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SOLUTIONS TO FRACTIONAL SOBOLEV-TYPE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES WITH OPERATOR PAIRS AND IMPULSIVE CONDITIONS

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Abstract. We are concerned with fractional Sobolev-type integro-differential equations in Banach spaces with operator pairs and impulsive conditions, where the operator pairs generate propagation families. With the help of the theory of propagation family and Laplace transforms, along with an estimate for a special sequence improved in this article, we introduce a definition of mild solutions to the impulsive problem for these abstract fractional Sobolev-type integro-differential equations and we establish general existence theorems and a continuous dependence theorem, which essentially extend some previous conclusions. In our results, the operator B could be unbounded, and the existence of an operator B^{-1} is not necessarily needed. Moreover, we give some examples to illustrate our main results.

1. Introduction

Consider the fractional Sobolev-type integro-differential equations in a Banach space X with operator pairs (A, B) and impulsive conditions

$$\begin{cases} {}^cD^q(Bu)(t) = Au(t) + Bf(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds), & t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

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where ${}^cD^q$, $q \in (0,1)$, is the Caputo fractional derivative of order q with the lower limit zero, A and B are closed (unbounded) linear operators with domains contained in X, the pair (A,B) generates a propagation family $\{W(t)\}_{t\geq 0}$ (see Definition 2.10, which was introduced by Liang and Xiao in [21]), $f: J\times X\times X \to D(B)$ with $J=[0,T]$ with $T>0$ a constant, $\rho: \Upsilon \to \mathbb{R}$ is continuous with $\Upsilon = \{(t,s)\in J\times J : t\geq s\}$, $h: \Upsilon\times X\to X$, $J' = J\setminus\{0\}$,

$$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T,$$

$\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t=t_k$—that is, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t=t_k$, respectively—and the impulsive functions $I_k : X\to D(B)$ are continuous, and $u_0 \in D(B)$.

There exist many mathematical models associated with practical problems in the real world that take the form of (1.1). For example, in physics, the following fractional diffusion equations of Sobolev type with impulsive conditions

$$\begin{cases} \frac{\partial^q}{\partial t^q}\big(u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2}\big) \\ \qquad = \frac{\partial^2 u(t,x)}{\partial x^2} + f\big(t,u(t,x),\int_0^t \rho(t,s)h(t,s,u(s,x))\,ds\big) \\ \qquad\quad - \frac{\partial^2}{\partial x^2} f\big(t,u(t,x),\int_0^t \rho(t,s)h(t,s,u(s,x))\,ds\big), \quad t\in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k,x)), \quad k=1,2,\ldots,m, \\ u(0,x) = u_0(x) \end{cases} \tag{1.2}$$

can be used to describe the behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle while the collective motion of micro-particles is acted upon by an external force at certain points and changes suddenly. If we set $A = \frac{\partial^2}{\partial x^2}$ and $B = I - A$, then the above system (1.2) can be rewritten as the abstract form (1.1).

In recent decades, many researchers have studied fractional integro-differential equations as well as impulsive differential equations of integer order or fractional order, and many results have been obtained. (For details on the theory and developments in related topics, we refer the reader, for example, to [1], [3], [4], [6]–[8], [10]–[12], [18], [20], [26], [25], [27], and especially to the most recent new work by Chalishajar and Karthikeyan [9]. Moreover, for contributions about Sobolev-type equations, we refer the reader, for example, to [2], [17], [19] and references therein.) Stimulated by these and other works such as [19], [21], and [23], in this paper we investigate the definition, existence, and continuous dependence of mild solutions of the fractional Sobolev-type integro-differential equations in a Banach space X with operator pairs and impulsive conditions (1.1), where the operator pairs generate propagation families. As one can see, our study of the system (1.1) makes no assumptions about whether B has bounded (or compact) inverse, or concerning the relation between $D(A)$ and $D(B)$.

The rest of this paper is organized as follows. In Section 2, we introduce some notation, recall some basic known results, and present a definition of mild solutions for the fractional Sobolev-type integro-differential equations in a Banach space X with operator pairs and impulsive conditions (1.1), where the operator pairs generate propagation families. In Section 3, we discuss the existence of mild solutions for the system (1.1) in the Lipschitz case and further in a more general case. Moreover, we prove the continuous dependence of solutions to the initial values, which implies that the solution is unique in the Lipschitz case. All our results are new even in the case of $B = I$ (the identity operator on X). In Section 4, some examples are given to illustrate our abstract results.

2. Preliminaries and definition of mild solutions

We begin with some notation. Throughout this paper, X is a Banach space with norm $\|\cdot\|$. We denote by $C(J, X)$ the space of all X-valued continuous functions on J with the natural norm $\|x\|_{C(J,X)} = \sup_{t\in J}\|x(t)\|$. Set

$$J_1 = [0, t_1], \qquad J_k = (t_{k-1}, t_k], \quad k = 2, 3, \ldots, m, \qquad J_{m+1} = (t_m, T].$$

Let $PC(J, X) = \{u : J \to X \mid u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$. Obviously, $PC(J, X)$ is a Banach space with norm $\|u\|_{PC} = \sup_{t\in J}\|u(t)\|$.

The *beta function* is defined by

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad p, q > 0.$$

The *gamma function* is defined by

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,dt, \quad p > 0.$$

It is well known that

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \Gamma(p+1) = p\Gamma(p).$$

A *binomial coefficient* is defined by

$$C_n^m = \frac{n!}{m!(n-m)!},$$

satisfying

$$C_n^m + C_n^{m-1} = C_{n+1}^m,$$

where $n! = n(n-1)(n-2)\cdots 1$.

Definition 2.1 ([1, p. 12], [18, p. 69]). The *fractional integral of order* q with the lower limit zero for a function $f \in AC[0, \infty)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)\,ds, \quad t > 0, 0 < q < 1.$$

Definition 2.2 ([1, p. 12], [18, p. 70]). The *Riemann–Liouville derivative of order* q with the lower limit zero for a function $f \in AC[0, \infty)$ can be written as

$${}^{\mathrm{RL}}D^q f(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^q}\,ds, \quad t > 0, 0 < q < 1.$$

Definition 2.3 ([1, p. 12], [18, p. 90]). The *Caputo derivative of order* q with the lower limit zero for a function $f \in AC[0, \infty)$ can be written as

$$ {}^{c}D^q f(t) = {}^{\mathrm{RL}}D^q\big(f(t) - f(0)\big), \quad t > 0, 0 < q < 1. $$

Remark 2.4. (1) If $f(t) \in C^1[0, \infty)$, then

$$ {}^{c}D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q}\, ds = I^{1-q} f'(t), \quad t > 0, 0 < q < 1. $$

(2) The Caputo derivative of a constant is equal to zero.

Next we introduce the Kuratowski measure of noncompactness $\mu(\cdot)$ defined on each bounded subset E in the Banach space X by

$$ \mu(E) = \inf\{d > 0 \mid E \text{ can be covered by a finite number of sets of diameter } < d\}. $$

Some basic properties of $\mu(\cdot)$ are listed in the following lemma.

Lemma 2.5 ([5, p. 134])**.** *Let $E, C \subset X$ be bounded sets. Then we have the following:*

(i) $\mu(E) = 0$ *if and only if E is relatively compact in X;*
(ii) $\mu(E) = \mu(\overline{E}) = \mu(\overline{co}E)$, *where $\overline{co}E$ is the closed convex hull of E;*
(iii) $\mu(E) \le \mu(C)$ *when $E \subseteq C$;*
(iv) $\mu(E + C) \le \mu(E) + \mu(C)$*;*
(v) $\mu(E \cup C) \le \max\{\mu(E), \mu(C)\}$*;*
(vi) $\mu(E(0, r)) = 2r$, *where $E(0, r) = \{x \in X \mid \|x\| \le r\}$, if $\dim X = +\infty$.*

Lemma 2.6 ([22, p. 281])**.** *Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \le 2\alpha(D_0)$.*

Lemma 2.7 ([15, p. 112])**.** *Let X be a Banach space, and let $\Omega \subset PC(J, X)$ be equicontinuous and bounded. Then $\alpha(\Omega(t)) \in PC(J, R^+)$, and $\alpha(\Omega) = \max_{t \in J} \alpha(\Omega(t))$.*

Lemma 2.8 ([16, Theorem 2.1], [22, p. 281])**.** *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X with $\|u_n(t)\| \le \tilde{m}(t)$ for almost all $t \in J$ and every $n \ge 1$, where $\tilde{m} \in L^1(J, R^+)$. Then the function $\psi(t) = \mu(\{u_n\}_{n=1}^{\infty})$ belongs to $L^1(J, R^+)$ and satisfies*

$$ \mu\Big(\Big\{\int_0^t u_n(s)\, ds : n \ge 1\Big\}\Big) \le 2 \int_0^t \psi(s)\, ds. $$

Sadovskii's fixed point theorem below will be used later.

Theorem 2.9 ([14, p. 133], [20, p. 477])**.** *Let X be a Banach space. Assume that $D \subset X$ is a bounded, closed and convex set on X and $Q : D \to D$ is a condensing mapping. Then Q has one fixed point on D.*

For the following abstract degenerate Cauchy problem (see [21, p. 398])

$$ \begin{cases} \frac{d}{dt} Bu(t) = Au(t), \quad t \in J, \\ Bu(0) = Bu_0, \end{cases} \tag{2.1} $$

we need the definition from [21].

Definition 2.10 ([21, p. 399]). A strongly continuous operator family $\{W(t)\}_{t\geq 0}$ of $D(B)$ to a Banach space X, satisfying the fact that $\{W(t)\}_{t\geq 0}$ is exponentially bounded, which means that for any $x \in D(B)$ there exist $a > 0$, $M > 0$ such that

$$\|W(t)x\| \leq Me^{at}\|x\|, \quad t \geq 0,$$

is called an *exponentially bounded propagation family* for (2.1) if, for $\lambda > a$,

$$(\lambda B - A)^{-1}Bx = \int_0^\infty e^{-\lambda t}W(t)x\,dt, \quad x \in D(B). \tag{2.2}$$

In this case, we also say that (2.1) has an exponentially bounded propagation family $\{W(t)\}_{t\geq 0}$. Moreover, if (2.2) holds, we also say that the pair (A, B) generates an exponentially bounded propagation family $\{W(t)\}_{t\geq 0}$.

In this paper, we assume that $\{W(t)\}_{t\geq 0}$ is a norm-continuous family for $t > 0$ and $\|W(t)\| \leq M$.

Definition 2.11 ([19, p.515]). By the *mild solution* of the following system

$$\begin{cases} {}^cD^q(Bu)(t) = Au(t) + Bg(t), & t \in J', \\ u(0) = u_0, \end{cases}$$

we mean that the function $u \in C(J, X)$ satisfies the integral equation

$$u(t) = Q(t)u_0 + \int_0^t (t-s)^{q-1}R(t-s)g(s)\,ds, \quad t \in J,$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma)W(t^q\sigma)\,d\sigma, \qquad R(t) = q\int_0^\infty \sigma\xi_q(\sigma)W(t^q\sigma)\,d\sigma,$$
$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0,$$
$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \sigma \in (0,\infty),$$

and ξ_q is a probability density function defined on $(0,\infty)$; that is,

$$\xi_q(\sigma) \geq 0, \quad \sigma \in (0,\infty) \qquad \text{and} \qquad \int_0^\infty \xi_q(\sigma)\,d\sigma = 1.$$

By [19], we know that

$$\|Q(t)\| \leq M, \qquad \|R(t)\| \leq \frac{M}{\Gamma(q)}, \quad t \geq 0. \tag{2.3}$$

The proof of the following lemma is obvious, so we omit it.

Lemma 2.12. *We have that $\{Q(t)\}_{t\geq 0}$ and $\{R(t)\}_{t\geq 0}$ are also norm-continuous for $t > 0$.*

For the rest of this section, we will derive a definition of mild solutions to the system (1.1). We first consider the following impulsive fractional system:

$$\begin{cases} {}^cD^q(Bu)(t) = Au(t) + Bg(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, m, \\ u(0) = u_0, \end{cases} \tag{2.4}$$

where $g(t) \in PC(J, D(B))$, $y_k \in D(B)$, $u_0 \in D(B)$.

We can decompose $u(\cdot)$ which is a solution of the system (2.4) to $v(\cdot) + w(\cdot)$, where v is a mild solution of

$$\begin{cases} {}^cD^q(Bv)(t) = Av(t) + Bg(t), \quad t \in J', \\ v(0) = u_0, \end{cases} \tag{2.5}$$

on J, and w is a mild solution of

$$\begin{cases} {}^cD^q(Bw)(t) = Aw(t), \quad t \in J', \\ \Delta w|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, m, \\ w(0) = 0. \end{cases} \tag{2.6}$$

By Definition 2.11, a mild solution of (2.5) is given by

$$v(t) = Q(t)u_0 + \int_0^t (t-s)^{q-1} R(t-s) g(s)\, ds, \quad t \in J.$$

Now we rewrite (2.6) in the following integral equation:

$$Bw(t) = \sum_{i=1}^m \chi_i(t) By_i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Aw(s)\, ds, \quad t \in J, \tag{2.7}$$

where

$$\chi_i(t) = \begin{cases} 0, \quad t \in [0, t_i), \\ 1, \quad t \in [t_i, T]. \end{cases}$$

We apply the Laplace transform for (2.7) to get

$$B\hat{w}(\lambda) = \sum_{i=1}^m \frac{e^{-t_i\lambda}}{\lambda} By_i + \frac{1}{\lambda^q} A\hat{w}(\lambda),$$

which implies that

$$\hat{w}(\lambda) = \sum_{i=1}^m e^{-t_i\lambda} \lambda^{q-1} (\lambda^q B - A)^{-1} By_i.$$

By [19], we know that the Laplace transform of $Q(t)y_i$ is $\lambda^{q-1}(\lambda^q B - A)^{-1}By_i$. Hence we derive the mild solution of (2.6) as

$$w(t) = \sum_{i=1}^m \chi_i(t) Q(t-t_i) y_i = \sum_{0<t_k<t} Q(t-t_k) y_k.$$

Summarizing, the mild solution of (2.4) is given by

$$u(t) = Q(t)u_0 + \int_0^t (t-s)^{q-1} R(t-s) g(s)\, ds + \sum_{0<t_k<t} Q(t-t_k) y_k.$$

According to the analysis above, we introduce the following definition of the mild solution to the system (1.1).

Definition 2.13. By a *mild solution* of the system (1.1), we mean a function $u \in PC(J, X)$ satisfying the following integral equation:

$$u(t) = Q(t)u_0 + \int_0^t (t-s)^{q-1} R(t-s) f\Big(s, u(s), \int_0^s \rho(s,\tau) h\big(s,\tau,u(\tau)\big)\, d\tau\Big)\, ds + \sum_{0<t_k<t} Q(t-t_k) I_k\big(u(t_k)\big).$$

3. Main results

We first consider the Lipschitz case for the system (1.1). In order to obtain a more general result, we first prove the following estimate for a special sequence S_n, which generalizes a result in [23], and will be used in the proof of our main results.

Theorem 3.1. *Suppose that* $0 < a,\ q < 1,\ b > 0$ *are constants. Let*

$$S_n = a^n + \frac{C_n^1 a^{n-1} b}{\Gamma(q+1)} + \frac{C_n^2 a^{n-2} b^2}{\Gamma(2q+1)} + \cdots + \frac{b^n}{\Gamma(nq+1)}, \quad n \in N.$$

Then

$$S_n = o\Big(\frac{1}{n^{d+1}}\Big), \quad n \to \infty,$$

for any real constant $d > 0$.

Proof. It is not difficult to verify that

$$\lim_{m\to\infty} \Big(a^{m-1} m \Big(\frac{m}{m-1}\Big)^{m-1}\Big)^{\frac{1}{m}} = a < 1.$$

Therefore, we can choose a positive integer $M > 2$, which is large enough such that

$$\Big(a^{M-1} M \Big(\frac{M}{M-1}\Big)^{M-1}\Big)^{\frac{1}{M}} \equiv w < 1. \tag{3.1}$$

For any positive integer $n > 2M$, set

$$n = Mj + p \quad (0 \le p < M),$$

where M is given above. Clearly, $j = [n/M] < [n/2]$. Therefore, for any positive integer n large enough (e.g., $n > 2M$), it follows from Stirling's formula

$$n! = \Big(\frac{n}{e}\Big)^n \sqrt{2\pi n}\Big(1 + O\Big(\frac{1}{n}\Big)\Big)$$

and the equality (3.1) that

$$
\begin{aligned}
\widetilde{S}_n &\equiv C_n^0 a^n + \frac{C_n^1 a^{n-1} b^1}{\Gamma(q+1)} + \frac{C_n^2 a^{n-2} b^2}{\Gamma(2q+1)} + \cdots + \frac{C_n^j a^{n-j} b^j}{\Gamma(jq+1)} \\
&\le C_n^j a^{n-j} \Big[1 + \frac{b^1}{\Gamma(q+1)} + \frac{b^2}{\Gamma(2q+1)} + \cdots + \frac{b^j}{\Gamma(jq+1)} \Big] \\
&= a^{n-j} \frac{n!}{j!(n-j)!} O(1) \\
&= \frac{O(1) a^{n-j} n^n \sqrt{2\pi n}}{j^j \sqrt{2\pi j} \sqrt{2\pi (n-j)} (n-j)^{n-j}} \\
&= O\Big(\frac{M^j}{\sqrt{j}}\Big) \Big(\frac{aM}{M-1}\Big)^{(M-1)j} \\
&= O\Big(\frac{(a^{M-1} M (\frac{M}{M-1})^{M-1})^j}{\sqrt{j}}\Big) \\
&= O\Big(\frac{w^{Mj}}{\sqrt{j}}\Big) \\
&= O\Big(\frac{w^n}{\sqrt{n}}\Big).
\end{aligned}
$$

On the other hand, without loss of generality, we can suppose that $b > 1$. By Stirling's formula

$$
\Gamma(z+1) = \sqrt{2\pi z} \Big(\frac{z}{e}\Big)^z \Big(1 + O\Big(\frac{1}{z}\Big)\Big), \quad z \to +\infty
$$

and (see [24])

$$
C_n^{[\frac{n}{2}]} = O(2^n/\sqrt{n}),
$$

we obtain

$$
\begin{aligned}
\widetilde{\widetilde{S}}_n &\equiv \frac{C_n^{j+1} a^{n-j-1} b^{j+1}}{\Gamma((j+1)q+1)} + \cdots + \frac{C_n^n a^{n-n} b^n}{\Gamma(nq+1)} \\
&\le \frac{1}{\Gamma((j+1)q+1)} C_n^{[\frac{n}{2}]} (a^{n-j-1} b^{j+1} + \cdots + a^{n-n} b^n) \\
&= \frac{O(\frac{2^n}{\sqrt{n}}) e^{(j+1)q} (a^{n-j-1} b^{j+1} + \cdots + a^{n-n} b^n)}{\sqrt{2\pi (j+1) q} ((j+1)q)^{(j+1)q} (1 + O(\frac{1}{(j+1)q}))} \\
&\le \frac{O(\frac{2^n}{\sqrt{n}}) e^{(j+1)q} (1 + b + b^2 + \cdots + b^{j+1} + \cdots + b^n)}{\sqrt{2\pi (j+1) q} ((j+1)q)^{(j+1)q}} \\
&\le \frac{O(1) 2^n e^{(j+1)q} b^{n+1}}{\sqrt{n} \sqrt{(j+1)q} ((j+1)q)^{(j+1)q}} \\
&\le \frac{O(1) 2^n (\frac{e}{q})^{(j+1)q} b^{n+1}}{j^{(j+1)q+1}}
\end{aligned}
$$

$$= o\Big(\frac{1}{j^{d+1}}\Big)$$
$$= o\Big(\frac{1}{n^{d+1}}\Big) \quad (n \to \infty),$$

where $d > 0$ can be any real constant.

Therefore,

$$\begin{aligned} S_n &= \widetilde{S}_n + \widetilde{\widetilde{S}}_n \\ &= O\Big(\frac{w^n}{\sqrt{n}}\Big) + o\Big(\frac{1}{n^{d+1}}\Big) \\ &= o\Big(\frac{1}{n^{d+1}}\Big)(n \to \infty), \end{aligned}$$

where $d > 0$ can be any real constant. This ends the proof of this theorem. □

Theorem 3.2. *Assume that the following conditions hold:*

(I_1) *$f : J \times X \times X \to D(B)$ is continuous, and there exist positive constants l_1, l_2 such that*

$$\begin{aligned} &\big\|f(t, x_1, x_2) - f(t, y_1, y_2)\big\| \\ &\quad \le l_1\|x_1 - y_1\| + l_2\|x_2 - y_2\|, \quad t \in J, x_i, y_i \in X, i = 1, 2; \end{aligned}$$

(I_2) *$h : \Upsilon \times X \to X$ is continuous, and there exists a positive constant l_3 such that*

$$\big\|h(t, s, x) - h(t, s, y)\big\| \le l_3\|x - y\|, \quad (t, s) \in \Upsilon, x, y \in X;$$

(I_3) *there exist positive constants c_k $(k = 1, \dots, m)$ such that*

$$\big\|I_k(x) - I_k(y)\big\| \le c_k\|x - y\|, \quad k = 1, \dots, m, x, y \in X;$$

(I_4) *we have*

$$M \sum_{k=1}^{m} c_k < 1.$$

Then the system (1.1) has a unique mild solution on J.

Proof. Define an operator P by

$$\begin{aligned} (Pu)(t) = Q(t)u_0 + \int_0^t (t - s)^{q-1} R(t - s) f\Big(s, u(s), \int_0^s \rho(s, \tau) h\big(s, \tau, u(\tau)\big)\, d\tau\Big)\, ds \\ + \sum_{0<t_k<t} Q(t - t_k) I_k\big(u(t_k)\big). \qquad (3.2) \end{aligned}$$

It is not difficult to verify that P maps $PC(J, X)$ into $PC(J, X)$. Set

$$\rho^* := \max_{(t,s)\in\Upsilon} \big|\rho(t, s)\big|.$$

Then we claim that for any $t \in J$, $u, v \in PC(J, X)$,

$$\begin{aligned}
&\big\|(P^n u)(t) - (P^n v)(t)\big\| \\
&\quad \le M^n \sum_{i=0}^{n} \frac{C_n^i (\sum_{k=1}^m c_k)^{n-i} (l_1 + \rho^* T l_2 l_3)^i (t^q)^i}{\Gamma(iq+1)} \\
&\qquad \times \|u - v\|_{PC}, \quad n = 1, 2, \ldots
\end{aligned} \tag{3.3}$$

Now we prove (3.3) by induction.

For $n = 1$, by (2.3), we get

$$\begin{aligned}
&\big\|(Pu)(t) - (Pv)(t)\big\| \\
&\quad \le \sum_{0<t_k<t} M c_k \big\|u(t_k) - v(t_k)\big\| \\
&\qquad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big[l_1 \|\big\|u(s) - v(s)\big\| + l_2 l_3 \rho^* \int_0^s \big\|u(\tau) - v(\tau)\big\| \, d\tau \Big] \, ds \\
&\quad \le M \Big[\sum_{k=1}^m c_k + \frac{1}{\Gamma(q+1)} (l_1 + \rho^* T l_2 l_3) t^q \Big] \|u - v\|_{PC}.
\end{aligned}$$

So, (3.3) holds for $n = 1$.

Suppose that (3.3) holds for $n = l$, that is,

$$\begin{aligned}
&\big\|(P^l u)(t) - (P^l v)(t)\big\| \\
&\quad \le M^l \sum_{i=0}^{l} \frac{C_l^i (\sum_{k=1}^m c_k)^{l-i} (l_1 + \rho^* T l_2 l_3)^i (t^q)^i}{\Gamma(iq+1)} \|u - v\|_{PC}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\big\|(P^{l+1} u)(t) - (P^{l+1} v)(t)\big\| \\
&\quad \le \sum_{0<t_k<t} M c_k \big\|(P^l u)(t_k) - (P^l v)(t_k)\big\| \\
&\qquad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big[l_1 \|\big\|(P^l u)(s) - (P^l v)(s)\big\| \\
&\qquad + l_2 l_3 \rho^* \int_0^s \big\|(P^l u)(\tau) - (P^l v)(\tau)\big\| \, d\tau \Big] \, ds \\
&\quad \le M \sum_{k=1}^m c_k M^l \sum_{i=0}^{l} \frac{C_l^i (\sum_{k=1}^m c_k)^{l-i} (l_1 + \rho^* T l_2 l_3)^i (t^q)^i}{\Gamma(iq+1)} \|u - v\|_{PC} \\
&\qquad + \frac{M (l_1 + \rho^* T l_2 l_3)}{\Gamma(q)} \\
&\qquad \times \int_0^t (t-s)^{q-1} M^l \sum_{i=0}^{l} \frac{C_l^i (\sum_{k=1}^m c_k)^{l-i} (l_1 + \rho^* T l_2 l_3)^i (s^q)^i}{\Gamma(iq+1)} \|u - v\|_{PC} \, ds.
\end{aligned}$$

In view of

$$\int_0^t (t-s)^{q-1} s^{mq} \, ds = t^{(m+1)q} B(q, mq+1), \quad m \in N,$$

and

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\quad (p,q>0),$$

we obtain

$$\begin{aligned}
&\big\|(P^{l+1}u)(t)-(P^{l+1}v)(t)\big\|\\
&\quad\le M^{l+1}\sum_{i=0}^{l}\frac{C_l^i(\sum_{k=1}^m c_k)^{l+1-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}\\
&\qquad+M^{l+1}\sum_{i=0}^{l}\frac{C_l^i(\sum_{k=1}^m c_k)^{l-i}(l_1+\rho^*Tl_2l_3)^{i+1}(t^q)^{i+1}}{\Gamma((i+1)q+1)}\|u-v\|_{PC}\\
&\quad=M^{l+1}\Big[\Big(\sum_{k=1}^m c_k\Big)^{l+1}+\sum_{i=1}^{l}\frac{C_l^i(\sum_{k=1}^m c_k)^{l-i+1}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u-v\|_{PC}\\
&\qquad+M^{l+1}\sum_{i=0}^{l}\frac{C_l^i(\sum_{k=1}^m c_k)^{l-i}(l_1+\rho^*Tl_2l_3)^{i+1}(t^q)^{i+1}}{\Gamma((i+1)q+1)}\|u-v\|_{PC}\\
&\quad=M^{l+1}\Big[\Big(\sum_{k=1}^m c_k\Big)^{l+1}+\sum_{i=1}^{l}\frac{C_l^i(\sum_{k=1}^m c_k)^{l-i+1}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u-v\|_{PC}\\
&\qquad+\ M^{l+1}\sum_{i=1}^{l+1}\frac{C_l^{i-1}(\sum_{k=1}^m c_k)^{l-i+1}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}\\
&\quad=M^{l+1}\Big[\Big(\sum_{k=1}^m c_k\Big)^{l+1}+\sum_{i=1}^{l}\frac{(C_l^i+C_l^{i-1})(\sum_{k=1}^m c_k)^{l-i+1}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\\
&\qquad\times\|u-v\|_{PC}+M^{l+1}\frac{(l_1+\rho^*Tl_2l_3)^{l+1}(t^q)^{l+1}}{\Gamma((l+1)q+1)}\|u-v\|_{PC}.
\end{aligned}$$

Since

$$C_m^i+C_m^{i-1}=C_{m+1}^i,$$

we have

$$\begin{aligned}
&\big\|(P^{l+1}u)(t)-(P^{l+1}v)(t)\big\|\\
&\quad\le M^{l+1}\Big[\Big(\sum_{k=1}^m c_k\Big)^{l+1}+\sum_{i=1}^{l}\frac{C_{l+1}^i(\sum_{k=1}^m c_k)^{l-i+1}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u-v\|_{PC}\\
&\qquad+M^{l+1}\frac{(l_1+\rho^*Tl_2l_3)^{l+1}(t^q)^{l+1}}{\Gamma((l+1)q+1)}\|u-v\|_{PC}\\
&\quad=M^{l+1}\sum_{i=0}^{l+1}\frac{C_{l+1}^i(\sum_{k=1}^m c_k)^{l+1-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}.
\end{aligned}$$

Hence, (3.3) holds for $n=l+1$.

Consequently, we see that

$$\begin{aligned}&\big\|(P^n u)(t) - (P^n v)(t)\big\| \\ &\qquad \le M^n \sum_{i=0}^{n} \frac{C_n^i (\sum_{k=1}^m c_k)^{n-i} (l_1 + \rho^* T l_2 l_3)^i (t^q)^i}{\Gamma(iq+1)} \|u-v\|_{PC}, \quad n = 1,2,\ldots.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|P^n u - P^n v\|_{PC} \\ &\qquad \le M^n \sum_{i=0}^{n} \frac{C_n^i (\sum_{k=1}^m c_k)^{n-i} (l_1 + \rho^* T l_2 l_3)^i (T^q)^i}{\Gamma(iq+1)} \|u-v\|_{PC}, \quad n = 1,2,\ldots.\end{aligned}$$

Write

$$a = M \sum_{k=1}^{m} c_k < 1, \qquad b = M(l_1 + \rho^* T l_2 l_3) T^q.$$

By Theorem 3.1, we know that there exists a positive integer n_0 such that

$$M^{n_0} \sum_{i=0}^{n_0} \frac{C_{n_0}^i (\sum_{k=1}^m c_k)^{n_0-i} (l_1 + \rho^* T l_2 l_3)^i (T^q)^i}{\Gamma(iq+1)} < 1,$$

that is, P^{n_0} is a contraction mapping on $PC(J,X)$. Then by a well-known extension of the Banach contraction mapping theorem, P has a unique fixed point $u(t)$ on $PC(J,X)$, which is the unique mild solution of the system (1.1). □

Next, we consider the continuous dependence of a mild solution to the system (1.1).

Theorem 3.3. *Suppose that the conditions* (I_1)–(I_4) *hold. Let* $u(t)$, $v(t)$ *be the unique mild solutions of the system (1.1) and the following system (3.4), respectively,*

$$\begin{cases} {}^cD^q(Bu)(t) = Au(t) + Bf(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds), \quad t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = v_0, \end{cases} \tag{3.4}$$

where $v_0 \in D(B)$*. Then there exists a constant* $\widetilde{M} > 0$ *such that*

$$\|u - v\|_{PC} \le \widetilde{M} \|u_0 - v_0\|.$$

Proof. Let $u(t)$ and $v(t)$ satisfy the following two equations, respectively:

$$\begin{aligned} u(t) = Q(t)u_0 &+ \int_0^t (t-s)^{q-1} R(t-s) f\Big(s, u(s), \int_0^s \rho(s,\tau) h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\,ds \\ &+ \sum_{0<t_k<t} Q(t-t_k) I_k\big(u(t_k)\big), \\ v(t) = Q(t)v_0 &+ \int_0^t (t-s)^{q-1} R(t-s) f\Big(s, v(s), \int_0^s \rho(s,\tau) h\big(s,\tau,v(\tau)\big)\,d\tau\Big)\,ds \\ &+ \sum_{0<t_k<t} Q(t-t_k) I_k\big(v(t_k)\big). \end{aligned}$$

Then we claim that

$$
\begin{aligned}
&\big\|u(t)-v(t)\big\| \\
&\quad\le M\Big[1+\sum_{j=1}^{n} M^{j}\sum_{i=0}^{j}\frac{C_j^i(\sum_{k=1}^m c_k)^{j-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u_0-v_0\| \\
&\qquad + M^{n+1}\sum_{i=0}^{n+1}\frac{C_{n+1}^i(\sum_{k=1}^m c_k)^{n+1-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)} \\
&\qquad \times \|u-v\|_{PC}, \quad n=1,2,\dots.
\end{aligned} \tag{3.5}
$$

Next, we prove (3.5) by induction. Clearly,

$$
\begin{aligned}
&\big\|u(t)-v(t)\big\| \\
&\quad\le M\|u_0-v_0\|+\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}(l_1+\rho^*Tl_2l_3)\|u-v\|_{PC}\,ds \\
&\qquad + M\sum_{k=1}^m c_k\|u-v\|_{PC} \\
&\quad= M\|u_0-v_0\|+M\Big[\sum_{k=1}^m c_k+\frac{(l_1+\rho^*Tl_2l_3)t^q}{\Gamma(q+1)}\Big]\|u-v\|_{PC}.
\end{aligned}
$$

For $n=1$,

$$
\begin{aligned}
&\big\|u(t)-v(t)\big\| \\
&\quad\le M\|u_0-v_0\|+\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}(l_1+\rho^*Tl_2l_3)\Big\{M\|u_0-v_0\| \\
&\qquad + M\Big[\sum_{k=1}^m c_k+\frac{(l_1+\rho^*Tl_2l_3)s^q}{\Gamma(q+1)}\Big]\|u-v\|_{PC}\Big\}\,ds \\
&\qquad + M\sum_{k=1}^m c_k\Big\{M\|u_0-v_0\|+M\Big[\sum_{k=1}^m c_k+\frac{(l_1+\rho^*Tl_2l_3)t^q}{\Gamma(q+1)}\Big]\|u-v\|_{PC}\Big\} \\
&\quad\le M\Big[1+M\Big(\sum_{k=1}^m c_k+\frac{(l_1+\rho^*Tl_2l_3)t^q}{\Gamma(q+1)}\Big)\Big]\|u_0-v_0\| \\
&\qquad + M^2\Big[\Big(\sum_{k=1}^m c_k\Big)^2 \\
&\qquad + \frac{2(\sum_{k=1}^m c_k)(l_1+\rho^*Tl_2l_3)t^q}{\Gamma(q+1)}+\frac{(l_1+\rho^*Tl_2l_3)^2(t^q)^2}{\Gamma(2q+1)}\Big]\|u-v\|_{PC}.
\end{aligned}
$$

So, (3.5) holds for $n=1$. Suppose that (3.5) is true for $n=l$, that is,

$$
\begin{aligned}
&\big\|u(t)-v(t)\big\| \\
&\quad\le M\Big[1+\sum_{j=1}^{l} M^{j}\sum_{i=0}^{j}\frac{C_j^i(\sum_{k=1}^m c_k)^{j-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u_0-v_0\|
\end{aligned}
$$

$$+ M^{l+1}\sum_{i=0}^{l+1}\frac{C_{l+1}^i(\sum_{k=1}^m c_k)^{l+1-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}.$$

Then we obtain

$$\begin{aligned}
&\big\|u(t)-v(t)\big\| \\
&\le M\|u_0-v_0\| + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}(l_1+\rho^*Tl_2l_3)\Big\{M\Big[1 \\
&\quad + \sum_{j=1}^{l} M^j \sum_{i=0}^{j}\frac{C_j^i(\sum_{k=1}^m c_k)^{j-i}(l_1+\rho^*Tl_2l_3)^i(s^q)^i}{\Gamma(iq+1)}\Big]\|u_0-v_0\| \\
&\quad + M^{l+1}\sum_{i=0}^{l+1}\frac{C_{l+1}^i(\sum_{k=1}^m c_k)^{l+1-i}(l_1+\rho^*Tl_2l_3)^i(s^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}\Big\}\,ds \\
&\quad + M\sum_{k=1}^{m} c_k\Big\{M\Big[1 \\
&\quad + \sum_{j=1}^{l} M^j \sum_{i=0}^{j}\frac{C_j^i(\sum_{k=1}^m c_k)^{j-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u_0-v_0\| \\
&\quad + M^{l+1}\sum_{i=0}^{l+1}\frac{C_{l+1}^i(\sum_{k=1}^m c_k)^{l+1-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}\Big\} \\
&= M\Big[1+\sum_{j=1}^{l+1} M^j \sum_{i=0}^{j}\frac{C_j^i(\sum_{k=1}^m c_k)^{j-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\Big]\|u_0-v_0\| \\
&\quad + M^{l+2}\sum_{i=0}^{l+2}\frac{C_{l+2}^i(\sum_{k=1}^m c_k)^{l+2-i}(l_1+\rho^*Tl_2l_3)^i(t^q)^i}{\Gamma(iq+1)}\|u-v\|_{PC}.
\end{aligned}$$

Hence, (3.5) holds for $n=l+1$. So, (3.5) is true for all natural numbers n.

Therefore,

$$\begin{aligned}
\|u-v\|_{PC} &\le M\Big[1+\sum_{j=1}^{n} M^j \sum_{i=0}^{j}\frac{C_j^i(\sum_{k=1}^m c_k)^{j-i}(l_1+\rho^*Tl_2l_3)^i(T^q)^i}{\Gamma(iq+1)}\Big]\|u_0-v_0\| \\
&\quad + M^{n+1}\sum_{i=0}^{n+1}\frac{C_{n+1}^i(\sum_{k=1}^m c_k)^{n+1-i}(l_1+\rho^*Tl_2l_3)^i(T^q)^i}{\Gamma(iq+1)} \\
&\quad \times\|u-v\|_{PC},\quad n=1,2,\ldots.
\end{aligned}$$

In view of Theorem 3.1, if we set

$$M_1=\sum_{n=1}^{\infty} M^n \sum_{i=0}^{n}\frac{C_n^i(\sum_{k=1}^m c_k)^{n-i}(l_1+\rho^*Tl_2l_3)^i(T^q)^i}{\Gamma(iq+1)},$$

then we get

$$\lim_{n\to\infty} M^{n+1}\sum_{i=0}^{n+1}\frac{C_{n+1}^{i}(\sum_{k=1}^{m} c_k)^{n+1-i}(l_1+\rho^* T l_2 l_3)^i (T^q)^i}{\Gamma(iq+1)}=0.$$

Hence,

$$\|u-v\|_{PC}\le M(1+M_1)\|u_0-v_0\|.$$

This implies the conclusion of Theorem 3.3. □

For the general case, we need the following assumptions.

$(H_1) Here$(i) $f: J\times X\times X\to D(B)\subset X$ satisfies that $f(\cdot,\nu,\omega): J\to D(B)\subset X$ is measurable for all $(\nu,\omega)\in X\times X$ and $f(t,\cdot,\cdot): X\times X\to D(B)\subset X$ is continuous for almost every $t\in J$, and there exist a function $\mu_1(\cdot)\in L^p(J,R^+)$ $(p>\frac{1}{q}>1)$ and a continuous function $\mu_1(\cdot)$ such that

$$\big\|f(t,\nu,\omega)\big\|\le \mu_1(t)\|\nu\|+\mu_2(t)\|\omega\|$$

for almost all $t\in J$.

(ii) There exists a function $\eta(\cdot)\in L^p(J,R^+)$ such that for any bounded sets $D_1,D_2\subset X$,

$$\alpha\big(f(t,D_1,D_2)\big)\le \eta(t)\big(\alpha(D_1)+\alpha(D_2)\big),\quad \text{a.e. } t\in J.$$

(H_2)(i) The function $h(t,s,\cdot): X\to X$ is continuous for almost every $(t,s)\in\Delta$, and for each $u\in X$, the function $h(\cdot,\cdot,u):\Delta\to X$ is measurable. Moreover, there exists a function $m:\Delta\to R^+$ with

$$\sup_{t\in J}\int_0^t m(t,s)\,ds := m^* < \infty$$

such that

$$\big\|h(t,s,u)\big\|\le m(t,s)\|u\|,\quad u\in X.$$

(ii) For any bounded set $D\subset X$ and $0\le s\le t\le T$, there exists a function $\zeta:\Delta\to R^+$ such that

$$\alpha\big(h(t,s,D)\big)\le \zeta(t,s)\alpha(D),$$

where

$$\sup_{t\in J}\int_0^t \zeta(t,s)\,ds := \zeta^* < \infty.$$

(H_3)(i) There exist positive constants L_k, N_k $(k=1,2,\ldots,m)$ such that

$$\big\|I_k(x)\big\|\le L_k\|x\|+N_k,\quad x\in X.$$

(ii) There exist positive constants M_k $(k=1,2,\ldots,m)$ such that, for any bounded set $D\subset X$,

$$\alpha\big(I_k(D)\big)\le M_k\alpha(D).$$

(H_4) We have

$$\frac{M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\mu_1\|_{L^p}+\frac{T^q\rho^* m^*\mu_2^*}{q}\Big)+M\sum_{k=1}^{m}L_k<1 \tag{3.6}$$

and

$$\frac{2M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\eta\|_{L^p}+\frac{2\rho^*\zeta^*T^q}{q}\Big)+M\sum_{k=1}^{m}M_k<\frac{1}{2}, \tag{3.7}$$

where

$$l_{p,q}:=\Big(\frac{p-1}{pq-1}\Big)^{\frac{p-1}{p}}, \qquad \mu_2^*=\sup_{t\in J}\mu_2(t).$$

Theorem 3.4. *Assume that* (H_1)–(H_4) *are satisfied. Then the system* (1.1) *has at least one mild solution on* J.

Proof. Consider the operator P defined in (3.2). P maps $PC(J,X)$ into $PC(J,X)$.

Set

$$B_r=\big\{u\in PC(J,X):\|u\|_{PC}\le r\big\}.$$

Step 1: We show that there exists some $r>0$ such that $P(B_r)\subset B_r$.

Suppose that this is not true. Then for each $r>0$, there exist $u_r(\cdot)\in B_r$ and some $t\in J$ such that

$$\big\|(Pu_r)(t)\big\|>r.$$

It follows from (H_1)(i), (H_2)(i), (H_3)(i) that

$$\begin{aligned}
r&<\big\|Q(t)u_0\big\|+\int_0^t(t-s)^{q-1}\Big\|R(t-s)f\Big(s,u_r(s),\int_0^s\rho(s,\tau)h\big(s,\tau,u_r(\tau)\big)\,d\tau\Big)\,ds\Big\|\\
&\quad+\sum_{0<t_k<t}\big\|Q(t-t_k)I_k\big(u_r(t_k)\big)\big\|\\
&\le M\|u_0\|+\frac{M}{\Gamma(q)}\int_0^t(t-s)^{q-1}\Big[\mu_1(s)r+\mu_2(s)\rho^*r\int_0^s m(s,\tau)\,d\tau\Big]\,ds\\
&\quad+M\sum_{k=1}^{m}(rL_k+N_k)\\
&\le M\|u_0\|+\frac{Mr}{\Gamma(q)}\Big(\int_0^t(t-s)^{q-1}\mu_1(s)\,ds+\frac{T^q\rho^*m^*\mu_2^*}{q}\Big)\\
&\quad+Mr\sum_{k=1}^{m}L_k+M\sum_{k=1}^{m}N_k.
\end{aligned}$$

Moreover, by the Hölder inequality, we have

$$\int_0^t(t-s)^{q-1}\mu_1(s)\,ds\le t^{\frac{pq-1}{p}}l_{p,q}\|\mu_1\|_{L^p}\le l_{p,q}T^{q-\frac{1}{p}}\|\mu_1\|_{L^p}.$$

Thus,

$$\begin{aligned}
r&<M\|u_0\|+\frac{Mr}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\mu_1\|_{L^p}+\frac{T^q\rho^*m^*\mu_2^*}{q}\Big)\\
&\quad+Mr\sum_{k=1}^{m}L_k+M\sum_{k=1}^{m}N_k.
\end{aligned}\tag{3.8}$$

Dividing both sides of (3.8) by r, and taking $r\to\infty$, we get

$$\frac{M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\mu_1\|_{L^p}+\frac{T^q\rho^*m^*\mu_2^*}{q}\Big)+M\sum_{k=1}^{m}L_k\geq 1.$$

This contradicts (3.6). Hence for some positive number r, $PB_r\subset B_r$.

Step 2: We show that P is continuous from B_r into B_r.

Let $u_n\in B_r$, $n=1,2,\dots$, be a sequence such that $u_n\to u\in B_r$ in $PC(J,X)$ as $n\to\infty$. By (H_1)(i) and (H_2)(i), we see that for almost every $t\in J$,

$$f\Big(t,u_n(t),\int_0^t\rho(t,s)h\big(t,s,u_n(s)\big)\,ds\Big)\to f\Big(t,u(t),\int_0^t\rho(t,s)h\big(t,s,u(s)\big)\,ds\Big),$$

as $n\to\infty$. Noting that $u_n\to u$ in $PC(J,X)$, we infer that there exists $\varepsilon>0$ such that $\|u_n-u\|_{PC}\leq\varepsilon$ for k sufficiently large. Therefore,

$$\begin{aligned}
&\Big\|f\Big(t,u_n(t),\int_0^t\rho(t,s)h\big(t,s,u_n(s)\big)\,ds\Big)-f\Big(t,u(t),\int_0^t\rho(t,s)h\big(t,s,u(s)\big)\,ds\Big)\Big\|\\
&\quad\leq\mu_1(t)\big(\|u_n(t)\|+\|u(t)\|\big)\\
&\qquad+\mu_2(t)\Big(\int_0^t\big\|\rho(t,s)h\big(t,s,u_n(s)\big)\big\|\,ds+\int_0^t\big\|\rho(t,s)h\big(t,s,u(s)\big)\big\|\,ds\Big)\\
&\quad\leq\mu_1(t)\big(\|u_n(t)-u(t)\|+2\|u(t)\|\big)\\
&\qquad+\mu_2(t)\rho^*\int_0^t m(t,s)\big(\|u_n(s)-u(s)\|+2\|u(s)\|\big)\,ds\\
&\quad\leq\big(\mu_1(t)+\mu_2^*\rho^*m^*\big)\big(\varepsilon+2\sup_{t\in J}\|u(t)\|\big).
\end{aligned}$$

The continuity of I_k ($k=1,\dots,m$) and Lebesgue's dominated convergence theorem imply that

$$\begin{aligned}
&\big\|(Pu_n)(t)-(Pu)(t)\big\|\\
&\quad\leq\int_0^t(t-s)^{q-1}\Big\|R(t-s)\Big[f\Big(t,u_n(t),\int_0^t\rho(t,s)h\big(t,s,u_n(s)\big)\,ds\Big)\\
&\qquad-f\Big(t,u(t),\int_0^t\rho(t,s)h\big(t,s,u(s)\big)\,ds\Big)\Big]\Big\|\,ds\\
&\qquad+\sum_{0<t_k<t}\big\|Q(t-t_k)\big[I_k\big(u_n(t_k)\big)-I_k\big(u(t_k)\big)\big]\big\|\\
&\quad\leq\frac{M}{\Gamma(q)}\int_0^t(t-s)^{q-1}\Big\|f\Big(t,u_n(t),\int_0^t\rho(t,s)h\big(t,s,u_n(s)\big)\,ds\Big)\\
&\qquad-f\Big(t,u(t),\int_0^t\rho(t,s)h\big(t,s,u(s)\big)\,ds\Big)\Big\|\,ds\\
&\qquad+M\sum_{0<t_k<t}\big\|I_k\big(u_n(t_k)\big)-I_k\big(u(t_k)\big)\big\|\to 0,\quad n\to\infty.
\end{aligned}$$

So,

$$\lim_{n\to\infty}\|Pu_n-Pu\|_{PC}=0,$$

meaning that P is continuous.

Step 3: We prove that $\{Pu : u \in B_r\}$ is a family of equicontinuous functions.

Since $W(t)$ is strongly continuous on $D(B)$ for $t \geq 0$, we know that $\{Q(\cdot)u_0 : \cdot \in J\}$ is equicontinuous. Without loss of generality, we suppose that $t_j \leq r_2 < r_1 < t_{j+1}$, $u \in B_r$. Then we obtain

$$\begin{aligned}
&\Big\|\int_0^{r_1}(r_1-s)^{q-1}R(r_1-s)f\Big(s,u(s),\int_0^s \rho(s,\tau)h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\,ds \\
&\qquad -\int_0^{r_2}(r_2-s)^{q-1}R(r_2-s)f\Big(s,u(s),\int_0^s \rho(s,\tau)h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\,ds\Big\| \\
&\quad \leq \int_0^{r_2}\Big\|\big[(r_1-s)^{q-1}R(r_1-s)-(r_2-s)^{q-1}R(r_2-s)\big] \\
&\qquad \times f\Big(s,u(s),\int_0^s \rho(s,\tau)h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\Big\|\,ds \\
&\qquad + \int_{r_2}^{r_1}(r_1-s)^{q-1}\big\|R(r_1-s)\big\|\Big\|f\Big(s,u(s),\int_0^s \rho(s,\tau)h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\Big\|\,ds \\
&\quad =: I_1+I_2,
\end{aligned}$$

$$\begin{aligned}
I_1 \leq\ & \int_0^{r_2}\big|(r_1-s)^{q-1}-(r_2-s)^{q-1}\big|\big\|R(r_1-s)\big\| \\
& \times \Big\|f\Big(s,u(s),\int_0^s \rho(s,\tau)h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\Big\|\,ds \\
& + \int_0^{r_2}(r_2-s)^{q-1}\big\|R(r_1-s)-R(r_2-s)\big\| \\
& \times \Big\|f\Big(s,u(s),\int_0^s \rho(s,\tau)h\big(s,\tau,u(\tau)\big)\,d\tau\Big)\Big\|\,ds \\
\leq\ & \frac{M}{\Gamma(q)}\int_0^{r_2}\big|(r_1-s)^{q-1}-(r_2-s)^{q-1}\big|\Big(\mu_1(s)\big\|u(s)\big\|+\mu_2^*\rho^* m^*\sup_{\tau\in[0,s]}\big\|u(\tau)\big\|\Big)\,ds \\
& + \int_0^{r_2}(r_2-s)^{q-1}\big\|R(r_1-s)-R(r_2-s)\big\|\Big(\mu_1(s)\big\|u(s)\big\| \\
& + \mu_2^*\rho^* m^*\sup_{\tau\in[0,s]}\big\|u(\tau)\big\|\Big)\,ds \\
\leq\ & \frac{Mr}{\Gamma(q)}\int_0^{r_2}\big|(r_1-s)^{q-1}-(r_2-s)^{q-1}\big|\mu_1(s)\,ds \\
& + r\int_0^{r_2}\mu_1(s)(r_2-s)^{q-1}\big\|R(r_1-s)-R(r_2-s)\big\|\,ds \\
& + \frac{M\rho^* m^*\mu_2^* r}{\Gamma(q)}\int_0^{r_2}\big|(r_1-s)^{q-1}-(r_2-s)^{q-1}\big|\,ds \\
& + \rho^* m^*\mu_2^* r\int_0^{r_2}(r_2-s)^{q-1}\big\|R(r_1-s)-R(r_2-s)\big\|\,ds \\
=:\ & I_3+I_4+I_5+I_6.
\end{aligned}$$

Clearly, I_3 tends to zero as $r_2 \to r_1$. Also, I_4 tends to zero as $r_2 \to r_1$ as a consequence of the continuity of $R(t)$ in the uniform operator topology for $t > 0$. Similarly, I_5 and I_6 tend to zero as $r_2 \to r_1$, respectively.

For I_2, we have

$$I_2 \le \frac{Mr}{\Gamma(q)}\Big[\int_{r_2}^{r_1}(r_1-s)^{q-1}\mu_1(s)\,ds+\rho^*m^*\mu_2^*\int_{r_2}^{r_1}(r_1-s)^{q-1}\,ds\Big]\to 0,\quad \text{as } r_2\to r_1.$$

In conclusion,

$$\big\|(Pu)(r_2)-(Pu)(r_1)\big\|\to 0,\quad \text{as } r_2\to r_1,$$

which means that the operator $P: B_r\to B_r$ is equicontinuous.

Let $H=\overline{co}Q(B_r)$. Then it is easy to see that P maps H into itself and $H\subset B_r$ is equicontinuous.

Step 4: Now we prove that $P: H\to H$ is a condensing operator.

For any $D\subset H$, by Lemma 2.6, there exists a countable set $D_0=\{u_n\}\subset D$ such that

$$\alpha\big(P(D)\big)\le 2\alpha\big(P(D_0)\big).$$

By the equicontinuity of H, we know that $D_0\subset D$ is also equicontinuous.

For $t\in J$, by Lemmas 2.5 and 2.8, we have

$$\begin{aligned}
&\alpha\big(P(D_0)(t)\big)\\
&\quad=\alpha\bigg(\Big\{Q(t)u_0\\
&\qquad+\int_0^t(t-s)^{q-1}R(t-s)f\Big(s,u_n(s),\int_0^s\rho(s,\tau)h\big(s,\tau,u_n(\tau)\big)\,d\tau\Big)\,ds\\
&\qquad+\sum_{0<t_k<t}Q(t-t_k)I_k\big(u_n(t_k)\big)\Big\}\bigg)\\
&\quad\le\frac{2M}{\Gamma(q)}\int_0^t(t-s)^{q-1}\alpha\bigg(\Big\{f\Big(s,u_n(s),\int_0^s\rho(s,\tau)h\big(s,\tau,u_n(\tau)\big)\,d\tau\Big)\Big\}\bigg)\,ds\\
&\qquad+M\sum_{0<t_k<t}\alpha\big(\big\{I_k\big(u_n(t_k)\big)\big\}\big)\\
&\quad\le\frac{2M}{\Gamma(q)}\int_0^t(t-s)^{q-1}\Big[\eta(s)\Big(\alpha(D_0)+2\rho^*\int_0^s\zeta(s,\tau)\,d\tau\alpha(D_0)\Big)\Big]\,ds\\
&\qquad+M\sum_{k=1}^m M_k\alpha(D_0)\\
&\quad\le\Big\{\frac{2M}{\Gamma(q)}\int_0^t(t-s)^{q-1}\big[\eta(s)+2\rho^*\zeta^*\big]\,ds+M\sum_{k=1}^m M_k\Big\}\alpha(D_0)\\
&\quad\le\Big[\frac{2M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\eta\|_{L^p}+\frac{2\rho^*\zeta^*T^q}{q}\Big)+M\sum_{k=1}^m M_k\Big]\alpha(D).
\end{aligned}$$

Since $P(D_0)\subset H$ is bounded and equicontinuous, we know from Lemma 2.7 that

$$\alpha\big(P(D_0)\big)=\max_{t\in J}\alpha\big(P(D_0)(t)\big).$$

Therefore, by (3.7) we have

$$\alpha\big(P(D)\big) \le 2\Big[\frac{2M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\eta\|_{L^p} + \frac{2\rho^*\zeta^*T^q}{q}\Big) + M\sum_{k=1}^{m} M_k\Big]\alpha(D) < \alpha(D).$$

Thus, $P : H \to H$ is a condensing mapping. It follows from Theorem 2.9 that P has at least one fixed point $u \in H$, which is just a mild solution of the system (1.1). □

When $B = I$, one has $D(B) = X$. We assume that A generates a norm-continuous semigroup $\{W(t)\}_{t\ge 0}$ of uniformly bounded linear operators on X. Then we have the following theorems.

Theorem 3.5. *Assume that (I_1)–(I_4) are satisfied. Then the system*

$$\begin{cases} {}^cD^q u(t) = Au(t) + f\big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds\big), & t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1,2,\dots,m, \\ u(0) = u_0, & u_0 \in X, \end{cases} \tag{3.9}$$

has a unique mild solution on J.

Theorem 3.6. *Assume that (I_1)–(I_4) are satisfied. Let $u(t)$, $v(t)$ be the unique mild solutions of the system (3.9) and the following system (3.10), respectively:*

$$\begin{cases} {}^cD^q u(t) = Au(t) + f\big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds\big), & t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1,2,\dots,m, \\ u(0) = v_0, & v_0 \in X. \end{cases} \tag{3.10}$$

Then there exists a constant $\widetilde{M} > 0$ such that

$$\|u - v\|_{PC} \le \widetilde{M}\|u_0 - v_0\|.$$

Theorem 3.7. *Assume that (H_1)–(H_4) are satisfied. Then the system (3.9) has at least one mild solution on J.*

4. Applications

In this section, we give some examples to illustrate the main results obtained above.

Example 4.1. Consider the problem (1.2) presented in Section 1. Let $X = L^2(\mathbb{R})$, $D(A) = D(B) = H^2(\mathbb{R})$, $Au = \frac{\partial^2 u}{\partial x^2}$, and $Bu = u - \frac{\partial^2 u}{\partial x^2}$. Then by Theorem 4.1 in [19], the pair (A, B) generates a uniformly bounded propagation family $W(t)$. As in the proofs of (2.15), (2.16), and (2.17) in [21], we see that $\{W(t)\}_{t\ge 0}$ is norm-continuous for $t > 0$ and $\|W(t)\| \le 1$. Thus, if conditions (H_1)–(H_3) hold, and

$$\max\Big\{\frac{T^{q-\frac{1}{p}}}{\Gamma(q+1)}\big(ql_{p,q}\|\mu_1\|_{L^p} + T^{\frac{1}{p}}\rho^* m^* \mu_2^*\big) + \sum_{k=1}^{m} L_k,$$
$$\frac{4T^{q-\frac{1}{p}}}{\Gamma(q+1)}\big(ql_{p,q}\|\eta\|_{L^p} + 2T^{\frac{1}{p}}\rho^*\zeta^*\big) + 2\sum_{k=1}^{m} M_k\Big\} < 1,$$

then by Theorem 3.4, we know that there exists at least one mild solution for the problem (1.2).

Example 4.2. Let $q = \frac{1}{2}$, $X = \mathbb{R}$, $A = 0$, $B = I$, $T = 1$, $f(t, x_1, x_2) = 2t + \frac{1}{2}x_1 + \frac{1}{2}x_2$, $\rho \equiv 1$, $h(t, s, x) = 4s + x$, $m = 2$, $t_1 = \frac{1}{3}$, $t_2 = \frac{2}{3}$, $I_1(x) = I_2(x) = \frac{1}{200}x$, and $u_0 = 0$. Then $M = 1$, $M(c_1 + c_2) = \frac{1}{100} < 1$. Hence, the conditions (I_1)–(I_4) in Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, the problem (1.1) has a unique mild solution on J. More specifically, we have

$$W_n = M^n \sum_{i=0}^{n} \frac{C_n^i (\sum_{k=1}^{m} c_k)^{n-i} (l_1 + \rho^* T l_2 l_3)^i (T^q)^i}{\Gamma(iq + 1)} = \sum_{i=0}^{n} \frac{C_n^i (\frac{1}{100})^{n-i}}{\Gamma(\frac{i}{2} + 1)},$$

$W_1 \approx 1.14$, $W_2 \approx 1.02$, and $W_3 \approx 0.78$. Hence, P^3 is a contraction mapping, although P is not a contraction mapping. So, $\lim_{n\to\infty} P^{3n} v$ converges to the unique fixed point of P, for any $v \in PC(J, X)$. This fixed point is the unique mild solution of the system (1.1).

Example 4.3. Consider the following impulsive Cauchy problem for fractional partial-integral differential equations:

$$\begin{cases} {}^cD^q(u - \triangle)(t) = \triangle u(t) + f(t, u(t), \int_0^t \rho(t,s) h(t, s, u(s))\, ds) - \triangle f, & t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $q = \frac{1}{2}$, $T = 1$, $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, $\rho(t, s) \equiv 1$,

$$h\big(t, s, u(s, x)\big) = \frac{1}{5} \sin s^3 \cdot \cos \frac{u(s, x)}{t^2}, \qquad f(t, u, v) = \frac{1}{5k \cdot \sqrt[k]{t}} u + \frac{t^3}{k^2} v,$$

$m = 3$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$, $I_1(u) = \frac{1}{50}u$, $I_2(u) = \frac{1}{50}\sin u$, $I_3(u) \equiv u_1$, and $u_0, u_1 \in H^2(\mathbb{R}^3)$.

Take $X = L^2(\mathbb{R}^3)$, and define

$$Au = \triangle u, \qquad Bu = u - \triangle u,$$

with $D(A) = D(B) = H^2(\mathbb{R}^3)$. Then problem (4.1) is transformed into (1.1). By [21], we know that the pair (A, B) generates a uniformly bounded propagation family $W(t)$. Moreover, by arguments similar to those in the proofs of (2.15), (2.16), and (2.17) in [21], we deduce that $\{W(t)\}_{t \geq 0}$ is norm-continuous for $t > 0$ and $\|W(t)\| \leq 1$. Clearly,

$$\big\|f(t, u, v)\big\| \leq \frac{1}{5k \cdot \sqrt[k]{t}} \|u\| + \frac{t^3}{k^2} \|v\| := \mu_1(t)\|u\| + \mu_2(t)\|v\|,$$

and if we take $k = 6$, then for any bounded sets $D_1, D_2 \subset X$,

$$f(t, D_1, D_2) \leq \frac{1}{5k \cdot \sqrt[k]{t}} \big(\alpha(D_1) + \alpha(D_2)\big) := \eta(t)\big(\alpha(D_1) + \alpha(D_2)\big), \quad t \in (0, 1].$$

Moreover,

$$\big\|h(t, s, u)\big\| \leq \frac{s^3}{5t^2} \|u\| := m(t, s)\|u\|$$

and

$$m^* := \sup_{t\in[0,1]} \int_0^t m(t,s)\,ds = \sup_{t\in[0,1]} \int_0^t \frac{s^3}{5t^2}\,ds = \frac{1}{20}.$$

For any $u_1, u_2 \in X$,

$$\big\|h(t,s,u_1) - h(t,s,u_2)\big\| \le \frac{s^3}{5t^2}\|u_1 - u_2\|.$$

So, for any bounded set $D \subset X$,

$$\alpha\big(h(t,s,D)\big) \le \frac{s^3}{5t^2}\alpha(D) := \zeta(t,s)\alpha(D)$$

and

$$\zeta^* := \sup_{t\in[0,1]} \int_0^t \zeta(t,s)\,ds = \sup_{t\in[0,1]} \int_0^t \frac{s^3}{5t^2}\,ds = \frac{1}{20}.$$

If we take $p = 5$, then

$$\|\mu_1\|_{L^p([0,1],\mathbb{R}^+)} = \|\eta\|_{L^p([0,1],\mathbb{R}^+)} = \frac{6^{\frac{1}{5}}}{30}, \qquad \mu_2^* = \frac{1}{36}.$$

Noting that

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}, \qquad l_{p,q} := \Big(\frac{p-1}{pq-1}\Big)^{\frac{p-1}{p}} = \Big(\frac{8}{3}\Big)^{\frac{4}{5}},$$

$$L_1 = M_1 = \frac{1}{50}, \qquad L_2 = M_2 = \frac{1}{50}, \qquad L_3 = M_3 = 0,$$

we have

$$\begin{aligned}&\frac{M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\mu_1\|_{L^p} + \frac{T^q\rho^* m^*\mu_2^*}{q}\Big) + M\sum_{k=1}^{m} L_k\\ &= \frac{1}{\sqrt{\pi}}\Big(\Big(\frac{8}{3}\Big)^{\frac{4}{5}}\cdot\frac{6^{\frac{1}{5}}}{30} + \frac{1}{360}\Big) + \frac{1}{25} \approx 0.1 < 1\end{aligned}$$

and

$$\frac{2M}{\Gamma(q)}\Big(l_{p,q}T^{q-\frac{1}{p}}\|\eta\|_{L^p} + \frac{2\rho^*\zeta^* T^q}{q}\Big) + M\sum_{k=1}^{m} M_k = \frac{2}{\sqrt{\pi}}\Big(\Big(\frac{8}{3}\Big)^{\frac{4}{5}}\cdot\frac{6^{\frac{1}{5}}}{30} + \frac{1}{5}\Big) + \frac{1}{25} \approx 0.38.$$

Hence, (H_1)–(H_4) are satisfied. This means that the system (4.1) has a mild solution by Theorem 3.4.

5. Conclusion

Making use of the theory of propagation family and Laplace transforms, we present a suitable definition of mild solutions for the impulsive Cauchy problem of some fractional Sobolev-type integro-differential equations in Banach spaces with operator pairs and impulsive conditions. Moreover, we provide some general sufficient conditions to ensure the existence of mild solutions for these equations without the assumption that the operator B has bounded or compact inverse. Finally, some applications of our abstract results are given.

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