N)_N$. In general, convergence in measure does not imply convergence pointwise a.e. But here the assumption guarantees that we have $g(x) := \lim_{N \rightarrow \infty} h_N(x)$ for a.e. x and, moreover, $\|g\|_{L^\infty[0, a]} \leq \sup_N \|h_N\|_{L^\infty[0, a]} < \infty$. Hence, $g \in L^\infty[0, a]$ and, since convergence pointwise a.e. implies convergence in measure, we conclude that $h = g$.

Conversely, if a bounded sequence $(h_N)_N$ in $L^\infty[0, a]$ (i.e., $\|h_n\|_{L^\infty[0, a]} \leq C$ for every N and some constant C) converges pointwise a.e. to a function $h \in L^\infty[0, a]$, then it also converges in measure and, since it is bounded, converges strongly to h .

Notice that in general (i.e., without the assumption that the sequence under consideration is essentially monotone) we cannot conclude that strong convergence in $L^\infty[0, a]$ implies convergence pointwise a.e. In this light it is useful to note the following observation.

Suppose that we have $f, g \in \ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])$, $f = (f_n)_n$, $g = (g_n)_n$. Then by the definition of the inner product in $\ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])$, we have $\langle f, g \rangle_{\ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])} = (\text{strong}) \sum_{n \in \mathbb{Z}} f_n \overline{g_n}$. Moreover, we claim that also $\langle f, g \rangle_{\ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])}(x) = \sum_{n \in \mathbb{Z}} f_n \overline{g_n}(x)$ for a.e. $x \in [0, a]$.

This follows from the polarization formula (that holds in every inner product module)

$$\langle f, g \rangle_{\ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])} = \sum_{k=0}^3 \frac{i^k}{4} \langle f + i^k g, f + i^k g \rangle_{\ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])}$$

and the first part of this remark. We also observe that convergence of the series $\langle f, g \rangle_{\ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a])}$ in the strong operator topology (and hence also pointwise) is unconditional.

For each $a \in \mathbb{R}$, let T_a denote translation by a ; that is, the operator given by $T_a f(x) = f(x - a)$, where f is any function on \mathbb{R} . Modulation by b is the operator

M_b defined by $M_b f(x) = e^{2\pi i b x} f(x)$. Note that T_a and M_b are unitary operators on $L^2(\mathbb{R})$ for all $a, b \in \mathbb{R}$.

Let us now fix $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. We denote by $G(g, a, b)$ the Gabor system generated by g with the lattice parameters a and b , i.e. the sequence $(M_{mb} T_{na} g)_{m, n \in \mathbb{Z}}$. For general facts about Gabor systems, we refer the reader to [9], [14], and [16]; see also [15].

Consider now an arbitrary function $f \in L^2(\mathbb{R})$ and $a > 0$. For each integer n and for every x , we denote by $(f_n(x))_n$ the sequence defined by $f_n(x) = f(x - na)$. Using the standard periodization trick (see, e.g., [14, Lemma 1.4.1])

$$\int_0^a \sum_{n \in \mathbb{Z}} |f(x - na)|^2 dx = \sum_{n \in \mathbb{Z}} \int_0^a |f(x - na)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx = \|f\|_{L^2(\mathbb{R})}^2 < \infty,$$

we conclude that $\sum_{n \in \mathbb{Z}} |f(x - na)|^2 < \infty$ a.e. which means that $(f_n(x))_n \in \ell^2_{\mathbb{Z}}$ for a.e. x . Moreover, by the general ℓ^2 -theory it now follows that the series

$$\langle f, g \rangle_a(x) = \sum_{n \in \mathbb{Z}} f(x - na) \overline{g(x - na)} \tag{50}$$

converges absolutely for a.e. x and for all f, g from $L^2(\mathbb{R})$. The resulting function $\langle f, g \rangle_a$ is a -periodic and belongs to $L^1[0, a]$. Thus, we have a map $\langle \cdot, \cdot \rangle_a : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1[0, a]$ and it is known that this map possesses all the properties of a vector-valued inner product. For details we refer the reader to [8]. We also point out that this map, under the name *bracket product*, has been successfully used in the study of shift-invariant systems (see [4], [5], [28], [29]).

However, $L^1[0, a]$ is not a C^* -algebra, and if we want to end up with a Hilbert C^* -module, then we must restrict ourselves to a suitable class of functions (as it has been done in [10]).

Definition 5.2. For any $a > 0$, let

$$L_a^\infty(\ell^2) = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : f \text{ measurable and } \|f\|_{L_a^\infty(\ell^2)}^2 = \operatorname{ess\,sup}_{x \in [0, a]} \sum_{n \in \mathbb{Z}} |f(x - na)|^2 < \infty \right\}. \tag{51}$$

First, we note that $C_c(\mathbb{R}) \subseteq L_a^\infty(\ell^2) \subseteq L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Second, one easily verifies that $L_a^\infty(\ell^2)$ is a vector space with pointwise operations. (Some work is needed to check that $L_a^\infty(\ell^2)$ is closed under addition, but we omit a verification.) It is also a left $L^\infty[0, a]$ -module with the action $(h, f) \mapsto h^a f$, $h \in L^\infty[0, a]$, $f \in L_a^\infty(\ell^2)$.

Having in mind (51) it is now natural to define a map

$$\langle \cdot, \cdot \rangle_{L_a^\infty(\ell^2)} : L_a^\infty(\ell^2) \times L_a^\infty(\ell^2) \rightarrow L^\infty[0, a]$$

by

$$\langle f, g \rangle_{L_a^\infty(\ell^2)}(x) = \sum_{n \in \mathbb{Z}} f(x - na) \overline{g(x - na)} = \sum_{n \in \mathbb{Z}} T_{na} f(x) \overline{T_{na} g(x)}. \tag{52}$$

Note that for $g = f$ this reduces to

$$\langle f, f \rangle_{L_a^\infty(\ell^2)}(x) = \sum_{n \in \mathbb{Z}} |f(x - na)|^2, \quad (53)$$

which is, by (51), a function from $L^\infty[0, a]$. Using polarization, we now conclude that $\langle f, g \rangle_{L_a^\infty(\ell^2)}$ introduced in (52) does belong to $L^\infty[0, a]$, for all $f, g \in L_a^\infty(\ell^2)$.

Note that the defining formulas for $\langle \cdot, \cdot \rangle_{L_a^\infty(\ell^2)}$ and $\langle \cdot, \cdot \rangle_a$ coincide; by writing $\langle \cdot, \cdot \rangle_{L_a^\infty(\ell^2)}$ we just emphasize the fact that both factors belong to $L_a^\infty(\ell^2)$ and hence the result is a function from $L^\infty[0, a]$.

As we already observed, $\langle \cdot, \cdot \rangle_{L_a^\infty(\ell^2)}$ has all the necessary properties of an $L^\infty[0, a]$ -valued inner product. So, in order to conclude that $L_a^\infty(\ell^2)$ is a Hilbert $L^\infty[0, a]$ -module, it only remains to see that $L_a^\infty(\ell^2)$ is complete with respect to $\| \cdot \|_{L_a^\infty(\ell^2)}$ introduced in (51). This is also already known (see [10]). However, in Theorem 5.6 below we will prove more. But first, three remarks are in order.

Remark 5.3. $L_a^\infty(\ell^2)$ embeds continuously in $L^2(\mathbb{R})$:

$$\|f\|_{L^2(\mathbb{R})} \leq \sqrt{a} \|f\|_{L_a^\infty(\ell^2)}, \quad \forall f \in L_a^\infty(\ell^2). \quad (54)$$

Indeed,

$$\begin{aligned} \|f\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} \int_0^a |f(x - na)|^2 dx \\ &= \int_0^a \sum_{n \in \mathbb{Z}} |f(x - na)|^2 dx \leq a \|f\|_{L_a^\infty(\ell^2)}^2 < \infty. \end{aligned}$$

Remark 5.4. Suppose that $G(g, a, b)$ is a Bessel sequence with a Bessel bound B . Then it is well known (see Theorem 11.6 in [16] or Proposition 8.3.2 in [9]) that $\sum_{n \in \mathbb{Z}} |g(x - na)|^2 < Bb$ a.e. Thus, $g \in L_a^\infty(\ell^2)$. Moreover, by Lemma 9.2.2 in [9] $G(g, a, b)$ is a Bessel sequence with a Bessel bound B if and only if $G(g, \frac{1}{b}, \frac{1}{a})$ is a Bessel sequence with a Bessel bound Bab . Hence, we also have $g \in L_{\frac{1}{b}}^\infty(\ell^2)$. This shows that all generators g of Gabor Bessel sequences $G(g, a, b)$ are contained in $L_a^\infty(\ell^2) \cap L_{\frac{1}{b}}^\infty(\ell^2)$.

Remark 5.5. In fact, $L_a^\infty(\ell^2)$ is the largest of four spaces that naturally appear in this context. For $a > 0$, consider the following spaces of measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$:

$$W_a(L^\infty, \ell^1) = \left\{ f : \sum_{n \in \mathbb{Z}} (\text{ess sup}_{x \in [0, a]} |f(x - na)|) < \infty \right\}, \quad (55)$$

$$W_a(L^\infty, \ell^2) = \left\{ f : \sum_{n \in \mathbb{Z}} (\text{ess sup}_{x \in [0, a]} |f(x - na)|)^2 < \infty \right\}, \quad (56)$$

$$X_a(L^\infty, \ell^2) = \left\{ f : \sum_{n \in \mathbb{Z}} |T_{-na} f \chi_{[0, a]}|^2 \text{ converges in norm in } L^\infty[0, a] \right\}. \quad (57)$$

The first two are well-known Wiener amalgam spaces. We claim that $W_a(L^\infty, \ell^1) \subseteq W_a(L^\infty, \ell^2) \subseteq X_a(L^\infty, \ell^2) \subseteq L_a^\infty(\ell^2)$.

It is clear that $W_a(L^\infty, \ell^1) \subseteq W_a(L^\infty, \ell^2)$. To prove that $W_a(L^\infty, \ell^2) \subseteq X_a(L^\infty, \ell^2)$, take any $f \in W_a(L^\infty, \ell^2)$ and observe that

$$\left(\operatorname{ess\,sup}_{x \in [0,a]} |f(x - na)|\right)^2 = \operatorname{ess\,sup}_{x \in [0,a]} |f(x - na)|^2, \quad \forall n \in \mathbb{Z}. \tag{58}$$

Now we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \left\| |T_{-n}f\chi_{[0,a]}|^2 \right\|_{L^\infty[0,a]} &= \sum_{n \in \mathbb{Z}} \operatorname{ess\,sup}_{x \in [0,a]} |T_{-n}f(x)|^2 \\ &= \sum_{n \in \mathbb{Z}} \operatorname{ess\,sup}_{x \in [0,a]} |f(x + na)|^2 \stackrel{(56), (58)}{<} \infty. \end{aligned}$$

This tells us that $\sum_{n \in \mathbb{Z}} |T_{-n}f\chi_{[0,a]}|^2$ is an absolutely convergent series in $L^\infty[0, a]$. Since $L^\infty[0, a]$ is a Banach space, we conclude that $\sum_{n \in \mathbb{Z}} |T_{-n}f\chi_{[0,a]}|^2$ converges in norm in $L^\infty[0, a]$; hence, $f \in X_a(L^\infty, \ell^2)$.

Finally, for each $f \in X_a(L^\infty, \ell^2)$ the series $\sum_{n \in \mathbb{Z}} |T_{-n}f\chi_{[0,a]}|^2$ converges in norm in $L^\infty[0, a]$ to a function, call it h , from $L^\infty[0, a]$. This implies pointwise a.e. convergence to h . But when the series $\sum_{n \in \mathbb{Z}} |T_{-n}f\chi_{[0,a]}|^2$ converges to a function from $L^\infty[0, a]$ pointwise a.e., then, clearly, $f \in L_a^\infty(\ell^2)$.

The functions

$$\begin{aligned} f_1 &= \sum_{n=0}^\infty \frac{1}{n+1} \chi_{[n+1-\frac{1}{2^n}, n+1-\frac{1}{2^{n+1}}]} \in W_1(L^\infty, \ell^2) \setminus W_1(L^\infty, \ell^1), \\ f_2 &= \sum_{n=0}^\infty \frac{1}{\sqrt{n+1}} \chi_{[n+1-\frac{1}{2^n}, n+1-\frac{1}{2^{n+1}}]} \in X_1(L^\infty, \ell^2) \setminus W_1(L^\infty, \ell^2), \end{aligned}$$

and

$$f_3 = \sum_{n=0}^\infty \chi_{[n+1-\frac{1}{2^n}, n+1-\frac{1}{2^{n+1}}]} \in L_1^\infty(\ell^2) \setminus X_1(L^\infty, \ell^2)$$

show us that the above inclusions are strict.

Observe that, for each $f \in X_a(L^\infty, \ell^2)$, the sequence $(T_{-n}f\chi_{[0,a]})_n$ belongs to $\ell^2_{\mathbb{Z}}(L^\infty[0, a])$. Conversely, given a sequence $(f_n)_n$ in $\ell^2_{\mathbb{Z}}(L^\infty[0, a])$, we can define a function $f \in L^\infty(\mathbb{R})$ by $f|_{[na, (n+1)a]} = T_n f_n$, for all $n \in \mathbb{Z}$. Note that f is in fact defined by $f(x + na) = f_n(x)$, for all $x \in [0, a]$ and $n \in \mathbb{Z}$. Then, clearly, $f \in X_a(L^\infty, \ell^2)$. This shows that $X_a(L^\infty, \ell^2)$ is a copy of $\ell^2_{\mathbb{Z}}(L^\infty[0, a])$ and hence can be endowed with the structure of a left Hilbert $L^\infty[0, a]$ -module. This brings us to the following theorem which shows the nature of $X_a(L^\infty, \ell^2)$ and $L_a^\infty(\ell^2)$ as Hilbert C^* -modules.

Theorem 5.6. $L_a^\infty(\ell^2)$, $a > 0$, is a Hilbert C^* -module over $L^\infty[0, a]$ with pointwise operations, the action of $L^\infty[0, a]$ defined by $(h, f) \mapsto h^a f$, $f \in L_a^\infty(\ell^2)$, $h \in L^\infty[0, a]$, and the inner product defined by (52):

$$\langle f, g \rangle_{L_a^\infty(\ell^2)}(x) = \sum_{n \in \mathbb{Z}} f(x - na) \overline{g(x - na)},$$

that contains $X_a(L^\infty, \ell^2)$ as a closed Hilbert $L^\infty[0, a]$ -submodule. The map

$$U_a : L_a^\infty(\ell^2) \rightarrow \ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a]), \quad U_a f = (T_{-n}f\chi_{[0,a]})_n, \tag{59}$$

is a unitary operator of Hilbert $L^\infty[0, a]$ -modules whose adjoint U_a^* is given by $U_a^*((f_n)_n) = f$, where f is defined by $f|_{[na, (n+1)a]} = T_{na}f_n$ for all n in \mathbb{Z} . Moreover,

$$U_a|_{X_a(L^\infty, \ell^2)} : X_a(L^\infty, \ell^2) \rightarrow \ell_{\mathbb{Z}}^2(L^\infty[0, a])$$

is also a unitary operator of Hilbert C^* -modules.

In particular, $L_a^\infty(\ell^2)$ is self-dual. Finally, $L_a^\infty(\ell^2)$ is naturally (antilinearly) isomorphic to the dual $X_a(L^\infty, \ell^2)'$ of $X_a(L^\infty, \ell^2)$.

Proof. First observe that

$$\operatorname{ess\,sup}_{x \in [0, a]} \sum_{n \in \mathbb{Z}} |f(x - na)|^2 < \infty$$

if and only if

$$\sup_{N \in \mathbb{N}} \operatorname{ess\,sup}_{x \in [0, a]} \sum_{n=-N}^N |f(x - na)|^2 < \infty,$$

in which case these two expressions coincide. (We omit a verification.) Therefore, we can rewrite (51) as

$$L_a^\infty(\ell^2) = \left\{ f : \|f\|_{L_a^\infty(\ell^2)}^2 = \sup_{N \in \mathbb{N}} \left\| \sum_{n=-N}^N |T_{-na}f\chi_{[0, a]}|^2 \right\|_{L^\infty[0, a]} < \infty \right\}.$$

This means that for each $f \in L_a^\infty(\ell^2)$ we have $U_a f = (T_{-na}f\chi_{[0, a]})_n \in \ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])$.

For the same reasons, the map $V : \ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a]) \rightarrow L_a^\infty(\ell^2)$, $V((f_n)_n) = f$, where f is given by $f|_{[na, (n+1)a]} = T_{na}f_n$ for all n in \mathbb{Z} , is also well defined. By a routine verification (which we also omit), one shows that both U_a and V are module maps and that $V = U_a^{-1}$.

Recall that for all $(f_n)_n \in \ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])$, we have

$$\langle (f_n)_n, (f_n)_n \rangle_{\ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])} = (\text{strong}) \sum_{n \in \mathbb{Z}} |f_n|^2, \tag{60}$$

where the series converges in the strong operator topology on $L^\infty[0, a]$. Using Remark 5.1, we see that the series that defines the inner product in $\ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])$, that is, $\langle (f_n)_n, (g_n)_n \rangle_{\ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])} = (\text{strong}) \sum_{n \in \mathbb{Z}} f_n \overline{g_n}$, also converges pointwise a.e.

On the other hand, we have

$$\langle U_a^{-1}(f_n)_n, U_a^{-1}(f_n)_n \rangle_{L_a^\infty(\ell^2)}(x) = \sum_{n \in \mathbb{Z}} |f(x - na)|^2 = \sum_{n \in \mathbb{Z}} |f_n(x)|^2, \tag{61}$$

where this series converges pointwise a.e. Hence, sums in (60) and (61) coincide. Thus, we have obtained

$$\begin{aligned} \langle U_a^{-1}(f_n)_n, U_a^{-1}(f_n)_n \rangle_{L_a^\infty(\ell^2)} &= \langle (f_n)_n, (f_n)_n \rangle_{\ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])}, \\ \forall (f_n)_n &\in \ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a]). \end{aligned} \tag{62}$$

This shows that U_a^{-1} is an isometry. In particular, since $\ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a])$ is complete, we conclude that $L_a^\infty(\ell^2)$ is also complete.

Furthermore, (62) by polarization implies that

$$\begin{aligned} \langle U_a^{-1}(f_n)_n, U_a^{-1}(g_n)_n \rangle_{L_a^\infty(\ell^2)} &= \langle (f_n)_n, (g_n)_n \rangle_{\ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a])}, \\ \forall (f_n)_n, (g_n)_n &\in \ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a]). \end{aligned}$$

If we now take $U_a^{-1}(f_n)_n = f$, this gives us

$$\begin{aligned} \langle f, U_a^{-1}(g_n)_n \rangle_{L_a^\infty(\ell^2)} &= \langle U_a f, (g_n)_n \rangle_{\ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a])}, \\ \forall f \in L_a^\infty(\ell^2), \forall (g_n)_n &\in \ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a]); \end{aligned}$$

thus, $U_a^{-1} = U_a^*$.

We now know that $L_a^\infty(\ell^2)$ and $\ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a])$ are unitarily equivalent Hilbert C^* -modules. Since $\ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a])$ is self-dual, $L_a^\infty(\ell^2)$ has the same property.

At the same time, we also see that the restriction of U_a to $X_a(L^\infty, \ell^2)$ is a bijection between the submodules $X_a(L^\infty, \ell^2)$ and $\ell_{\mathbb{Z}}^2(L^\infty[0, a])$. In particular, $X_a(L^\infty, \ell^2)$ is complete and hence a Hilbert C^* -module over $L^\infty[0, a]$. Finally, using unitary equivalence of $X_a(L^\infty, \ell^2)$ and $L_a^\infty(\ell^2)$ with $\ell_{\mathbb{Z}}^2(L^\infty[0, a])$ and $\ell_{\mathbb{Z},\text{strong}}^2(L^\infty[0, a])$, respectively, we conclude that the dual of $X_a(L^\infty, \ell^2)$ is $L_a^\infty(\ell^2)$. \square

Remark 5.7. Note that the norm on $X_a(L^\infty, \ell^2)$ arising from its inner product is given by

$$\|f\|_{X_a(L^\infty, \ell^2)} = \|\langle f, f \rangle_{X_a(L^\infty, \ell^2)}\|_{L^\infty[0, a]}^{\frac{1}{2}} = \left\| \sum_{n \in \mathbb{Z}} |T_{-na} f \chi_{[0, a]}|^2 \right\|_{L^\infty[0, a]}^{\frac{1}{2}}, \quad (63)$$

which can be written as

$$\|f\|_{X_a(L^\infty, \ell^2)} = \left(\text{ess sup}_{x \in [0, a]} \sum_{n \in \mathbb{Z}} |f(x - na)|^2 \right)^{\frac{1}{2}}.$$

This, of course, agrees with (51), but the difference is that here, for functions in $X_a(L^\infty, \ell^2)$, we have norm-convergence of the series, while the series in (51) converges only pointwise a.e. Precisely the same is true for inner products.

Observe that this reflects the analogous situation in ℓ^2 -modules. The inner product on $\ell_{\mathbb{Z},\text{strong}}^2(\mathbf{A})$ is given by $\langle (f_n)_n, (g_n)_n \rangle_{\ell_{\mathbb{Z},\text{strong}}^2(\mathbf{A})} = (\text{strong}) \sum_{n \in \mathbb{Z}} f_n g_n^*$, but this series converges actually in norm in the underlying algebra \mathbf{A} for all $(f_n)_n$ and $(g_n)_n$ from $\ell_{\mathbb{Z}}^2(\mathbf{A})$.

Remark 5.8. Recall that $W_a(L^\infty, \ell^1) \subseteq W_a(L^\infty, \ell^2) \subseteq X_a(L^\infty, \ell^2) \subseteq L_a^\infty(\ell^2)$ and that

$$\|f\|_{W_a(L^\infty, \ell^2)} = \left(\sum_{k \in \mathbb{Z}} \text{ess sup}_{x \in [0, a]} |f(x - ka)|^2 \right)^{\frac{1}{2}} \quad (64)$$

is a norm on $W_a(L^\infty, \ell^2)$. It is easy to see that $\|f\|_{L_a^\infty(\ell^2)} \leq \|f\|_{W_a(L^\infty, \ell^2)}$. One can also check that $\|f\|_{W_a(L^\infty, \ell^1)} \leq \|f\|_{W_a(L^\infty, \ell^2)}$ for all f from $W_a(L^\infty, \ell^1)$.

Note also that $W_a(L^\infty, \ell^1)$ is dense in $X_a(L^\infty, \ell^2)$ with respect to $\|\cdot\|_{L^\infty(\ell^2)}$. In fact, its subspace consisting of all essentially bounded functions with compact support is already dense in $X_a(L^\infty, \ell^2)$ with respect to $\|\cdot\|_{L^\infty(\ell^2)}$. Observe that the set of all essentially bounded functions with compact support is via the unitary operator U_a from Theorem 5.6 in a bijective correspondence with the set of all finite sequences in $\ell_{\mathbb{Z}}^2(L^\infty[0, a]) \subseteq \ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, a])$.

We are now ready for the main result in this section in which we establish a correspondence of Gabor Bessel sequences (weak frames) and weak Bessel sequences (frames) of translates in our Hilbert C^* -module $L_a^\infty(\ell^2)$. In fact, it turns out that the translation parameter a continues to play the same role, while the modulation parameter b determines the ambient module.

Theorem 5.9. *Let $g \in L^2(\mathbb{R})$, and let $a, b > 0$. Then $G(g, a, b)$ is a Bessel sequence in $L^2(\mathbb{R})$ with a Bessel bound B (a frame with frame bounds A and B) if and only if the sequence $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a weak Bessel sequence with a Bessel bound B (a weak frame with frame bounds A and B) in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$.*

Proof. Recall that for all $f, g \in L^2(\mathbb{R})$ and $a > 0$, we have a function $\langle f, g \rangle_a \in L^1[0, a]$ defined by (50).

Let us take arbitrary $g \in L^2(\mathbb{R})$ and $a, b > 0$. By Corollary 4.6.17 from [8], $G(g, a, b)$ is a frame with frame bounds A and B if and only if

$$A \langle f, f \rangle_{\frac{1}{b}}(x) \leq \sum_{n \in \mathbb{Z}} \left| \left\langle f, T_{na} \left(\frac{1}{\sqrt{b}} g \right) \right\rangle_{\frac{1}{b}}(x) \right|^2 \leq B \langle f, f \rangle_{\frac{1}{b}}(x),$$

for a.e. $x, \forall f \in L^2(\mathbb{R})$. (65)

By Theorem 5.5 from [10], (65) is equivalent to

$$A \langle f, f \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) \leq \sum_{n \in \mathbb{Z}} \left| \left\langle f, T_{na} \left(\frac{1}{\sqrt{b}} g \right) \right\rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) \right|^2 \leq B \langle f, f \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x),$$

for a.e. $x, \forall f \in L_{\frac{1}{b}}^\infty(\ell^2)$. (66)

It should be mentioned, however, that the factor $\frac{1}{\sqrt{b}}$ that multiplies g in (65) and (66) is missing in both Corollary 4.6.17 from [8] and Theorem 5.5 from [10].¹

Having obtained (66), we are just one step from the end of the proof. First, in the sum over $n \in \mathbb{Z}$ in (66) we have convergence pointwise a.e. Observe that this sum is, by the right-hand side inequality, an essentially bounded function. Since pointwise a.e. convergence in $L^\infty[0, \frac{1}{b}]$ implies convergence in measure, which is in turn equivalent to convergence in the strong operator topology in $L^\infty[0, \frac{1}{b}]$, we can rewrite (66) in the form

¹This really simple mistake—a missing \sqrt{b} in the denominator—actually originates from Theorem 4.6.3 in [8]. Unfortunately, the key result (Corollary 4.6.17) from [8] is quoted and used in [10] in this wrong form without the factor $\frac{1}{\sqrt{b}}$.

$$A\langle f, f \rangle_{L^\infty(\ell^2)} \leq (\text{strong}) \sum_{n \in \mathbb{Z}} \left| \left\langle f, T_{na} \left(\frac{1}{\sqrt{b}} g \right) \right\rangle_{L^\infty(\ell^2)} \right|^2 \leq B\langle f, f \rangle_{L^\infty(\ell^2)},$$

$$\forall f \in L^\infty(\ell^2). \quad (67)$$

Thus, by Definition 3.2 the sequence $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a weak frame in $L^\infty(\ell^2)$ with frame bounds A and B . Conversely, if we have (67), then we conclude using Remark 5.1 that the series over $n \in \mathbb{Z}$ in (67) converges also pointwise a.e. and this gives us (66). \square

6. Weak frames in Gabor analysis

We open the section by demonstrating a couple of useful properties of the Hilbert C^* -module $L^\infty_a(\ell^2)$, $a > 0$. For any $r > 0$, we denote by D_r the dilation operator defined by $D_r f(x) = f(rx)$. Observe that here we work with the dilation operator D_r without the normalizing factor that is usually used when the dilation is regarded as an operator on $L^2(\mathbb{R})$.

Proposition 6.1. *Let $a, c > 0$. Then $D_{\frac{a}{c}} : L^\infty[0, a] \rightarrow L^\infty[0, c]$, $D_{\frac{a}{c}} f(x) = f(\frac{a}{c}x)$, is an isomorphism of von Neumann algebras and also $D_{\frac{a}{c}} : L^\infty_a(\ell^2) \rightarrow L^\infty_c(\ell^2)$ is a unitary operator of Hilbert C^* -modules. Finally, the restriction $D_{\frac{a}{c}}|_{X_a(L^\infty, \ell^2)}$ is also a unitary operator of Hilbert C^* -modules $X_a(L^\infty, \ell^2)$ and $X_c(L^\infty, \ell^2)$.*

Proof. One checks that $D_{\frac{a}{c}} : L^\infty[0, a] \rightarrow L^\infty[0, c]$, $D_{\frac{a}{c}} f(x) = f(\frac{a}{c}x)$ is an isomorphism of von Neumann algebras by an easy verification.

Now observe that the map $\varphi : \ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, a]) \rightarrow \ell^2_{\mathbb{Z}, \text{strong}}(L^\infty[0, c])$ defined by $\varphi((f_n)_n) = (D_{\frac{a}{c}} f_n)_n$ is a unitary operator (actually a $D_{\frac{a}{c}}$ -morphism) of Hilbert C^* -modules. This is also seen by a routine verification which we omit; however, let us only mention that here we use again (as in the proof of Proposition 3.18) the fact that $D_{\frac{a}{c}}$ is, being an isomorphism of von Neumann algebras, a normal map.

Applying Theorem 5.6, we now conclude that the map $\Phi = U_c^* \varphi U_a : L^\infty_a(\ell^2) \rightarrow L^\infty_c(\ell^2)$ is a unitary operator. We claim that Φ acts as dilation by $\frac{a}{c}$. First recall that $U_a f = (f_n)_n$, where $f_n = T_{-na} f \chi_{[0, a]}$, that is, $f_n(x) = f(x + na)$. Then $\varphi U_a f = (D_{\frac{a}{c}} f_n)_n$, where $(D_{\frac{a}{c}} f_n)(x) = f_n(\frac{a}{c}x) = f(\frac{a}{c}x + na)$, $x \in [0, c]$. Finally, $\Phi(f) = U_c^* \varphi U_a(f)$ means that $\Phi(f)(x + nc) = f(\frac{a}{c}x + na) = D_{\frac{a}{c}} f(x + nc)$, for all $x \in [0, c]$ and $n \in \mathbb{Z}$; thus, $\Phi = D_{\frac{a}{c}}$.

The last assertion of the proposition follows from the fact that the restriction $\varphi|_{\ell^2_{\mathbb{Z}}(L^\infty[0, a])}$ is a unitary operator of Hilbert C^* -modules $\ell^2_{\mathbb{Z}}(L^\infty[0, a])$ and $\ell^2_{\mathbb{Z}}(L^\infty[0, c])$ together with the corresponding statement of Theorem 5.6. \square

Proposition 6.2. *Let $a, c > 0$. Then $W_a(L^\infty, \ell^1) = W_c(L^\infty, \ell^1)$ and $W_a(L^\infty, \ell^2) = W_c(L^\infty, \ell^2)$. If $\frac{a}{c} \in \mathbb{Q}$, then we also have $X_a(L^\infty, \ell^2) = X_c(L^\infty, \ell^2)$ and $L^\infty_a(\ell^2) = L^\infty_c(\ell^2)$ and the corresponding norms on $L^\infty_a(\ell^2)$ and $L^\infty_c(\ell^2)$ are equivalent. Conversely, each of last two equalities implies that $\frac{a}{c} \in \mathbb{Q}$.*

Proof. The first two equalities for Wiener amalgam spaces are well known (see, e.g., [16, Section 11.4]). This allows us to write $W(L^\infty, \ell^1)$ and $W(L^\infty, \ell^2)$ without

specifying any particular parameter a and we will adopt this convention in the rest of the paper.

Suppose now that $\frac{a}{c} = \frac{p}{q}$ with $p, q \in \mathbb{N}$, $(p, q) = 1$. For $f \in L_a^\infty(\ell^2)$, put $\text{ess sup}_{x \in [0, a]} \sum_{l \in \mathbb{Z}} |f(x - la)|^2 = C$. Take any $y \in [0, c]$. Then the numbers $y, c + y, 2c + y, \dots, (p - 1)c + y$ are contained in the interval $[0, pc] = [0, qa]$ and are therefore of the form $x_j + k_j a$, $j = 1, \dots, p$ for some $x_1, \dots, x_p \in [0, a]$ and $k_1, \dots, k_p \in \{0, 1, \dots, q - 1\}$. Hence,

$$\sum_{k=0}^{p-1} |f(kc + y)|^2 = \sum_{j=1}^p |f(x_j + k_j a)|^2.$$

Now we consider next cycle. We have $y + pc = x_1 + k_1 a + qa$, $y + (p + 1)c = x_2 + k_2 a + qa$, \dots , $y + (2p - 1)c = x_p + k_p a + qa$, which together with the preceding equality gives us

$$\sum_{k=0}^{2p-1} |f(kc + y)|^2 = \sum_{j=1}^p (|f(x_j + k_j a)|^2 + |f(x_j + k_j a + qa)|^2).$$

It now follows that for a.e. y , we have

$$\sum_{k \in \mathbb{Z}} |f(kc + y)|^2 \leq \sum_{l \in \mathbb{Z}} |f(x_1 + la)|^2 + \sum_{l \in \mathbb{Z}} |f(x_2 + la)|^2 + \dots + \sum_{l \in \mathbb{Z}} |f(x_p + la)|^2 \leq pC;$$

thus,

$$\|f\|_{L_c^\infty(\ell^2)} \leq \sqrt{p} \|f\|_{L_a^\infty(\ell^2)}.$$

By symmetry, we also conclude that

$$\|f\|_{L_a^\infty(\ell^2)} \leq \sqrt{q} \|f\|_{L_c^\infty(\ell^2)}.$$

To prove that $\frac{a}{c} \in \mathbb{Q}$ is also a necessary condition for the equality $L_a^\infty(\ell^2) = L_c^\infty(\ell^2)$, we first claim that

$$L_a^\infty(\ell^2) = L_c^\infty(\ell^2) \Rightarrow L_{ar}^\infty(\ell^2) = L_{cr}^\infty(\ell^2), \quad \forall r > 0. \tag{68}$$

To see this, take any $r > 0$ and the corresponding dilation $D_r f(x) = f(rx)$, and observe that

$$\sum_{n \in \mathbb{Z}} |f(x + ncr)|^2 = \sum_{n \in \mathbb{Z}} \left| f\left(r\left(\frac{x}{r} + nc\right)\right) \right|^2 = \sum_{n \in \mathbb{Z}} \left| D_r f\left(\frac{x}{r} + nc\right) \right|^2.$$

This shows that $f \in L_{cr}^\infty(\ell^2)$ if and only if $D_r f \in L_c^\infty(\ell^2)$ or, equivalently, $D_r^{-1} L_c^\infty(\ell^2) = L_{cr}^\infty(\ell^2)$. For the same reason we also have $D_r^{-1} L_a^\infty(\ell^2) = L_{ar}^\infty(\ell^2)$, and these two equalities prove (68).

Suppose now that we have $L_a^\infty(\ell^2) = L_c^\infty(\ell^2)$ for a, c such that $\frac{a}{c} \notin \mathbb{Q}$. Then (68) implies $L_1^\infty(\ell^2) = L_{\frac{a}{c}}^\infty(\ell^2)$. But this is impossible as demonstrated by an example from [11] (see the proof of Proposition 9.6.2 in [9]).

We now turn to submodules $X_a(L^\infty, \ell^2)$ and $X_c(L^\infty, \ell^2)$. Suppose first that $\frac{a}{c} \in \mathbb{Q}$, and take any $f \in X_a(L^\infty, \ell^2)$. Then by Remark 5.8 there is a sequence $(f_n)_n$ of bounded functions with compact support such that $\|f - f_n\|_{L_a^\infty(\ell^2)} \rightarrow 0$.

Since $\|\cdot\|_{L_a^\infty(\ell^2)}$ and $\|\cdot\|_{L_c^\infty(\ell^2)}$ are equivalent norms on $L_a^\infty(\ell^2) = L_c^\infty(\ell^2)$, we also have $\|f - f_n\|_{L_c^\infty(\ell^2)} \rightarrow 0$. By Remark 5.8, this implies $f \in X_c(L^\infty, \ell^2)$.

To end the proof we need to show that $X_a(L^\infty, \ell^2) = X_c(L^\infty, \ell^2)$ implies $\frac{a}{c} \in \mathbb{Q}$. Suppose that $X_a(L^\infty, \ell^2) = X_c(L^\infty, \ell^2)$. Since $X_a(L^\infty, \ell^2)$ and $X_c(L^\infty, \ell^2)$ are by Proposition 6.1 unitarily equivalent modules, we have two equivalent norms— $\|\cdot\|_{L_a^\infty(\ell^2)}$ and $\|\cdot\|_{L_c^\infty(\ell^2)}$ —on the same set $X_a(L^\infty, \ell^2) = X_c(L^\infty, \ell^2)$. Consider, for any $h \in L_a^\infty(\ell^2)$, the map l_h on $X_a(L^\infty, \ell^2)$ defined by $l_h(f) = \langle f, h \rangle_{L_a^\infty(\ell^2)}$. This map is also a bounded module map on $X_c(L^\infty, \ell^2)$ and therefore there exists $\tilde{h} \in L_c^\infty(\ell^2)$ such that $l_h(f) = \langle f, \tilde{h} \rangle_{L_c^\infty(\ell^2)}$, for all $f \in X_c(L^\infty, \ell^2)$. So, we have $\sum_{n \in \mathbb{Z}} f(x - na) \overline{h(x - na)} = \sum_{n \in \mathbb{Z}} f(x - nc) \overline{\tilde{h}(x - nc)}$ for all f in $X_a(L^\infty, \ell^2) = X_c(L^\infty, \ell^2)$; in particular, for all $f \in C_c(\mathbb{R})$. This is enough to conclude that $h = \tilde{h}$; that is, $h \in L_c^\infty(\ell^2)$ which proves $L_a^\infty(\ell^2) \subseteq L_c^\infty(\ell^2)$. The reverse inclusion is proved in the same way; hence, $L_a^\infty(\ell^2) = L_c^\infty(\ell^2)$. By the preceding part of the proof this implies $\frac{a}{c} \in \mathbb{Q}$. \square

It is well known that the Wiener spaces $W(L^\infty, \ell^1)$ and $W(L^\infty, \ell^2)$ are translation-invariant. The same is true for our Hilbert C^* -modules $X_a(L^\infty, \ell^2)$ and $L_a^\infty(\ell^2)$.

Proposition 6.3. *Let $a > 0$. Then $X_a(L^\infty, \ell^2)$ and $L_a^\infty(\ell^2)$ are invariant under all translations T_c , $c \in \mathbb{R}$. Each translation T_c is an isometry on $L_a^\infty(\ell^2)$.*

Proof. Fix $a > 0$ and $c \in \mathbb{R}$. Let $f \in L_a^\infty(\ell^2)$ and $x \in [0, a]$. Consider $x - c$, and find an integer n_0 with the property $x - c - n_0a = y \in [0, a]$. Then we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |T_c f(x - na)|^2 &= \sum_{n \in \mathbb{Z}} |f(x - c - n_0a + n_0a - na)|^2 = \sum_{n \in \mathbb{Z}} |f(y - (n - n_0)a)|^2 \\ &= \sum_{n' \in \mathbb{Z}} |f(y - n'a)|^2. \end{aligned}$$

From this we conclude that $T_c f \in L_a^\infty(\ell^2)$ and $\|T_c f\|_{L_a^\infty(\ell^2)} \leq \|f\|_{L_a^\infty(\ell^2)}$. The same conclusion applied to T_{-c} and $T_c f$ gives us the opposite inequality: $\|f\|_{L_a^\infty(\ell^2)} = \|T_{-c} T_c f\|_{L_a^\infty(\ell^2)} \leq \|T_c f\|_{L_a^\infty(\ell^2)}$.

Let us now take any $f \in X_a(L^\infty, \ell^2)$. This means that the series $\sum_{n \in \mathbb{Z}} |T_{-na} f \chi_{[0,a]}|^2$ converges in norm in $L^\infty[0, a]$ and we need to show that this implies norm-convergence in $L^\infty[0, a]$ of the series $\sum_{n \in \mathbb{Z}} |T_{-na} T_c f \chi_{[0,a]}|^2$. Recall that is enough to consider only symmetric partial sums since if a series of positive elements in a C^* -algebra converges, it converges unconditionally.

In the n th summand we have the function $|f(x + na - c)|^2$ for $x \in [0, a]$; that is, $|f(y)|^2$ for $na - c \leq y \leq (n + 1)a - c$. Observe that

$$[na - c, (n + 1)a - c] \subseteq [(n' - 1)a, n'a] \cup [n'a, (n' + 1)a], \tag{69}$$

where $n - n' = d_{a,c}$ and the integer $d_{a,c}$ is independent of n and depends only on a and c .

Let $\epsilon > 0$ be given. Since $\sum_{n \in \mathbb{Z}} |T_{-na} f \chi_{[0,a]}|^2$ converges in norm in $L^\infty[0, a]$, there exists a natural number N_0 with the property

$$N_2 > N_1 \geq N_0 \implies \left\| \sum_{N_1+1 \leq |n| \leq N_2} |T_{-na}f\chi_{[0,a]}|^2 \right\|_{L^\infty[0,a]} < \epsilon.$$

From this we conclude, by letting $N_2 \rightarrow \infty$, that

$$\left\| \sum_{|n| > N_0} |T_{-na}f\chi_{[0,a]}|^2 \right\|_{L^\infty[0,a]} \leq \epsilon.$$

Now for $N_2 > N_1 \geq N_0 + |d_{a,c}|$, we have

$$\begin{aligned} & \left\| \sum_{N_1+1 \leq |n| \leq N_2} |T_{-na}T_c f\chi_{[0,a]}|^2 \right\|_{L^\infty[0,a]} \\ & \leq \left\| 2 \sum_{N_1+1-d_{a,c} \leq |n'| \leq N_2-d_{a,c}} |T_{-n'a}f\chi_{[0,a]}|^2 \right\|_{L^\infty[0,a]} \\ & \leq \left\| 2 \sum_{|n'| > N_0} |T_{-n'a}f\chi_{[0,a]}|^2 \right\|_{L^\infty[0,a]} \\ & \leq 2\epsilon. \end{aligned} \quad \square$$

Remark 6.4. We note that T_c for $c \neq na, n \in \mathbb{Z}$, is not a module map on $L_a^\infty(\ell^2)$. In particular, this shows us that T_c cannot be an adjointable operator on the Hilbert C^* -module $L_a^\infty(\ell^2)$, except for those c 's that are integer multiples of a .

We are now in position to apply our results from Sections 3 and 4 in Gabor analysis.

Suppose that we are given a function $g \in L^2(\mathbb{R})$ and $a, b > 0$. Let

$$\Gamma_{jk}(x) = \langle T_{ka}g, T_{ja}g \rangle_{\frac{1}{b}}(x) = \sum_{l \in \mathbb{Z}} g\left(x - ka - \frac{l}{b}\right) \overline{g\left(x - ja - \frac{l}{b}\right)}, \quad j, k \in \mathbb{Z}, \quad (70)$$

and

$$\Gamma_j(x) = \Gamma_{j0}(x) = \langle g, T_{ja}g \rangle_{\frac{1}{b}}(x) = \sum_{l \in \mathbb{Z}} g\left(x - \frac{l}{b}\right) \overline{g\left(x - ja - \frac{l}{b}\right)}, \quad j \in \mathbb{Z}. \quad (71)$$

In addition, let

$$G_k(x) = \langle g, T_{\frac{k}{b}}g \rangle_a(x) = \sum_{n \in \mathbb{Z}} g(x - na) \overline{g\left(x - na - \frac{k}{b}\right)}, \quad k \in \mathbb{Z}. \quad (72)$$

As we observed in Section 5 (recall (50)), the functions Γ_{jk}, Γ_j , and G_k are well defined for a.e. x .

The following theorem is known. We include the proof to demonstrate how the theory of weak Bessel modular sequences applies.

Theorem 6.5. *Suppose that for $g \in L^2(\mathbb{R})$ and $a, b > 0$, either of the following two conditions is satisfied:*

$$\sup_{N \in \mathbb{N}} \left\| \sum_{j=-N}^N |\Gamma_j| \right\|_{L^\infty[0, \frac{1}{b}]} \leq B, \tag{73}$$

$$\sup_{N \in \mathbb{N}} \left\| \sum_{k=-N}^N |G_k| \right\|_{L^\infty[0, a]} \leq B. \tag{74}$$

Then $G(g, a, b)$ is a Bessel sequence in $L^2(\mathbb{R})$ with a Bessel bound $\frac{1}{b}B$.

Proof. Suppose that (73) is satisfied. By Theorem 5.9, we need to show that $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a weak Bessel sequence with a Bessel bound $\frac{1}{b}B$ in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L^\infty_{\frac{1}{b}}(\ell^2)$. This is equivalent to the property that $(T_{na}g)_{n \in \mathbb{Z}}$ is a weak Bessel sequence with a Bessel bound B . By Proposition 4.1, it suffices to prove that the Gram matrix Γ of the sequence $(T_{na}g)_{n \in \mathbb{Z}}$ defines a bounded module map $\ell^2(L^\infty[0, \frac{1}{b}]) \rightarrow \ell^2_{\text{strong}}(L^\infty[0, \frac{1}{b}])$ with a bound B .

Notice that (73) implies in particular that $\Gamma_0 \in L^\infty[0, \frac{1}{b}]$, which means that $g \in L^\infty_{\frac{1}{b}}(\ell^2)$. This in turn implies by Proposition 6.3 that all $T_{na}g$'s belong to $L^\infty_{\frac{1}{b}}(\ell^2)$ and hence formulas (70) and (71) can be rewritten with $\langle \cdot, \cdot \rangle_{L^\infty_{\frac{1}{b}}(\ell^2)}$ instead of $\langle \cdot, \cdot \rangle_{\frac{1}{b}}$.

Observe now that the matrix coefficients of the Gram matrix Γ are

$$\langle T_{ka}g, T_{ja}g \rangle_{L^\infty_{\frac{1}{b}}(\ell^2)}(x) = \sum_{l \in \mathbb{Z}} g\left(x - ka - \frac{l}{b}\right) \overline{g\left(x - ja - \frac{l}{b}\right)} = \Gamma_{jk}(x), \quad j, k \in \mathbb{Z}.$$

By Proposition 4.4, it is enough to see that the matrix coefficients Γ_{jk} satisfy condition (39):

$$\begin{aligned} & \left\| (\text{strong}) \sum_{j \in \mathbb{Z}} |\Gamma_{jk}| \right\|_{L^\infty[0, \frac{1}{b}]} \leq B, \quad \forall k \in \mathbb{Z} \quad \text{and} \\ & \left\| (\text{strong}) \sum_{k \in \mathbb{Z}} |\Gamma_{jk}| \right\|_{L^\infty[0, \frac{1}{b}]} \leq B, \quad \forall j \in \mathbb{Z}. \end{aligned} \tag{75}$$

First observe that $\Gamma_{kj} = \overline{\Gamma_{jk}}$ for all k and j . Therefore, it is enough to check the first inequality in (75). Second, we have

$$\begin{aligned} \Gamma_{jk}(x) &= \langle T_{ka}g, T_{ja}g \rangle_{L^\infty_{\frac{1}{b}}(\ell^2)}(x) \\ &= \sum_{l \in \mathbb{Z}} g\left(x - ka - \frac{l}{b}\right) \overline{g\left(x - ja - \frac{l}{b}\right)} \\ &= \sum_{l \in \mathbb{Z}} g\left(x - ka - \frac{l}{b}\right) \overline{g\left(x - ka - (j - k)a - \frac{l}{b}\right)} \\ &= \Gamma_{j-k}(x - ka). \end{aligned}$$

Since all Γ_{j-k} 's are periodic functions, our assumption (73) now implies the desired first inequality in (75).

Let us now assume (74). Observe that this means that the function g satisfies (73) with parameters $\frac{1}{b}$ and $\frac{1}{a}$ playing the roles of a and b , respectively. Hence, by

the first part of the proof the sequence $(T_{\frac{n}{b}}(\sqrt{a}g))_{n \in \mathbb{Z}}$ is a weak Bessel sequence with a Bessel bound aB in the Hilbert $L^\infty[0, a]$ -module $L_a^\infty(\ell^2)$. By Theorem 5.9, this means that the sequence $G(g, \frac{1}{b}, \frac{1}{a})$ is Bessel with a Bessel bound aB . Finally, by Lemma 9.2.2 from [9] we conclude that $G(g, a, b)$ is Bessel with a Bessel bound $\frac{1}{ab}aB = \frac{1}{b}B$. \square

Remark 6.6. Suppose that g satisfies (74). This means that the sequence of symmetric partial sums of the series $\sum_{k \in \mathbb{Z}} |G_k|$ is bounded, which is equivalent to the strong convergence of this series, which is in turn, by Remark 5.1, equivalent to its pointwise a.e. convergence to a function in $L^\infty[0, a]$. Hence, we may rewrite (74) in the form

$$\operatorname{ess\,sup}_{x \in [0, a]} \sum_{k \in \mathbb{Z}} |G_k(x)| \leq B, \tag{76}$$

which is what is usually called the *CC condition* (see [9, Section 9.1]). Therefore, if g satisfies (74) with the parameters a and b , then the above theorem actually restates the first assertion of Theorem 8.4.4 from [9].

We also note that a stronger assumption, namely,

$$\sum_{k \in \mathbb{Z}} \|G_k\|_{L^\infty[0, a]} \leq B, \tag{77}$$

ensures the same conclusion. To see this, simply observe that (77) implies (76) or use Proposition 4.2. The condition (77) is well known (cf. equation (8.13) and Theorem 8.4.1 in [9]); it was first used in the late 1980s by I. Daubechies.

We now turn to the frame operator. Suppose that $G(g, a, b)$ and $G(h, a, b)$ are Bessel sequences for some $g, h \in L^2(\mathbb{R})$ and $a, b > 0$. Then $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ are weak Bessel sequences in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$. Denote by U and V the corresponding analysis operators. To avoid confusion we will denote by U_0 and V_0 the analysis operators of the original sequences $G(g, a, b)$ and $G(h, a, b)$. By Theorem 3.4, we have

$$V^*Uf = (\text{weak-strong}) \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} T_{na}h. \tag{78}$$

This means that

$$\begin{aligned} \langle V^*Uf, p \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} &= (\text{strong}) \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} \langle T_{na}h, p \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}, \\ \forall f, p &\in L_{\frac{1}{b}}^\infty(\ell^2). \end{aligned} \tag{79}$$

By Remark 5.1, we obtain the pointwise a.e. convergence for all $f, p \in L_{\frac{1}{b}}^\infty(\ell^2)$:

$$\begin{aligned} \langle V^*Uf, p \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) \langle T_{na}h, p \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) \\ \text{for a.e. } x &\in \left[0, \frac{1}{b}\right]. \end{aligned} \tag{80}$$

Denote by e the constant function 1 on the interval $[0, \frac{1}{b}]$. Since e is the unit element in our von Neumann algebra $L^\infty[0, \frac{1}{b}]$, we know from Remark 3.8 that the sequence $(e^{(n)})_{n \in \mathbb{Z}}$ is the canonical weak basis for $\ell_{\mathbb{Z}, \text{strong}}^2(L^\infty[0, \frac{1}{b}])$. Theorem 5.6 tells us now that the sequence $(\chi_{[\frac{n}{b}, \frac{n+1}{b}]})_{n \in \mathbb{Z}}$ is the canonical weak basis in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$. In our next proposition we compute the matrix of the frame operator in this basis.

Let

$$G_k^{h,g}(x) = \left(\sum_{n \in \mathbb{Z}} T_{na} h T_{\frac{k}{b}} T_{na} \bar{g} \right)(x) = \sum_{n \in \mathbb{Z}} h(x - na) \overline{g\left(x - na - \frac{k}{b}\right)}, \quad k \in \mathbb{Z}. \quad (81)$$

Recall from Remark 5.4 that $g, h \in L_a^\infty(\ell^2) \cap L_{\frac{1}{b}}^\infty(\ell^2)$, so we can write

$$G_k^{h,g}(x) = \langle h, T_{\frac{k}{b}} g \rangle_{L_a^\infty(\ell^2)}, \quad k \in \mathbb{Z}. \quad (82)$$

Observe also that for $h = g$ we have $G_k^{h,g} = G_k$, where the G_k 's are introduced in (72).

Proposition 6.7. *Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be such that $G(g, a, b)$ and $G(h, a, b)$ are Bessel sequences in $L^2(\mathbb{R})$. Denote by U and V the analysis operators of weak Bessel sequences $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$. Let (m_{kj}) be the matrix of the operator V^*U in the canonical weak basis $(\chi_{[\frac{n}{b}, \frac{n+1}{b}]})_{n \in \mathbb{Z}}$ of $L_{\frac{1}{b}}^\infty(\ell^2)$. Then*

$$m_{kj}(x) = \frac{1}{b} G_{k-j}^{h,g}\left(x + \frac{k}{b}\right), \quad x \in \left[0, \frac{1}{b}\right], k, j \in \mathbb{Z}. \quad (83)$$

In particular, if (s_{kj}) is the matrix of the frame operator $S = U^*U$ of $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ with respect to $(\chi_{[\frac{n}{b}, \frac{n+1}{b}]})_{n \in \mathbb{Z}}$, then

$$s_{kj}(x) = \frac{1}{b} G_{k-j}\left(x + \frac{k}{b}\right), \quad x \in \left[0, \frac{1}{b}\right], k, j \in \mathbb{Z}. \quad (84)$$

Proof. We first note that, for any $f \in L_{\frac{1}{b}}^\infty(\ell^2)$ and $j \in \mathbb{Z}$,

$$\langle f, \chi_{[\frac{j}{b}, \frac{j+1}{b}]} \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) = f\left(x + \frac{j}{b}\right), \quad x \in \left[0, \frac{1}{b}\right]. \quad (85)$$

Indeed, we have

$$\langle f, \chi_{[\frac{j}{b}, \frac{j+1}{b}]} \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) = \sum_{k \in \mathbb{Z}} f\left(x - \frac{k}{b}\right) \chi_{[\frac{j}{b}, \frac{j+1}{b}]}(x - \frac{k}{b}), \quad x \in \left[0, \frac{1}{b}\right].$$

Clearly, for all $k \neq -j$ the corresponding terms vanish, while for $k = -j$ we get $f(x + \frac{j}{b})$.

Now we have for all $k, j \in \mathbb{Z}$ and $x \in [0, \frac{1}{b}]$,

$$\begin{aligned}
 m_{kj}(x) &= \langle V^*U\chi_{[\frac{j}{b}, \frac{j+1}{b}]}, \chi_{[\frac{k}{b}, \frac{k+1}{b}]} \rangle_{L^{\infty}(\ell^2)}(x) \\
 &\stackrel{(80)}{=} \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle \chi_{[\frac{j}{b}, \frac{j+1}{b}]}, T_{na}g \rangle_{L^{\infty}(\ell^2)}(x) \langle T_{na}h, \chi_{[\frac{k}{b}, \frac{k+1}{b}]} \rangle_{L^{\infty}(\ell^2)}(x) \\
 &\stackrel{(85)}{=} \frac{1}{b} \sum_{n \in \mathbb{Z}} h\left(x - na + \frac{k}{b}\right) \overline{g\left(x - na + \frac{j}{b}\right)} \\
 &\stackrel{(81)}{=} \frac{1}{b} G_{k-j}^{h,g}\left(x + \frac{k}{b}\right). \quad \square
 \end{aligned}$$

Remark 6.8. Now one can easily reobtain a well-known characterization of Parseval Gabor frames. By a result from [29] (see also Theorem 3.2 in [6]), a sequence $G(g, a, b)$ is a Parseval frame in $L^2(\mathbb{R})$ if and only if the following two conditions are satisfied:

- (i) $G_0(x) = b$ a.e.,
- (ii) $G_k(x) = 0$ a.e. for all $k \neq 0$.

To see this, assume first that $G(g, a, b)$ is a Parseval frame. Then by Theorem 5.9, $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a weak Parseval frame in $L^{\infty}(\ell^2)$. This implies that U^*U is the identity operator on $L^{\infty}(\ell^2)$ and hence its matrix coefficients s_{jk} satisfy $s_{jk} = \delta_{jk}e$ for all j, k (with the Kronecker δ_{jk}). Now the preceding proposition gives us (i) and (ii).

Conversely, if (i) and (ii) are satisfied, we first conclude using Theorem 6.5 that $G(g, a, b)$ is a Bessel sequence and hence, by Theorem 5.9, $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a weak Bessel sequence. In particular, we have a well-defined and bounded analysis operator U , and now using (i) and (ii) and the preceding proposition we see that the frame operator $S = U^*U$ is the identity operator. Hence $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a weak Parseval frame, and by applying Theorem 5.9 again we conclude that $G(g, a, b)$ is a Parseval frame.

Another important feature of the correspondence of Bessel Gabor sequences in $L^2(\mathbb{R})$ and weak Bessel sequences in $L^{\infty}(\ell^2)$ is that it preserves duality. We first prove a lemma that is important in its own right.

Lemma 6.9. *Let $G(g, a, b)$ and $G(h, a, b)$ be Gabor Bessel sequences in $L^2(\mathbb{R})$ with the analysis operators U_0 and V_0 . Denote by U and V the analysis operators of the corresponding weak Bessel sequences $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$. Then $V_0^*U_0f = V^*Uf$ for every $f \in L^{\infty}(\ell^2)$.*

Proof. Consider again equation (80):

$$\begin{aligned}
 \langle V^*Uf, p \rangle_{L^{\infty}(\ell^2)}(x) &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L^{\infty}(\ell^2)}(x) \langle T_{na}h, p \rangle_{L^{\infty}(\ell^2)}(x) \\
 &\text{for a.e. } x \in \left[0, \frac{1}{b}\right],
 \end{aligned}$$

which holds for all $f, p \in L^{\infty}(\ell^2)$. If we take $p = \chi_{[\frac{j}{b}, \frac{j+1}{b}]}$, then, using (85), we get for all $f \in L^{\infty}(\ell^2)$ and every $j \in \mathbb{Z}$,

$$V^*Uf\left(x + \frac{j}{b}\right) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L^\infty(\ell^2)}(x) T_{na}h\left(x + \frac{j}{b}\right), \quad \text{for a.e. } x \in \left[0, \frac{1}{b}\right].$$

Since $\langle f, T_{na}g \rangle_{L^\infty(\ell^2)}$ is $\frac{1}{b}$ -periodic, we can write $\langle f, T_{na}g \rangle_{L^\infty(\ell^2)}(x) = \langle f, T_{na}g \rangle_{L^\infty(\ell^2)}(x + \frac{j}{b})$ so that the preceding equality can be written as

$$V^*Uf(x) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L^\infty(\ell^2)}(x) T_{na}h(x), \quad \text{for a.e. } x \in \mathbb{R}, \forall f \in L^\infty(\ell^2). \quad (86)$$

We now recall Theorem 4.6.8 from [8], which states that

$$V_0^*U_0f = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{\frac{1}{b}} T_{na}h, \quad \forall f \in L^2(\mathbb{R}) \quad (87)$$

unconditionally. In fact, this formula is stated and proved in [8] only for $h = g$, but an inspection of the proof shows that the same result is valid in this more general form for two Gabor windows g and h . Observe a subtle difference between (78) (which led us to (86)) and (87). In (87) we have convergence in $\|\cdot\|_{L^2(\mathbb{R})}$ but we cannot conclude a similar relation in $\|\cdot\|_{L^\infty(\ell^2)}$ since $\|f\|_{L^2(\mathbb{R})} \leq \frac{1}{b} \|f\|_{L^\infty(\ell^2)}$ for all f in $L^\infty(\ell^2)$. In addition, note that in (87) we have the $\frac{1}{b}$ -product $\langle f, T_{na}g \rangle_{\frac{1}{b}}$ defined in (50). Although the defining formula for $\langle \cdot, \cdot \rangle_{L^\infty(\ell^2)}$ and $\langle \cdot, \cdot \rangle_{\frac{1}{b}}$ is the same, $\langle \cdot, \cdot \rangle_{\frac{1}{b}}$ makes sense for all functions from $L^2(\mathbb{R})$, but the result is a function in $L^1[0, \frac{1}{b}]$ and not necessarily in $L^\infty[0, \frac{1}{b}]$.

However, by comparing (86) and (87) we conclude that (87) also holds pointwise for all functions from $L^\infty(\ell^2)$, that is,

$$V_0^*U_0f(x) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{\frac{1}{b}}(x) T_{na}h(x), \quad \text{for a.e. } x \in \mathbb{R}, \forall f \in L^\infty(\ell^2). \quad (88)$$

A comparison of the last equality and (86) gives us the desired conclusion. \square

Remark 6.10. Let us maintain the notation from the preceding proof. We claim that (88) holds for all $f \in L^2(\mathbb{R})$, that is, that

$$V_0^*U_0f(x) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{\frac{1}{b}}(x) T_{na}h(x), \quad \text{for a.e. } x \in \mathbb{R}, \forall f \in L^2(\mathbb{R}). \quad (89)$$

To see this, take any $f \in L^2(\mathbb{R})$, and define

$$f_0(x) = \begin{cases} 0 & \text{if } \langle f, f \rangle_{\frac{1}{b}}(x) = 0, \\ \frac{f(x)}{\langle f, f \rangle_{\frac{1}{b}}(x)^{1/2}} & \text{if } \langle f, f \rangle_{\frac{1}{b}}(x) \neq 0. \end{cases}$$

Obviously, $f_0 \in L^\infty(\ell^2)$ and (88) applied to f_0 yields (89).

Proposition 6.11. *Let $G(g, a, b)$ and $G(h, a, b)$ be Gabor Bessel sequences in $L^2(\mathbb{R})$. $G(g, a, b)$ and $G(h, a, b)$ are mutually dual if and only if the weak Bessel sequences $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ are dual to each other.*

Proof. Denote again by U_0 and V_0 the original analysis operators, and denote by U and V the corresponding modular analysis operators. We must show that $V^*U = I_{L^\infty(\ell^2)}$ if and only if $V_0^*U_0 = I_{L^2(\mathbb{R})}$, where I denotes the identity operator on the indicated ambient space.

Suppose first that $G(g, a, b)$ and $G(h, a, b)$ are dual to each other. Then we have $V_0^*U_0 = I_{L^2(\mathbb{R})}$, which by Lemma 6.9 immediately implies $V^*U = I_{L^\infty(\ell^2)}$.

Conversely, if $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ are dual to each other, then we have $V^*U = I_{L^\infty(\ell^2)}$. By Lemma 6.9, we conclude that $V_0^*U_0f = f$ for all $f \in L^\infty(\ell^2)$. This is enough to conclude that $V_0^*U_0 = I_{L^2(\mathbb{R})}$ since these two bounded operators coincide on $L^\infty(\ell^2)$, which is a dense subspace of $(L^2(\mathbb{R}))$ with respect to $\|\cdot\|_{L^2(\mathbb{R})}$. □

Remark 6.12. We are now again in position to show how an important classical result can easily be reobtained by passing from Gabor Bessel sequences to the corresponding weak Bessel sequences.

The Wexler–Raz theorem (see [9, Theorem 9.3.5]) states that Gabor Bessel sequences $G(g, a, b)$ and $G(h, a, b)$ in $L^2(\mathbb{R})$ are dual to each other if and only if the following two conditions are satisfied:

- (i) $\langle h, M^m_a T^k_b g \rangle_{L^2(\mathbb{R})} = 0$ for all $m, k \in \mathbb{Z}$ such that $m^2 + k^2 > 0$,
- (ii) $\langle h, g \rangle_{L^2(\mathbb{R})} = ab$.

To see this, first recall a result from [4] (see also Proposition 4.4.5 in [8]): if g, h are in $L^2(\mathbb{R})$, then

$$\text{span}\{M^m_a h : m \in \mathbb{Z}\} \perp \text{span}\{M^m_a g : m \in \mathbb{Z}\} \Leftrightarrow \langle h, g \rangle_a = 0 \quad \text{a.e.}, \tag{90}$$

where \perp indicates orthogonality with respect to the inner product in $L^2(\mathbb{R})$. To see this, just observe that using the periodization trick we have

$$\langle h, M^m_a g \rangle_{L^2(\mathbb{R})} = \int_0^a \langle h, g \rangle_a(x) e^{-2\pi i \frac{m}{a} x} dx.$$

Suppose now that $G(g, a, b)$ and $G(h, a, b)$ are Bessel sequences dual to each other. Then by the preceding proposition weak frames $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ in $L^\infty(\ell^2)$ are dual to each other. If U and V again denote the corresponding analysis operators, then we have $V^*U = I_{L^\infty(\ell^2)}$. By Proposition 6.7, we conclude that

$$\frac{1}{b} G^{h,g}_{k-j} \left(x + \frac{k}{b} \right) = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j, \end{cases} \quad \text{for a.e. } x \in \left[0, \frac{1}{b} \right],$$

where $G_k^{h,g}(x) = \sum_{n \in \mathbb{Z}} h(x - na) \overline{g(x - na - \frac{k}{b})} = \langle h, T^k_b g \rangle_a$ are the functions introduced in (81). This implies that $G_k^{h,g}(x) = 0$ a.e. for all $k \neq 0$ and $G_0^{h,g}(x) = b$ a.e. Using (90), we immediately get (i) and (ii).

Conversely, (i) and (ii) imply via (90) that $G_k^{h,g}(x) = 0$ a.e. for all $k \neq 0$. For $k = 0$, we have $\int_0^a G_0^{h,g}(x) dx = ab$ and $\int_0^a G_0^{h,g}(x) e^{-2\pi i \frac{m}{a} x} dx = ab$ for all $m \neq 0$.

This is enough to conclude that $G_0^{h,g}(x) = b$ a.e. Proposition 6.7 now implies that $V^*U = I_{L_{\frac{1}{b}}^\infty(\ell^2)}$.

We now turn to the Walnut representation. Suppose again that $G(g, a, b)$ and $G(h, a, b)$ are Bessel sequences in $L^2(\mathbb{R})$ for some $g, h \in L^2(\mathbb{R})$ and $a, b > 0$. Then $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ are weak Bessel sequences in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$. Denote by U and V the corresponding analysis operators. We first recall equation (78):

$$V^*Uf = (\text{weak-strong}) \frac{1}{b} \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} T_{\frac{k}{b}} f T_{\frac{k}{b}} T_{na} \bar{g} \right) T_{na} h, \quad \forall f \in L_{\frac{1}{b}}^\infty(\ell^2).$$

Since here both series converge unconditionally, we may write

$$V^*Uf = (\text{weak-strong}) \frac{1}{b} \sum_{k \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} T_{na} h T_{\frac{k}{b}} T_{na} \bar{g} \right) T_{\frac{k}{b}} f, \quad \forall f \in L_{\frac{1}{b}}^\infty(\ell^2). \tag{91}$$

Using (81), we may rewrite (91) as

$$V^*Uf = (\text{weak-strong}) \frac{1}{b} \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f, \quad \forall f \in L_{\frac{1}{b}}^\infty(\ell^2). \tag{92}$$

We know from (82) that $G_k^{h,g} \in L^\infty[0, a]$. Extended by a -periodicity, these functions can be viewed as elements in $L^\infty(\mathbb{R})$. In particular, in (92) the functions $G_k^{h,g}$ are understood as elements of $L^\infty[0, \frac{1}{b}]$ acting on $T_{\frac{k}{b}} f$ in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$.

In this way we have obtained a general modular form of the modular Walnut representation.

Theorem 6.13. *Let $G(g, a, b)$ and $G(h, a, b)$ be Bessel sequences in $L^2(\mathbb{R})$. Denote by U and V the analysis operators of the weak Bessel sequences $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L_{\frac{1}{b}}^\infty(\ell^2)$. Then*

$$\langle V^*Uf, p \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} = (\text{strong}) \frac{1}{b} \left\langle \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f, p \right\rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}, \quad \forall f, p \in L_{\frac{1}{b}}^\infty(\ell^2), \tag{93}$$

and

$$\langle V^*Uf, p \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x) = \frac{1}{b} \left\langle \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f, p \right\rangle_{L_{\frac{1}{b}}^\infty(\ell^2)}(x),$$

$$\text{for a.e. } x \in \left[0, \frac{1}{b}\right], \forall f, p \in L_{\frac{1}{b}}^\infty(\ell^2). \tag{94}$$

Proof. The first formula is obtained directly from (92) and the second formula follows from Remark 5.1. □

As an immediate consequence, we get the following result concerning pointwise convergence of the Walnut representation for Gabor Bessel sequences in $L^2(\mathbb{R})$.

Corollary 6.14. *Let $G(g, a, b)$ and $G(h, a, b)$ be Bessel sequences in $L^2(\mathbb{R})$ with the analysis operators U_0 and V_0 . Then*

$$V_0^*U_0f(x) = \frac{1}{b} \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f(x), \quad \text{for a.e. } x \in \mathbb{R}, \forall f \in L^2(\mathbb{R}). \tag{95}$$

Proof. Denote by U and V the analysis operators of the weak Bessel sequences $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$ in the Hilbert $L^\infty[0, \frac{1}{b}]$ -module $L^\infty_{\frac{1}{b}}(\ell^2)$. The equality (94) with $p = \chi_{[\frac{j}{b}, \frac{j+1}{b}]}$ for all $j \in \mathbb{Z}$ gives us

$$V^*Uf(x) = \frac{1}{b} \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f(x), \quad \text{for a.e. } x \in \mathbb{R}, \forall f \in L^\infty(\ell^2). \tag{96}$$

By Lemma 6.9, we then also have

$$V_0^*U_0f(x) = \frac{1}{b} \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f(x), \quad \text{for a.e. } x \in \mathbb{R}, \forall f \in L^\infty_{\frac{1}{b}}(\ell^2). \tag{97}$$

Take any $f \in L^2(\mathbb{R})$. As in Remark 6.10, let

$$f_0(x) = \begin{cases} 0 & \text{if } \langle f, f \rangle_{\frac{1}{b}}(x) = 0, \\ \frac{f(x)}{\langle f, f \rangle_{\frac{1}{b}}(x)^{1/2}} & \text{if } \langle f, f \rangle_{\frac{1}{b}}(x) \neq 0. \end{cases}$$

Obviously, $f_0 \in L^\infty_{\frac{1}{b}}(\ell^2)$ and (97) applied to f_0 gives us (95). □

In particular, when we take $h = g$ in the preceding corollary we get a formula for the frame operator S . Recall the classical Walnut representation for Gabor Bessel sequences (see [9, Theorem 9.2.1]):

$$Sf = \frac{1}{b} \sum_{k \in \mathbb{Z}} G_k T_{\frac{k}{b}} f, \quad \forall f \in L^2(\mathbb{R}). \tag{98}$$

This was originally proved by D. Walnut for Bessel Gabor sequences with the generating function g from the Wiener space $W(L^\infty, \ell^1)$ and then extended to Bessel Gabor sequences whose generators g satisfy (72) (i.e., the CC condition; see [7]). The series in (98) converges absolutely when $g \in W(L^\infty, \ell^1)$ and unconditionally in the case that g satisfies (72). A natural question arises whether the Walnut representation in the modular setting can be obtained with some stronger form of convergence than in (93).

It turns out that the answer is positive for those weak Bessel sequences in $L^\infty_{\frac{1}{b}}(\ell^2)$ that are standard in $X_{\frac{1}{b}}(L^\infty, \ell^2)$.

Proposition 6.15. *Let $G(g, a, b)$ and $G(h, a, b)$ be Bessel sequences in $L^2(\mathbb{R})$. Denote by U and V the analysis operators of the weak Bessel sequences $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ and $(T_{na}(\frac{1}{\sqrt{b}}h))_{n \in \mathbb{Z}}$. Suppose that $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a standard Bessel sequence in $X_{\frac{1}{b}}(L^\infty, \ell^2)$. Then*

$$V^*Uf = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{L^\infty(\ell^2)} T_{na}h, \quad \forall f \in X_{\frac{1}{b}}(L^\infty, \ell^2), \tag{99}$$

and

$$V^*Uf = \frac{1}{b} \sum_{k \in \mathbb{Z}} G_k^{h,g} T_{\frac{k}{b}} f, \quad \forall f \in X_{\frac{1}{b}}(L^\infty, \ell^2), \tag{100}$$

where both series converge unconditionally in norm in $L_{\frac{1}{b}}^\infty(\ell^2)$.

Proof. Since $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a standard Bessel sequence in $X_{\frac{1}{b}}(L^\infty, \ell^2)$, we know that

$$Uf = \left(\left\langle f, \frac{1}{\sqrt{b}} T_{na} g \right\rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} \right)_n \in \ell_{\mathbb{Z}}^2 \left(L^\infty \left[0, \frac{1}{b} \right] \right), \quad \forall f \in X_{\frac{1}{b}}(L^\infty, \ell^2).$$

Using Remark 3.7, we conclude (99). Since the series in (99) converges unconditionally, we can write

$$V^*Uf = \frac{1}{b} \sum_{k \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} T_{na} h T_{\frac{k}{b}} T_{na} \bar{g} \right) T_{\frac{k}{b}} f, \quad \forall f \in X_{\frac{1}{b}}(L^\infty, \ell^2),$$

which is in fact the desired equality (100). □

As a corollary, we now have the Walnut representation of the frame operator of Gabor Bessel sequences in $L^2(\mathbb{R})$ generated by windows which need not belong to the Wiener space or necessarily satisfy the CC condition.

Theorem 6.16. *Let $G(g, a, b)$ be a Bessel sequence in $L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$, $a, b > 0$, with the frame operator S . If $g \in X_{\frac{1}{b}}(L^\infty, \ell^2) \cap X_a(L^\infty, \ell^2)$, then*

$$Sf = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na} g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} T_{na} g, \quad \forall f \in X_{\frac{1}{b}}(L^\infty, \ell^2), \tag{101}$$

where this series converges unconditionally in norm in $L^2(\mathbb{R})$.

Proof. Denote by U the analysis operator of the corresponding weak Bessel sequence $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$, and denote by U_0 the analysis operator of $G(g, a, b)$. Notice that $S = U_0^*U_0$. Since by Lemma 6.9 U^*U and $U_0^*U_0$ coincide on $L_{\frac{1}{b}}^\infty(\ell^2)$ and since $\|f\|_{L^2(\mathbb{R})} \leq \frac{1}{b} \|f\|_{L_{\frac{1}{b}}^\infty(\ell^2)}$ for all $f \in L_{\frac{1}{b}}^\infty(\ell^2)$, it is enough to see that

$$U^*Uf = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na} g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} T_{na} g, \quad \forall f \in X_{\frac{1}{b}}(L^\infty, \ell^2)$$

in norm in $L_{\frac{1}{b}}^\infty(\ell^2)$. This in turn will follow from (99) in Proposition 6.15 if we can prove that $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ is a standard Bessel sequence in $X_{\frac{1}{b}}(L^\infty, \ell^2)$.

To this end, first observe that the assumption $g \in X_{\frac{1}{b}}(L^\infty, \ell^2)$ implies by Proposition 6.3 that the whole sequence $(T_{na}(\frac{1}{\sqrt{b}}g))_{n \in \mathbb{Z}}$ belongs to $X_{\frac{1}{b}}(L^\infty, \ell^2)$. By Proposition 3.24, it suffices to show that $(\langle f, T_{na} g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)})_n \in \ell_{\mathbb{Z}}^2(L^\infty[0, \frac{1}{b}])$ for all f from a dense subset of $X_{\frac{1}{b}}(L^\infty, \ell^2)$.

We will show that

$$\left(\langle \chi_{[\frac{j}{b}, \frac{j+1}{b}]} , T_{na} g \rangle_{L_{\frac{1}{b}}^\infty(\ell^2)} \right)_n \in \ell_{\mathbb{Z}}^2 \left(L^\infty \left[0, \frac{1}{b} \right] \right), \quad \forall j \in \mathbb{Z}.$$

Take any $j \in \mathbb{Z}$. We must show that the series

$$\sum_{n \in \mathbb{Z}} \left| \langle T_{na}g, \chi_{[\frac{j}{b}, \frac{j+1}{b}]} \rangle_{L^\infty(\ell^2)} \right|^2$$

converges in norm in $L^\infty[0, \frac{1}{b}]$. Using (85), the above series can be written as

$$\sum_{n \in \mathbb{Z}} \left| g\left(x - na + \frac{j}{b}\right) \right|^2. \tag{102}$$

This is a series of positive elements in a C^* -algebra; thus, it is enough to consider symmetric partial sums.

Now we use the other part of our assumption, namely, that $g \in X_a(L^\infty, \ell^2)$. Again by Proposition 6.3 we have $T_{-\frac{j}{b}}g \in X_a(L^\infty, \ell^2)$. Thus, the series

$$\langle T_{-\frac{j}{b}}g, T_{-\frac{j}{b}}g \rangle_{L^\infty(\ell^2)} = \sum_{n \in \mathbb{Z}} \left| g\left(x - na + \frac{j}{b}\right) \right|^2$$

converges in norm in $L^\infty[0, a]$. In particular, the sequence of partial sums is Cauchy. For $\epsilon > 0$, there exists N_0 such that

$$M > N \geq N_0 \Rightarrow \left\| \sum_{n=-M}^M \left| g\left(x - na + \frac{j}{b}\right) \right|^2 - \sum_{n=-N}^N \left| g\left(x - na + \frac{j}{b}\right) \right|^2 \right\|_{L^\infty[0, a]} < \epsilon,$$

whence

$$\operatorname{ess\,sup}_{x \in [0, a]} \sum_{|n| \geq N_0} \left| g\left(x - na + \frac{j}{b}\right) \right|^2 \leq \epsilon. \tag{103}$$

If $\frac{1}{b} < a$, then this obviously implies

$$\operatorname{ess\,sup}_{x \in [0, \frac{1}{b}]} \sum_{|n| \geq N_0} \left| g\left(x - na + \frac{j}{b}\right) \right|^2 \leq \epsilon.$$

If $a < \frac{1}{b}$, let N_1 be the greatest natural number for which $N_1a < b$. Then for every $x \in [0, \frac{1}{b}]$ there exists a natural number $N(x) \in \{0, 1, \dots, N_1\}$ such that $x - N(x)a \in [0, a]$. This allows us to conclude from (103) that

$$\operatorname{ess\,sup}_{x \in [0, \frac{1}{b}]} \sum_{|n| \geq N_0 + N_1} \left| g\left(x - na + \frac{j}{b}\right) \right|^2 \leq \epsilon.$$

Thus, the sequence of partial sums of the series (102) is Cauchy in $L^\infty[0, \frac{1}{b}]$. \square

Remark 6.17. Observe that (101) extends by Corollary 6.14 to all functions from $L^2(\mathbb{R})$, but only in the sense of pointwise a.e. convergence for those f 's that do not belong to $X_{\frac{1}{b}}(L^\infty, \ell^2)$.

The condition $g \in X_{\frac{1}{b}}(L^\infty, \ell^2) \cap X_a(L^\infty, \ell^2)$ from Theorem 6.16 is satisfied for all $g \in W(L^\infty, \ell^1)$ —this follows immediately from Remark 5.5. Also, if $g \in X_{\frac{1}{b}}(L^\infty, \ell^2)$ or $g \in X_a(L^\infty, \ell^2)$ with the additional assumption $ab \in \mathbb{Q}$, then Proposition 6.2 implies $g \in X_{\frac{1}{b}}(L^\infty, \ell^2) \cap X_a(L^\infty, \ell^2)$.

Conclusion. In the first part of the paper the dual of the standard Hilbert C^* -module over an arbitrary C^* -algebra is described. In particular, a concrete description is obtained when the underlying algebra is a von Neumann algebra. As pointed out by M. Frank in private communication, some of the results concerning modules over von Neumann algebras can be extended at least to the class of Hilbert C^* -modules over monotone complete C^* -algebras. This is left for future investigation.

We have introduced a concept of a weak Bessel sequence and a weak frame in Hilbert C^* -modules over von Neumann algebras. We obtained fundamental properties of such systems. It turned out that such weak modular systems behave similarly to that of standard Bessel sequences and frames with respect to certain weak topology. Moreover, if the underlying von Neumann algebra is commutative, then these weak modular systems are naturally described and represented by their Gram matrices.

Weak modular Bessel sequences and frames naturally appear in Gabor analysis. In fact, standard Gabor Bessel sequences and Gabor frames in $L^2(\mathbb{R})$ may be interpreted as weak Bessel sequences (resp., weak frames) of translates in certain Hilbert C^* -modules over the commutative von Neumann algebra $L^\infty(I)$, where $I = [0, \frac{1}{b}] \subseteq \mathbb{R}$ is the interval determined by the modulation parameter b . This correspondence enabled us to reobtain (and to reinterpret) in a natural way some of the classical results from Gabor analysis. Some of the results, for example, those concerned with the Walnut representation, appear to broaden the scope of the corresponding classical results on Gabor systems. Certainly, this line of investigation owes very much to the approach of P. Casazza and M. C. Lammers and uses the ideas, but in a different language, from the work of A. Ron and Z. Shen. However, the simplicity of these new proofs of some of the classical results (whose original proofs are very involved), suggests that our Hilbert C^* -module technique might serve as a promising tool in the study of Gabor systems.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA CESTA 30, 10000 ZAGREB, CROATIA.

E-mail address: bakic@math.hr