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ON SOLUTIONS OF AN INFINITE SYSTEM OF NONLINEAR INTEGRAL EQUATIONS ON THE REAL HALF-AXIS

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Abstract. We investigate the existence of solutions of an infinite system of integral equations of Volterra–Hammerstein type on the real half-axis. The method applied in our study is connected with the construction of a suitable measure of noncompactness in the space of continuous and bounded functions defined on the real half-axis with values in the space c_0 consisting of real sequences converging to zero and equipped with the classical supremum norm. We use the mentioned measure of noncompactness together with a fixed point theorem of Darbo type. Our investigations are illustrated by an example.

1. Introduction

The paper is devoted to the study of the solvability of an infinite system of nonlinear integral equations of Volterra–Hammerstein type. We will look for solutions of that system of integral equations in the Banach space of continuous and bounded functions defined on the real half-axis $\mathbb{R}_+$ and taking values in the classical sequence space c_0. Obviously, the space c_0 consists of real sequences converging to zero, and it is normed via the classical supremum norm.

It is well known that integral equations play an essential role in the applications to the description of numerous events in the real world (see, e.g., [13], [22], [24]–[26]). Apart from that, integral equations create an important branch of nonlinear functional analysis (see, e.g., [9], [13], [14]). Obviously, the theory of

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integral equations is closely related to the theory of differential equations (see, e.g., [7], [12], [25], [26]). On the basis of the above relations, we can consider infinite systems of differential equations as well as infinite systems of integral equations applying similar methods and tools, even though these theories are generally different (see [7]). We call attention to the fact that, despite their stated importance, theories of differential and integral equations create very young mathematical disciplines for which there exist only a few papers and monographs devoted to the study of these theories (see, e.g., [7], [8], [12], [13], [21]).

The present paper is thoroughly dedicated to the study of infinite systems of nonlinear integral equations. As we mentioned above, we will look for solutions of infinite systems of integral equations in the space $BC(\mathbb{R}_+, c_0)$ of continuous and bounded functions defined on the real half-axis and with values in the sequence space c_0. The basic role in our study will be played by a suitable measure of noncompactness constructed in the indicated space $BC(\mathbb{R}_+, c_0)$. Such a measure of noncompactness has not been constructed to date. Thus, from a priority point of view, the present paper is the first one in which the theory of measures of noncompactness in the space $BC(\mathbb{R}_+, c_0)$ is considered.

In the construction of the mentioned measure of noncompactness in the space $BC(\mathbb{R}_+, c_0)$, we utilized some results obtained earlier by Nussbaum, Goebel, and the first author (see, e.g., [4], [23]). Obviously, the mentioned measure of noncompactness will be applied in the present paper together with the fixed point property due to Darbo [11] (see also [4]).

As we pointed out above, the results of the paper are new and simultaneously create a generalization of many of the results obtained in [4], [7], [8], [12], and [21], for example. It is also worth mentioning that solutions of infinite systems of integral equations defined on the half-axis $\mathbb{R}_+$ have not, as far as we know, been previously studied. All considerations concerning solutions of infinite systems of differential and integral equations and put forward up to now have been located in the spaces of functions defined on a bounded interval (see, e.g., [7], [8], [12], [21]).

2. Notation, definitions, and auxiliary facts

This section contains some basic facts that will be used in our study. First, we establish some notation, namely, we denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{N}$ the set of natural numbers (positive integers). We put $\mathbb{R}_+ = [0, \infty)$. Further, let E be a Banach space with norm $\|\cdot\|_E$ and zero element θ. Then we denote by $B(x, r)$ the closed ball centered at x with radius r. We write B_r to denote the ball $B(\theta, r)$.

If X, Y are subsets of the Banach space E, then we use the standard symbols $X + Y$, λX ($\lambda \in \mathbb{R}$) to denote the algebraic operations on subsets of E. For a subset X of E, we denote by $\overline{X}$ the closure of X and by $\operatorname{Conv} X$ the closed convex hull of the set X. Moreover, if we assume that X is a nonempty and bounded set in E, then the diameter of X will be denoted by $\operatorname{diam} X$. The symbol $\|X\|_E$ will stand for the norm of X; that is, we have $\|X\|_E = \sup\{\|x\|_E : x \in X\}$. Next, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. Now, we recall the definition of the fundamental concept used in the monograph [4].

Definition 2.1 ([4, Definition 3.1.2]). A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ will be called a *measure of noncompactness* in E if it satisfies the following conditions.

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii) $\mu(\overline{X}) = \mu(X)$.
(iv) $\mu(\operatorname{Conv} X) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The set $\ker \mu$ defined in axiom (i) is called the *kernel of the measure of noncompactness* μ. Note that the intersection set X_∞ appearing in axiom (vi) is an element of the family $\ker \mu$ (see [4]). This simple observation plays an essential role in our further considerations.

Furthermore, let us assume that μ is a measure of noncompactness in the space E. We say that μ is *sublinear* (see [4]) if it satisfies the following additional conditions:

(vii) $\mu(X + Y) \leq \mu(X) + \mu(Y)$,
(viii) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

Moreover, we consider that a measure of noncompactness μ has *maximum property* if

(ix) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

If the measure of noncompactness μ is such that

(x) $\ker \mu = \mathfrak{N}_E$,

then it is called *full*. A sublinear measure of noncompactness which is full and has maximum property is said to be a *regular measure of noncompactness* (see [4]). Note that the first measure of noncompactness was defined by Kuratowski [17] in the following way:

$$\alpha(X) = \inf\{\varepsilon > 0 : X \text{ can be covered by a finite family of sets } X_1, X_2, \ldots, X_m \text{ such that } \operatorname{diam} X_i \leq \varepsilon \text{ for } i = 1, 2, \ldots, m\}.$$

The measure $\alpha(X)$ is called the *Kuratowski measure of noncompactness*. It can be shown that the Kuratowski measure of noncompactness is regular (see [4]).

Another important measure of noncompactness, called the *Hausdorff measure of noncompactness*, was defined as follows (see [15], [16]):

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}.$$

It can be shown that the Hausdorff measure of noncompactness χ is regular and it is equivalent to the Kuratowski measure $\alpha(X)$. More precisely, for an arbitrary set $X \in \mathfrak{M}_E$, the following inequalities hold (see [4]):

$$\chi(X) \leq \alpha(X) \leq 2\chi(X). \tag{2.1}$$

On the other hand, it turns out that the Hausdorff measure of noncompactness χ is more convenient in applications than the Kuratowski measure α. This is mainly due to the fact that in some classical Banach spaces there are known formulas expressing the Hausdorff measure with help of the structure of those spaces, but such formulas are not known for the Kuratowski measure of noncompactness in any Banach space (see [4]).

For example, in Banach spaces c_0, l_p $(1 \leq p < \infty)$ and $C([a,b])$, we know formulas expressing the Hausdorff measure of noncompactness χ, while in the spaces c and $L^p([a,b])$ we can give formulas for regular measures of noncompactness equivalent to the Hausdorff measure χ (see [4], [7]). Taking into account our further goals, we recall a few mentioned formulas.

Thus, let c_0 denote the space of real sequences $x = (x_n)$ converging to zero and normed by the norm

$$\|x\|_{c_0} = \big\|(x_n)\big\|_{c_0} = \sup\big\{|x_n| : n = 1, 2, \ldots\big\}.$$

Then, for an arbitrary set $X \in \mathfrak{M}_{c_0}$, we have (see [7])

$$\chi(X) = \lim_{n\to\infty}\Big\{\sup_{x=(x_i)\in X}\big\{\sup\big\{|x_k| : k \geq n\big\}\big\}\Big\}. \tag{2.2}$$

It can also be shown that, in the classical function space $C = C([a,b])$ consisting of real functions defined and continuous on the interval $[a,b]$ and normed via the maximum norm, the Hausdorff measure of noncompactness of an arbitrary set $X \in \mathfrak{M}_C$ can be expressed by the formula

$$\chi(X) = \frac{1}{2}\omega_0(X),$$

where the quantity $\omega_0(X)$ denotes the "limit value" of the modulus of continuity of the set X (see [4]).

In what follows, let us assume that μ is a regular measure of noncompactness in a Banach space E. Then, it can be easily shown (see [4]) that for an arbitrary set $X \in \mathfrak{M}_E$, we have the estimate

$$\mu(X) \leq c_1\chi(X), \tag{2.3}$$

where $c_1 = \mu(B_E)$. Thus, we can raise the question of whether an arbitrary regular measure of noncompactness μ is equivalent to the Hausdorff measure χ. Taking into account estimate (2.3), the above question can be formulated equivalently in the following manner: Does there exist a constant $c_2 > 0$ such that

$$c_2\chi(X) \leq \mu(X)$$

for $X \in \mathfrak{M}_E$?

It turns out that the answer to the above question is negative (see [1]); that is, in an arbitrary infinite-dimensional Banach space E there exists a regular measure of noncompactness which is not equivalent to the Hausdorff measure χ (see also [5], [10], [19], [20]). Later, we formulate a fixed point theorem which comes back to Darbo [11]. That theorem was formulated for the Kuratowski measure of noncompactness α. The general version presented below comes from [4] (see also [2]).

Theorem 2.2 ([4, Theorem 5.1]). *Let μ be an arbitrary measure of noncompactness in the Banach space E. Assume that Ω is a nonempty, bounded, closed, and convex subset of E and that $Q : \Omega \to \Omega$ is a continuous operator such that there exists a constant $k \in [0, 1)$ for which $\mu(QX) \le k\mu(X)$ for an arbitrary nonempty subset X of Ω. Then the operator Q has at least one fixed point in the set Ω.*

Remark 2.3. Under assumptions of Theorem 2.2 it can be shown that the set $\operatorname{Fix} Q$ consisting of all fixed points of the operator Q belonging to Ω is an element of the kernel $\ker \mu$.

This observation is very important in the characterization of solutions of an operator equation having the form $x = Qx$ (see [2]–[4], [7]).

3. Measures of noncompactness in the space $BC(\mathbb{R}_+, c_0)$

First, let us assume that E is a given Banach space with the norm $\|\cdot\|_E$. Throughout this paper we will assume that E is an infinite-dimensional space. Next, consider the Banach space $BC(\mathbb{R}_+, E)$ consisting of functions $x = x(t)$ defined on $\mathbb{R}_+$ with values in E which are continuous and bounded on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+, E)$ will be equipped with the supremum norm $\|\cdot\|_\infty$ defined as

$$\|x\|_\infty = \sup\{\|x(t)\|_E : t \in \mathbb{R}_+\}.$$

Moreover, for a given $T > 0$ we will also consider the space $C_T = C([0, T], E)$ consisting of functions defined and continuous on the interval $[0, T]$ with values in the space E. The space C_T will also be endowed with the supremum norm which will be denoted here by $\|\cdot\|_T$; that is, $\|x\|_T = \sup\{\|x(t)\|_E : t \in [0, T]\}$.

Now, let us fix a set $X \in \mathfrak{M}_{BC(\mathbb{R}_+,E)}$. Take $T > 0$ and $\varepsilon > 0$. For an arbitrary function $x \in X$, we define the modulus of continuity $\omega^T(x, \varepsilon)$ of the function x on the interval $[0, T]$ by putting

$$\omega^T(x, \varepsilon) = \sup\{\|x(t) - x(s)\|_E : t, s \in [0, T], |t - s| \le \varepsilon\}.$$

Next, we define

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\},$$
$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon).$$

Notice that the above limit exists and is finite since the function $\varepsilon \to \omega^T(X, \varepsilon)$ is nondecreasing and nonnegative for $\varepsilon > 0$. Finally, we put

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X). \tag{3.1}$$

The existence of the above limit is a consequence of the fact that the function $T \to \omega_0^T(X)$ is nondecreasing and the inequality

$$\omega^T(X, \varepsilon) \le 2\|X\|_{BC(\mathbb{R}_+,E)}$$

holds for each $\varepsilon > 0$ and $T > 0$. Hence, we have $\omega_0(X) \le 2\|X\|_{BC(\mathbb{R}_+,E)}$, where the norm $\|X\|_{BC(\mathbb{R}_+,E)}$ of the set X was defined earlier.

Further, assume that $\gamma = \gamma(X)$ is a given measure of noncompactness in the Banach space E. For an arbitrarily fixed number $t \in \mathbb{R}_+$ denote by $X(t)$ the cross section of the set X at t; that is, $X(t) = \{x(t) : x \in X\}$. Obviously, $X(t)$ is a subset of the space E.

Next, for a fixed $T > 0$, let us put

$$\overline{\gamma}_T(X) = \sup\{\gamma(X(t)) : t \in [0,T]\}. \tag{3.2}$$

Observe that the function $T \to \overline{\gamma}_T(X)$ is nondecreasing and bounded from above since the set X is a bounded subset of the space $BC(\mathbb{R}_+, E)$. Indeed, we have

$$\|X(t)\|_E \le \|X\|_{BC(\mathbb{R}_+,E)} < \infty$$

for any $t \in \mathbb{R}_+$. Consecutively, we define the following quantity

$$\overline{\gamma}_\infty(X) = \lim_{T\to\infty} \overline{\gamma}_T(X). \tag{3.3}$$

Next, for a fixed $T > 0$, let us put

$$a_T(X) = \sup_{x\in X}\{\sup\{\|x(t)\|_E : t \ge T\}\}.$$

Observe that the function $T \to a_T(X)$ is nonincreasing, and thus there exists the finite limit

$$a_\infty(X) = \lim_{T\to\infty} a_T(X). \tag{3.4}$$

Further, we can also define a few other quantities measuring the behavior of functions from the set X at infinity. Namely, for a fixed $T > 0$, define the function $b_T = b_T(X)$ by putting

$$b_T(X) = \sup_{x\in X}\{\sup\{\|x(t) - x(s)\|_E : t, s \ge T\}\}.$$

Obviously, the function $T \to b_T(X)$ is nonincreasing on $\mathbb{R}_+$, which implies that there exists the finite limit $\lim_{T\to\infty} b_T(X)$. We put

$$b_\infty(X) = \lim_{T\to\infty} b_T(X). \tag{3.5}$$

Let us indicate the following equality

$$b_\infty(X) = \inf\{b_T(X) : T \in \mathbb{R}_+\}.$$

Further, for a fixed $t \in \mathbb{R}_+$ we put

$$\operatorname{diam} X(t) = \sup\{\|x(t) - y(t)\|_E : x, y \in X\}.$$

Now, let us define

$$c(X) = \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{3.6}$$

In what follows let us consider the functions γ_a, γ_b, γ_c defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,E)}$ in the following way:

$$\gamma_a(X) = \omega_0(X) + \overline{\gamma}_\infty(X) + a_\infty(X), \tag{3.7}$$

$$\gamma_b(X) = \omega_0(X) + \overline{\gamma}_\infty(X) + b_\infty(X), \tag{3.8}$$

$$\gamma_c(X) = \omega_0(X) + \overline{\gamma}_\infty(X) + c(X). \tag{3.9}$$

We show that under some assumptions concerning the measure of noncompactness γ, the functions γ_a, γ_b, γ_c defined by the formulas (3.7)–(3.9) are measures of noncompactness in the space $BC(\mathbb{R}_+, E)$. To this end we first recall some results due to Nussbaum [23] which will be utilized in our reasoning process (we note that this result of Nussbaum was originally shown in a more general setting).

Lemma 3.1 ([23, Theorem 1])**.** *Let $\alpha_T = \alpha_T(X)$ denote the Kuratowski measure of noncompactness in the space $C_T = C([0,T], E)$. Then*

$$\max\Big\{\frac{1}{2}\omega_0^T(X), \overline{\alpha}_T(X)\Big\} \leq \alpha_T(X) \leq 2\omega_0^T(X) + \overline{\alpha}_T(X), \tag{3.10}$$

where the quantity $\overline{\alpha}_T$ was defined by (3.2).

In what follows let us notice that linking inequalities (3.10) and (2.1), we derive the estimates

$$\frac{1}{4}\Big[\frac{1}{2}\omega_0^T(X) + \overline{\chi}_T(X)\Big] \leq \chi_T(X) \leq 2\big[\omega_0^T(X) + \overline{\chi}_T(X)\big] \tag{3.11}$$

for any $T > 0$. We now formulate the earlier announced result.

Theorem 3.2. *Assume that γ is the Hausdorff measure of noncompactness in the Banach space E; that is, $\gamma = \chi_E$. Then, the functions χ_a, χ_b, and χ_c defined by (3.7)–(3.9) are measures of noncompactness in the space $BC(\mathbb{R}_+, E)$ such that*

$$\chi(X) \leq 2\chi_b(X), \tag{3.12}$$

$$\chi(X) \leq 4\chi_c(X), \tag{3.13}$$

$$\chi_b(X) \leq 2\chi_a(X), \qquad \chi_c(X) \leq 2\chi_a(X) \tag{3.14}$$

for an arbitrary set $X \in \mathfrak{M}_{BC(\mathbb{R}_+,E)}$, where χ denotes the Hausdorff measure of noncompactness in the space $BC(\mathbb{R}_+, E)$.

Proof. First we prove inequality (3.12). To this end, fix a set $X \in \mathfrak{M}_{BC(\mathbb{R}_+,E)}$, and denote $r = \chi_b(X)$. Let us also put $r_1 = \omega_0(X)$, $r_2 = \overline{\chi}_\infty(X)$, $r_3 = b_\infty(X)$. Obviously we have $r_1 + r_2 + r_3 = r$. On the basis of the definitions (3.1) and (3.2) of quantities ω_0 and $\overline{\chi}_T$, we have

$$\omega_0^T(X) \leq r_1, \tag{3.15}$$

$$\overline{\chi}_T(X) \leq r_2 \tag{3.16}$$

for a fixed $T > 0$. On the other hand, taking an arbitrarily fixed number $\varepsilon > 0$ and utilizing (3.5), we find a number $T_0 > 0$ such that for an arbitrary $T \geq T_0$ we obtain

$$b_T(X) \leq r_3 + \varepsilon. \tag{3.17}$$

Using (3.17) and the definition of the function b_T, we infer that

$$\sup\big\{\big\|x(t) - x(s)\big\|_E : t, s \geq T_0\big\} \leq r_3 + \varepsilon \tag{3.18}$$

for an arbitrary function $x \in X$.

In the rest of this article, let us fix an arbitrary number $T, T \geq T_0$. Then, keeping in mind estimate (3.11) and inequalities (3.15) and (3.16), we obtain the following inequality:

$$\chi_T(X) \leq 2r_1 + r_2.$$

Hence we infer that, for an arbitrarily fixed number $\delta > 0$, we can find $(2r_1 + r_2 + \delta)$-net $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_m$ of the set X in the space $C([0,T],E)$. This means that for an arbitrary function $x \in X$ there exists $k \in \{1,2,\dots,m\}$ such that

$$\left\|x(t) - \overline{x}_k(t)\right\|_E \le 2r_1 + r_2 + \delta \tag{3.19}$$

for $t \in [0,T]$.

Now, consider the extension x_k of the function $\overline{x}_k$ $(k = 1,2,\dots,m)$ on the interval $\mathbb{R}_+$ defined in the following way:

$$x_k(t) = \begin{cases} \overline{x}_k(t) & \text{for } t \in [0,T], \\ \overline{x}_k(T) & \text{for } t > T. \end{cases}$$

Obviously we have $x_k \in BC(\mathbb{R}_+, E)$ $(k = 1,2,\dots,m)$. Further, utilizing (3.18) and (3.19), for an arbitrary $t \ge T$ we get

$$\begin{aligned} \left\|x(t) - x_k(t)\right\|_E &\le \left\|x(t) - x(T)\right\|_E + \left\|x(T) - x_k(t)\right\|_E \\ &\le r_3 + \varepsilon + \left\|x(T) - \overline{x}_k(T)\right\|_E \le r_3 + \varepsilon + 2r_1 + r_2 + \delta \\ &\le 2r_1 + 2r_2 + 2r_3 + \varepsilon + \delta \le 2r + \varepsilon + \delta. \end{aligned}$$

From the above estimate stems the fact that the functions $x_1, x_2, \dots, x_m$ form a finite $(2r + \varepsilon + \delta)$-net of the set X in the space $BC(\mathbb{R}_+, E)$. Consequently, we have

$$\chi(X) \le 2r + \varepsilon + \delta.$$

Hence, taking into account the fact that ε and δ were chosen arbitrarily, we obtain

$$\chi(X) \le 2\chi_b(x).$$

This proves inequality (3.12).

In order to prove (3.13) similarly as in the preceding inequality, let us put $r = \chi_c(X)$, $r_1 = \omega_0(X)$, $r_2 = \overline{\chi}_\infty(X)$, $r_3 = c(X)$. Obviously we have $r = r_1 + r_2 + r_3$. Next, take an arbitrary number $\varepsilon > 0$. Then, we can find a number $T_0 > 0$ such that for $t \ge T_0$ the following inequality is satisfied:

$$\operatorname{diam} X(t) \le r_3 + \varepsilon. \tag{3.20}$$

Furthermore, arguing in the same way as previously, we deduce that, for an arbitrarily fixed number $T \ge T_0$, the set X considered in the space $C([0,T],E)$, that is, the set

$$\overline{X}_T = \{x|_{[0,T]} : x \in X\},$$

has, for an arbitrary $\delta > 0$, a finite $(2r_1 + r_2 + \delta)$-net composed by functions $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_m$ belonging to the space $C([0,T],E)$.

Now, let us choose arbitrary functions $y_1, y_2, \dots, y_m \in X$ such that, for any $i \in \{1,2,\dots,m\}$, the inequality

$$\left\|y_i(t) - \overline{x}_i(t)\right\|_E \le 2r_1 + r_2 + \delta \tag{3.21}$$

is satisfied for $t \in [0,T]$.

Further, take an arbitrary function $x \in X$. Then we find $i \in \{1,2,\dots,m\}$ such that

$$\big\|x(t)-\overline{x}_i(t)\big\|_E \le 2r_1+r_2+\delta \tag{3.22}$$

for an arbitrary number $t\in[0,T]$. Next taking into account (3.21) and (3.22), we get

$$\big\|x(t)-y_i(t)\big\|_E \le \big\|x(t)-\overline{x}_i(t)\big\|_E + \big\|\overline{x}_i(t)-y_i(t)\big\|_E \le 2(r_1+r_2)+2\delta \tag{3.23}$$

for an arbitrary $t\in[0,T]$.

Now, combining (3.20) and (3.23), for an arbitrary number $t\in\mathbb{R}_+$ we obtain

$$\begin{aligned}\big\|x(t)-y_i(t)\big\|_E &\le \max\big\{2(2r_1+r_2)+2\delta, r_3+\varepsilon\big\}\\ &\le 4r_1+2r_2+r_3+\varepsilon+2\delta \le 4r+\varepsilon+2\delta.\end{aligned}$$

From the above estimate we deduce that the functions $y_1,y_2,\ldots,y_m$ form a finite $(4r+\varepsilon+2\delta)$-net of the set X in the space $BC(\mathbb{R}_+,E)$. Thus, we have

$$\chi(X)\le 4\chi_c(X)+\varepsilon+2\delta.$$

Hence, taking into account the arbitrariness of the numbers ε and δ, we derive inequality (3.13). Furthermore, let us observe that the first inequality of (3.14) follows immediately from the estimate $\|x(t)-x(s)\|_E\le\|x(t)\|_E+\|x(s)\|_E$, which is satisfied for arbitrary numbers $t,s\in\mathbb{R}_+$. The second inequality of (3.14) is a simple consequence of the inequality

$$\big\|x(t)-y(t)\big\|_E \le \big\|x(t)\big\|_E + \big\|y(t)\big\|_E,$$

which holds for arbitrary functions $x,y\in BC(\mathbb{R}_+,E)$ and for any $t\in\mathbb{R}_+$.

Further, let us observe that, in view of estimates (3.12)–(3.14), we can easily deduce that the functions χ_a, χ_b, and χ_c satisfy axioms (i) and (vi) of Definition 2.1. Moreover, taking into account the properties of the components ω_0, $\overline{\chi}_\infty$, a_∞, b_∞, and c, we conclude that the mentioned functions satisfy the remaining axioms of Definition 2.1, that is, axioms (ii)–(v). Thus, the discussed functions χ_a, χ_b, and χ_c are measures of noncompactness in the space $BC(\mathbb{R}_+,E)$. The proof is complete. □

Let us also mention that measures χ_a and χ_b are sublinear and have the maximum property. On the other hand, χ_c is *sublinear* but does not have the maximum property. It is easily seen that the measures of noncompactness χ_a, χ_b, and χ_c are not full since

$$\ker\mu_{\chi_z}\subsetneq\mathfrak{N}_{BC(\mathbb{R}_+,E)}$$

for $z=a$, $z=b$, or $z=c$. Examples showing that the above assertion is valid can be easily constructed even in the case $E=\mathbb{R}$ (see [6]).

It is worth mentioning that the kernel $\ker\chi_a$ of the measure of noncompactness χ_a defined by (3.7) consists of all bounded subsets X of the space $BC(\mathbb{R}_+,E)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and tend to zero at infinity with the same rate. Additionally, all cross sections $X(t)$ of the set X are relatively compact in Banach space E. Similarly, the kernel $\ker\chi_b$ of the measure χ_b defined by (3.8) consists of all bounded subsets X of $BC(\mathbb{R}_+,E)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$, and all cross sections $X(t)$ of the set X are relatively compact in E. Moreover, all functions from X tend to limits uniformly with respect to the set X.

Finally, to describe the kernel $\ker \chi_c$ of the measure of noncompactness χ_c defined by (3.9), note that it contains all bounded subsets X of $BC(\mathbb{R}_+, E)$ which are locally equicontinuous on $\mathbb{R}_+$ and such that the cross sections $X(t)$ of X are relatively compact in E for any $t \in \mathbb{R}_+$. Apart from this, the thickness of the bundle formed by graphs of functions from X tends to zero at infinity.

Remark 3.3. Observe that Theorem 3.2 remains valid if, in the construction of the component $\overline{\gamma}_\infty$ defined by (3.3), we replace the Hausdorff (or Kuratowski) measure of noncompactness χ (or α) by an arbitrary regular measure of noncompactness μ which is equivalent to the Hausdorff measure χ (see Section 2). The proof of the fact that such new functions μ_a, μ_b, and μ_c defined by (3.7)–(3.9) are measures of noncompactness can be conducted in the same way as the proof of Theorem 3.2 and is therefore omitted.

In what follows, we are going to give formulas expressing measures of noncompactness defined by (3.7)–(3.9) in the case when we take the space c_0 as the simplest Banach sequence space. We remark that the fact that we consider Banach sequence spaces like c_0, c, l_1, and l_p is dictated by the need to study infinite systems of integral equations. In such a situation, Banach sequence spaces seem ideally suited to conduct the study in question.

Thus, let us consider the space $BC(\mathbb{R}_+, c_0)$ consisting of functions $x : \mathbb{R}_+ \to c_0$ which are continuous and bounded on $\mathbb{R}_+$. Obviously, such a function can be represented in the form

$$x(t) = \big(x_n(t)\big) = \big(x_1(t), x_2(t), \ldots\big)$$

for any $t \in \mathbb{R}_+$, where the sequence $(x_n(t))$ is an element of the space c_0 for any fixed t. This means that $\lim_{n\to\infty} x_n(t) = 0$ for $t \in \mathbb{R}_+$. The norm of the function $x = x(t) = (x_n(t))$ is defined by the equality

$$\|x\| = \sup_{t\in\mathbb{R}_+} \big\{\sup\big\{\big|x_n(t)\big| : n = 1, 2, \ldots\big\}\big\}.$$

Now, we are going to present explicitly the formulas expressing the components of the measure of noncompactness $\chi_a(X)$ for $X \in \mathfrak{M}_{BC(\mathbb{R}_+, c_0)}$. To this end, take arbitrary numbers $T > 0$ and $\varepsilon > 0$. Next, for $x = x(t) = (x_n(t)) \in X$, we obtain

$$\begin{aligned}\omega^T(x, \varepsilon) &= \sup\big\{\big\|x(t) - x(s)\big\|_{c_0} : t, s \in [0, T], |t - s| \le \varepsilon\big\}\\ &= \sup\big\{\sup\big\{\big|x_n(t) - x_n(s)\big| : n = 1, 2, \ldots\big\} : t, s \in [0, T], |t - s| \le \varepsilon\big\}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}&\omega^T(X, \varepsilon)\\ &\quad= \sup_{x\in X}\big\{\sup\big\{\sup\big\{\big|x_n(t) - x_n(s)\big| : n \in \mathbb{N}\big\} : t, s \in [0, T], |t - s| \le \varepsilon\big\}\big\}.\end{aligned} \tag{3.24}$$

Further steps in the construction of the component $\omega_0(X)$ are the same as those presented above in a general situation.

In order to create the second component related to the measures of noncompactness χ_a, χ_b, and χ_c, let us take the Hausdorff measure χ in the space c_0.

Recall that, according to (2.2), for an arbitrary bounded subset X of the space c_0 we have

$$\chi(X) = \lim_{n\to\infty}\Big\{\sup_{x=(x_k)\in X}\big\{\sup\{|x_k| : k \geq n\}\big\}\Big\}.$$

Thus, if we take an arbitrary set $X \in \mathfrak{M}_{BC(\mathbb{R}_+,c_0)}$, then for an arbitrary fixed $t \in \mathbb{R}_+$ we have

$$\chi\big(X(t)\big) = \lim_{n\to\infty}\Big\{\sup_{x=(x_k)\in X}\big\{\sup\{\big|x_k(t)\big| : k \geq n\}\big\}\Big\}.$$

Consequently, for a fixed $T > 0$ we get

$$\begin{aligned}\overline{\chi}_T(X) &= \sup\big\{\chi\big(X(t)\big) : t \in [0,T]\big\}\\ &= \sup_{t\in[0,T]}\Big\{\lim_{n\to\infty}\Big\{\sup_{(x_k(t))\in X(t)}\big\{\sup\{\big|x_k(t)\big| : k \geq n\}\big\}\Big\}\Big\}.\end{aligned}$$

Finally, we derive the following expression:

$$\begin{aligned}\overline{\chi}_\infty(X) &= \lim_{T\to\infty}\overline{\chi}_T(X)\\ &= \lim_{T\to\infty}\Big\{\sup_{t\in[0,T]}\Big\{\lim_{n\to\infty}\Big\{\sup_{x=(x_k)\in X}\big\{\sup\{\big|x_k(t)\big| : k \geq n\}\big\}\Big\}\Big\}\Big\}. \qquad (3.25)\end{aligned}$$

Now, we proceed to the construction of the components $a_\infty(X)$, $b_\infty(X)$, and $c(X)$ defined by (3.4), (3.5), and (3.6), respectively. Thus, fix an arbitrary number $T > 0$. Then, we have

$$\begin{aligned}a_T(X) &= \sup_{x\in X}\big\{\sup\{\big\|x(t)\big\|_{c_0} : t \geq T\}\big\}\\ &= \sup_{x=(x_n)\in X}\big\{\sup\big\{\sup\{\big|x_n(t)\big| : n \in \mathbb{N}\big\} : t \in T\big\}\big\}.\end{aligned}$$

Finally, we get

$$a_\infty(X) = \lim_{T\to\infty}\Big\{\sup_{x=(x_n)\in X}\big\{\sup_{t\geq T}\big\{\sup\{\big|x_n(t)\big| : n \in \mathbb{N}\}\big\}\big\}\Big\}. \qquad (3.26)$$

Similarly, keeping in mind formula (3.5), we subsequently obtain

$$\begin{aligned}b_T(X) &= \sup_{x=(x_n)\in X}\big\{\sup\{\big\|x(t)-x(s)\big\|_{c_0} : t,s \geq T\}\big\}\\ &= \sup_{x=(x_n)\in X}\big\{\sup\{\sup_{n\in\mathbb{N}}\big|x_n(t)-x_n(s)\big| : t,s \geq T\}\big\},\\ b_\infty(X) &= \lim_{T\to\infty}\Big\{\sup_{x=(x_n)\in X}\big\{\sup\{\sup_{n\in\mathbb{N}}\big|x_n(t)-x_n(s)\big| : t,s \in T\}\big\}\Big\}. \qquad (3.27)\end{aligned}$$

In the same way, taking into account formula (3.6), we get

$$\begin{aligned}c(X) &= \limsup_{t\to\infty}\big\{\sup\{\big\|x(t)-y(t)\big\|_{c_0} : x,y \in X\}\big\}\\ &= \limsup_{t\to\infty}\big\{\sup\{\sup_{n\in\mathbb{N}}\big|x_n(t)-y_n(t)\big| : x=(x_n), y=(y_n) \in X\}\big\}. \qquad (3.28)\end{aligned}$$

Constructions of measures of noncompactness in the case when $E = c$ or $E = l^p$ can be realized in the same fashion. We will not provide details of those constructions in this paper since in our further considerations we confine ourselves to the case of measures of noncompactness in the space $BC(\mathbb{R}_+, c_0)$ only.

4. Theorems on the existence of solutions of infinite systems of integral equations on the real half-axis

We will consider the infinite system of nonlinear quadratic integral equations of Volterra–Hammerstein type having the form

$$x_n(t) = a_n(t) + f_n\big(t, x_n(t), x_{n+1}(t), \dots\big) \int_0^t k_n(t,s) g_n\big(s, x_1(s), x_2(s), \dots\big)\, ds \tag{4.1}$$

for $t \in \mathbb{R}_+$ and for $n = 1, 2, \dots$.

Our considerations concerning the solvability of the infinite system of integral equations (4.1) will be preceded by two lemmas which will be used in our later arguments. In our further investigations, the space $BC(\mathbb{R}_+, c_0)$ will be denoted by BC_0.

Lemma 4.1. *Let the function $x(t) = (x_n(t))$ be an element of the space BC_0. Then the sequence (x_n) is equibounded and locally equicontinuous on $\mathbb{R}_+$.*

Proof. First, let us note that the function $x = x(t)$ acts continuously from the interval $\mathbb{R}_+$ into c_0. Hence, we deduce that, for each $T > 0$, the function $x(t)$ is uniformly continuous on the interval $[0, T]$. Thus for a given $\varepsilon > 0$, we can choose $\delta > 0$ such that the inequality $|t_2 - t_1| \le \delta$ for $t_1, t_2 \in [0, T]$ implies that

$$\big\|x(t_2) - x(t_1)\big\|_{c_0} = \sup\big\{\big|x_n(t_2) - x_n(t_1)\big| : n = 1, 2, \dots\big\} \le \varepsilon.$$

This means that $|x_n(t_2) - x_n(t_1)| \le \varepsilon$ for $n = 1, 2, \dots$.

Summing up, we conclude that for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for arbitrary $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \le \delta$ and for each $n = 1, 2, \dots$, we have $|x_n(t_2) - x_n(t_1)| \le \varepsilon$. Thus, the function sequence (x_n) is equicontinuous on the interval $[0, T]$. Hence it follows that the mentioned function sequence (x_n) is locally equicontinuous on $\mathbb{R}_+$.

On the other hand, the fact that the function $x = x(t)$ is bounded on $\mathbb{R}_+$ implies that there exists a constant $M > 0$ such that $\|x(t)\|_{c_0} \le M$ for $t \in \mathbb{R}_+$. Hence we derive $|x_n(t)| \le M$ for $n = 1, 2, \dots$ and for each $t \in \mathbb{R}_+$. Thus, we obtain the desired equiboundedness of the sequence (x_n) on the interval $\mathbb{R}_+$. □

Lemma 4.2. *Let $x = (x_n) \in BC_0$. Then the function sequence (x_n) is nearly uniformly convergent to zero on the interval $\mathbb{R}_+$.*

Proof. Fix arbitrarily $T > 0$, and consider the function sequence $(x_n(t))$ on the interval $[0, T]$. By imposed assumptions, we infer that for any $t \in [0, T]$ the sequence $(x_n(t))$ converges to zero. Apart from this, in view of Lemma 4.1 we deduce that the function sequence $(x_n(t))$ is equicontinuous on the interval $[0, T]$. Linking the above facts and applying a result from [18], we infer that this sequence converges uniformly to zero on the interval $[0, T]$. The proof is complete. □

Remark 4.3. The interval of the uniform convergence to zero of the function sequence $(x_n(t))$ in Lemma 4.2 cannot be extended to the whole interval $\mathbb{R}_+$. Indeed, consider the function sequence (x_n), where $x_n = x_n(t)$ $(n = 1, 2, \ldots)$ is defined on $\mathbb{R}_+$ in the following way:

$$x_n(t) = \begin{cases} |\sin t| & \text{for } t \in [(n-1)\pi, n\pi], \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that the sequence $(x_n(t))$ converges to zero on every interval of the form $[0, T]$, but this convergence is not uniform on $\mathbb{R}_+$.

Now we formulate assumptions under which the infinite system of integral equations (4.1) will be investigated.

(i) The sequence $(a_n(t))$ is an element of the space BC_0 such that $\lim_{t\to\infty} a_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$; that is, the following condition is satisfied (see Lemma 4.2):

$$\forall_{\varepsilon>0}\exists_{T>0}\forall_{t\geq T}\forall_{n\in\mathbb{N}}\ \big|a_n(t)\big| \leq \varepsilon.$$

(ii) The functions $k_n(t,s) = k_n : \mathbb{R}_+^2 \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+^2$ $(n = 1, 2, \ldots)$. Moreover, the functions $t \to k_n(t,s)$ are locally equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $s \in \mathbb{R}_+$; that is, the following condition is satisfied:

$$\forall_{T>0}\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{n\in\mathbb{N}}\forall_{s\in\mathbb{R}_+}\forall_{t_1,t_2\in[0,T]}\ \big[|t_2 - t_1| \leq \delta \Rightarrow \big|k_n(t_2,s) - k_n(t_1,s)\big| \leq \varepsilon\big].$$

(iii) There exists a constant $K_1 > 0$ such that

$$\int_0^t \big|k_n(t,s)\big|\, ds \leq K_1$$

for any $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$.

(iv) The sequence $(k_n(t,s))$ is equibounded on $\mathbb{R}_+^2$; that is, there exists a constant $K_2 > 0$ such that $|k_n(t,s)| \leq K_2$ for $t, s \in \mathbb{R}_+$ and $n = 1, 2, \ldots$.

(v) The function f_n is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values for $n = 1, 2, \ldots$. Moreover, the function $t \to f_n(t, x_n, x_{n+1}, \ldots)$ is continuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_n) \in c_0$ and uniformly with respect to $n \in \mathbb{N}$; that is, the following condition is satisfied:

$$\forall_{\varepsilon>0}\forall_{t_0\in\mathbb{R}_+}\exists_{\delta>0}\forall_{(x_i)\in c_0}\forall_{t\in\mathbb{R}_+}\forall_{n\in\mathbb{N}}\ \big[|t - t_0| \leq \delta \Rightarrow \big|f_n(t, x_n, x_{n+1}, \ldots) - f_n(t_0, x_n, x_{n+1}, \ldots)\big| \leq \varepsilon\big].$$

(vi) There exists a function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that l is nondecreasing on $\mathbb{R}_+$, continuous at 0, and the following condition is satisfied:

$$\big|f_n(t, x_n, x_{n+1}, \ldots) - f_n(t, y_n, y_{n+1}, \ldots)\big| \leq l(r) \sup\big\{|x_i - y_i| : i \geq n\big\}$$

for all $x = (x_i)$, $y = (y_i) \in c_0$ such that $\|x\|_{c_0} \leq r$, $\|y\|_{c_0} \leq r$ and for any $n = 1, 2, \ldots$.

(vii) The sequence of functions $(\overline{f}_n)$, where $\overline{f}_n(t) = |f_n(t, 0, 0, \dots)|$, is an element of the space BC_0 such that $\lim_{t\to\infty} \overline{f}_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$ (cf. assumption (i)).

Observe that on the basis of assumption (vii) we can define the finite constant $\overline{F} = \sup\{\overline{f}_n(t) : t \in \mathbb{R}_+, n = 1, 2, \dots\}$.

Now, we can formulate our further assumptions concerning infinite system (4.1).

(viii) The function g_n is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values for $n = 1, 2, \dots$. Moreover, the operator g defined on the space $\mathbb{R}_+ \times c_0$ by the formula

$$(gx)(t) = \big(g_n(t, x)\big) = \big(g_1(t, x), g_2(t, x), \dots\big)$$

transforms the space $\mathbb{R}_+ \times c_0$ into c_0 and is such that the family of functions $\{(gx)(t)\}_{t\in\mathbb{R}_+}$ is equicontinuous at every point of the space c_0; that is, for each arbitrarily fixed $x \in c_0$ and for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big\|(gy)(t) - (gx)(t)\big\|_{c_0} \leq \varepsilon$$

for every $t \in \mathbb{R}_+$ and for any $g \in c_0$ such that $\|y - x\|_{c_0} \leq \delta$.

(ix) The operator g defined in assumption (viii) is bounded on the space $\mathbb{R}_+ \times c_0$. More precisely, there exists a positive constant $\overline{G}$ such that $\|(gx)(t)\|_{c_0} \leq \overline{G}$ for any $x \in c_0$ and for each $t \in \mathbb{R}_+$.

(x) There exists a positive solution r_0 of the inequality

$$A + \overline{FG}K_1 + \overline{G}K_1 r l(r) \leq r$$

such that $\overline{G}K_1 l(r_0) < 1$, where the constants $\overline{F}$, $\overline{G}$, K_1 were defined above and the constant A is defined in the following way:

$$A = \sup\big\{\big|a_n(t)\big| : t \in \mathbb{R}_+, n = 1, 2, \dots\big\}.$$

Obviously, in view of assumption (i) and Lemma 4.1, we have $A < \infty$.

Now, we are prepared to formulate an existence theorem concerning the infinite system of integral equations (4.1).

Theorem 4.4. *Under assumptions (i)–(x), infinite system (4.1) has at least one solution $x(t) = (x_n(t))$ in the space $BC_0 = BC(\mathbb{R}_+, c_0)$.*

Proof. First let us consider three operators F, V, Q defined on the space BC_0 in the following way:

$$(Fx)(t) = \big((F_n x)(t)\big) = \big(f_n\big(t, x(t)\big)\big) = \big(f_n\big(t, x_n(t), x_{n+1}(t), \dots\big)\big),$$

$$(Vx)(t) = \big((V_n x)(t)\big) = \Big(\int_0^t k_n(t, s) g_n\big(s, x_1(s), x_2(s), \dots\big)\, ds\Big),$$

$$(Qx)(t) = \big((Q_n x)(t)\big) = \big(a_n(t) + (F_n x)(t)(V_n x)(t)\big).$$

We start by showing that the operator F transforms the space BC_0 into itself.

Thus, let us take $x = x(t) = (x_n(t)) \in BC_0$. This implies that $\lim_{n\to\infty} x_n(t) = 0$ for arbitrarily fixed $t \in \mathbb{R}_+$. On the other hand, since the function $x : \mathbb{R}_+ \to c_0$ is continuous, we infer that the following condition is satisfied:

$$\forall_{t_0\in\mathbb{R}_+}\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{t\in\mathbb{R}_+}\ \big[|t-t_0|\le\delta \Rightarrow \big\|x(t)-x(t_0)\big\|_{c_0}\le\varepsilon\big]. \tag{4.2}$$

Now, fix $n \in \mathbb{N}$, and take a number $t \in \mathbb{R}_+$. Then, in view of the imposed assumptions, we get

$$\begin{aligned}\big|(F_nx)(t)\big| &\le \big|f_n\big(t,x_n(t),x_{n+1}(t),\dots\big)-f_n(t,0,0,\dots)\big|+\big|f_n(t,0,0,\dots)\big|\\ &\le l\big(\big\|x(t)\big\|_{c_0}\big)\sup\big\{\big|x_i(t)\big| : i\ge n\big\}+\overline{f}_n(t). \end{aligned}\tag{4.3}$$

From the above estimate, in view of Lemma 4.2 and assumption (vii), we deduce that $\lim_{n\to\infty}(F_nx)(t) = 0$. Thus, $Fx \in c_0$. Moreover, from (4.3) we conclude that the function Fx is bounded on $\mathbb{R}_+$.

In order to prove the continuity of the function Fx on the interval $\mathbb{R}_+$, take $\varepsilon > 0$. Next, let us fix an arbitrary number $t_0 \in \mathbb{R}_+$ and choose $\delta > 0$ according to condition (4.2). Then, for $t \in \mathbb{R}_+$ such that $|t - t_0| \le \delta$, we obtain

$$\begin{aligned}&\big|(F_nx)(t)-(F_nx)(t_0)\big|\\ &\quad\le \big|f_n\big(t,x_n(t),x_{n+1}(t),\dots\big)-f_n\big(t_0,x_n(t),x_{n+1}(t),\dots\big)\big|\\ &\qquad + l\big(\sup\big\{\big\|x(t)\big\|_{c_0} : t\in\mathbb{R}_+\big\}\big)\sup\big\{\big|x_i(t)-x_i(t_0)\big| : i\ge n\big\}\\ &\quad\le \big|f_n\big(t,x_n(t),x_{n+1}(t),\dots\big)-f_n\big(t_0,x_n(t),x_{n+1}(t),\dots\big)\big|+l\big(\|x\|_{BC_0}\big)\varepsilon. \end{aligned}\tag{4.4}$$

Now, keeping in mind assumption (v), we can choose a number $\delta > 0$ in such a way that

$$\big|f_n\big(t,x_n(t),x_{n+1}(t),\dots\big)-f_n\big(t_0,x_n(t),x_{n+1}(t),\dots\big)\big|\le\varepsilon$$

for $|t - t_0| \le \delta$ and for $n = 1, 2, \dots$.

Hence, by virtue of (4.4), we derive the estimate

$$\big|(F_nx)(t)-(F_nx)(t_0)\big|\le\big(1+l\big(\|x\|_{BC_0}\big)\big)\varepsilon$$

for any $t \in \mathbb{R}_+$ and for $n = 1, 2, \dots$. But this means that the operator F transforms the space BC_0 into itself.

In what follows, we are going to show that the operator V defined above acts from the space BC_0 into itself. To this end, similarly as above, take an arbitrary function $x = x(t) = (x_n(t)) \in BC_0$. Then, for arbitrarily fixed numbers $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, in view of assumptions (iii) and (ix), we obtain

$$\begin{aligned}\big|(V_nx)(t)\big| &\le \int_0^t\big|k_n(t,s)\big|\big|g_n\big(s,x_1(s),x_2(s),\dots\big)\big|\,ds\\ &\le \int_0^t\big|k_n(t,s)\big|\overline{G}\,ds\le\overline{G}\int_0^t\big|k_n(t,s)\big|\,ds\le\overline{G}K_1. \end{aligned}\tag{4.5}$$

In particular, the above estimate yields that the function Vx is bounded on the interval $\mathbb{R}_+$.

Next, fix $\varepsilon > 0$ and $T > 0$. Let $t_0 \in [0, T)$. Choose a number $\delta > 0$ according to the continuity of the function $x = x(t)$ at the point t_0. Then, utilizing assumption (viii), for $t \in [0, T)$ such that $|t - t_0| \leq \delta$ (we can assume without loss of generality that $t > t_0$) we obtain

$$\begin{aligned}
&\big|(V_n x)(t) - (V_n x)(t_0)\big| \\
&\quad \leq \Big| \int_0^t k_n(t,s) g_n\big(s, x_1(s), x_2(s), \ldots\big)\, ds \\
&\qquad - \int_0^t k_n(t_0,s) g_n\big(s, x_1(s), x_2(s), \ldots\big)\, ds \Big| \\
&\qquad + \Big| \int_0^t k_n(t_0,s) g_n\big(s, x_1(s), x_2(s), \ldots\big)\, ds \\
&\qquad - \int_0^{t_0} k_n(t_0,s) g_n\big(s, x_1(s), x_2(s), \ldots\big)\, ds \Big| \\
&\quad \leq \int_0^t \big|k_n(t,s) - k_n(t_0,s)\big| \big|g_n\big(s, x_1(s), x_2(s), \ldots\big)\big|\, ds \\
&\qquad + \int_{t_0}^t \big|k_n(t_0,s)\big| \big|g_n\big(s, x_1(s), x_2(s), \ldots\big)\big|\, ds \\
&\quad \leq \int_0^t \omega_k^T(\varepsilon) \big|g_n\big(s, x_1(s), x_2(s), \ldots\big)\big|\, ds \\
&\qquad + \int_{t_0}^t K_2 \big|g_n\big(s, x_1(s), x_2(s), \ldots\big)\big|\, ds,
\end{aligned} \tag{4.6}$$

where K_2 is a constant from assumption (iv) and $\omega_k^T(\varepsilon)$ denotes a common modulus of continuity of the sequence of functions $t \to k_n(t,s)$ on the set $[0,T]$ (according to assumption (iii)). Obviously, $\omega_k^T(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Further, taking into account estimate (4.6) and assumption (ix), we get

$$\big|(V_n x)(t) - (V_n x)(t_0)\big| \leq T\overline{G}\omega_k^T(\varepsilon) + K_2\overline{G}\varepsilon.$$

Hence we conclude that the function Vx is continuous on the interval $[0,T)$. In view of the arbitrariness of T this yields to the conclusion that the function Vx is continuous on the interval $\mathbb{R}_+$.

Utilizing the fact that the space $BC_0 = BC(\mathbb{R}_+, c_0)$ is a Banach algebra with respect to the coordinatewise multiplication of function sequences and taking into account the definition of the operator Q, as well as keeping in mind assumption (i), we infer that for an arbitrary fixed function $x = x(t) \in BC_0$ the function $(Qx)(t) = ((Q_n x)(t)) = (a_n(t) + (F_n x)(t)(V_n x)(t))$ transforms the interval $\mathbb{R}_+$ into the space c_0.

Indeed, in view of the fact that $((F_n x)(t)) \in c_0$ for any $t \in \mathbb{R}_+$ and in light of estimate (4.5), we get

$$\big|(Q_n x)(t)\big| \leq \big|a_n(t)\big| + \overline{G}K_1\big|(F_n x)(t)\big|.$$

Hence, by (4.3) we infer that $(Qx)(t) = ((Q_n x)(t)) \in c_0$ for each $t \in \mathbb{R}_+$.

Next, let us observe that the continuity of the function Qx on $\mathbb{R}_+$ is a simple consequence of the fact that both the function Fx and the function Vx are continuous on the interval $\mathbb{R}_+$. In a similar way we can derive the fact that the function Qx is bounded on $\mathbb{R}_+$; to this end, it is only sufficient to take into account assumption (i) and Lemma 4.1. Finally, combining all the above established properties of the function Qx, we derive that the operator Q transforms the space BC_0 into itself.

Now, let us observe that, based on estimates (4.3) and (4.5) for arbitrarily fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we get

$$\begin{aligned} \big|(Q_n x)(t)\big| &\le \big|a_n(t)\big| + \big|(F_n x)(t)\big|\big|(V_n x)(t)\big| \\ &\le A + \big[l\big(\|x(t)\|_{c_0}\big)\sup\big\{\big|x_i(t)\big| : i \ge n\big\} + \overline{f}_n(t)\big]\overline{G}K_1 \\ &\le A + \big[l\big(\|x\|_{BC_0}\big)\|x\|_{BC_0} + \overline{F}\big]\overline{G}K_1 \\ &\le A + \overline{FG}K_1 + \overline{G}K_1 l\big(\|x\|_{BC_0}\big)\|x\|_{BC_0}. \end{aligned}$$

Hence we obtain the following estimate:

$$\|Qx\|_{BC_0} \le A + \overline{FG}K_1 + \overline{G}K_1 l\big(\|x\|_{BC_0}\big)\|x\|_{BC_0}.$$

From this estimate and assumption (x), we deduce that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} into itself.

In the rest of our proof, we show that the operator Q is continuous on the ball B_{r_0}. Obviously, in view of the representation of the operator Q given at the beginning of the proof, it is sufficient to show the continuity of the operators F and V separately.

Thus, fix arbitrarily $\varepsilon > 0$ and $x \in B_{r_0}$. Next, take an arbitrary point $y \in B_{r_0}$ such that $\|x - y\|_{BC_0} \le \varepsilon$. Then, for fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, by virtue of assumption (vi), we get

$$\begin{aligned} &\big|(F_n x)(t) - (F_n y)(t)\big| \\ &\quad = \big|f_n\big(t, x_n(t), x_{n+1}(t), \dots\big) - f_n\big(t, y_n(t), y_{n+1}(t), \dots\big)\big| \\ &\quad \le l(r_0)\sup\big\{\big|x_i(t) - y_i(t)\big| : i \ge n\big\} \\ &\quad \le l(r_0)\|x - y\|_{BC_0} \le l(r_0)\varepsilon. \end{aligned}$$

Consequently, we obtain

$$\|Fx - Fy\|_{BC_0} \le l(r_0)\varepsilon.$$

This estimate proves the desired continuity of the operator F on the ball B_{r_0}. Furthermore, based on assumption (viii), we can define the function $\delta(\varepsilon)$ by putting

$$\delta(\varepsilon) = \sup\big\{\big|g_n(t,x) - g_n(t,y)\big| : x, y \in c_0, \|x - y\|_{c_0} \le \varepsilon, t \in \mathbb{R}_+, n \in \mathbb{N}\big\}.$$

Observe that in view of assumption (viii), we have that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, assuming similarly as above that $x, y \in B_{r_0}$ are such that $\|x - y\|_{BC_0} \le \varepsilon$, for $t \in \mathbb{R}_+$ and for an arbitrarily fixed $n \in \mathbb{N}$, we get

$$\big|(V_n x)(t) - (V_n y)(t)\big| \le \int_0^t \big|k_n(t,s)\big|\delta(\varepsilon)\, ds \le K_1\delta(\varepsilon).$$

Consequently, we obtain

$$\|Vx - Vy\|_{BC_0} \le K_1\delta(\varepsilon).$$

Hence we infer that the operator V is continuous on the ball B_{r_0}. Linking this fact with the earlier established continuity of the operator F, we derive the continuity of the operator Q on the ball B_{r_0}.

In what follows, let us fix arbitrary numbers $\varepsilon > 0$ and $T > 0$. Next, choose $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$, and take a nonempty subset X of the ball B_{r_0}. Then, for a function $x = x(t) = (x_n(t)) \in X$ and for a fixed natural number n, estimating similarly as in (4.4) we get

$$\begin{aligned} &\big|(F_n x)(t) - (F_n x)(s)\big| \\ &\quad \le l(r_0)\sup\big\{\big|x_i(t) - x_i(s)\big| : i \ge n\big\} \\ &\qquad + \sup\big\{\big|f_n(t, x_n, x_{n+1}, \ldots) - f_n(s, x_n, x_{n+1}, \ldots)\big| : t, s \in [0, T], \\ &\qquad |t - s| \le \varepsilon, \|x\|_{c_0} = \big\|(x_n)\big\|_{c_0} \le r_0\big\} \\ &\quad \le l(r_0)\omega^T(x, \varepsilon) + \omega^1(f, \varepsilon), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \omega^1(f, \varepsilon) = \sup_{n\in\mathbb{N}}\big\{&\sup\big\{\big|f_n(t, x_n, x_{n+1}, \ldots) - f_n(s, x_n, x_{n+1}, \ldots)\big| : t, s \in [0, T], \\ &|t - s| \le \varepsilon, \|x\|_{c_0} = \big\|(x_n)\big\|_{c_0} \le r_0\big\}\big\}. \end{aligned}$$

Obviously, in view of assumption (v) we have that $\omega^1(f, \varepsilon) \to 0$ as $\varepsilon \to 0$. Further, from estimate (4.7) we derive the following:

$$\omega^T(Fx, \varepsilon) \le l(r_0)\omega^T(x, \varepsilon) + \omega^1(f, \varepsilon). \tag{4.8}$$

Now, let us observe that under the same assumptions as above, assuming additionally that $t > s$, in a similar way as in (4.6) we can obtain the estimate

$$\begin{aligned} \big|(V_n x)(t) - (V_n x)(s)\big| &\le \int_0^t \omega_k^T(\varepsilon)\overline{G}\, ds + K_2\overline{G}\varepsilon \\ &\le T\overline{G}\omega_k^T(\varepsilon) + \overline{G}K_2\varepsilon, \end{aligned}$$

where the function $\omega_k^T(\varepsilon)$ was introduced earlier. From the above estimate we get

$$\omega^T(Vx, \varepsilon) \le T\overline{G}\omega_k^T(\varepsilon) + \overline{G}K_2\varepsilon. \tag{4.9}$$

Now, for a function $x \in X$ and for arbitrary $t, s \in \mathbb{R}_+$, keeping in mind the representation of the operator Q given at the beginning of the proof, we obtain

$$\begin{aligned} \big\|(Qx)(t) - (Qx)(s)\big\|_{c_0} \le \big\|a(t) - a(s)\big\|_{c_0} &\\ + \big\|(Vx)(t)\big\|_{c_0}&\big\|(Fx)(t) - (Fx)(s)\big\|_{c_0} \\ + \big\|(Fx)(s)\big\|_{c_0}&\big\|(Vx)(t) - (Vx)(s)\big\|_{c_0}, \end{aligned}$$

where we denoted $a(t) = (a_n(t))$. Assuming that $t, s \in [0, T]$, $|t - s| \le \varepsilon$ and utilizing estimates (4.8), (4.9), (4.3), and (4.5), from the above inequality we get

$$\begin{aligned}\omega^T(Qx,\varepsilon) &\le \omega^T(a,\varepsilon)+\overline{G}K_1\omega^T(Fx,\varepsilon)\\&\quad+\big(l(r_0)r_0+\overline{F}\big)\big(T\overline{G}\omega_k^T(\varepsilon)+\overline{G}K_2\varepsilon\big)\\&\le \omega^T(a,\varepsilon)+\overline{G}K_1\big\{l(r_0)\omega^T(x,\varepsilon)+\omega^1(f,\varepsilon)\big\}\\&\quad+\big(l(r_0)r_0+\overline{F}\big)\big(T\overline{G}\omega_k^T(\varepsilon)+\overline{G}K_2\varepsilon\big).\end{aligned}$$

Hence, we derive the following estimate:

$$\begin{aligned}\omega^T(QX,\varepsilon) &\le \omega^T(a,\varepsilon)+\overline{G}K_1\big\{l(r_0)\omega^T(X,\varepsilon)+\omega^1(f,\varepsilon)\big\}\\&\quad+\big(l(r_0)r_0+\overline{F}\big)\big(T\overline{G}\omega_k^T(\varepsilon)+\overline{G}K_2\varepsilon\big).\end{aligned}$$

Consequently, bearing in mind the above-established properties of the functions $\varepsilon\to\omega^1(f,\varepsilon)$ and $\varepsilon\to\omega_k^T(\varepsilon)$, we obtain

$$\omega_0^T(QX)\le\overline{G}K_1l(r_0)\omega_0^T(X).$$

Passing with $T\to\infty$, we derive the inequality

$$\omega_0(QX)\le\overline{G}K_1l(r_0)\omega_0(X). \tag{4.10}$$

In what follows, we are going to estimate the second term of the measure of noncompactness $\chi_a=\chi_a(X)$ (see formula (3.7) and Theorem 3.2) expressed by formula (3.25). To this end, similarly as before, fix a set $X\subset B_{r_0}$ and a function $x\in X$. Take an arbitrary number $T>0$. Then, for a fixed natural number n and for $t\in[0,T]$, on the basis of estimates (4.3) and (4.5), for $i\ge n$ we obtain

$$\begin{aligned}\big|(Q_ix)(t)\big| &\le \big|a_i(t)\big|\\&\quad+\big|f_i\big(t,x_i(t),x_{i+1}(t),\dots\big)\big|\int_0^t\big|k_i(t,s)\big|\big|g_i\big(s,x_1(s),x_2(s),\dots\big)\big|\,ds\\&\le\big|a_i(t)\big|+\big[l(r_0)\sup\big\{\big|x_i(t)\big|:i\ge n\big\}+\overline{f}_i(t)\big]\overline{G}K_1.\end{aligned}$$

Hence, taking the supremum over all $x=(x_i)\in X$, we get

$$\begin{aligned}\sup_{x=(x_i)\in X}\big|(Q_ix)(t)\big| &\le \big|a_i(t)\big|\\&\quad+\overline{G}K_1l(r_0)\Big\{\sup_{x=(x_i)\in X}\big\{\sup\big\{\big|x_i(t)\big|:i\ge n\big\}\big\}\Big\}+\overline{G}K_1\overline{f}_i(t).\end{aligned}$$

Next, keeping in mind formula (3.25) and assumptions (i) and (vii), we derive the following estimate:

$$\begin{aligned}&\lim_{n\to\infty}\Big\{\sup_{x=(x_i)\in X}\big|(Q_ix)(t)\big|\Big\}\\&\quad\le\overline{G}K_1l(r_0)\lim_{n\to\infty}\Big\{\sup_{x=(x_i)\in X}\big\{\sup\big\{\big|x_i(t)\big|:i\ge n\big\}\big\}\Big\}.\end{aligned}$$

Finally, taking the supremum over $t\in[0,T]$ on both sides of the above inequality and next passing with $T\to\infty$, in view of formula (3.25) we deduce the following estimate:

$$\overline{\chi}_\infty(QX)\le\overline{G}K_1l(r_0)\overline{\chi}_\infty(X). \tag{4.11}$$

To evaluate the last term $a_\infty(X)$ of the measure of noncompactness χ_a (see formula (3.7)) expressed by formula (3.26), let us take a nonempty set X, $X\subset$

B_{r_0}, and choose a function $x \in X$. Next, fix arbitrarily $T > 0$. Then, taking $t \geq T$ and keeping in mind the previously utilized estimates (4.3) and (4.5), we get

$$\begin{aligned}
&\sup\{|(Q_n x)(t)| : n \in \mathbb{N}\} \\
&\quad \leq \sup\{|a_n(t)| : n \in \mathbb{N}\} \\
&\qquad + \sup\Big\{|f_n(t, x_n(t), x_{n+1}(t), \ldots)| \\
&\qquad \times \int_0^t |k_n(t,s)||g_n(s, x_1(s), x_2(s), \ldots)|\, ds : n \in \mathbb{N}\Big\} \\
&\quad \leq \sup\{|a_n(t)| : n \in \mathbb{N}\} + \sup\big[l(r_0)\sup\{|x_i(t)| : i \geq n\} + \overline{f}_n(t)\big]\overline{G}K_1 \\
&\quad \leq \sup\{|a_n(t)| : n \in \mathbb{N}\} + l(r_0)\overline{G}K_1 \sup_{n\in\mathbb{N}}\{\sup\{|x_i(t)| : i \geq n\}\} \\
&\qquad + \overline{G}K_1 \sup\{\overline{f}_n(t) : n \in \mathbb{N}\} \\
&\quad \leq \sup\{|a_n(t)| : n \in \mathbb{N}\} + l(r_0)\overline{G}K_1 \sup\{|x_n(t)| : n \in \mathbb{N}\} \\
&\qquad + \overline{G}K_1 \sup\{\overline{f}_n(t) : n \in \mathbb{N}\}.
\end{aligned}$$

Now, taking in the above estimate the supremum over $t \geq T$ and next, over $x = (x_n) \in X$, we get

$$\begin{aligned}
&\sup_{x=(x_n)\in X}\Big\{\sup_{t\geq T}\{\sup\{|(Q_n x)(t)| : n \in \mathbb{N}\}\}\Big\} \\
&\quad \leq \sup_{t\geq T}\{\sup\{|a_n(t)| : n \in \mathbb{N}\}\} \\
&\qquad + l(r_0)\overline{G}K_1\Big\{\sup_{x=(x_n)\in X}\Big\{\sup_{t\geq T}\{\sup\{|x_n(t)| : n \in \mathbb{N}\}\}\Big\}\Big\} \\
&\qquad + \overline{G}K_1\Big\{\sup_{t\geq T}\{\sup\{\overline{f}_n(t) : n \in \mathbb{N}\}\}\Big\}.
\end{aligned}$$

Next, passing with $T \to \infty$ and keeping in mind assumptions (i) and (vii), we arrive at the following estimate:

$$a_\infty(QX) \leq l(r_0)\overline{G}K_1 a_\infty(X). \tag{4.12}$$

Finally, linking estimates (4.10)–(4.12) and taking into account formula (3.7) with $\gamma = \chi$, we obtain the following inequality for an arbitrary nonempty subset X of the ball B_{r_0}:

$$\chi_a(QX) \leq l(r_0)\overline{G}K_1 \chi_a(X).$$

Hence, in view of the previously established fact that the operator Q is a continuous self-mapping of the ball B_{r_0}, utilizing Theorem 2.2 we conclude that the infinite system of Volterra–Hammerstein integral equations (4.1) has at least one solution $x(t) = (x_n(t))$ in the space $BC_0 = BC(\mathbb{R}_+, c_0)$. Obviously $x \in B_{r_0}$. The proof is complete. □

Now, we provide a remark connected with assumption (x) imposed in the above theorem.

Remark 4.5. Let us observe that the inequality $\overline{G}K_1 l(r_0) < 1$ is not a strong constraint in assumption (x). More precisely, it follows "almost immediately" from the inequality

$$A + \overline{FG}K_1 + \overline{G}K_1 r_0 l(r_0) \le r_0$$

which is imposed in assumption (x). Indeed, dividing both sides of the above inequality by r_0, we obtain

$$\frac{A}{r_0} + \frac{\overline{FG}K_1}{r_0} + \overline{G}K_1 l(r_0) \le 1.$$

Hence we deduce that $\overline{G}K_1 l(r_0) \le 1$. Even more, if $A \ne 0$ or $\overline{FG}K_1 \ne 0$, then the inequality $\overline{G}K_1 l(r_0) < 1$ is automatically satisfied.

5. An example

In this section we are going to provide an example illustrating the existence result proved in Section 4.

Example 5.1. Let us consider the following infinite system of nonlinear Volterra–Hammerstein integral equations of the form

$$x_n(t) = \frac{1}{12}\sin\Big(\frac{t}{n^2+t^2+1}\Big) + \Big(\frac{t}{n+4t^2} + \frac{x_n(t)}{1+x_n^2(t)} + \frac{x_{n+1}^2(t)}{n}\Big) \times \int_0^t \frac{s}{1+n(s^2+t^2)} \arctan\Big(\frac{x_1(s)+x_n(s)}{n+s}\Big)\, ds \tag{5.1}$$

for $n = 1, 2, \ldots$ and $t \in \mathbb{R}_+$. Observe that infinite system (5.1) is a particular case of system (4.1) if we put

$$a_n(t) = \frac{1}{12}\sin\Big(\frac{t}{n^2+t^2+1}\Big), \tag{5.2}$$

$$f_n(t, x_n, x_{n+1}\ldots) = \frac{t}{n+4t^2} + \frac{x_n}{1+x_n^2} + \frac{x_{n+1}^2}{n}, \tag{5.3}$$

$$k_n(t,s) = \frac{s}{1+n(s^2+t^2)}, \tag{5.4}$$

$$g_n(t, x_1, x_2, \ldots) = \arctan\Big(\frac{x_1+x_n}{n+t}\Big) \tag{5.5}$$

for $n = 1, 2, \ldots$ and $t, s \in \mathbb{R}_+$.

Thus, in order to show that the infinite system of integral equations (5.1) has a solution in the Banach space $BC_0 = BC(\mathbb{R}_+, c_0)$ it is sufficient to apply Theorem 4.4. To this end, we have to show that functions (5.2)–(5.5) satisfy assumptions (i)–(x) of Theorem 4.4.

First, let us observe that for any $t \in \mathbb{R}_+$ we have $\lim_{n\to\infty} a_n(t) = 0$. Moreover, it is easy to check that $|a_n(t) - a_n(s)| \le \frac{1}{12n^2}|t-s|$ for $t, s \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. This shows that the sequence $(a_n(t))$ is an element of the space BC_0.

Apart from this we have

$$\big|a_n(t)\big| \le \frac{1}{12}\frac{t}{n^2+t^2+1} \le \frac{1}{12t}$$

for $t > 0$. This proves the second part of assumption (i) and shows that the sequence $(a_n(t))$ defined by (5.2) satisfies assumption (i). Moreover, we have

$$A = \sup\{|a_n(t)| : t \in \mathbb{R}_+, n \in \mathbb{N}\} \le \frac{1}{12}.$$

Furthermore, let us observe that the function $k_n(t,s)$ defined by (5.4) ($n = 1, 2, \ldots$) is continuous on $\mathbb{R}_+^2$. Apart from this, using standard tools of differential calculus it is easy to verify that

$$|k_n(t_2,s) - k_n(t_1,s)| \le \frac{1}{n}|t_2 - t_1|$$

for $n = 1, 2, \ldots$ and for $t_1, t_2, s \in \mathbb{R}_+$. Thus the sequence of functions $(k_n(\cdot, s))$ is equicontinuous on $\mathbb{R}_+$ uniformly with respect to $s \in \mathbb{R}_+$. Summing up, we derive that assumption (ii) is satisfied.

Next, let us note that, for any $n \in \mathbb{N}$ and for arbitrary $t, s \in \mathbb{R}_+$, we have the estimate

$$|k_n(t,s)| \le \frac{s}{1+ns^2} \le \frac{s}{1+s^2} \le \frac{1}{2}.$$

This allows us to assert that the sequence $(k_n(t,s))$ is equibounded on $\mathbb{R}_+^2$ with the constant $K_2 = \frac{1}{2}$. Hence we see that assumption (iv) is satisfied.

On the other hand, we get

$$\int_0^t |k_n(t,s)|\, ds = \frac{1}{2n} \ln\Big(\frac{1+2nt^2}{t+nt^2}\Big) \le \frac{1}{2n}\ln 2 \le \frac{1}{2}\ln 2.$$

This yields that the function sequence $(k_n(t,s))$ satisfies assumption (iii) with the constant $K_1 = \frac{1}{2}\ln 2$.

Now, consider the function $f_n = f_n(t, x_n, x_{n+1}, \ldots)$ defined by (5.3) for $n = 1, 2, \ldots$. Fix arbitrary $t_1, t_2 \in \mathbb{R}_+$ and $x = (x_n) \in c_0$. Then, we obtain

$$|f_n(t_2, x_n, x_{n+1}, \ldots) - f_n(t_1, x_n, x_{n+1}, \ldots)| \le \frac{1}{n}|t_2 - t_1|$$

for $n = 1, 2, \ldots$. This shows that the functions f_n ($n = 1, 2, \ldots$) satisfy assumption (v).

To verify assumption (vi), let us fix a number $r > 0$ and choose arbitrary $x = (x_i)$, $y = (y_i) \in c_0$ such that $\|x\|_{c_0} \le r$, $\|y\|_{c_0} \le r$. Then, for a fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
&|f_n(t, x_n, x_{n+1}, \ldots) - f_n(t, y_n, y_{n+1}, \ldots)| \\
&\quad \le \left|\frac{x_n}{1+x_n^2} - \frac{y_n}{1+y_n^2}\right| + \frac{1}{n}|x_{n+1}^2 - y_{n+1}^2| \\
&\quad \le |x_n - y_n| + \frac{1}{n}|x_{n+1} - y_{n+1}|\big(|x_{n+1}| + |y_{n+1}|\big) \\
&\quad \le |x_n - y_n| + \frac{2r}{n}|x_{n+1} - y_{n+1}| \le \max\{1, 2r\} \sup\{|x_i - y_i| : i \ge n\}.
\end{aligned}$$

The above estimate proves that the function f_n ($n = 1, 2, \ldots$) satisfies assumption (vi) with the function $l(r) = \max\{1, 2r\}$.

Furthermore, let us take into account the function $\overline{f}_n(t) = |f_n(t,0,0,\ldots)| = \frac{t}{n+4t^2}$ for $n = 1,2,\ldots$ and $t \in \mathbb{R}_+$. Obviously we have that $\lim_{n\to\infty} \overline{f}_n(t) = 0$ for any $t \in \mathbb{R}_+$. Moreover, in view of the inequality

$$\big|\overline{f}_n(t) - \overline{f}_n(s)\big| = \Big|\frac{t}{n+4t^2} - \frac{s}{n+4s^2}\Big| \le \frac{1}{n}|t-s|$$

for all $t,s \in \mathbb{R}_+$, we see that the function $\overline{f} = (\overline{f}_n)$ maps continuously $\mathbb{R}_+$ into the space c_0. This allows us to infer that $(\overline{f}_n) \in BC_0$.

Finally, let us notice that the following estimate holds

$$\overline{f}_n(t) = \frac{t}{n+4t^2} \le \frac{t}{1+4t^2}$$

for $t \in \mathbb{R}_+$ and $n = 1,2,\ldots$. Hence we deduce that $\lim_{t\to\infty} \overline{f}_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$.

The arguments outlined above show that assumption (vii) is satisfied. Moreover, we have

$$\overline{F} = \sup\big\{\overline{f}_n(t) : t \in \mathbb{R}_+, n \in \mathbb{N}\big\} = \frac{1}{4}.$$

Now, we proceed to verify assumption (viii). To this end, fix arbitrarily $n \in \mathbb{N}$ and consider the function $g_n(t,x) = g_n(t,x_1,x_2,\ldots)$ defined by (5.5); that is,

$$g_n(t,x_1,x_2,\ldots) = \arctan\Big(\frac{x_1+x_n}{n+t}\Big).$$

From the estimate

$$\big|g_n(t,x_1,x_2,\ldots)\big| \le \frac{|x_1|+|x_n|}{n+t} \le \frac{|x_1|+|x_n|}{n}$$

we conclude that the operator g defined in assumption (viii) by the equality

$$(gx)(t) = \big(g_n(t,x)\big) = \big(g_1(t,x), g_2(t,x), \ldots\big)$$

transforms the space $\mathbb{R} \times c_0$ into c_0.

Furthermore, fix $t \in \mathbb{R}_+$ and take $x = (x_n)$, $y = (y_n) \in c_0$. Then we have

$$\begin{aligned}&\big|g_n(t,x) - g_n(t,y)\big| \\ &\qquad \le \frac{|x_1-y_1|}{n+t} + \frac{|x_n-y_n|}{n+t} \le \frac{|x_1-y_1|}{n} + \frac{|x_n-y_n|}{n}.\end{aligned}$$

Hence we derive the following estimate:

$$\begin{aligned}\big\|(gx)(t) - (gy)(t)\big\|_{c_0} &= \sup\big\{\big|g_n(t,x) - g_n(t,y)\big| : n \in \mathbb{N}\big\} \\ &\le 2\sup\Big\{\frac{|x_n-y_n|}{n} : n \in \mathbb{N}\Big\} \le 2\|x-y\|_{c_o}.\end{aligned}$$

From the above estimate it follows that the operator g satisfies assumption (viii).

Apart from this, for an arbitrary $x \in c_0$ and for $t \in \mathbb{R}_+$ we obtain

$$\big\|(gx)(t)\big\|_{c_0} = \sup\big\{\big|g_n(t,x)\big| : n \in \mathbb{N}\big\} \le \frac{\pi}{2}.$$

Thus, the operator g satisfies assumption (ix) with the constant $\overline{G} = \frac{\pi}{2}$.

Finally, let us consider the inequality

$$A + \overline{FG}K_1 + \overline{G}K_1 r l(r) \le r$$

from assumption (x). Gathering all the above evaluations of the constants appearing in the above inequality and taking into account the form of the function $l = l(r)$, we see that this inequality can be written in the form

$$\frac{1}{12} + \frac{\pi}{16}\ln 2 + \frac{\pi}{4} r \max\{1, 2r\} \ln 2 \le r.$$

It is easy to check that any number $r \in [\frac{1}{2}, 0.556)$ satisfies the above inequality. Taking, for example, $r_0 = 0.55$, we have that $\overline{G}K_1 l(r_0) = 0.5988\ldots < 1$. Thus we see that assumption (x) is satisfied.

In view of the arguments set forth above and in Theorem 4.4, we infer that the infinite system of integral equations (5.1) has a solution $x(t) = (x_n(t))$ belonging to the ball B_{r_0} in the space $BC_0 = BC(\mathbb{R}_+, c_0)$.

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