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CARLESON MEASURES ON CIRCULAR DOMAINS AND CANONICAL EMBEDDINGS OF HARDY SPACES INTO FUNCTION LATTICES

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Abstract. We study general variants of spaces of holomorphic functions on circular domains on the complex plane. We define Hardy-type spaces generated by Banach function lattices, for which we prove the Carleson theorem. We also analyze canonical embeddings of such spaces into appropriate function lattices. Finally, we study composition operators on Hardy-type spaces on circular domains and characterize order-boundedness of such maps.

1. Introduction

Various types of spaces of holomorphic functions on domains in the complex plane have been extensively studied over the last hundred years. Most attention has been focused on the disk functions (see, e.g., Duren's monograph [7]), even though the foundations of the theory for different domains had already been established in the 1950s (see Rudin's article [18]; see also Fisher's book [8]).

Let Ω be a multiply connected domain in $\mathbb{C}$, and let $H(\Omega)$ denote the set of holomorphic functions on Ω. The Hardy space $H^p(\Omega)$, where $p \in [1, \infty)$, contains all functions $f \in H(\Omega)$ such that there exists a harmonic majorant of $|f|^p$ (see [18]). These spaces were studied mainly within the context of operator theory and related topics. We mention here Sarason's essay [21] and, more recently,

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Qui's paper [16], Chalendar and Partington's article [4], and the second author's works [19], [20].

The purpose of the present paper is to consider general variants of Hardy spaces on multiply connected domains. An extension of Rudin's original ideas has led us to a definition of the abstract Hardy space $HX(\Omega)$, generated by a rearrangement-invariant Banach lattice X. Then, using techniques from both harmonic and functional analysis, we prove the main structural properties of these spaces, including a direct sum representation and a variant of the Fatou–Riesz theorem. Finally, we study canonical embeddings of the mentioned spaces into appropriate function lattices and prove a variant of Carleson's famous theorem. At the end of the present article, we characterize order-bounded composition operators on abstract Hardy spaces on multiply connected domains. The latter part of this article is, in a sense, a domain counterpart of Mastyło and Rodríguez-Piazza's work [14] on the disk.

2. Preliminaries

We will use techniques from the theory of Banach spaces and interpolation theory, along with selected methods from harmonic analysis, to study Hardy spaces on circular domains and related issues. In order to make the paper as self-contained as possible, we start with an introductory section.

2.1. Banach lattices. Let (Ω, Σ, μ) be a complete σ-finite measure space, and let $L^0(\Omega) := L^0(\Omega, \Sigma, \mu)$ denote the space of real-valued measurable functions on Ω with the topology of convergence in measure on μ-finite sets. The order $|f| \leq |g|$ means that $|f(\omega)| \leq |g(\omega)|$ for μ-almost all $\omega \in \Omega$. If a real Banach space $X \subset L^0(\Omega)$ is such that there exists $f \in X$ with $f > 0$ μ-a.e. on Ω and for all $h \in L^0(\Omega)$, if $|h| \leq |g|$ with $g \in X$ implies that $h \in X$ and $\|h\|_X \leq \|g\|_X$, then X is said to be a *Banach lattice* (on Ω or on (Ω, μ)).

The Köthe dual space X' of a normed space $X \hookrightarrow L^0(\Omega)$ is defined as the space of all $f \in L^0(\Omega)$ such that $fg \in L^1(\Omega)$ for every $g \in X$. Note that X' is a Banach lattice on (Ω, Σ, μ) when equipped with the norm $\|f\|_{X'} := \sup\{fg : \|g\|_X \leq 1\}$. A Banach lattice X is said to be *maximal* if $X'' = X$. Equivalently, we say that X possesses the *Fatou property* if for any sequence $(f_n) \subset X$ such that $f_n \to f$ a.e. and $\sup\{\|f_n\|_X : n \in \mathbb{N}\} < \infty$, we have $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$ (see [17, p. 102]).

An important class of Banach lattices are rearrangement-invariant spaces. Given $f \in L^0(\Omega)$, its *distribution function* is defined by $\mu_f(\lambda) = \mu(\{t \in \Omega : |f(t)| > \lambda\})$, $\lambda \geq 0$. A Banach lattice X is said to be *rearrangement-invariant* (*r.i. space*, for short) if, for any $f \in X$ and $g \in L^0(\Omega)$ such that $\mu_f = \mu_g$, we have $g \in X$ and $\|f\|_X = \|g\|_X$. It is well known that if X is an r.i. space for some finite measure space Ω, then $L^\infty(\Omega) \hookrightarrow X \hookrightarrow L^1(\Omega)$ (see [17]).

Throughout the paper we will consider *complex* r.i. spaces. The term *complex r.i. space* refers to the *complexification* of a real r.i. space; that is, if X denotes the (real) r.i. space, then the complexification $X(\mathbb{C})$ of X is the Banach space of all complex-valued measurable functions f on Ω such that the element $|f|$ defined by $|f|(\omega) = |f(\omega)|$ for $\omega \in \Omega$ is in X and $\|f\| = \||f|\|_X$. For simplicity of presentation, we will often write *r.i. space* instead of *complex r.i. space* and avoid using the symbol $X(\mathbb{C})$.

If X is an r.i. space, then for any measurable set A, the expression $\|\chi_A\|$ depends only on $\mu(A)$. Thus, for every $t \in \{\mu(A) : A \in \Sigma\}$, we define a function ϕ_X by the formula $\phi_X(t) = \|\chi_A\|$ where A is any measurable set with $\mu(A) = t$. This function is called the *fundamental function* of X. If X is an r.i. function space on a nonatomic measure space (Ω, Σ, μ), then ϕ_X is quasiconcave on $[0, \tau)$ with $\tau = \mu(\Omega)$; that is, $\phi_X(0) = 0$, ϕ_X is positive, nondecreasing, and $t \mapsto \phi_X(t)/t$ is nonincreasing on $(0, \tau)$ (see [1]). Note that, for a quasiconcave function ψ, there exists a concave function $\overline{\psi}$, given by $\overline{\psi}(t) = \inf\{(1 + \frac{t}{s})\psi(s) : s \in (0, \tau)\}$, such that $\psi(t) \le \overline{\psi}(t) \le 2\psi(t)$ for $t \in [0, \tau)$. Thus, we may assume that ϕ_X is a concave function. Note also that ϕ_X is continuous at zero if and only if $X \neq L^\infty$.

We will give two important examples of r.i. spaces: Lorentz spaces and Orlicz spaces. Let ψ be a positive, increasing, and concave function on $[0, \tau)$. Let (Ω, Σ, μ) be a nonatomic measure space. The Lorentz space $\Lambda(\psi)$ consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{\Lambda(\psi)} = \int_0^\tau f^*(t)\, d\psi(t) < \infty.$$

It is well known that the fundamental function of the Lorentz space $\Lambda(\psi)$ equals ψ. Let Φ be an Orlicz function; that is, $\Phi\colon [0, \infty) \to [0, \infty)$, continuous, and convex. Then the Orlicz space L^Φ consists of the elements from $L^0(\mu)$ such that

$$\|f\|_{L^\Phi} := \inf\Big\{\varepsilon > 0 : \int_\Omega \Phi\Big(\frac{|f|}{\varepsilon}\Big)\, d\mu \le 1\Big\} < \infty.$$

2.2. Interpolation theory and Riesz–Fischer spaces. One of the main tools we use below is *interpolation theory* (see [1], [2], and [10]).

Let E_0 and E_1 be Banach spaces. The pair $\vec{E} = (E_0, E_1)$ is called a *Banach couple* if there exists a Hausdorff topological vector space $\mathcal{X}$ such that $E_j \hookrightarrow \mathcal{X}$, $j = 0, 1$. A Banach space E is called an *intermediate space* with respect to $\vec{E}$ if $E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1$.

A Banach space E, intermediate with respect to the Banach couple $\vec{E}$, is called a *C-interpolation* (an exact interpolation provided $C = 1$) between E_0 and E_1 if, for every operator $T\colon \vec{E} \to \vec{E}$, we have $T(E) \subset E$ and $\|T\|_{E\to E} \le C \max\{\|T|_{E_0}\|_{E_0\to E_0}, \|T|_{E_1}\|_{E_1\to E_1}\}$.

A mapping $\mathcal{F}$ from the category of all couples of Banach spaces into the category of all Banach spaces is said to be a *bounded interpolation functor* if there exists a constant $C > 0$ such that for every Banach couple $\vec{E}$, $\mathcal{F}(\vec{E})$ is a Banach space intermediate with respect to $\vec{E}$, and, for any $T\colon \vec{E} \to \vec{F}$, $T(\mathcal{F}(\vec{E})) \subset \mathcal{F}(\vec{F})$ with

$$\|T\|_{\mathcal{F}(\vec{E})\to\mathcal{F}(\vec{F})} \le C \max_{i=0,1} \|T|_{E_i}\|_{E_i\to F_i}.$$

If $C = 1$, then $\mathcal{F}$ is said to be an *exact* interpolation functor.

Let (Ω, μ) be a σ-finite and complete measure space. Recall that every maximal r.i. space $X = X(\Omega, \mu)$ is a so-called *monotone Riesz–Fischer* space (see [1, pp. 116, 304–305]); namely, the norm $\|\cdot\|_X$ is represented by the Riesz–Fischer norm $\|\cdot\|_E$, where $E = E([0,\mu(\Omega)])$ with the Lebesgue measure is an r.i. space, in the following way:

$$\|f\|_X = \|f^*_\mu\|_E, \quad f \in X.$$

Here g^* denotes the decreasing rearrangement of $g \in L^0(\Omega,\mu)$; that is, $g^*(t) := g^*_\mu(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}$, for all $t \geq 0$. In Section 4 we construct r.i. spaces on different measure spaces for a given maximal r.i. space $X(\Omega,\mu)$ using the Riesz–Fischer norm. Namely, if (Ω,ν) is any other measure space, then we define $X(\Omega,\nu) = \{f \in L^0(\Omega,\nu) \colon \|f\|_{X(\Omega,\nu)} = \|f^*_\nu\|_E < \infty\}$, where $\|\cdot\|_E$ is the Riesz–Fischer norm representing $\|\cdot\|_{X(\Omega,\mu)}$.

It is also convenient to consider, together with the space X, the space $X([0,\mu(\Omega)],m)$, where m is the Lebesgue measure on $[0,\mu(\Omega)]$. This space consists of all $h \in L^0([0,\mu(\Omega)],m)$ such that $f^*_\mu = h^*_m$ for some $f \in X$. If there exists a measure-preserving isomorphism $\theta \colon (\Omega,\mu) \to ([0,\mu(\Omega)],m)$, then the map $f \mapsto f \circ \theta^{-1}$ induces the isometric isomorphism between Banach lattices X and $X([0,\mu(\Omega)],m)$, and $X([0,\mu(\Omega)],m)$ is also an r.i. space (see [17]).

2.3. Harmonic measures. Let Ω be a domain on the Riemann sphere for which the Dirichlet problem is solvable (we write $\Omega \in (SDP)$), and let $p \in \Omega$. For $u \in C(\partial\Omega)$, denote by $\widetilde{u} \colon \overline{\Omega} \to \mathbb{R}$ the harmonic extension of u. Then the map $u \mapsto \widetilde{u}(p)$ is linear and bounded by the maximum modulus principle. The Riesz representation theorem implies that there is the unique real measure $\omega_p := \omega_{\Omega,p}$ on $\partial\Omega$ such that

$$\widetilde{u}(p) = \int_{\partial\Omega} u \, d\omega_p. \tag{2.1}$$

This measure is called a *harmonic measure* on $\partial\Omega$ for p. Note that ω_p is a probability measure which has no atoms.

Now we recall the important properties of harmonic measures. First, observe that the measure ω_p depends on the point $p \in \Omega$, but it can be shown that, for any compact subset K of Ω and $p, q \in K$, there exists a positive constant $M = M_K$ such that

$$\frac{1}{M}\omega_q(E) \leq \omega_p(E) \leq M\omega_q(E)$$

for all Borel subsets E of $\partial\Omega$ (see [8, Theorem 1.6.1, p. 19]).

Suppose now that Ω_1 and Ω_2 are two domains and that f is a holomorphic function which maps $\overline{\Omega}_1$ onto $\overline{\Omega}_2$ homeomorphically. If $\Omega_1 \in (SDP)$, then also $\Omega_2 \in (SDP)$. Let $p_1 \in \Omega_1$ and $p_2 = f(p_1)$. Denote by ω_1 the harmonic measure on $\partial\Omega_1$ for p_1, and define a measure μ on $\partial\Omega_2$ as follows:

$$\mu(E) = \omega_1\big(f^{-1}(E)\big), \quad E \subset \partial\Omega_2.$$

Then μ is a harmonic measure on $\partial\Omega_2$ for p_2.

Let $\Omega_1, \Omega_2 \in (SDP)$, $\Omega_1 \subset \Omega_2$. Let $p \in \Omega_1$, and let ω_1 and ω_2 be harmonic measures on $\partial\Omega_1$ and $\partial\Omega_2$, respectively, for p. Then for each compact set $E \subset \partial\Omega_1 \cap \partial\Omega_2$ we have $\omega_1(E) \leq \omega_2(E)$.

Suppose that Ω is bounded by a finite number of disjoint analytic Jordan curves. Then (see [8, Theorem 1.6.4.]) for each $z \in \Omega$ we have

$$d\omega_z = -\frac{1}{2\pi}\frac{\partial}{\partial n} g(\cdot, z)\, ds,$$

where $g(\cdot, z)$ is Green's function for Ω with a pole at z, $\frac{\partial}{\partial n}$ is a derivative in the outward direction normal at $\partial\Omega$, and ds is the arc length. The function $P_z(\zeta) := \frac{d\omega_z}{ds}(\zeta) = -\frac{1}{2\pi}\frac{\partial}{\partial n} g(\zeta, z)$, where $\zeta \in \partial\Omega$ and $z \in \Omega$, is called the *Poisson kernel* for Ω. It satisfies the inequalities

$$c_1 \leq \frac{d\omega_z}{ds} \leq c_2,$$

for positive constants c_1, c_2.

2.4. Hardy spaces on the disk. The prototypes of Hardy spaces on domains are obviously the classical Hardy spaces $H^p(\mathbb{D})$ on the unit disk, $p \in [1, \infty)$. We recall here that there are at least two equivalent ways of defining the mentioned disk spaces. On the one hand, a function $f \in H(\mathbb{D})$ belongs to the Hardy space $H^p(\mathbb{D})$ if there exists a harmonic majorant of $|f|^p$ on $\mathbb{D}$. On the other hand, a function $f \in H(\mathbb{D})$ belongs to $H^p(\mathbb{D})$ if a radial limit function f_* exists a.e. on $\mathbb{T}$ and $f_* \in L^p(\mathbb{T})$, where for almost all $t \in [0, 1)$ (see [7])

$$f_*(t) := \lim_{r\to 1^-} f(re^{2\pi i t}).$$

Below we will refer to the above definitions in a more general context.

Let X be a Banach lattice on $[0, 1)$. A function $f \in H(\mathbb{D})$ belongs to the Hardy space $HX(\mathbb{D})$ if

$$\sup\big\{\|f_r\|_X : r \in [0, 1)\big\} < \infty,$$

where $f_r(t) = f(re^{2\pi i t})$, $t \in [0, 1)$. Let $H_*(\mathbb{D})$ denote the set of functions $f \in H(\mathbb{D})$ having a radial limit a.e. on $\mathbb{T}$. The Hardy space $HX(\mathbb{T})$ consists of those $f \in H_*(\mathbb{D})$ for which $f_* \in X$. When X is a maximal r.i. space on $[0, 1)$, then $HX(\mathbb{D}) = HX(\mathbb{T})$ (see [13]).

For particular spaces X, the above method produces variants of Hardy spaces widely studied in the literature. For example, if $X = L^p$, $p \in [1, \infty]$, then HX is the standard Hardy space H^p (see [6]). In a similar way we define Hardy–Lorentz (see [12]) and Hardy–Orlicz (see [11]) spaces.

3. Abstract Hardy spaces on circular domains

In this section we define and prove the basic properties of a general variant of Hardy spaces on finitely connected domains generated by r.i. spaces. Let us note that although "boundary" Hardy spaces on circular domains were studied in [15], a corresponding theory of harmonic majorants for abstract variants of Hardy spaces has not been thoroughly investigated yet. The aim of this section is to bridge this gap.

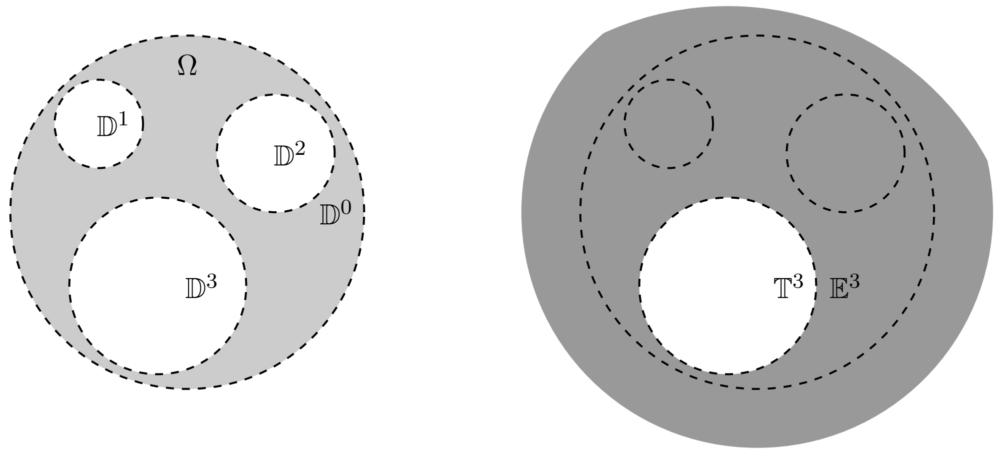

FIGURE 1. A circular domain Ω.

3.1. Circular domains. Let Ω be a bounded domain whose boundary consists of $m+1$ disjoint analytic Jordan curves; that is,

$$\partial\Omega = \bigcup_{k=0}^{m} \Gamma^k,$$

where Γ^k is an analytic Jordan curve and $\Gamma^k \cap \Gamma^j = \emptyset$ for $k \neq j$. Assume that Γ^0 is the boundary of the unbounded component of the complement of Ω. Denote by $\mathbb{E}^0$ the bounded component of $\mathbb{C}^* \setminus \Gamma^0$, and denote by $\mathbb{E}^k$ the unbounded component of $\mathbb{C}^* \setminus \Gamma^k$, for $k \in \{1, 2, \ldots, m\}$, where $\mathbb{C}^*$ is the Riemann sphere. Notice that any such set is conformally equivalent to some *circular domain*, that is, the unit disk with m smaller closed and not intersecting disks removed (see [5] and Figure 1). Therefore, in the sequel we will assume that Ω is a circular domain.

For $i = 1, \ldots, m$, $\mathbb{D}^i$ denotes the disk with radius $r_i \in (0,1)$ centered at $a_i \in \mathbb{D}$. The symbol $\mathbb{T}^i$ will stand for the boundary of $\mathbb{D}^i$. Sometimes it will be convenient to write $\mathbb{D}^0$ instead of $\mathbb{D}$ and $\mathbb{T}^0$ instead of $\mathbb{T}$. The complement of $\mathbb{D}^i \cup \mathbb{T}^i$ in $\mathbb{C}^*$ is denoted by $\mathbb{E}^i$ (see Figure 1).

Since $\mathbb{E}^i$, $i \in \{1, \ldots, m\}$, and $\mathbb{D}$ are simply connected, there exist conformal maps $\eta_i \colon \mathbb{D} \to \mathbb{E}^i$ between these regions, namely:

$$\eta_i(z) = \begin{cases} \frac{r_i}{z} + a_i & \text{for } z \in \mathbb{D} \setminus \{0\}, \\ \infty & \text{for } z = 0, \end{cases}$$

$$\eta_i^{-1}(z) = \begin{cases} \frac{r_i}{z - a_i} & \text{for } z \in \mathbb{E}^i \setminus \{\infty\}, \\ 0 & \text{for } z = \infty. \end{cases}$$

Let $\{\Omega_n\}$ be a *regular exhaustion* of Ω; that is, a sequence $\{\Omega_n\}_{n=1}^{\infty}$ of subdomains of Ω which satisfy the following conditions:

(i) $\overline{\Omega}_n \subset \Omega_{n+1}$ for $n \in \mathbb{N}$;
(ii) $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$;

(iii) every component of $\partial\Omega_n$ is formed by an analytic Jordan curve, for each $n \in \mathbb{N}$;
(iv) every element of the sequence $\{\Omega_n\}_{n=1}^{\infty}$ is $(m+1)$-connected.

3.2. Hardy spaces on circular domains. Let $u\colon \Omega \to (-\infty, \infty]$ be any subharmonic and continuous function. The symbol u_n will stand for the solution of the Dirichlet problem for the function u restricted to $\overline{\Omega}_n$, where Ω_n is an element of a regular exhaustion of Ω. Note that, since u is subharmonic, the sequence $\{u_n\}$ is nondecreasing, meaning that $u_n \leq u_{n+1}$ on Ω_n. By Harnack's theorem either $\{u_n\}$ converges uniformly on compact subsets of Ω to harmonic function v, or $u_n(z) \to \infty$ for every $z \in \Omega$. From the construction it follows that v (if it exists) is the *least harmonic majorant* of subharmonic function u.

Examples 3.1. Using the integral representation (2.1) of evaluation of harmonic functions at a given point and the construction of harmonic majorants, we obtain some standard examples of Hardy-type spaces on Ω.

(i) Put $X_n(\partial\Omega_n, \omega_{n,z}) = L^p(\partial\Omega_n, \omega_{n,z})$, where $p \geq 1$, $\omega_{n,z} := \omega_{\Omega_n,z}$ and $n \in \mathbb{N}$. For any $f \in H(\Omega)$ we have

$$\begin{aligned}\int_{\partial\Omega_n} |f|^p \, d\omega_{n,z} &= \int_{\partial\Omega_n} u_{f,n,p} \, d\omega_{n,z} = \int_{\partial\Omega_{n-1}} u_{f,n,p} \, d\omega_{n-1,z} \\ &\geq \int_{\partial\Omega_{n-1}} |f|^p \, d\omega_{n-1,z},\end{aligned}$$

where $u_{f,n,p}$ is the solution of the Dirichlet problem for the subharmonic function $|f|^p$ restricted to $\partial\Omega_n$. Note that the second equality follows from the definition of harmonic measure (general version of the mean-value property). The inequality follows from the subharmonicity of $|f|^p$. The sequence $\{\|f\|_{X_n(\partial\Omega_n,\omega_{n,z})}\}$ is nondecreasing so if the limit is finite, then, by Harnack's theorem, it is equal to $v_f(z)^{1/p}$, where v_f is the least harmonic majorant of $|f|^p$ and the value $v_f(z)^{1/p}$ is independent of the choice of $\{\Omega_n\}$. In this case

$$\|f\|_{H^p(\Omega)} = \lim_{n\to+\infty} \|f\|_{X_n(\partial\Omega_n,\omega_{n,z})}.$$

We can also use Harnack's inequality to prove that two different points $z_1, z_2 \in \Omega$ generate equivalent norms. In the case of $\Omega = \mathbb{D}$ and $z = 0$ we have the classical definition of H^p.

(ii) Let $X_n(\partial\Omega_n, \omega_{n,z}) = L^{\Phi}(\partial\Omega_n, \omega_{n,z})$, where Φ is a convex Orlicz function. In a similar way, we see that the equality

$$\begin{aligned}\|f\|_{H^{\Phi}(\Omega)} &:= \lim_{n\in\mathbb{N}} \|f|_{\partial\Omega_n}\|_{X_n(\partial\Omega_n,\omega_{n,z})} = \lim_{n\to+\infty} \|f_n\|_{X_n(\partial\Omega_n,\omega_{n,z})} \\ &= \lim_{n\to\infty} \inf\Big\{\varepsilon > 0 : \int_{\partial\Omega_n} \Phi\Big(\frac{|f|}{\varepsilon}\Big) \, d\omega_{n,z} \leq 1\Big\}\end{aligned}$$

defines the Hardy–Orlicz norm of $f \in H(\Omega)$. In terms of harmonic majorants, $H^{\Phi}(\Omega)$ is a set of all holomorphic functions f for which there exists $\lambda > 0$ such that the subharmonic function $\Phi(\lambda|f|)$ has a harmonic majorant. Moreover,

$$\|f\|_{H^{\Phi}(\Omega)} = \inf\big\{\varepsilon > 0 : v_{f,\varepsilon}(z) \leq 1\big\},$$

where $v_{f,\varepsilon}$ is the least harmonic majorant of $\Phi(\frac{|f|}{\varepsilon})$. The norm is independent of the choice of $\{\Omega_n\}$, and two different points $z_1, z_2 \in \Omega$ generate equivalent norms.

Let us note that it is not at all obvious that taking as X_n Lorentz (or Marcinkiewicz) spaces on the measure spaces $(\partial\Omega_n, \omega_{n,z})$ we get that the sequence $\{\|f_n\|\}$ is nondecreasing.

In fact, in the above examples we constructed Hardy-type spaces of holomorphic functions of Ω using sequences $\{X_n\}$ of Banach lattices of the "same type" (Lebesgue or Orlicz spaces) defined on different measure spaces $(\partial\Omega_n, \omega_{n,z})$. However, this procedure cannot be applied in the abstract case (i.e, r.i. space in general) because we define the spaces X_n in each case on different measure spaces.

We have to proceed in a slightly different manner. Namely, for each measure space, construct a measure-preserving isomorphism to the fixed measure space which induces an isometric isomorphism between appropriate Banach lattices. To be precise, let X be an r.i. space on the measure space $([0,1), m)$, where $m := m_1$ is the Lebesgue measure on $[0,1)$. Note that, for each $n \in \mathbb{N}$, there exists a measure-preserving isomorphism $\theta_n := \theta_{\Omega_n} \colon (\partial\Omega_n, \omega_{n,z}) \to ([0,1), m)$, where $\omega_{n,z}$ is a harmonic measure on $\partial\Omega_n$ with respect to some $z \in \Omega_1$. Indeed, for each $n \in \mathbb{N}$, choose any points $x_n^0, \dots, x_n^m$ and $x^0, \dots, x^m$ such that $x_n^0 \in \Gamma_n^0, \dots, x_n^m \in \Gamma_n^m$, $x^k \in \Gamma^k$, $k \in \{0, \dots, m\}$, and

$$\lim_{n\to\infty} x_n^k = x^k,$$

where Γ_n^k is a component of $\partial\Omega_n$ and Γ^k is a component of $\partial\Omega$. For $\zeta \in \Gamma_n^k$, we denote by (x_n^k, ζ) an arc on Γ_n^k (with the standard orientation). For $\zeta \in \Gamma_n^k$, we put

$$\theta_n(\zeta) := \sum_{i=0}^{k-1} \omega_{n,z}(\Gamma_n^i) + \omega_{n,z}\big((x_n^k, \zeta)\big). \tag{3.1}$$

Similarly, we define the measure-preserving isomorphism $\theta \colon (\partial\Omega, \omega_z) \to ([0,1), m)$,

$$\theta(\zeta) := \sum_{i=0}^{k-1} \omega_z(\Gamma^i) + \omega_z\big((x^k, \zeta)\big),$$

for $\zeta \in \Gamma^k$. Note that θ_n and θ are in fact analogues of the standard distribution function defined by formula $d(x) = \mu((-\infty, x))$, $x \in \mathbb{R}$, where μ is a measure defined on the σ-algebra of Borel subsets of $\mathbb{R}$. Since harmonic measure has no atoms, we see that $\theta_n(x_n^0) = 0$. For any $\xi \in \Gamma_n^0$, $\xi \neq x^0$, we have $\theta_n(\xi) = \omega_{n,z}((x_n^0, \xi))$. Furthermore, $\theta_n(x_n^1) = \omega_{n,z}(\Gamma_n^0)$, and for $\xi \in \Gamma_n^1$, $\xi \neq x_n^1$, we have $\theta_n(\xi) = \omega_{n,z}(\Gamma_n^0) + \omega_{n,z}((x_n^1, \xi))$, and so on. It is easy to see that, for every $n \in \mathbb{N}$, the function $\theta_n \colon \partial\Omega_n \to [0,1)$ is a bijection with possible discontinuities at $x_n^1, x_n^2, \dots, x_n^m$ and it is a measure-preserving isomorphism because harmonic measure is a probability measure. Thus, for every $n \in \mathbb{N}$ and every holomorphic function $f \in H(\Omega)$, we put $\|f|_{\partial\Omega_n}\|_{X(\partial\Omega_n, \omega_{n,z})} := \|f|_{\partial\Omega_n} \circ \theta_n^{-1}\|_X$. We define $HX(\Omega)$ as a set of all holomorphic functions $f \in H(\Omega)$ such that

$$\|f\|_{HX(\Omega)} := \sup_{n\in\mathbb{N}} \|f|_{\partial\Omega_n}\|_{X(\partial\Omega, \omega_{n,z})} = \sup_{n\in\mathbb{N}} \|f|_{\partial\Omega_n}\|_{X(\partial\Omega, \omega_{n,z})} < \infty.$$

If G is a subdomain of $\mathbb{C}^*$ with $\infty \in G$, then by $HX_0(G)$ we denote the subspace of $HX(G)$ which contains only those elements of $HX(G)$ that vanish at ∞.

Let us note that every regular exhaustion $\{\Omega_n\}$ of Ω induces appropriate regular exhaustions $\{E_n^i\}$ of $\mathbb{E}^i$, $i \in \{0, \dots, m\}$, $n \in \mathbb{N}$. Indeed, denoting by Γ_n^i the ith component of $\partial\Omega_n$, we define E_n^0 as the bounded component of $\mathbb{C}^* \setminus \Gamma_n^0$ and E_n^i as the unbounded component of $\mathbb{C}^* \setminus \Gamma_n^i$, $i \in \{1, \dots, m\}$. Denote by $\omega_{E_n^i,z}$ and $\omega_{\mathbb{E}^i,z}$ the harmonic measures on $\Gamma_n^i = \partial E_n^i$ and on $\Gamma^i = \partial \mathbb{E}^i$, respectively, for a given $z \in E_n^i$ (or $z \in \mathbb{E}^i$). If $z \in \Omega_1$, then for every Borel set $A \subset \Gamma_n^i$ we have that

$$\omega_{\Omega_n,z}(A) \leq \omega_{E_n^i,z}(A) \leq C_n \omega_{\Omega_n,z}(A),$$

where $C_n > 0$ depends only on Ω_n and $z \in \Omega_1$. Using the fact that the sequence $\{\omega_{\Omega_n,z}\}$ is weak*-convergent to $\omega_{\Omega,z}$ in $C^*(\overline{\Omega})$ and, similarly, that the sequence $\{\omega_{E_n^i,z}\}$ is weak*-convergent to $\omega_{\mathbb{E}^i,z}$ in $C^*(\overline{\mathbb{E}^i})$, $i \in \{0, \dots, m\}$ (see [9, pp. 68, 89–90]), we deduce that there exists a general constant $C > 0$ (independent of n) such that, for any sequence $\{A_n\}$, $A_n \subset \Gamma_n^i$, we have

$$\omega_{\Omega_n,z}(A_n) \leq \omega_{E_n^i,z}(A_n) \leq C \omega_{\Omega_n,z}(A_n). \tag{3.2}$$

The following theorem plays a crucial role in the further study of abstract Hardy spaces on circular domains and their operators. The case of the standard Hardy spaces H^p was proved in [3, Lemma 2.1].

Theorem 3.2 (Direct sum representation)**.** *Let $X = X([0,1), m)$ be a maximal r.i. space.*

(i) *Any $f \in HX(\Omega)$ can be decomposed as*

$$f(z) = f^0(z) + f^1(z) + \cdots + f^m(z), \quad z \in \Omega,$$

where $f^k \in HX_0(\mathbb{E}^k)$, for any $k \in \{1, \dots, n\}$, and $f^0 \in HX(\mathbb{E}^0)$. Maps $f \mapsto f^k \colon HX(\Omega) \to HX_0(\mathbb{E}^k)$, $k = \{1, \dots, n\}$, and $f \mapsto f^0 \colon HX(\Omega) \to HX(\mathbb{E}^0)$ are bounded linear projections.

(ii) *Moreover,*

$$HX(\Omega) \cong HX(\mathbb{E}^0) \oplus HX_0(\mathbb{E}^1) \oplus \cdots \oplus HX_0(\mathbb{E}^m).$$

Proof. (i) Fix $f \in HX(\Omega)$, $z \in \Omega$, and let $C_0, \dots, C_m$ be smooth Jordan curves so close to $\mathbb{T}^0, \dots, \mathbb{T}^m$, respectively, that z is exterior to $C_1, \dots, C_m$ and interior to C_0. Now, for each $0 \leq k \leq m$, set

$$f^k(z) := \frac{1}{2\pi i} \int_{C_k} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

It is clear that $f = f^0 + f^1 + \cdots + f^m$ and $f \mapsto f^k$ is a linear operator. Thus $f^k \in H(\mathbb{E}^k)$ and it is independent of the choice of C_k, $k \in \{0, \dots, m\}$. We have also $f^k(\infty) = 0$, for each $k \in \{1, \dots, m\}$. From Cauchy's theorem it follows that, if $f \in HX(\Omega)$ and f extends holomorphically to $\mathbb{E}^k$ for some k, with $f(\infty) = 0$, then $f^l = 0$ for $l \neq k$ and hence $f = f^k$. All that remains to show is that $\|f^k\|_{HX(\mathbb{E}^k)} \leq C\|f^k\|_{HX(\Omega)}$, for some $C > 0$. Let $T_n^i \colon L^0(\partial\Omega_n, \omega_{\Omega_n,z}) \to L^0(\partial E_n^i, \omega_{E_n^i,z})$, $i \in \{0, \dots, m\}$, be the restriction operator defined by

$$T_n^i f = f|_{\partial E_n^i}, \quad f \in L^0(\partial\Omega_n, \omega_{\Omega_n,z}).$$

Note that each T_n^i induces the operator $\widetilde{T}_n^i\colon L^0([0,1),m)\to L^0([0,1),m)$ which is defined as

$$\widetilde{T}_n^i f\circ\theta_{\Omega_n}^{-1} = f|_{\partial E_n^i}\circ\theta_{E_n^i}^{-1},\quad f\in L^0(\partial\Omega_n,\omega_{\Omega_n,z}),$$

where $\theta_{\partial\Omega_n}\colon(\partial\Omega_n,\omega_{\Omega_n,z})\to([0,1),m)$ and $\theta_{\partial E_n^i}\colon(\partial E_n^i,\omega_{E_n^i,z})\to([0,1),m)$ are measure-preserving isomorphisms defined by (3.1). It is easy to show that

$$\|\widetilde{T}_n^i\|_{L^\infty([0,1))\to L^\infty([0,1))}=1.$$

Using (3.2) it is also easy to deduce that

$$\|\widetilde{T}_n^i\|_{L^1([0,1))\to L^1([0,1))}\le C.$$

Since every maximal r.i. space is an exact interpolation space between L^1 and L^∞, we get that $\|T_n^i\|_{X(\partial\Omega_n,\omega_{\Omega_n,z})\to X(\partial E_n^i,\omega_{E_n^i,z})}=\|\widetilde{T}_n^i\|_{X([0,1))\to X([0,1))}\le C$, for $i\in\{0,\dots,m\}$, $n\in\mathbb{N}$, and this implies that $\|f^k\|_{HX(\mathbb{E}^k)}\le C'\|f^k\|_{HX(\Omega)}$ for some constant $C'>0$.

(ii) From the proof of (i) it follows that the operator $(f^0,\dots,f^m)\mapsto f$ is a bounded linear bijection from $(HX_0(\mathbb{E}^0)\oplus\cdots\oplus HX_0(E^m))$ onto $HX(\Omega)$. By the open mapping theorem, the inverse operator is also continuous. □

The above theorem has important consequences.

Corollary 3.3. *Let X be a maximal r.i. space. Then $HX(\Omega)$ is a Banach space.*

Proof. Theorem 3.2 implies that $HX(\Omega)$ is isomorphic to $HX(\mathbb{E}^0)\oplus HX_0(\mathbb{E}^1)\oplus\cdots\oplus HX_0(\mathbb{E}^m)$. On the other hand, $HX_0(\mathbb{E}^i)$ is isometric to the subspace of $HX(\mathbb{D})$ containing those holomorphic functions that vanish at zero, which is clearly a Banach space. □

From Theorem 3.2 and [15, Theorem 4.4] we obtain an important corollary on the interpolation of Hardy spaces on circular domains (for the disk version of this result, see [22]). It will allow us to use interpolation theory as a handy tool in the study of canonical embeddings in Section 4.

Corollary 3.4. *For any exact interpolation functor $\mathcal{F}$, the following formula holds with equivalence of norms:*

$$\mathcal{F}\big(H^1(\Omega),H^\infty(\Omega)\big)=H\mathcal{F}\big(L^1(\partial\Omega),L^\infty(\partial\Omega)\big)(\Omega).$$

The main result of this section is contained in the theorem below. For the disk version of this result, see [13] (see also [8]).

Theorem 3.5. *Suppose that X is an r.i. space on $[0,1)$. For each $f\in H(\Omega)$, the following assertions are equivalent:*

(i) *$f\in HX(\Omega)$;*

(ii) *there exists a function $f_*\in X''(\partial\Omega)$ such that*

$$f(z)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f_*(w)}{w-z}\,dw,\quad z\in\Omega, \tag{3.3}$$

$$0=\int_{\partial\Omega}\frac{f_*(w)}{w-z}\,dw,\quad z\notin\overline{\Omega}, \tag{3.4}$$

$$f(z) = \int_{\partial\Omega} f_*(\zeta)\, d\omega_z(\zeta), \quad z \in \Omega. \tag{3.5}$$

If these conditions are satisfied, then $\|f_*\|_{X''(\partial\Omega,\omega_z)} = \|f\|_{HX(\Omega)}$.

Proof. First, we show that condition (i) implies (ii). Let $f \in HX(\Omega)$. Without loss of generality, we may assume that $X \neq L^\infty([0,1))$. Since $X \hookrightarrow L^1([0,1))$ implies that $f \in H^1(\Omega)$, so in particular, by [8, Theorem 4.4.4], the boundary function f_* exists ω_z-a.e. on $\partial\Omega$. Note that $|f|_{\partial\Omega_n}|$ is continuous and $X^\circ := X^\circ([0,1))$ (the closure of $L^\infty([0,1))$ in X) is a minimal r.i. space. Since $X \neq L^\infty([0,1))$, we have that $\|f|_{\partial\Omega_n}\|_{X(\partial\Omega_n,\omega_{n,z})} = \|f|_{\partial\Omega_n}\|_{X''(\partial\Omega_n,\omega_{n,z})} = \|f|_{\partial\Omega_n} \circ \theta_n^{-1}\|_{X''}$ (see [1, Theorem 5.5]), for every $n \in \mathbb{N}$. Using the fact that X'' has the Fatou property, we get that $f_* \in X''(\partial\Omega, \omega_z)$ and

$$\|f_*\|_{X''(\partial\Omega,\omega_z)} = \|f_* \circ \theta^{-1}\|_{X''} \le \sup_{n\in\mathbb{N}} \|f|_{\partial\Omega_n}\|_{X''(\partial\Omega_n,\omega_{n,z})}.$$

Since $f_* \in H^1(\Omega)$, f_* satisfies conditions (3.3), (3.4), and (3.5).

To prove the converse, suppose that $f_* \in X(\partial\Omega)$ satisfies (3.3)–(3.5). In particular, $f_* \in L^1(\partial\Omega, \omega_z)$ and by [8, Theorem 4.4.4] we have that $f \in H^1(\Omega)$. Define the operator $T_n\colon L^1([0,1)) \to L^1([0,1))$, $n \in \mathbb{N}$, by the formula

$$T_n g = g_n \circ \theta_n^{-1}, \quad g \in L^1\big([0,1)\big),$$

where the function g_n is defined, for every $\zeta \in \partial\Omega_n$, as follows:

$$g_n(\zeta) = \int_{\partial\Omega} g \circ \theta\, d\omega_\zeta.$$

We show that $\|T_n\|_{L^1\to L^1} = 1$ and $\|T_n\|_{L^\infty\to L^\infty} = 1$. Since $g \circ \theta \in L^1(\partial\Omega, \omega_z)$, the function $\zeta \mapsto \int_{\partial\Omega} g \circ \theta\, d\omega_\zeta$ is harmonic in Ω and has a (nontangential) boundary value equal to $g(\xi)$ for ω_z-almost all $\xi \in \partial\Omega$. Furthermore, we have

$$\begin{aligned}
\|g_n \circ \theta_n^{-1}\|_{L^1([0,1))} = \|g_n\|_{L^1(\partial\Omega_n,\omega_{n,z})} &= \int_{\partial\Omega_n} \Big| \int_{\partial\Omega} g \circ \theta(\xi)\, d\omega_\zeta(\xi) \Big|\, d\omega_{n,z}(\zeta) \\
&\le \lim_{n\to+\infty} \int_{\partial\Omega_n} \Big| \int_{\partial\Omega} g \circ \theta(\xi)\, d\omega_\zeta(\xi) \Big|\, d\omega_{n,z}(\zeta) \\
&\le \|g \circ \theta\|_{L^1(\partial\Omega,\omega_z)} = \|g\|_{L^1([0,1))}.
\end{aligned}$$

To see that $\|T_n\|_{L^1\to L^1} = 1$ it is enough to consider $g = 1$. Similarly, we can show that $\|T_n\|_{L^\infty\to L^\infty} = 1$. Now recall that every maximal r.i. space is an interpolation between L^1 and L^∞ and that X'' is maximal. Hence, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
\big\| |f|_{\partial\Omega_n}| \big\|_{X(\partial\Omega_n,\omega_{n,z})} &= \big\| |f|_{\partial\Omega_n} \circ \theta_n^{-1}| \big\|_X = \big\| |f|_{\partial\Omega_n} \circ \theta_n^{-1}| \big\|_{X''} \\
&\le \big\| T_n\big(|f_* \circ \theta^{-1}|\big) \big\|_{X''} \le \big\| |f_* \circ \theta^{-1}| \big\|_{X''} \\
&= \big\| |f_*| \big\|_{X''(\partial\Omega,\omega_z)},
\end{aligned} \tag{3.6}$$

which implies that

$$\sup_{n\in\mathbb{N}} \|f|_{\partial\Omega_n}\|_{X(\partial\Omega_n,\omega_{n,z})} \le \|f_*\|_{X''(\partial\Omega,\omega_z)},$$

and we are done. □

Notice that the second part of the proof implies that, for a maximal r.i. space, the sequence $\{\|f|_{\partial\Omega_n}\|_{X(\partial\Omega_n,\omega_{n,z})}\}$ is nondecreasing. To show this, it is enough to apply inequality (3.6) with $\Omega = \Omega_{n+1}$ and $\omega_z = \omega_{n+1,z}$ for $n \in \mathbb{N}$. In this case (similarly as for Hardy and Hardy–Orlicz spaces; see [19]) the norm is independent of the choice of a regular exhaustion of Ω. It allows us to look at the construction of an $HX(\Omega)$-norm as an abstract case of constructing a harmonic majorant. We note that the norm of $HX(\Omega)$ depends on the choice of $z \in \Omega_1$, but by Harnack's inequality it follows that for two different points we have equivalent topologies. We finish off this part of the paper by showing an estimate on the norm of the evaluation functionals in Proposition 3.6. To this end, we use a special family of functions (see [11]). For $a \in \partial\mathbb{D}$ and $r \in (0,1)$,

$$u_{a,r}(z) = \left(\frac{1-r}{1-\bar{a}rz}\right)^2$$

is a holomorphic function on $\mathbb{D}$ with $\|u_{a,r}\|_{H^1} \leq 1-r$, $\|u_{a,r}\|_{H^\infty} = 1$. In [14, Proof of Lemma 1.2], it was shown that for any r.i. space X with the fundamental function ϕ_X, the inequality

$$\|u_{a,r}\|_{HX(\mathbb{D})} \leq \phi_X\left(\frac{1-r}{1+r}\right)$$

holds for every $a \in \mathbb{T}$ and every $r \in (0,1)$.

For $i \in \{1,\ldots,m\}$, $a \in \mathbb{T}$, and $r \in (0,1)$, we define

$$u^i_{a,r}(z) = (u_{a,r} \circ \eta_i^{-1})(z) = \left(\frac{1-r}{1-\frac{\bar{a}rr_i}{z-a_i}}\right)^2, \quad z \in \Omega,$$

and $u^0_{a,r} = u_{a,r}|_\Omega$, where the η_i's are defined on page 869. Note that, for each $i \in \{0,\ldots,m\}$, the function $u^i_{a,r}$ extends to a holomorphic function on $\mathbb{E}^i$. Using the fact that conformal maps generate isometries in the class of Hardy spaces and Theorem 3.2, it is also clear that we have an analogous norm estimation: $\|u^i_{a,r}\|_{H^1(\mathbb{E}^i)} \leq 1-r$, $\|u^i_{a,r}\|_{H^\infty(\mathbb{E}^i)} = 1$, $\|u^i_{a,r}\|_{H^1(\Omega)} \leq C(1-r)$, $\|u^i_{a,r}\|_{H^\infty(\Omega)} \approx 1$, and

$$\|u^i_{a,r}\|_{HX(\Omega)} \leq C\phi_X\left(\frac{1-r}{1+r}\right)$$

for some constant $C > 0$.

Recall that, for an r.i. space X and its fundamental function ϕ_X, the following estimates on evaluation functionals δ_z on $HX(\mathbb{D})$ were proved in [14, Lemma 1.2]:

$$\frac{1}{4\phi_X(1-|z|)} \leq \|\delta_z\|_{(HX)^*} \leq \frac{2}{\phi_X(1-|z|)}, \quad z \in \mathbb{D}. \tag{3.7}$$

Proposition 3.6. *Let Ω be a circular domain, and let X be an r.i. space. Then, for each $z \in \Omega$, we have the following estimation:*

$$\|\delta_z\|_{HX(\Omega)} \approx \frac{1}{\phi_X(\operatorname{dist}(z,\partial\Omega))}. \tag{3.8}$$

Proof. First we prove that if $w \in \mathbb{E}^i$, $1 \le i \le m$, then, for every $g \in B_{HX(\mathbb{E}^i)}$, we have

$$\big|g(w)\big| \le C\frac{1}{\phi_X(\mathrm{dist}(w, \partial\mathbb{E}^i))} \tag{3.9}$$

for a constant $C > 0$. Indeed, put $f = g \circ \eta_i$. We have $\|f\|_{HX(\mathbb{D})} = \|g\|_{HX(\mathbb{E}^i)}$, and so, if $z = \eta_i^{-1}(w) = \frac{r_i}{w-a_i}$, then

$$\big|g(w)\big| = \big|(f \circ \eta_i^{-1})(w)\big| = \Big|f\Big(\frac{r_i}{w-a_i}\Big)\Big|.$$

Now using (3.7), we get

$$\begin{aligned}\big|g(w)\big| &\le \frac{2}{\phi_X(1 - |\frac{r_i}{w-a_i}|)} = \frac{2}{\phi_X(\frac{|w-a_i|-r_i}{|w-a_i|})}\\ &\le \frac{4}{\phi_X(\mathrm{dist}(w, \partial\mathbb{E}^i))}.\end{aligned}$$

Fix $z \in \Omega$. By Theorem 3.2 and inequality (3.9), for each $f \in B_{HX(\Omega)}$, we have

$$\begin{aligned}\big|f(z)\big| \le \big|f^0(z)\big| + \cdots + \big|f^m(z)\big| &\le 4\sum_{i=0}^{m}\frac{1}{\phi_X(\mathrm{dist}(z, \partial\mathbb{E}^i))}\\ \le 4\sum_{i=0}^{m}\frac{1}{\phi_X(\mathrm{dist}(z, \partial\Omega))}.\end{aligned}$$

This proves the upper estimate in (3.8) with a constant $C = 4(m + 1)$. To prove the lower estimate, let us notice that if $z \in \Omega$, then $\mathrm{dist}(\partial\Omega, z) = \mathrm{dist}(\Gamma^i, z)$ for some $i \in \{0, \ldots, m\}$. If $i = 0$, then we can write $z \in \Omega$ in the form $z = sp$, where $|z| = s$, $|p| = 1$ with $s \in (0, 1)$. Similarly, if $i \in \{1, \ldots, m\}$, then we can assume that $z = a_i + \frac{r_i}{s}\overline{p}$ with $|p| = 1$ and $s \in (\frac{r_i}{2}, 1)$. Now we easily deduce that

$$\frac{1}{4} \le \frac{1}{(1+s)^2} \le \big|u^i_{p,s}(z)\big| \le \|\delta_z\|_{HX(\Omega)^*}\|u^i_{p,s}\|_{HX(\Omega)} \le C\phi_X(1-s)\|\delta_z\|_{HX(\Omega)^*}.$$

This implies the thesis in the case when $\mathrm{dist}(\partial\Omega, z) = \mathrm{dist}(\Gamma^0, z) = 1 - s$, $s \in (0, 1)$. If $\mathrm{dist}(\partial\Omega, z) = \mathrm{dist}(\Gamma^i, z)$ when $i \ge 1$, then we have $\mathrm{dist}(\partial\Omega, z) = r_i\frac{|1-s|}{|s|}$, where $s \in (\frac{r_i}{2}, 1)$. Using this fact and the inequality

$$\begin{aligned}\phi_X(1-s) &= \phi_X\Big(\frac{r_i(1-s)}{s}\cdot\frac{s}{r_i}\Big)\\ &\le \frac{1}{r_i}\phi_X\big(\mathrm{dist}(z, \Gamma^i)\big),\end{aligned}$$

it is easy to obtain the lower estimation in (3.8). □

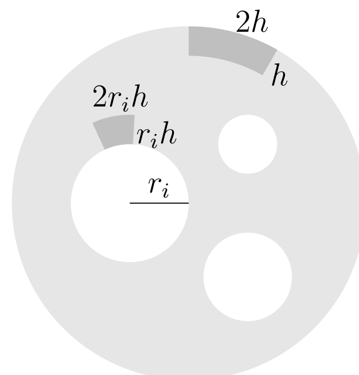

FIGURE 2. Carleson windows.

4. Canonical embeddings and Carleson measures

In this section we study canonical embeddings of Hardy spaces into the spaces of measurable functions and Carleson measures for circular domains.

Let μ be a positive finite Borel measure on Ω, and let $E \subset H(\Omega)$ be a Banach space. We define $j_\mu \colon E \to L^0(\Omega, \mu)$ by

$$j_\mu f(z) = f(z), \quad z \in \Omega.$$

We will study measures for which $j_\mu \colon HX(\Omega) \to X(\Omega, \mu)$ is a bounded operator, where $X(\Omega, \mu)$ is generated by $X([0,1), m)$.

Let $h \in (0, \min_{i \neq j} \operatorname{dist}(\Gamma^i, \Gamma^j))$. For $i = 0, \dots, m$, we define a Carleson window $W^i(\zeta, h)$, where $\zeta = a_i + r_i\xi \in \Gamma^i$, $\xi \in \mathbb{T}$, by the following formula (see Figure 2):

$$W^i(\zeta, h) = \begin{cases} \{z \in \overline{\Omega} : 1 - h < |z|, |\arg(\overline{\xi} z)| < h\} & i = 0, \\ \{z \in \overline{\Omega} : |z - a_i| < \frac{r_i}{1-h}, |\arg(\xi z)| < h\} & i \geq 1. \end{cases}$$

Note that $\eta_i(W^0(\zeta, h)) = W^i(\eta_i(\zeta), h)$.

For a Borel measure μ on Ω, we define a function $\rho_\mu \colon (0, \min_{i \neq j} \operatorname{dist}(\Gamma^i, \Gamma^j)) \to [0, \infty)$ by

$$\rho_\mu(h) := \max_{0 \leq i \leq m} \sup\bigl\{\mu\bigl(W_i(\xi, h)\bigr) : \xi \in \Gamma^i\bigr\}.$$

We consider a Borel measure μ on Ω a Carleson measure if there exists $C > 0$ such that $\rho_\mu(h) \leq Ch$. The famous Carleson theorem states that the canonical inclusion map $j_\mu \colon H^p(\mathbb{D}) \to L^p(\overline{\mathbb{D}}, \mu)$ is bounded if and only if μ is a Carleson measure. Note also that in [16] the Carleson theorem was proved for the spaces $H^p(\Omega)$. The aim of this section is to extend these results to the general framework of abstract Hardy spaces. It is worth mentioning that we do not mimic the proof of the result from [16]. Instead, we apply the tool of interpolation theory to obtain a very general version of the Carleson result for circular domains (see [14] for the Carleson theorem for abstract Hardy spaces on the disk).

Qui proved in [16, Theorem 1.1] that μ is a Carleson measure on Ω if and only if $\mu \circ \eta_i$ is a Carleson measure on $\mathbb{D}$, for every $i = 0, \dots, m$. Let us clarify that the measure $\mu \circ \eta_i$ is defined for any Borel subset A of the unit disk $\mathbb{D}$ by the formula

$$\mu \circ \eta_i(A) = \mu\bigl(\eta_i(A) \cap \Omega)\bigr).$$

Since $\eta_i\colon \mathbb{D} \to \mathbb{E}^i$ is a holomorphic bijection and $\bigcap_{i=0}^m \mathbb{E}^i = \Omega$, one might expect some mutual dependence between Carleson measures on $\mathbb{D}$ and Carleson measures on Ω. However, the equivalence from [16, Theorem 1.1] does not establish any mutual correspondence between Carleson measures on the disk and on circular domains, as the following example shows.

Example 4.1. Let Ω be a circular domain whose boundary is formed by the curves $\Gamma_0 = \mathbb{T}$, $\Gamma_1 = \frac{2}{3} + \frac{5}{24}\mathbb{T}$, $\Gamma_2 = \frac{11}{12} + \frac{1}{36}\mathbb{T}$. Define the measure μ on the Borel subsets of $\mathbb{D}$ by the formula

$$\mu(B) = m(\overline{B} \cap \mathbb{T}) + \nu_1(\overline{B} \cap \Gamma') + \nu_2(\overline{B} \cap \Gamma'') + \nu_3(\overline{B} \cap \Gamma'''),$$

where $\Gamma' = \frac{11}{12} + \frac{1}{36}\mathbb{T}$, $\Gamma'' = \frac{1}{4} + \frac{3}{8}\mathbb{T}$, $\Gamma''' = -\frac{12}{33} + \frac{10}{33}\mathbb{T}$, m is the Lebesgue measure on $\mathbb{T}$, and ν_i, $i = 1, 2, 3$, are calculated in the following way. If $(\frac{11}{12} + \frac{1}{36}e^{i\alpha}, \frac{11}{12} + \frac{1}{36}e^{i\beta}) \subset \Gamma'$ is an arc, then

$$\nu_1\Big(\Big(\frac{11}{12} + \frac{1}{36}e^{i\alpha}, \frac{11}{12} + \frac{1}{36}e^{i\beta}\Big)\Big) = \int_\alpha^\beta |e^{it} - 1|^{-1/4}\, dt.$$

Similarly, we define ν_2 and ν_3 for arcs contained in Γ'' and Γ''', respectively. Since ν_i is a finite measure, for $i = 1, 2, 3$, and the circles Γ', Γ'', and Γ''' do not intersect $\mathbb{T}$, then μ is a Carleson measure on $\mathbb{D}$.

Nonetheless, for $i \in \{0, 1, 2\}$, consider the measures $\mu \circ \eta_i^{-1}$ defined, for any Borel subset A of Ω, as

$$\mu \circ \eta_i^{-1}(A) := \mu\big(\eta_i^{-1}(A)\big),$$

and observe that $\eta_0(\Gamma') = \Gamma' = \Gamma_2$, $\eta_1(\Gamma'') = \Gamma_0 = \mathbb{T}$, $\eta_2(\Gamma''') = \Gamma_1$. Since

$$\int_{-h}^h |e^{it} - 1|^{-1/4}\, dt \approx h^{3/4}$$

for small $h > 0$, we deduce that $\mu \circ \eta_1^{-1}(W^0(1, h)) \approx h^{3/4}$, $\mu \circ \eta_0^{-1}(W^2(\frac{34}{36}, h)) \approx h^{3/4}$, and $\mu \circ \eta_2^{-1}(W^1(\frac{21}{24}, h)) \approx h^{3/4}$, for h small enough. Therefore, $\mu \circ \eta_i^{-1}$ is not a Carleson measure for any $i \in \{0, 1, 2\}$.

In the theorem below we show, among other things, that there still is a certain correspondence between Carleson measures on the disk and on circular domains. We will need a special measure $\widetilde{\mu}$ on $\mathbb{D}$ associated with a measure on Ω. Let μ be a probabilistic Borel measure on Ω, and let $\delta = \frac{1}{2}\min\{\operatorname{dist}(\Gamma^i, \Gamma^j) : i \neq j\}$. For $i = 0, 1, \ldots, m$, put $\mathbb{E}^i_\delta = \{z \in \mathbb{E}^i : |z - a_i| < r_i + \delta\}$ and define a measure μ_i on $\mathbb{E}^i$ in the following way: $\mu_i = \mu$ on $\mathbb{E}^i_\delta$ and $\mu_i = 0$ on $\mathbb{E}^i \setminus \mathbb{E}^i_\delta$. We specify the measure $\widetilde{\mu}$ on $\mathbb{D}$ by the formula

$$\widetilde{\mu}(A) := \frac{1}{m+1}\sum_{i=0}^m \mu_i \circ \eta_i(A) = \frac{1}{m+1}\sum_{i=0}^m \mu_i\big(\eta_i(A) \cap \mathbb{E}^i\big),$$

for any Borel set $A \subset \mathbb{D}$. It can be observed that if μ is a Carleson measure on Ω, then $\widetilde{\mu}$ is a Carleson measure on $\mathbb{D}$ (see [16, Lemma 1.1]).

Recall that the fundamental function ϕ_X of an r.i. space X is called *regular* if there exists a constant $C > 0$ such that $2\phi_X(t/C) \leq \phi_X(t)$, for all t from a domain of ϕ_X.

Theorem 4.2. *Let μ be a probability Borel measure on a circular domain Ω, and let $X(\Omega,\mu)$ be an r.i. space generated by a maximal r.i. space X. If μ is nonatomic and the fundamental function ϕ_X is regular, then the following statements are equivalent:*

(i) *μ is a Carleson measure on Ω,*
(ii) *the operator j_μ is bounded from $HX(\Omega)$ into $X(\Omega,\mu)$,*
(iii) *the operator $j_{\widetilde{\mu}}$ is bounded from $HX(\mathbb{D})$ into $X(\mathbb{D},\widetilde{\mu})$,*
(iv) *$\widetilde{\mu}$ is a Carleson measure on $\mathbb{D}$.*

The proofs of the above statements require the following lemma.

Lemma 4.3. *Let X be a maximal r.i. space. Let $(\Omega,\mathcal{B},\mu)$ and $(\Omega,\mathcal{B},\nu)$ be measure spaces, where $\mathcal{B}$ is a σ-algebra of Borel subsets of Ω. If there exists $C>0$ such that $\mu(A)\le C\nu(A)$ for every $A\in\mathcal{B}$, then $X(\Omega,\nu)\hookrightarrow X(\Omega,\mu)$, where $X(\Omega,\nu)$ and $X(\Omega,\mu)$ are r.i. spaces generated by $X(\partial\Omega,\omega_z)$.*

Proof. Since X is maximal, it is an exact interpolation space between L^1 and L^∞. Thus, there exists an exact interpolation functor $\mathcal{F}$ such that $X=\mathcal{F}(L^1,L^\infty)$. Put

$$X(\Omega,\mu)=\mathcal{F}\big(L^1(\Omega,\mu),L^\infty(\Omega,\mu)\big)\qquad\text{and}\qquad X(\nu)=\mathcal{F}\big(L^1(\Omega,\nu),L^\infty(\Omega,\nu)\big).$$

It follows that $X(\Omega,\mu)$ and $X(\Omega,\nu)$ are relative interpolation spaces with respect to $(L^1(\Omega,\mu),L^\infty(\Omega,\mu))$ and $(L^1(\Omega,\nu),L^\infty(\Omega,\nu))$, respectively. We need to show that $L^1(\Omega,\nu)\hookrightarrow L^1(\Omega,\mu)$ and $L^\infty(\Omega,\nu)\hookrightarrow L^\infty(\Omega,\mu)$ are bounded operators. Since μ is absolutely continuous with respect to ν, then $L^\infty(\Omega,\nu)\hookrightarrow L^\infty(\Omega,\mu)$ with the norm equal to 1. To calculate the norm of $L^1(\Omega,\nu)\hookrightarrow L^1(\Omega,\mu)$ it is enough to observe that $\mu_f(\lambda)\le C\nu_f(\lambda)$, for every $f\in L^1(\Omega,\nu)$. Hence

$$\|f\|_{L^1(\Omega,\mu)}=\int_0^\infty\mu_f(\lambda)\,d\lambda\le C\int_0^\infty\nu_f(\lambda)\,d\lambda=C\|f\|_{L^1(\Omega,\nu)}.$$

By interpolation, $X(\Omega,\nu)\hookrightarrow X(\Omega,\mu)$. □

Proof of Theorem 4.2. The equivalence of (iii) and (iv) was proved in [14, Theorem 2.2].

(iii) ⇒ (ii): Let $f\in HX(\Omega)$. By Theorem 3.2, f can be written as a sum $f=f^0+\cdots+f^m$, where $f^i\in HX_0(\mathbb{E}^i)$, $i=1,\dots,m$, and $f^0\in HX(\mathbb{D})$. Then we have

$$\|f\|_{X(\Omega,\mu)}\le\|f^0|_\Omega\|_{X(\Omega,\mu)}+\cdots+\|f^m|_\Omega\|_{X(\Omega,\mu)}.$$

Let us note that, if we extend the measure μ to $\mathbb{E}^i$ by putting $\mu(\mathbb{E}^i\setminus\Omega)=0$, then the last sum is equal to

$$\|f^0\|_{X(\mathbb{E}^0,\mu)}+\cdots+\|f^m\|_{X(\mathbb{E}^m,\mu)}=\|f^0\circ\eta_0\|_{X(\mathbb{D},\mu\circ\eta_0)}+\cdots+\|f^m\circ\eta_m\|_{X(\mathbb{D},\mu\circ\eta_m)},$$

since the correspondence $g\mapsto g\circ\eta_i$ is an isometric isomorphism between $X(\mathbb{E}^i,\mu)$ and $X(\mathbb{D},\mu\circ\eta_i)$. Since $\mu\circ\eta_i\le(m+1)\widetilde{\mu}$, for small h, and $\widetilde{\mu}$ is a Carleson measure on $\mathbb{D}$ (see [14, Theorem 2.2]), then $\mu\circ\eta_i$ is a Carleson measure on $\mathbb{D}$ for $i\in\{0,\dots,m\}$. Hence, by [14, Theorem 2.2],

$$\|f\|_{X(\Omega,\mu)}\le C\big(\|f^0\|_{HX(\mathbb{D})}+\cdots+\|f^m\|_{HX(\mathbb{D})}\big),$$

for some constant $C > 0$. Now, by Theorem 3.2, we obtain $\|f\|_{X(\mu)} \le C'\|f\|_{HX(\Omega)}$, for a certain constant $C' > 0$, meaning that j_μ is bounded from $HX(\Omega)$ into $X(\Omega, \mu)$.

(ii) $\Rightarrow$ (iii): Let $f \in HX(\mathbb{D})$. Then $f|_\Omega \in HX(\Omega)$. By Theorem 3.2, $f^0 = f$ and the $HX(\mathbb{D})$-norm and the $HX(\Omega)$-norm are equivalent for elements in $HX(\mathbb{D})$. Let us note that the map $g \mapsto g \circ \eta_i^{-1}$ is an isometric isomorphism between $HX(\mathbb{D})$ and $HX(\mathbb{E}^i)$. From this and our assumption, we have

$$\|f \circ \eta_i^{-1}\|_{X(\Omega,\mu)} \le C\|f \circ \eta_i^{-1}\|_{HX(\Omega)} \le C'\|f\|_{HX(\mathbb{D})}.$$

It is obvious that $\mu_i \le \mu$, for each $i \in \{0, \dots, m\}$, so, by Lemma 4.3, we have that

$$\|f\|_{X(\mathbb{D},\mu_i\circ\eta_i)} = \|f \circ \eta_i^{-1}\|_{X(\mathbb{E}^i,\mu_i)} \le \|f \circ \eta_i^{-1}\|_{X(\Omega,\mu)} \le C'\|f\|_{HX(\mathbb{D})},$$

for every $i \in \{0, \dots, m\}$. We will be done if we prove that

$$\|f\|_{X(\mathbb{D},\widetilde{\mu})} \le C'' \sum_{i=0}^{m} \|f\|_{X(\mathbb{D},\mu_i\circ\eta_i)},$$

because Lemma 4.3 implies that the last sum is not greater than $C'''(m + 1)\|f\|_{X(\mathbb{D},\widetilde{\mu})}$. First, observe that, for any Borel set $A \subset \mathbb{D}$, there exists $i \in \{0, \dots, m\}$ such that $\widetilde{\mu}(A) \le \mu_i \circ \eta_i(A)$. Thus, for every $\lambda > 0$, there exists $i \in \{0, \dots, m\}$ for which we have $\widetilde{\mu}_f(\lambda) \le (\mu_i \circ \eta_i)_f(\lambda)$. Hence, for any $t > 0$,

$$f^*_{\widetilde{\mu}}(t) \le \sum_{i=0}^{m} f^*_{\mu_i\circ\eta_i}(t).$$

The last inequality implies that

$$\|f\|_{X(\mathbb{D},\widetilde{\mu})} = \|f^*_{\widetilde{\mu}}\|_X \le \Big\|C'' \sum_{i=0}^{m} f^*_{\mu_i\circ\eta_i}\Big\|_X \le C'' \sum_{i=0}^{m} \|f^*_{\mu_i\circ\eta_i}\|_X = C'' \sum_{i=0}^{m} \|f\|_{X(\mathbb{D},\mu_i\circ\eta_i)}.$$

(i) $\Leftrightarrow$ (iv): This equivalence follows from the construction of $\widetilde{\mu}$ and the definition of Carleson windows on Ω. Indeed, if $h > 0$ is small enough, then, for every $i \in \{0, \dots, m\}$ and every $\zeta \in \partial\mathbb{D}$, we have $\eta_i(W^0(\zeta, h)) \subset \mathbb{E}^i_\delta$. In this case $\rho_{\widetilde{\mu}}(h) \le \rho_\mu(h) \le (m + 1)\rho_{\widetilde{\mu}}(h)$. □

5. Order-bounded composition operators

In this section we characterize order-bounded composition operators on abstract Hardy spaces on circular domains. We recall that an operator $T: X \to Z$ from a Banach space X into a Banach subspace Z of a Banach lattice Y is *order bounded* if there is some positive $y \in Y$ such that $|Tx| \le y$ for every x in the unit ball B_X of X.

Let Ω be a circular domain. The symbol $\Upsilon := \Upsilon_\Omega$ will denote the set of holomorphic maps $\varphi: \Omega \to \Omega$. For $\varphi \in \Upsilon$, a *composition operator* $C_\varphi: H(\Omega) \to H(\Omega)$ is defined by the formula

$$(C_\varphi f)(z) := (f \circ \varphi)(z), \quad z \in \Omega.$$

We show that $C_\varphi\colon HX(\Omega)\to HX(\Omega)$ is a bounded linear operator whenever X is an r.i. space. Clearly, C_φ is linear. It is obvious that $C_\varphi\colon H^\infty(\Omega)\to H^\infty(\Omega)$ is bounded. To show that $C_\varphi\colon H^1(\Omega)\to H^1(\Omega)$ is a bounded operator, denote by $v_f(z)$ the value of the least harmonic majorant of $f\in H^1(\Omega)$ at point $z\in\Omega$ that we used to define the $H^1(\Omega)$-norm. By Harnack's inequality, there exists $C>0$ such that $u(\varphi(z))\le Cu(z)$ for any function u harmonic in Ω. Thus

$$\|C_\varphi f\|_{H^1(\Omega)} = v_f\big(\varphi(z)\big)\le Cv_f(z) = C\|f\|_{H^1(\Omega)}.$$

Now, by interpolation we easily deduce that $C_\varphi\colon HX(\Omega)\to HX(\Omega)$ is bounded.

For $z\in\Omega$, we denote by $\omega:=\omega_z$ the harmonic measure on $\partial\Omega$ with respect to z. By Theorem 3.5, we conclude that, for every $f\in HX(\Omega)$, the boundary function f_* exists ω-a.e. on $\partial\Omega$ and $f_*\in X(\partial\Omega,\omega)$. Let φ be a holomorphic self-map of Ω, and let $C_\varphi\colon HX(\Omega)\to HX(\Omega)$ be a composition operator. Since each such operator is bounded, the operator $\widetilde{C}_\varphi\colon HX(\Omega)\to HX(\partial\Omega,\omega)$, given by $\widetilde{C}_\varphi f=(C_\varphi f)_*$, is well defined, where $HX(\partial\Omega)$ denotes the subspace of $X(\partial\Omega,\omega)$ that consists of those functions $f_*\in X(\partial\Omega,\omega)$ which are boundary functions of $f\in HX(\Omega)$.

Theorem 5.1. *Let X be a maximal r.i. space on $[0,1)$, $X\neq L^\infty$, and let $\varphi\in\Upsilon$. The composition operator $C_\varphi\colon HX(\Omega)\to HX(\Omega)$ induces an operator $\widetilde{C}_\varphi\colon HX(\Omega)\to HX(\partial\Omega)$ which is order bounded if and only if* $\operatorname{dist}(\varphi_*,\partial\Omega)>0$ *ω_z-a.e. and the function $[\phi_X(\operatorname{dist}(\varphi_*,\partial\Omega))]^{-1}\in X(\partial\Omega,\omega_z)$.*

Proof. Suppose that $g:=[\phi_X(\operatorname{dist}(\varphi_*,\partial\Omega))]^{-1}\in X(\partial\Omega,\omega_z)$. Then $\operatorname{dist}(\varphi_*,\partial\Omega)>0$ ω_z-a.e. and, by the estimate (3.8), we conclude that there exists a constant $C>0$ such that, for every $f\in B_{HX(\Omega)}$, the inequality

$$\big|(C_\varphi f)_*(\xi)\big| = \big|f\big(\varphi_*(\xi)\big)\big|\le\|\delta_{\varphi_*(\xi)}\|_{(HX(\Omega))^*}\le Cg(\xi)$$

is satisfied for ω-almost all $\xi\in\partial\Omega$; that is, $\widetilde{C}_\varphi\colon HX(\Omega)\to HX(\partial\Omega)$ is order bounded.

Assume, conversely, that $\widetilde{C}_\varphi\colon HX(\Omega)\to HX(\partial\Omega)$ is order bounded and $X\neq L^\infty$. Then, by the definition, there exists a function $g\in X(\partial\Omega,\omega_z)$ such that $|\widetilde{C_\varphi f}|\le g$ for every $f\in B_{HX(\Omega)}$. Using the harmonic measure ω_z to express $|f(\varphi(z))|$, we obtain that

$$|\delta_{\varphi(z)}f| = \big|f\big(\varphi(z)\big)\big| = \Big|\int_{\partial\Omega}(C_\varphi f)_*\,d\omega_z\Big| = \Big|\int_{\partial\Omega}\widetilde{C_\varphi f}\,d\omega_z\Big|\le\int_{\partial\Omega}g\,d\omega_z,$$

for every $z\in\Omega$ and every $f\in B_{HX(\Omega)}$. Now, taking the supremum over $f\in B_{HX(\Omega)}$, we get

$$\|\delta_{\varphi(z)}\|\le\int_{\partial\Omega}g\,d\omega_z,$$

and, by (3.8), we obtain that

$$\big[\phi_X\big(\operatorname{dist}\big(\varphi(z),\partial\Omega\big)\big)\big]^{-1}\le C\int_{\partial\Omega}g\,d\omega_z,\quad z\in\Omega.$$

Notice that the function $z \mapsto \int_{\partial\Omega} g\, d\omega_z$ has the boundary function g for ω_z-almost all $\xi \in \partial\Omega$. Since $X \neq L^\infty$, we have $\lim_{t\to 0^+} \phi_X(t) = 0$ and using the fact that ϕ_X is continuous, we obtain that $\operatorname{dist}(\varphi_*(\xi), \partial\Omega) > 0$ for ω_z-almost all $\xi \in \partial\Omega$ and $[\phi_X(\operatorname{dist}(\varphi_*(\xi), \partial\Omega))]^{-1} \leq Cg(\xi)$ ω_z-a.e.; that is, $[\phi_X(\operatorname{dist}(\varphi_*, \partial\Omega))]^{-1} \in X(\partial\Omega, \omega_z)$. □

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