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INTRINSIC SQUARE FUNCTION CHARACTERIZATIONS OF WEAK MUSIELAK–ORLICZ HARDY SPACES

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Abstract. Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfy that, for any given $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight uniformly in $t \in (0, \infty)$. In this article, via using the atomic and Littlewood–Paley function characterizations of the weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$, for any $\alpha \in (0, 1]$ and $s \in \mathbb{Z}_+$, we first establish its s-order intrinsic square function characterizations in terms of the intrinsic Lusin area function $S_{\alpha,s}$, the intrinsic g-function $g_{\alpha,s}$ and the intrinsic g^*_λ-function $g^*_{\lambda,\alpha,s}$ with the best known range $\lambda \in (2 + 2(\alpha + s)/n, \infty)$.

1. Introduction

As a natural generalization of the Lebesgue space $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum and Orlicz in [3] and [29], which is defined via an Orlicz function. Recall that a Musielak–Orlicz space is defined via a Musielak–Orlicz function (see, e.g., [28]), and a Musielak–Orlicz function is a natural generalization of an Orlicz function. Observe that, unlike Orlicz functions, Musielak–Orlicz functions may also vary with respect to spatial variables. Musielak–Orlicz spaces include many function spaces far beyond $L^p(\mathbb{R}^n)$, and the motivation to study function spaces of Musielak–Orlicz type comes from various applications to mathematics and physics (see, e.g., [5], [6], [22] and references therein). Recently, Ky [22] introduced a new Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which generalizes both the Orlicz–Hardy space (see, e.g.,[20]) and the weighted Hardy space (see,

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e.g., [15]), and hence has a wide generality. Observe also that the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ includes some new Hardy-type spaces (see, e.g., [18], [22]), which are not covered by known Hardy spaces, and that it appears naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [5], [6]), and the endpoint estimates for both the div-curl lemma and the commutators of Calderón–Zygmund operators (see [4], [5], [8], [21]). More applications are referred to in [10], [14], [17], [18], [23], [26], and [38].

It is well known that, when studying the boundedness of some operators in the critical case, the weak Hardy space $WH^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is a good substitute of the Hardy space $H^p(\mathbb{R}^n)$ (see [2]). Moreover, the space $WH^p(\mathbb{R}^n)$ was proved as the intermediate space in the real interpolation between the Hardy space $H^p(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ (see [11]). Motivated by the above, Liang, Yang, and Jiang [27] first introduced the weak Musielak–Orlicz Hardy spaces $WH^{\varphi}(\mathbb{R}^n)$ via the grand maximal function and characterized these spaces by means of maximal functions, atoms, molecules, and Littlewood–Paley functions. As an application, the authors in [27] proved the boundedness of Calderón–Zygmund operators from the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ to the space $WH^{\varphi}(\mathbb{R}^n)$ including the critical case.

On the other hand, the study of intrinsic square functions on several function spaces has recently attracted much attention. To be precise, Wilson [33] first introduced intrinsic square functions to settle a conjecture proposed by Fefferman and Stein [12] on the boundedness of the Lusin area function $S(f)$ from the weighted Lebesgue space $L^2_{\mathcal{M}(v)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_v(\mathbb{R}^n)$, where $0 \le v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{M}(v)$ denotes the Hardy–Littlewood maximal function of v. The boundedness of these intrinsic square functions on the weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$, when $p \in (1, \infty)$ and w belongs to Muckenhoupt weights $A_p(\mathbb{R}^n)$, was proved by Wilson in [34]. Later, Huang and Liu in [19] obtained the intrinsic square function characterizations of the weighted Hardy space $H^1_w(\mathbb{R}^n)$ under the additional assumption that $f \in L^1_w(\mathbb{R}^n)$, which was further generalized to the weighted Hardy space, $H^p_w(\mathbb{R}^n)$ with $p \in (n/(n + \alpha), 1)$ and $\alpha \in (0, 1)$ by Wang and Liu in [32], under the additional assumption that $f \in (\mathrm{Lip}(\alpha, 1, 0))^*$, where $(\mathrm{Lip}(\alpha, 1, 0))^*$ denotes the dual space of the Lipschitz space $\mathrm{Lip}(\alpha, 1, 0)$. Recently, Liang and Yang in [26] established the s-order intrinsic square function characterizations of the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ in terms of the intrinsic Lusin area function, the intrinsic g-function, and the g^*_λ-function with the best known range $\lambda \in (2 + 2(\alpha + s)/n, \infty)$, which essentially improved the known results in [19] and [32]. Motivated by [26], Zhuo, Yang, and Liang [39] generalized the corresponding results in [26] to the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with $\lambda \in (3 + 2(\alpha + s)/n, \infty)$. In addition, Wang [31] obtained the boundedness of the intrinsic square function including the g^*_λ-function on the weighted weak Hardy space $WH^p_w(\mathbb{R}^n)$ with $\lambda \in (3 + 2\alpha/n, \infty)$. Very recently, Yan [37] established the intrinsic square function characterizations including the g^*_λ-function of the variable weak Hardy space $WH^{p(\cdot)}(\mathbb{R}^n)$ with $\lambda \in (3 + 2(\alpha + s)/n, \infty)$. More applications of such intrinsic square functions were also given by Wilson [35], [36] and Lerner [23], [24].

In this article, we establish the intrinsic square function characterizations of the weak Musielak–Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$ introduced by Liang, Yang, and Jiang in [27], including the intrinsic Littlewood–Paley g-function, the intrinsic Lusin area function, and the intrinsic g_λ^*-function, by using the atomic and Littlewood–Paley function characterization theorems of the space $WH^{\varphi}(\mathbb{R}^n)$ obtained in [27]. We point out that our characterizations of $WH^{\varphi}(\mathbb{R}^n)$, even when $s = 0$,

$$\varphi(x,t) := w(x)t^p \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in [0,\infty),$$

with $p \in (n/(n+\alpha), 1]$ and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$, also essentially improve the known results in [31] by the range of λ to $\lambda \in (2 + 2\alpha/n, \infty)$.

This article is organized as follows. Section 2 is devoted to the proofs of Theorems 1.6 and 1.7. Using some ideas from [27], the fact that the intrinsic square functions are pointwise comparable as was proved in [26] (see also Lemmas 2.4 and 2.5 below), the weak type superposition principle, and the atomic decomposition and Littlewood–Paley g-function characterizations of $WH^{\varphi}(\mathbb{R}^n)$ obtained in [27] (see also Propositions 2.1 and 2.2 below), we first establish the intrinsic Lusin area function and g-function characterizations of $WH^{\varphi}(\mathbb{R}^n)$ (see Theorem 1.6 below). In the proof of Theorem 1.7, by borrowing some ideas from Folland and Stein [13] and Aguilera and Segovia [1], we prove the key technical Lemma 2.8. We use this, together with Theorem 1.6 and the Littlewood-Paley g_λ^*-function characterizations of $WH^{\varphi}(\mathbb{R}^n)$ obtained in [27, Theorem 4.13] (see also Proposition 2.3 below), to further establish the intrinsic g_λ^*-function characterizations of $WH^{\varphi}(\mathbb{R}^n)$ with the best known range $\lambda \in (2 + 2(\alpha + s)/n, \infty)$.

To state the results, we first establish some notational conventions. Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote by C a *positive constant* which is independent of the main parameters, but which may vary from line to line. We use $C_{(\alpha,\dots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \dots$. The *symbol* $A \lesssim B$ means $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of $\mathbb{R}^n$, then we denote by χ_E its *characteristic function* and by $E^{\complement}$ the set $\mathbb{R}^n \backslash E$. For all $r \in (0,\infty)$ and $x \in \mathbb{R}^n$, denote by $B(x,r)$ the ball centered at x with radius r, namely, $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. For any ball B, we use x_B to denote its center and r_B its radius, and denote by λB for any $\lambda \in (0,\infty)$ the ball concentric with B having the radius λr_B. For any index $q \in [1,\infty]$, we denote by q' its *conjugate index*, namely, $1/q + 1/q' = 1$.

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0,\infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$. The function Φ is said to be of *upper* (resp., *lower*) type p for some $p \in [0,\infty)$ if there exists a positive constant C such that, for all $s \in [1,\infty)$ (resp., $s \in [0,1]$) and $t \in [0,\infty)$, $\Phi(st) \le Cs^p\Phi(t)$.

For a given function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of *uniformly upper* (resp., *uniformly lower*) type p for some $p \in [0,\infty)$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $s \in [1,\infty)$ (resp., $s \in [0,1]$), and $t \in [0,\infty)$, $\varphi(x,st) \le Cs^p\varphi(x,t)$. The *critical uniformly lower type index* and the *critical uniformly upper type index* of φ are, respectively, defined by

$$i(\varphi) := \sup\big\{p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p\big\} \tag{1.1}$$

and

$$I(\varphi) := \inf\big\{p \in (0, \infty) : \varphi \text{ is of uniformly upper type } p\big\}. \tag{1.2}$$

Observe that $i(\varphi)$ and $I(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ or of uniformly upper type $I(\varphi)$ (see [25]).

The function $\varphi(\cdot, t)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1, \infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1, \infty)$,

$$[\varphi]_{\mathbb{A}_q(\mathbb{R}^n)} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x, t)\, dx \Big\{ \int_B \big[\varphi(y, t)\big]^{-q'/q} dy \Big\}^{q/q'} < \infty, \tag{1.3}$$

or, when $q = 1$,

$$[\varphi]_{\mathbb{A}_1(\mathbb{R}^n)} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x, t)\, dx \Big(\operatorname*{ess\,sup}_{y\in B} \big[\varphi(y, t)\big]^{-1} \Big) < \infty.$$

Here the first suprema are taken over all $t \in (0, \infty)$ and the second ones are taken over all balls $B \subset \mathbb{R}^n$.

Recalling that $\mathbb{A}_q(\mathbb{R}^n)$ with $q \in (1, \infty)$ was introduced by [22], we have the following properties for the functions in $\mathbb{A}_q(\mathbb{R}^n)$, which are from [22, Lemma 4.5] and [25, Lemma 2.2], respectively.

Lemma 1.1. *Let $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ with $q \in (1, \infty)$. Then the following statements hold true:*

(i) *There exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$, a measurable set $S \subset B$, and $t \in (0, \infty)$,*

$$\frac{\varphi(B, t)}{\varphi(S, t)} \le C \Big[\frac{|B|}{|S|}\Big]^q.$$

(ii) *There exists $p \in (1, q)$ such that $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$.*

Let $\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n)$. The *critical weight index* of $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ is defined as follows:

$$q(\varphi) := \inf\big\{q \in [1, \infty) : \varphi \in \mathbb{A}_q(\mathbb{R}^n)\big\}. \tag{1.4}$$

Now we recall the notion of growth functions (see [22]).

Definition 1.2. A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a Musielak–Orlicz function, namely,
  (a) the function $\varphi(x, \cdot) : [0, \infty) \to [0, \infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
  (b) the function $\varphi(\cdot, t)$ is a measurable function on $\mathbb{R}^n$ for all $t \in [0, \infty)$;

(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$;

(iii) φ is of positive uniformly lower type p for some $p \in (0, 1]$ and of uniformly upper type 1.

Throughout the article, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0,\infty)$, we denote $\int_E \varphi(x,t)\,dx$ simply by $\varphi(E,t)$. Let us now introduce the Musielak–Orlicz space and the weak Musielak–Orlicz space.

Definition 1.3. Let φ be a growth function.

(i) The *Musielak–Orlicz space* $L^\varphi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx < \infty$, equipped with the *quasinorm*

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

(ii) The *weak Musielak–Orlicz space* $WL^\varphi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\sup_{\alpha\in(0,\infty)} \varphi(\{x \in \mathbb{R}^n : |f(x)| > \alpha\},\alpha) < \infty$, equipped with the *quasinorm*

$$\|f\|_{WL^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \mathbb{R}^n : \left|f(x)\right| > \alpha\right\}, \frac{\alpha}{\lambda}\right) \le 1\right\}.$$

The following lemma is from [7, Lemma 7.13]. When $\varphi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$ with $p \in (0,1)$, it goes back to the well-known superposition principle of weak type estimates obtained by Stein, Taibleson, and Weiss [30, Lemma (1.8)].

Lemma 1.4. *Let φ be as in Definition 1.2 satisfying $I(\varphi) \in (0,1)$, where $I(\varphi)$ is as in (1.2). Assume that $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of measurable functions such that*

$$\sum_{k\in\mathbb{N}} \sup_{\lambda\in(0,\infty)} \varphi\left(\left\{x \in \mathbb{R}^n : \left|f_k(x)\right| > \lambda\right\}, \lambda\right) < \infty.$$

Then, for any $\eta \in (0,\infty)$,

$$\varphi\left(\left\{x \in \mathbb{R}^n : \sum_{k\in\mathbb{N}} \left|f_k(x)\right| > \eta\right\}, \eta\right) \lesssim \sum_{k\in\mathbb{N}} \sup_{\lambda\in(0,\infty)} \varphi\left(\left\{x \in \mathbb{R}^n : \left|f_k(x)\right| > \lambda\right\}, \lambda\right),$$

where the implicit equivalent positive constant is independent of η.

In what follows, denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space* equipped with the weak-$*$ topology. For any $m \in \mathbb{N}$, let

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{\psi \in \mathcal{S}(\mathbb{R}^n) : \sum_{\beta\in\mathbb{Z}_+^n, |\beta|\le m+1} \sup_{x\in\mathbb{R}^n} \left[\left(1+|x|\right)^{(m+2)(n+1)} \left|D^\beta \psi(x)\right|\right] \le 1\right\},$$

where, for any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$,

$$|\beta| := \beta_1 + \cdots + \beta_n \qquad \text{and} \qquad D^\beta := \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}.$$

Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *nontangential grand maximal function* f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi\in\mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t, t\in(0,\infty)} \left|f * \psi_t(y)\right|,$$

here and hereafter, for all $t \in (0,\infty)$, $\psi_t(\cdot) := t^{-n}\psi(\cdot/t)$. When

$$m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor, \tag{1.5}$$

where $i(\varphi)$ and $q(\varphi)$ are, respectively, as in (1.1) and (1.4), we denote $f^*_{m(\varphi)}$ simply by f^*. Here and hereafter, for any $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor$ denotes the maximal integer not larger than α.

For any measurable set $E \subset \mathbb{R}^n$ and $r \in (0,\infty)$, let $L^r(E)$ be the set of all measurable functions f such that

$$\|f\|_{L^r(E)} := \left[\int_E |f(x)|^r \, dx\right]^{1/r} < \infty.$$

For $r \in (0,\infty)$, denote by $L^r_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all r-locally integrable functions on $\mathbb{R}^n$. Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy, \tag{1.6}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Now we recall the definitions of the Musielak–Orlicz Hardy space (see [22], [25], [26]) and the weak Musielak–Orlicz Hardy space (see [27]).

Definition 1.5. Let φ be as in Definition 1.2.

(i) The *Musielak–Orlicz Hardy space* $H^\varphi(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$, equipped with the quasinorm

$$\|f\|_{H^\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}.$$

(ii) The *weak Musielak–Orlicz Hardy space* $WH^\varphi(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in WL^\varphi(\mathbb{R}^n)$, equipped with the quasinorm

$$\|f\|_{WH^\varphi(\mathbb{R}^n)} := \|f^*\|_{WL^\varphi(\mathbb{R}^n)}.$$

For any $s \in \mathbb{Z}_+$, $C^s(\mathbb{R}^n)$ denotes the *set of all functions having continuous classical derivatives up to an order of not more than* s. For $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, let $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ be the *set of functions* $\phi \in C^s(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : |x| \le 1\}$, $\int_{\mathbb{R}^n} \phi(x)x^\gamma \, dx = 0$ for all $\gamma \in \mathbb{Z}^n_+$ and $|\gamma| \le s$, and, for all $x_1,\ x_2 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}^n_+$ with $|\nu| = s$,

$$|D^\nu \phi(x_1) - D^\nu \phi(x_2)| \le |x_1 - x_2|^\alpha. \tag{1.7}$$

For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$, let

$$A_{\alpha,s}(f)(y,t) := \sup_{\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)} |f * \phi_t(y)|.$$

Then the *intrinsic g-function*, the *intrinsic Lusin area integral*, and the *intrinsic* g^*_λ*-function* of f are, respectively, defined by setting, for all $x \in \mathbb{R}^n$ and $\lambda \in (0,\infty)$,

$$g_{\alpha,s}(f)(x) := \left\{\int_0^\infty \left[A_{\alpha,s}(f)(x,t)\right]^2 \frac{dt}{t}\right\}^{1/2},$$

$$S_{\alpha,s}(f)(x) := \left\{\int_{\Gamma(x)} \left[A_{\alpha,s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2},$$

and

$$g^*_{\lambda,\alpha,s}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left[A_{\alpha,s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2},$$

here and hereafter, $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$.

We also recall another kind of similar-looking square functions, defined via convolutions with kernels that have unbounded supports. For $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (0,\infty)$, let $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ be the *set of functions* $\phi \in C^s(\mathbb{R}^n)$ such that, for all $x \in \mathbb{R}^n$, $\gamma \in \mathbb{Z}^n_+$ and $|\gamma| \le s$, $|D^\gamma \phi(x)| \le (1+|x|)^{-n-\epsilon}$, $\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$ and, for all x_1, $x_2 \in \mathbb{R}^n$, $\nu \in \mathbb{Z}^n_+$ with $|\nu| = s$,

$$\left|D^\nu \phi(x_1) - D^\nu \phi(x_2)\right| \le |x_1 - x_2|^\alpha \left[\left(1+|x_1|\right)^{-n-\epsilon} + \left(1+|x_2|\right)^{-n-\epsilon}\right].$$

We remark that, in what follows, the parameter ϵ usually has to be chosen large enough. For all f satisfying

$$\left|f(\cdot)\right|\left(1+|\cdot|\right)^{-n-\epsilon} \in L^1(\mathbb{R}^n) \tag{1.8}$$

and $(y,t) \in \mathbb{R}^{n+1}_+$, let

$$\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) := \sup_{\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)} \left|f * \phi_t(y)\right|.$$

Then, for all $x \in \mathbb{R}^n$ and $\lambda \in (0,\infty)$, we let

$$\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) := \left\{\int_0^\infty \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(x,t)\right]^2 \frac{dt}{t}\right\}^{1/2},$$

$$\widetilde{S}_{(\alpha,\epsilon),s}(f)(x) := \left\{\int_{\Gamma(x)} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2},$$

and

$$\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

These intrinsic square functions, when $s = 0$, were originally introduced by Wilson [33] and later further generalized to $s \in \mathbb{Z}_+$ by Liang and Yang [26].

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$ (see, e.g., [13, p. 50]).

Now we state the main results of this article.

Theorem 1.6. *Let φ be a growth function. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$.*

(i) *If* $f \in \mathcal{S}'(\mathbb{R}^n)$, f *vanishes weakly at infinity, and* $g_{\alpha,s}(f) \in WL^{\varphi}(\mathbb{R}^n)$, *then* $f \in WH^{\varphi}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C, *independent of* f, *such that*

$$\|f\|_{WH^{\varphi}(\mathbb{R}^n)} \le C\big\|g_{\alpha,s}(f)\big\|_{WL^{\varphi}(\mathbb{R}^n)}.$$

(ii) *If* $f \in (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$, *with* $(\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$ *being the dual space of* $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, *then there exists a positive constant* C *such that, for all* $f \in WH^{\varphi}(\mathbb{R}^n)$, *it holds true that*

$$\big\|g_{\alpha,s}(f)\big\|_{WL^{\varphi}(\mathbb{R}^n)} \le C\|f\|_{WH^{\varphi}(\mathbb{R}^n)}.$$

The same is true if $g_{\alpha,s}(f)$ *is replaced, respectively, by* $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$ *and* $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ *with* $\epsilon \in (\max\{\alpha, s\}, \infty)$.

Theorem 1.7. *Let* φ *be a growth function. Assume that* $\alpha \in (0, 1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n + \alpha + s), 1]$, $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$, *and* $\lambda \in (2 + 2(\alpha + s)/n, \infty)$.

(i) *If* $f \in \mathcal{S}'(\mathbb{R}^n)$, f *vanishes weakly at infinity, and* $g^*_{\lambda,\alpha,s}(f) \in WL^{\varphi}(\mathbb{R}^n)$, *then* $f \in WH^{\varphi}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C, *independent of* f, *such that*

$$\|f\|_{WH^{\varphi}(\mathbb{R}^n)} \le C\big\|g^*_{\lambda,\alpha,s}(f)\big\|_{WL^{\varphi}(\mathbb{R}^n)}.$$

(ii) *If* $f \in (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$, *then there exists a positive constant* C *such that, for all* $f \in WH^{\varphi}(\mathbb{R}^n)$, *it holds true that*

$$\big\|g^*_{\lambda,\alpha,s}(f)\big\|_{WL^{\varphi}(\mathbb{R}^n)} \le C\|f\|_{WH^{\varphi}(\mathbb{R}^n)}.$$

The same is true if $g^*_{\lambda,\alpha,s}(f)$ *is replaced by* $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ *with* $\epsilon \in (\max\{\alpha, s\}, \infty)$.

Remark 1.8.

(i) We point out that there exists a positive constant C such that, for all $\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)$, $C\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ and hence $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ is a subset of $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$. Thus, the intrinsic square functions are well defined for functionals in $(\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$.

(ii) Recall that, for $\alpha \in (0, 1]$, $p \in (n/(n + \alpha), 1]$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$, Wang [31] established the boundedness of the intrinsic Littlewood–Paley g^*_λ-function on the weighted weak Hardy space $WH^p_w(\mathbb{R}^n)$ with $\lambda \in (3 + 2\alpha/n, \infty)$. This corresponds to the case when $s = 0$ of Theorem 1.7(ii), in which we improve the range of λ to $\lambda \in (2 + 2\alpha/n, \infty)$, which coincides with the best known range of λ as in [26].

(iii) Here and hereafter, $q_0 := p(1 + (\alpha + s)/n)$. Since $\alpha \in (0, 1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n + \alpha + s), 1]$, and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$, it follows that $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$, $q_0 \in (q(\varphi), \infty)$, and

$$\begin{aligned} m(\varphi) &= \big\lfloor n\big[q(\varphi)/i(\varphi) - 1\big]\big\rfloor \le n\big[q(\varphi)/i(\varphi) - 1\big] \\ &< n(q_0/p - 1) = \alpha + s, \end{aligned} \tag{1.9}$$

where $m(\varphi)$ is defined as in (1.5).

2. Proofs of main results

For any measurable subset E of $\mathbb{R}^n$, the *space* $L^q_\varphi(E)$ for $q\in[1,\infty]$ is defined as the set of measurable functions f on E such that

$$\|f\|_{L^q_\varphi(E)} := \begin{cases} \sup_{t\in(0,\infty)}[\frac{1}{\varphi(E,t)}\int_E |f(x)|^q\varphi(x,t)\,dx]^{1/q}<\infty & \text{when } q\in[1,\infty),\\ \|f\|_{L^\infty(E)}<\infty & \text{when } q=\infty,\end{cases}$$

where $\varphi(E,t)$ denotes $\int_E \varphi(x,t)\,dx$.

We first recall the weak atomic Musielak–Orlicz Hardy space introduced in [27]. Let φ be as in Definition 1.2, let $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$, and let $q\in(q(\varphi),\infty]$. A measurable function a on $\mathbb{R}^n$ is called a (φ,q,s)-*atom* if there exists a ball B such that

(i) $\operatorname{supp} a\subset B$;
(ii) $\|a\|_{L^q_\varphi(B)}\le \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx=0$ for all $\alpha\in\mathbb{Z}^n_+$ with $|\alpha|\le s$.

The *weak atomic Musielak–Orlicz Hardy space*, denoted by $WH^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)$, is defined as the space of all $f\in\mathcal{S}'(\mathbb{R}^n)$ which can be decomposed as

$$f=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j}\quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of (φ,q,s)-atoms, which is associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, satisfying that there exists a positive constant C such that, for all $x\in\mathbb{R}^n$ and $i\in\mathbb{Z}$, $\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}(x)\le C$ and, for all $i\in\mathbb{Z}$ and $j\in\mathbb{N}$, $\lambda_{i,j}:=\widetilde{C}2^i\|\chi_{B_{i,j}}\|_{L^\varphi(\mathbb{R}^n)}$ with $\widetilde{C}$ being a positive constant independent of i and j.

Moreover, for any $f\in WH^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)$, define

$$\|f\|_{WH^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)} := \inf\left\{\inf\left[\lambda\in(0,\infty):\sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{N}}\varphi\left(B_{i,j},\frac{2^i}{\lambda}\right)\right\}\le 1\right]\right\},$$

where the first infimum is taken over all decompositions of f as above.

The following atomic function characterization of the weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ is just from [27, Theorem 3.5].

Proposition 2.1. *Let φ be as in Definition 1.2, and let $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$ and $q\in(q(\varphi),\infty]$. Then $WH^\varphi(\mathbb{R}^n)=WH^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)$ with equivalent quasinorms.*

Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying

$$\operatorname{supp}\phi\subset\left\{x\in\mathbb{R}^n:|x|\le 1\right\},$$

$$\int_{\mathbb{R}^n}\phi(x)x^\gamma\,dx=0\quad\text{for all } \gamma\in\mathbb{Z}^n_+ \text{ with } |\gamma|\le m(\varphi)$$

and

$$\int_0^\infty \left|\widehat{\phi}(\xi t)\right|^2\frac{dt}{t}=1\quad \text{for all } \xi\in\mathbb{R}^n\setminus\{0_n\},$$

where, $0_n:=(0,\dots,0)$.

Recall that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the Littlewood–Paley *g-function* and the *g_λ^*-function* of f with $\lambda \in (0,\infty)$ are defined, respectively, by setting, for all $x \in \mathbb{R}^n$,

$$g(f)(x) := \left[\int_0^\infty \left|f * \phi_t(x)\right|^2 \frac{dt}{t}\right]^{1/2}$$

and

$$g_\lambda^*(f)(x) := \left[\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|f * \phi_t(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2}.$$

The following Propositions 2.2 and 2.3 are, respectively, from [27, Theorem 4.8] and [27, Theorem 4.13].

Proposition 2.2. *Let φ be as in Definition 1.2. Then $f \in WH^\varphi(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g(f) \in WL^\varphi(\mathbb{R}^n)$. Moreover, for all $f \in WH^\varphi(\mathbb{R}^n)$,*

$$C^{-1}\left\|g(f)\right\|_{WL^\varphi(\mathbb{R}^n)} \le \|f\|_{WH^\varphi(\mathbb{R}^n)} \le C\left\|g(f)\right\|_{WL^\varphi(\mathbb{R}^n)},$$

where C is a positive constant independent of f.

Proposition 2.3. *Let φ be as in Definition 1.2, and let $q \in [1,\infty)$, $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, and $\lambda \in (2q/p,\infty)$. Then $f \in WH^\varphi(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g_\lambda^*(f) \in WL^\varphi(\mathbb{R}^n)$. Moreover, for all $f \in WH^\varphi(\mathbb{R}^n)$,*

$$C^{-1}\left\|g_\lambda^*(f)\right\|_{WL^\varphi(\mathbb{R}^n)} \le \|f\|_{WH^\varphi(\mathbb{R}^n)} \le C\left\|g_\lambda^*(f)\right\|_{WL^\varphi(\mathbb{R}^n)},$$

where C is a positive constant independent of f.

Lemmas 2.4 and 2.5 below are, respectively, [26, Proposition 2.4] and [26, Theorem 2.6], which, in the case when $s = 0$, were first proved by Wilson [33].

Lemma 2.4. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (0,\infty)$. Then, for all f satisfying (1.8) and $x \in \mathbb{R}^n$, it holds true that*

$$g_{\alpha,s}(f)(x) \sim S_{\alpha,s}(f)(x)$$

and

$$\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \sim \widetilde{S}_{(\alpha,\epsilon),s}(f)(x)$$

with the implicit positive constants independent of f.

Lemma 2.5. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (\max\{\alpha,s\},\infty)$. Then there exists a positive constant C such that, for all f satisfying (1.8) and $x \in \mathbb{R}^n$,*

$$\frac{1}{C} g_{\alpha,s}(f)(x) \le \widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \le C g_{\alpha,s}(f)(x).$$

The following lemma was proved in [26, Proposition 3.2].

Lemma 2.6. *Let $q \in (1,\infty)$, let $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ be a growth function, and let $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$. Then there exists a positive constant C such that, for all $t \in (0,\infty)$ and measurable functions f,*

$$\int_{\mathbb{R}^n} \left[g_{\alpha,s}(f)(x)\right]^q \varphi(x,t)\,dx \le C \int_{\mathbb{R}^n} \left|f(x)\right|^q \varphi(x,t)\,dx.$$

By an argument similar to that used in the proof of [27, Lemma 4.4], we obtain the following estimate, which plays a key role in the proof of Theorem 1.6.

Lemma 2.7. *Let φ be as in Definition 1.2, and let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$, and $\widetilde{q} \in (q(\varphi), q_0)$. For a sequence of multiples of (φ, q_0, s)-atoms, $\{\lambda_{i,j}a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, satisfying that, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\|\lambda_{i,j}a_{i,j}\|_{L^{q_0}_\varphi(B_{i,j})} \lesssim 2^i$ and, for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \lesssim 1$, then there exists a positive constant C, independent of $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, such that, for all $i \in \mathbb{Z}$ and $\lambda \in (0,\infty)$,*

$$\int_{\mathbb{R}^n} \Big[\sum_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x)\Big]^{\widetilde{q}} \varphi(x,\lambda)\,dx \le C \sum_{j\in\mathbb{N}} 2^{i\widetilde{q}} \varphi(B_{i,j},\lambda).$$

Proof. For any $k \in \mathbb{Z}_+$, let

$$U_k(B_{i,j}) := \begin{cases} B_{i,j} & \text{when } k = 0, \\ (2^k B_{i,j}) \setminus (2^{k-1} B_{i,j}) & \text{when } k > 0. \end{cases}$$

By $(L^{\widetilde{q}}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n))^* = L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)$ with $1/\widetilde{q} + 1/\widetilde{q}' = 1$ and the Hölder inequality, we know that, for all $i \in \mathbb{Z}$ and $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big[\sum_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x)\Big]^{\widetilde{q}} \varphi(x,\lambda)\,dx \\
&\quad = \sup_{\|g\|_{L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)}=1} \Big\{ \int_{\mathbb{R}^n} \Big[\sum_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x)\Big] g(x)\varphi(x,\lambda)\,dx \Big\}^{\widetilde{q}} \\
&\quad \le \sup_{\|g\|_{L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)}=1} \Bigg\{ \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}_+} \Big[\int_{U_k(B_{i,j})} \big[\lambda_{i,j} g_{\alpha,s}(a_{i,j})(x)\big]^{q_0} \varphi(x,\lambda)\,dx\Big]^{1/q_0} \\
&\qquad \times \Big[\int_{U_k(B_{i,j})} \big|g(x)\big|^{q_0{}'} \varphi(x,\lambda)\,dx\Big]^{1/q_0{}'} \Bigg\}^{\widetilde{q}} \\
&\quad \le \sup_{\|g\|_{L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)}=1} \Bigg\{ \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}_+} \big\|\lambda_{i,j} g_{\alpha,s}(a_{i,j})\big\|_{L^{q_0}_\varphi(U_k(B_{i,j}))} \big[\varphi(2^k B_{i,j},\lambda)\big]^{1/q_0} \\
&\qquad \times \Big[\int_{U_k(B_{i,j})} \big|g(x)\big|^{q_0{}'} \varphi(x,\lambda)\,dx\Big]^{1/q_0{}'} \Bigg\}^{\widetilde{q}}. \qquad (2.1)
\end{aligned}$$

For all $\lambda \in (0,\infty)$, measurable functions f, and $x \in \mathbb{R}^n$, let

$$M_\varphi(f)(x) := \sup_{B\ni x} \frac{1}{\varphi(B,\lambda)} \int_B |f(y)|\varphi(y,\lambda)\,dy, \tag{2.2}$$

where the supremum is taken over all balls B containing x. Then we know that, for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $k \in \mathbb{Z}_+$, and $\lambda \in (0,\infty)$,

$$\begin{aligned}
I &:= \Big[\int_{U_k(B_{i,j})} |g(x)|^{q_0'} \varphi(x,\lambda)\,dx\Big]^{1/q_0'} \\
&\le \big[\varphi(2^k B_{i,j},\lambda)\big]^{1/q_0'} \Big[\frac{1}{\varphi(2^k B_{i,j},\lambda)} \int_{2^k B_{i,j}} |g(y)|^{q_0'} \varphi(y,\lambda)\,dy\Big]^{1/q_0'} \\
&\lesssim \big[\varphi(2^k B_{i,j},\lambda)\big]^{1/q_0'} \inf_{x\in B_{i,j}} \big\{M_\varphi\big(|g|^{q_0'}\big)(x)\big\}^{1/q_0'} \\
&\lesssim \big[\varphi(2^k B_{i,j},\lambda)\big]^{1/q_0'} \Big\{\frac{1}{\varphi(B_{i,j},\lambda)} \int_{B_{i,j}} \big[M_\varphi\big(|g|^{q_0'}\big)(x)\big]^{\widetilde{q}'/q_0'} \varphi(x,\lambda)\,dx\Big\}^{1/\widetilde{q}'}, \quad (2.3)
\end{aligned}$$

where M_φ is defined as in (2.2).

Next we estimate $\|\lambda_{i,j} g_{\alpha,s}(a_{i,j})\|_{L^{q_0}_\varphi(U_k(B_{i,j}))}$. From $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ and Lemma 2.6, we deduce that, for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$, and $k \in \{0,1\}$,

$$\big\|\lambda_{i,j} g_{\alpha,s}(a_{i,j})\big\|_{L^{q_0}_\varphi(U_k(B_{i,j}))} \lesssim \lambda_{i,j} \|a_{i,j}\|_{L^{q_0}_\varphi(B_{i,j})} \lesssim 2^i. \tag{2.4}$$

On the other hand, by $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$ and (1.3), we conclude that, for all $\lambda \in (0,\infty)$,

$$\int_B \varphi(y,\lambda)\,dy \Big\{\int_B \big[\varphi(y,\lambda)\big]^{-1/(q_0-1)}\,dy\Big\}^{q_0-1} \lesssim |B|^{q_0}. \tag{2.5}$$

Let $\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)$, and, for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, let $B_{i,j} := B(x_{i,j}, r_{i,j})$ with some $x_{i,j} \in \mathbb{R}^n$ and $r_{i,j} \in (0,\infty)$. From the vanishing moment of $a_{i,j}$, Taylor's remainder theorem, (1.7), the Hölder inequality, and (2.5), it follows that, for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $t \in (0,\infty)$, and $x \in (2B_{i,j})^\complement$,

$$\begin{aligned}
&\big|a_{i,j} * \phi_t(x)\big| \\
&\quad = \Big|\int_{B_{i,j}} a_{i,j}(y) \Big[\phi\Big(\frac{x-y}{t}\Big) - \sum_{|\gamma|\le s} \frac{D^\gamma \phi(\frac{x-x_{i,j}}{t})}{\gamma!} \Big(\frac{x_{i,j}-y}{t}\Big)^\gamma\Big] \frac{dy}{t^n}\Big| \\
&\quad \sim \Big|\int_{B_{i,j}} a_{i,j}(y) \Big\{\sum_{|\gamma|=s} \Big[D^\gamma\phi(\xi) - D^\gamma \phi\Big(\frac{x-x_{i,j}}{t}\Big)\Big] \Big(\frac{x_{i,j}-y}{t}\Big)^\gamma\Big\} \frac{dy}{t^n}\Big| \\
&\quad \sim \Big|\int_{B_{i,j}} a_{i,j}(y) \Big\{\sum_{|\gamma|=s} \Big(\frac{\theta|x_{i,j}-y|}{t}\Big)^\alpha \Big(\frac{x_{i,j}-y}{t}\Big)^\gamma\Big\} \frac{dy}{t^n}\Big| \\
&\quad \lesssim \frac{{r_{i,j}}^{\alpha+s}}{t^{n+\alpha+s}} \Big[\int_{B_{i,j}} |a_{i,j}(y)|^{q_0} \varphi(y,\lambda)\,dy\Big]^{1/q_0} \Big(\int_{B_{i,j}} \big[\varphi(y,\lambda)\big]^{-1/(q_0-1)}\,dy\Big)^{(q_0-1)/q_0} \\
&\quad \lesssim \frac{{r_{i,j}}^{\alpha+s}}{t^{n+\alpha+s}} \Big[\frac{1}{\varphi(B_{i,j},\lambda)} \int_{B_{i,j}} |a_{i,j}(y)|^{q_0} \varphi(y,\lambda)\,dy\Big]^{1/q_0}
\end{aligned}$$

$$\times \left\{\varphi(B_{i,j},\lambda)\Big[\int_{B_{i,j}}\big[\varphi(y,\lambda)\big]^{-1/(q_0-1)}\,dy\Big]^{q_0-1}\right\}^{1/q_0}$$
$$\lesssim \Big(\frac{r_{i,j}}{t}\Big)^{n+\alpha+s}\|a_{i,j}\|_{L^{q_0}_{\varphi}(B_{i,j})}, \tag{2.6}$$

where $\xi = \frac{(x-x_{i,j})+\theta(x_{i,j}-y)}{t}$ for some $\theta\in(0,1)$.

Notice that $\operatorname{supp}\phi\subset\{x\in\mathbb{R}^n:|x|\le 1\}$. If $x\in(2B_{i,j})^\complement$ and $a_{i,j}*\phi_t(x)\neq 0$, then there exists $y\in B_{i,j}$ such that $|x-y|/t\le 1$ and hence $t\ge|x-y|\ge |x-x_{i,j}|-|x_{i,j}-y|\ge|x-x_{i,j}|/2$. From this and (2.6), we deduce that

$$\begin{aligned}g_{\alpha,s}(a_{i,j})(x) &= \left\{\int_0^\infty\Big[\sup_{\phi\in\mathcal{C}_{\alpha,s}(\mathbb{R}^n)}\big|a_{i,j}*\phi_t(x)\big|\Big]^2\frac{dt}{t}\right\}^{\frac12}\\ &\lesssim \|a_{i,j}\|_{L^{q_0}_{\varphi}(B_{i,j})}{r_{i,j}}^{n+\alpha+s}\left\{\int_{\frac{|x-x_{i,j}|}{2}}^\infty t^{-2(n+\alpha+s)-1}\,dt\right\}^{\frac12}\\ &\lesssim \|a_{i,j}\|_{L^{q_0}_{\varphi}(B_{i,j})}\Big(\frac{r_{i,j}}{|x-x_{i,j}|}\Big)^{n+\alpha+s}.\end{aligned} \tag{2.7}$$

Thus, from (2.7) and the fact that, for all $k\in\mathbb{Z}_+\cap[2,\infty)$ and $x\in U_k(B_{i,j})$, $|x-x_{i,j}|\sim 2^k r_{i,j}$. We deduce that, for all $i\in\mathbb{Z}$, $j\in\mathbb{N}$ and $k\in\mathbb{Z}_+\cap[2,\infty)$,

$$\big\|\lambda_{i,j}g_{\alpha,s}(a_{i,j})\big\|_{L^{q_0}_{\varphi}(U_k(B_{i,j}))}\lesssim 2^{-k(n+\alpha+s)}\lambda_{i,j}\|a_{i,j}\|_{L^{q_0}_{\varphi}(B_{i,j})}\lesssim 2^{i-k(n+\alpha+s)}. \tag{2.8}$$

By $q_0>1$, $\varphi\in\mathbb{A}_{q_0}(\mathbb{R}^n)$ and Lemma 1.1(ii), we know that there exists $q_1\in(1,q_0)$, such that $\varphi\in\mathbb{A}_{q_1}(\mathbb{R}^n)$. From this, (2.1), (2.3), (2.4), (2.8), Lemma 1.1(i), the Hölder inequality, $\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}(x)\lesssim 1$ for all $x\in\mathbb{R}^n$ and $i\in\mathbb{Z}$, and the boundedness on $L^{\widetilde{q}'/q_0'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)$ of M_φ (see, e.g., [9, p. 624]), it follows that, for all $i\in\mathbb{Z}$ and $\lambda\in(0,\infty)$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}g_{\alpha,s}(a_{i,j})(x)\Big]^{\widetilde{q}}\varphi(x,\lambda)\,dx\\ &\quad\lesssim \sup_{\|g\|_{L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)}=1}\left\{\sum_{j\in\mathbb{N}}2^i\sum_{k\in\mathbb{Z}_+}2^{-k(n+\alpha+s-nq_1)}\varphi(B_{i,j},\lambda)\right.\\ &\quad\quad\left.\times\left\{\frac{1}{\varphi(B_{i,j},\lambda)}\int_{B_{i,j}}\big[M_\varphi\big(|g|^{q_0'}\big)(x)\big]^{\widetilde{q}'/q_0'}\varphi(x,\lambda)\,dx\right\}^{1/\widetilde{q}'}\right\}^{\widetilde{q}}\\ &\quad\lesssim \sup_{\|g\|_{L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)}=1}\left\{\Big[\sum_{j\in\mathbb{N}}2^{i\widetilde{q}}\varphi(B_{i,j},\lambda)\Big]^{1/\widetilde{q}}\right.\\ &\quad\quad\left.\times\Big[\sum_{j\in\mathbb{N}}\int_{B_{i,j}}\big[M_\varphi\big(|g|^{q_0'}\big)(x)\big]^{\widetilde{q}'/q_0'}\varphi(x,\lambda)\,dx\Big]^{1/\widetilde{q}'}\right\}^{\widetilde{q}}\end{aligned}$$

$$\lesssim \left[\sum_{j\in\mathbb{N}} 2^{i\widetilde{q}}\varphi(B_{i,j},\lambda)\right] \sup_{\|g\|_{L^{\widetilde{q}'}_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)}=1} \left[\int_{\mathbb{R}^n} |g(x)|^{\widetilde{q}'}\varphi(x,\lambda)\,dx\right]^{\widetilde{q}/\widetilde{q}'}$$

$$\lesssim \sum_{j\in\mathbb{N}} 2^{i\widetilde{q}}\varphi(B_{i,j},\lambda),$$

which completes the proof of Lemma 2.7. □

Proof of Theorem 1.6. For $\epsilon \in (\max\{\alpha, s\}, \infty)$, by Lemmas 2.4 and 2.5, we see that $g_{\alpha,s}(f)$, $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$, and $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ are pointwise comparable. Thus, to prove this theorem, we only need to consider $g_{\alpha,s}(f)$ in our proof.

From (1.9) and the definitions of Littlewood–Paley g-function and intrinsic g-function, we deduce that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$g(f)(x) \lesssim g_{\alpha,s}(f)(x).$$

From this, the fact that f vanishes weakly at infinity, $g_{\alpha,s}(f) \in WL^{\varphi}(\mathbb{R}^n)$, and Proposition 2.2, we deduce that $f \in WH^{\varphi}(\mathbb{R}^n)$ and

$$\|f\|_{WH^{\varphi}(\mathbb{R}^n)} \lesssim \|g(f)\|_{WL^{\varphi}(\mathbb{R}^n)} \lesssim \|g_{\alpha,s}(f)\|_{WL^{\varphi}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.6(i).

It therefore remains to prove Theorem 1.6(ii). Let $f \in WH^{\varphi}(\mathbb{R}^n)$. By $q_0 \in (q(\varphi), \infty)$, (1.9), and Proposition 2.1, we know that there exists a sequence of (φ, q_0, s)-atoms $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, and positive constants $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that

$$f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j} \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

$\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \lesssim 1$ for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\|\lambda_{i,j}a_{i,j}\|_{L^{q_0}_{\varphi}(B_{i,j})} \lesssim 2^i$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, and

$$\|f\|_{WH^{\varphi}(\mathbb{R}^n)} \sim \|f\|_{WH^{\varphi,q_0,s}_{\text{atom}}(\mathbb{R}^n)} \sim \inf\left\{\lambda \in (0,\infty) : \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{N}} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\} \le 1\right\}.$$

Thus, to show that $g_{\alpha,s}(f) \in WL^{\varphi}(\mathbb{R}^n)$, it suffices to prove that, for all $\beta, \lambda \in (0,\infty)$ and $f \in WH^{\varphi}(\mathbb{R}^n)$,

$$\varphi\left(\{x \in \mathbb{R}^n : g_{\alpha,s}(f)(x) > \beta\}, \frac{\beta}{\lambda}\right) \lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{N}} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\}. \tag{2.9}$$

To prove (2.9), we choose $i_0 \in \mathbb{Z}$ such that $2^{i_0} \le \beta < 2^{i_0+1}$ and write

$$f = \sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j} + \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j} =: f_1 + f_2.$$

Since $q_0 \in (q(\varphi), \infty)$, it follows that there exists $\widetilde{q} \in (q(\varphi), q_0)$. Let $a \in (0, 1 - \frac{1}{\widetilde{q}})$ be a positive constant. By the Hölder inequality, we find that, for all $\lambda \in (0,\infty)$,

$$
\begin{aligned}
&\varphi\Big(\big\{x\in\mathbb{R}^n:g_{\alpha,s}(f_1)(x)>2^{i_0}\big\},\frac{2^{i_0}}{\lambda}\Big)\\
&\quad\lesssim\int_{\mathbb{R}^n}2^{-i_0\widetilde{q}}\Big[\sum_{i=-\infty}^{i_0-1}2^{ia}2^{-ia}\sum_{j\in\mathbb{N}}\lambda_{i,j}g_{\alpha,s}(a_{i,j})(x)\Big]^{\widetilde{q}}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\\
&\quad\lesssim\int_{\mathbb{R}^n}2^{-i_0\widetilde{q}}\Big(\sum_{i=-\infty}^{i_0-1}2^{ia\widetilde{q}'}\Big)^{\widetilde{q}/\widetilde{q}'}\sum_{i=-\infty}^{i_0-1}2^{-ia\widetilde{q}}\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}g_{\alpha,s}(a_{i,j})(x)\Big]^{\widetilde{q}}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\\
&\quad\lesssim\int_{\mathbb{R}^n}2^{-i_0\widetilde{q}(1-a)}\sum_{i=-\infty}^{i_0-1}2^{-ia\widetilde{q}}\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}g_{\alpha,s}(a_{i,j})(x)\Big]^{\widetilde{q}}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx.
\end{aligned}
$$

From this, Lemma 2.7, the uniformly upper type 1 property of φ, and $a\in(0,1-\frac{1}{\widetilde{q}})$, it follows that, for all $\lambda\in(0,\infty)$,

$$
\begin{aligned}
&\varphi\Big(\big\{x\in\mathbb{R}^n:g_{\alpha,s}(f_1)(x)>2^{i_0}\big\},\frac{2^{i_0}}{\lambda}\Big)\\
&\quad\lesssim 2^{-i_0\widetilde{q}(1-a)}\sum_{i=-\infty}^{i_0-1}2^{-ia\widetilde{q}}\sum_{j\in\mathbb{N}}2^{i\widetilde{q}}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\\
&\quad\lesssim 2^{i_0[1-\widetilde{q}(1-a)]}\sum_{i=-\infty}^{i_0-1}2^{i[\widetilde{q}(1-a)-1]}\sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{N}}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\}\\
&\quad\sim\sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{N}}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\}. \qquad (2.10)
\end{aligned}
$$

Let us now deal with f_2. Let $A_{i_0}:=\bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{N}}(2B_{i,j})$. Since φ is of uniformly lower type p, it follows that, for all $\lambda\in(0,\infty)$,

$$
\begin{aligned}
\varphi\Big(A_{i_0},\frac{2^{i_0}}{\lambda}\Big)&\lesssim\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\\
&\lesssim\sum_{i=i_0}^{\infty}2^{(i_0-i)p}\sum_{j\in\mathbb{N}}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\lesssim\sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{N}}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\}. \qquad (2.11)
\end{aligned}
$$

By (1.9), it follows that there exist $q_2\in(q(\varphi),q_0)$ and $p_2\in(p,i(\varphi))$ such that $s>n(\frac{q_2}{p_2}-1)-\alpha$ and hence $p_2>\frac{nq_2}{n+\alpha+s}$. For $x\in\mathbb{R}^n$ and $t\in(0,\infty)$, let $\widetilde{\varphi}(x,t):=\varphi(x,t)t^{\frac{nq_2}{n+\alpha+s}-p_2}$. Then $\widetilde{\varphi}$ is a Musielak–Orlicz function of uniformly lower type $\frac{nq_2}{n+\alpha+s}$. By this, (2.7), $\varphi\in\mathbb{A}_{q_2}(\mathbb{R}^n)$ (which is guaranteed by $\varphi\in\mathbb{A}_\infty(\mathbb{R}^n)$ and $q_2\in(q(\varphi),\infty)$), Lemma 1.1(i), and the uniformly lower type p_2 property of φ, we conclude that, for all $i\in\mathbb{Z}\cap[i_0,\infty)$, $j\in\mathbb{N}$ and $\lambda\in(0,\infty)$,

$$
\begin{aligned}
&\widetilde{\varphi}\Big(\big\{x\in A_{i_0}^{\complement}:\lambda_{i,j}g_{\alpha,s}(a_{i,j})(x)>2^{i_0}\big\},\frac{2^{i_0}}{\lambda}\Big)\\
&\quad\lesssim\varphi\Big(\Big\{x\in A_{i_0}^{\complement}:2^i\Big(\frac{r_{i,j}}{|x_{i,j}-x|}\Big)^{n+\alpha+s}>2^{i_0}\Big\},\frac{2^{i_0}}{\lambda}\Big)\Big(\frac{2^{i_0}}{\lambda}\Big)^{\frac{nq_2}{n+\alpha+s}-p_2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \varphi\Big([2^{i-i_0}]^{\frac{1}{n+\alpha+s}} B_{i,j}, \frac{2^{i_0}}{\lambda}\Big)\Big(\frac{2^{i_0}}{\lambda}\Big)^{\frac{nq_2}{n+\alpha+s}-p_2}\\
&\lesssim \varphi\Big(B_{i,j}, \frac{2^{i}}{\lambda}\Big)(2^{i-i_0})^{\frac{nq_2}{n+\alpha+s}}(2^{i_0-i})^{p_2}\Big(\frac{2^{i_0}}{\lambda}\Big)^{\frac{nq_2}{n+\alpha+s}-p_2}\\
&\sim \varphi\Big(B_{i,j}, \frac{2^{i}}{\lambda}\Big)\Big(\frac{2^{i}}{\lambda}\Big)^{\frac{nq_2}{n+\alpha+s}-p_2},
\end{aligned}
$$

which, together with $I(\widetilde{\varphi}) \le 1 + \frac{nq_2}{n+\alpha+s} - p_2 \in (0,1)$ and Lemma 1.4, further implies that, for all $\lambda \in (0,\infty)$,

$$
\begin{aligned}
&\varphi\Big(\Big\{x \in A_{i_0}^{\complement} : \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x) > 2^{i_0}\Big\}, \frac{2^{i_0}}{\lambda}\Big)\\
&\quad = \widetilde{\varphi}\Big(\Big\{x \in A_{i_0}^{\complement} : \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x) > 2^{i_0}\Big\}, \frac{2^{i_0}}{\lambda}\Big)\Big(\frac{2^{i_0}}{\lambda}\Big)^{p_2-\frac{nq_2}{n+\alpha+s}}\\
&\quad \lesssim \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}} \widetilde{\varphi}\Big(\big\{x \in A_{i_0}^{\complement} : \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x) > 2^{i_0}\big\}, \frac{2^{i_0}}{\lambda}\Big)\Big(\frac{2^{i_0}}{\lambda}\Big)^{p_2-\frac{nq_2}{n+\alpha+s}}\\
&\quad \lesssim \Big(\frac{2^{i_0}}{\lambda}\Big)^{p_2-\frac{nq_2}{n+\alpha+s}} \sum_{i=i_0}^{\infty}\Big(\frac{2^{i}}{\lambda}\Big)^{\frac{nq_2}{n+\alpha+s}-p_2} \sum_{j\in\mathbb{N}} \varphi\Big(B_{i,j}, \frac{2^{i}}{\lambda}\Big)\\
&\quad \lesssim \sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{N}} \varphi\Big(B_{i,j}, \frac{2^{i}}{\lambda}\Big)\Big\}. \qquad (2.12)
\end{aligned}
$$

Combining (2.10), (2.11), and (2.12), we obtain (2.9) and hence complete the proof of Theorem 1.6(ii). □

For all $\beta \in (0,\infty)$, $f \in (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$, and $x \in \mathbb{R}^n$, let

$$
\begin{aligned}
&\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)\\
&\qquad := \Big\{\int_0^{\infty}\int_{\{y\in\mathbb{R}^n:|y-x|<\beta t\}} \big[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\big]^2 (\beta t)^{-n} \frac{dy\,dt}{t}\Big\}^{1/2}.
\end{aligned}
$$

To prove Theorem 1.7, we need the following technical lemma. The idea behind the proof was motivated by Folland and Stein [13, p. 218, Theorem 7.1] and Aguilera and Segovia [1, Theorem 1].

Lemma 2.8. *Let $q \in [1,\infty)$, and let $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ be as in Definition 1.2. Then there exists a positive constant C such that, for all $\beta \in (1,\infty)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$
\begin{aligned}
&\sup_{\lambda\in(0,\infty)} \varphi\big(\big\{x \in \mathbb{R}^n : \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) > \lambda\big\}, \lambda\big)\\
&\qquad \le C\beta^{n(q-p/2)} \sup_{\lambda\in(0,\infty)} \varphi\big(\big\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \lambda\big\}, \lambda\big).
\end{aligned}
$$

Proof. For all $\beta \in (1,\infty)$, $\lambda \in (0,\infty)$, and $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$E_\lambda := \big\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \lambda\beta^{n/2}\big\}$$

and

$$U := \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{E_\lambda})(x) > (4\beta)^{-n}\},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function as in (1.6). Since $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, from the boundedness of $\mathcal{M}$ on $L^q_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)$ (see, e.g., [16, Theorem 9.1.9]), it follows that, for all $\beta \in (1,\infty)$, $\lambda \in (0,\infty)$, and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned}\varphi(U,\lambda) &= \varphi\big(\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{E_\lambda})(x) > (4\beta)^{-n}\},\lambda\big)\\ &\lesssim (4\beta)^{nq}\|\chi_{E_\lambda}\|^q_{L^q_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)} \sim \beta^{nq}\varphi(E_\lambda,\lambda)\end{aligned} \tag{2.13}$$

and, by [26, p. 3249, (3.9)], we know that

$$\beta^{n(1-q)}\int_{U^\complement}\big[\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)\big]^2\varphi(x,\lambda)\,dx \lesssim \int_{E^\complement_\lambda}\big[\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\big]^2\varphi(x,\lambda)\,dx. \tag{2.14}$$

Thus, from the uniformly lower type p and the uniformly upper type 1 properties of φ, (2.13), and (2.14), it follows that, for all $\beta \in (1,\infty)$, $\lambda \in (0,\infty)$, and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned}&\varphi\big(\{x \in \mathbb{R}^n : \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) > \lambda\},\lambda\big)\\ &\quad\le \varphi(U,\lambda) + \varphi\big(U^\complement \cap \{x \in \mathbb{R}^n : \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) > \lambda\},\lambda\big)\\ &\quad\lesssim \beta^{nq}\varphi(E_\lambda,\lambda) + \lambda^{-2}\int_{U^\complement}\big[\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)\big]^2\varphi(x,\lambda)\,dx\\ &\quad\lesssim \beta^{nq}\varphi(E_\lambda,\lambda) + \beta^{n(q-1)}\lambda^{-2}\int_{E^\complement_\lambda}\big[\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\big]^2\varphi(x,\lambda)\,dx\\ &\quad\sim \beta^{n(q-p/2)}\varphi\big(\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \lambda\beta^{n/2}\},\lambda\beta^{n/2}\big)\\ &\qquad + \beta^{n(q-1)}\lambda^{-2}\int_0^{\lambda\beta^{n/2}} t\varphi\big(\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\},\lambda\big)\,dt\\ &\quad\lesssim \left\{\beta^{n(q-p/2)} + \beta^{n(q-1)}\lambda^{-2}\left[\int_0^\lambda t\left(\frac{\lambda}{t}\right)^1 dt + \int_\lambda^{\lambda\beta^{n/2}} t\left(\frac{\lambda}{t}\right)^p dt\right]\right\}\\ &\qquad \times \sup_{\gamma\in(0,\infty)}\varphi\big(\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \gamma\},\gamma\big)\\ &\quad\sim \beta^{n(q-p/2)}\sup_{\gamma\in(0,\infty)}\varphi\big(\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \gamma\},\gamma\big).\end{aligned}$$

This finishes the proof of Lemma 2.8. □

Proof of Theorem 1.7. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ vanish weakly at infinity, and let $g^*_{\lambda,\alpha,s}(f) \in WL^\varphi(\mathbb{R}^n)$. By $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$, and $\lambda \in (2+2(\alpha+s)/n,\infty)$, we obtain that $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$, $q_0 > 1$, and

$$\lambda > 2+2(\alpha+s)/n = 2\big(1+(\alpha+s)/n\big) = 2q_0/p.$$

From this, the fact that for all $x \in \mathbb{R}^n$, $g^*_\lambda(f)(x) \lesssim g^*_{\lambda,\alpha,s}(f)(x) \lesssim \widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)$, and Proposition 2.3, we deduce that $f \in WH^\varphi(\mathbb{R}^n)$ and

$$\|f\|_{WH^\varphi(\mathbb{R}^n)} \lesssim \big\|g^*_\lambda(f)\big\|_{WL^\varphi(\mathbb{R}^n)} \lesssim \big\|g^*_{\lambda,\alpha,s}(f)\big\|_{WL^\varphi(\mathbb{R}^n)} \lesssim \big\|\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)\big\|_{WL^\varphi(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.7(i).

It therefore remains to prove Theorem 1.7(ii). Let $f \in WH^{\varphi}(\mathbb{R}^n)$. Then, by an argument similar to that used in [26, p. 3252, (3.12)], we conclude that, for all $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (0,\infty)$, $\lambda \in (1,\infty)$, and $x \in \mathbb{R}^n$,

$$\left[\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)\right]^2 \lesssim \sum_{k=0}^{\infty} 2^{-kn(\lambda-1)}\left[\widetilde{S}_{2^k,(\alpha,\epsilon),s}(f)(x)\right]^2. \tag{2.15}$$

Since $\lambda \in (2q_0/p,\infty)$, it follows that there exists $r \in (0,1)$ such that $\lambda - \frac{2r}{n} \in (2q_0/p,\infty)$. Let $C_{(r)} := \frac{1}{1-2^{-r}}$. Then, from (2.15), $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$, Lemma 2.8, and the uniformly lower type p property of φ, we deduce that, for all $\gamma \in (0,\infty)$,

$$\begin{aligned}
&\varphi\left(\left\{x \in \mathbb{R}^n : g^*_{\lambda,\alpha,s}(f)(x) > \gamma\right\},\gamma\right)\\
&\quad\lesssim \varphi\left(\left\{x \in \mathbb{R}^n : \widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x) > \gamma\right\},\gamma\right)\\
&\quad\lesssim \varphi\left(\left\{x \in \mathbb{R}^n : \sum_{k=0}^{\infty} 2^{-kn(\lambda-1)/2}\widetilde{S}_{2^k,(\alpha,\epsilon),s}(f)(x) > \frac{1}{C_{(r)}}\sum_{k=0}^{\infty} 2^{-kr}\gamma\right\},\gamma\right)\\
&\quad\lesssim \sum_{k=0}^{\infty}\varphi\left(\left\{x \in \mathbb{R}^n : \widetilde{S}_{2^k,(\alpha,\epsilon),s}(f)(x) > 2^{kn(\lambda-1)/2-kr}\frac{1}{C_{(r)}}\gamma\right\},\gamma\right)\\
&\quad\lesssim \sum_{k=0}^{\infty} 2^{[-kn(\lambda-1)/2+kr]p}2^{kn(q_0-p/2)}\sup_{\beta\in(0,\infty)}\varphi\left(\left\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \beta\right\},\beta\right)\\
&\quad\lesssim \sup_{\beta\in(0,\infty)}\varphi\left(\left\{x \in \mathbb{R}^n : \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \beta\right\},\beta\right).
\end{aligned}$$

Thus, by this, the fact that $\epsilon \in (\max\{\alpha,s\},\infty)$, and Theorem 1.6(ii), we see that

$$\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{WL^{\varphi}(\mathbb{R}^n)} \lesssim \left\|\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)\right\|_{WL^{\varphi}(\mathbb{R}^n)} \lesssim \left\|\widetilde{S}_{(\alpha,\epsilon),s}(f)\right\|_{WL^{\varphi}(\mathbb{R}^n)} \lesssim \|f\|_{WH^{\varphi}(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.7(ii). □

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References

1. N. Aguilera and C. Segovia, *Weighted norm inequalities relating the g^*_λ and the area functions*, Studia Math. **61** (1977), no. 3, 293–303. Zbl 0375.28015. MR0492276. DOI 10.4064/sm-61-3-293-303. 971, 984
2. J. Alvarez and M. Milman, *H^p continuity properties of Calderón-Zygmund-type operators*, J. Math. Anal. Appl. **118** (1986), no. 1, 63–79. Zbl 0596.42006. MR0849442. DOI 10.1016/0022-247X(86)90290-8. 970
3. Z. Birnbaum and W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Math. **3** (1931), no. 1, 1–67. Zbl 0003.25202. 969
4. A. Bonami, J. Feuto, and S. Grellier, *Endpoint for the DIV-CURL lemma in Hardy spaces*, Publ. Mat. **54** (2010), no. 2, 341–358. Zbl 1205.42020. MR2675927. DOI 10.5565/PUBLMAT_54210_03. 970
5. A. Bonami, S. Grellier, and L. D. Ky, *Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets*, J. Math. Pures Appl. (9) **97** (2012), no. 3, 230–241. Zbl 1241.47028. MR2887623. DOI 10.1016/j.matpur.2011.06.002. 969, 970

6. A. Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister, *On the product of functions in BMO and H^1*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 5, 1405–1439. Zbl 1132.42010. MR2364134. 969, 970
7. T. A. Bui, J. Cao, L. D. Ky, D. Yang, and S. Yang, *Musielak-Orlicz-Hardy spaces associated with operators satisfying reinforced off-diagonal estimates*, Anal. Geom. Metr. Spaces **1** (2013), 69–129. Zbl 1261.42034. MR3108869. DOI 10.2478/agms-2012-0006. 973
8. Z. Cai, C. Chu, and C. Lei, *Existence of positive solutions for a fractional elliptic problems with the Hardy-Sobolev-Maz'ya potential and critical nonlinearities*, Bound. Value Probl. **2017**, no 181. Zbl 1378.35006. MR3736613. DOI 10.1186/s13661-017-0912-8. 970
9. R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. (N.S.) **83** (1977), no. 4, 569–645. Zbl 0358.30023. MR0447954. DOI 10.1090/S0002-9904-1977-14325-5. 981
10. F. Deringoz, V. S. Guliyev, and M. A. Ragusa, *Intrinsic square functions on vanishing generalized Orlicz-Morrey spaces*, Set-Valued Var. Anal. **25** (2017), no. 4, 807–828. Zbl 1382.42009. MR3740518. DOI 10.1007/s11228-017-0422-y. 970
11. C. Fefferman, N. M. Rivière and Y. Sagher, *Interpolation between H^p spaces: The real method*, Trans. Amer. Math. Soc. **191** (1974), 75–81. Zbl 0285.41006. MR0388072. DOI 10.2307/1996982. 970
12. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3–4, 137–193. Zbl 0257.46078. MR0447953. DOI 10.1007/BF02392215. 970
13. G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes **28**, Princeton Univ. Press, Princeton, 1982. Zbl 0508.42025. MR0657581. 971, 975, 984
14. X. Fu and D. Yang, *Wavelet characterizations of Musielak-Orlicz Hardy spaces*, Banach J. Math. Anal. **12** (2018), no. 4, 1017–1046. Zbl 1402.42046. MR3858759. DOI 10.1215/17358787-2018-0010. 970
15. J. García-Cuerva, *Weighted H^p spaces*, Dissertationes Math. **162** (1979), 1–63. Zbl 0434.42023. MR0549091. 970
16. L. Grafakos, *Modern Fourier Analysis*, 3rd ed., Grad. Texts in Math. **250**, Springer, New York, 2014. Zbl 1304.42002. MR3243741. DOI 10.1007/978-1-4939-1230-8. 985
17. V. S. Guliyev, M. N. Omarova, M. A. Ragusa and A. Scapellato, *Commutators and generalized local Morrey spaces*, J. Math. Anal. Appl. **457** (2018), no. 2, 1388–1402. Zbl 1376.42021. MR3705360. DOI 10.1016/j.jmaa.2016.09.070. 970
18. S. Hou, D. Yang, and S. Yang, *Lusin area function and molecular characterizations of Musielak-Orlicz Hardy spaces and their applications*, Commun. Contemp. Math. **15** (2013), no. 6, art. ID 1350029. Zbl 1285.42020. MR3139410. DOI 10.1142/S0219199713500296. 970
19. J. Huang and Y. Liu, *Some characterizations of weighted Hardy spaces*, J. Math. Anal. Appl. **363** (2010), no. 1, 121–127. Zbl 1246.42021. MR2559046. DOI 10.1016/j.jmaa.2009.07.054. 970
20. R. Jiang and D. Yang, *New Orlicz-Hardy spaces associated with divergence form elliptic operators*, J. Funct. Anal. **258** (2010), no. 4, 1167–1224. Zbl 1205.46014. MR2565837. DOI 10.1016/j.jfa.2009.10.018. 969
21. A. Kalybay, D. Karatayeva, R. Oinarov, and A. Temirkhanova, *Oscillation of a second order half-linear difference equation and the discrete Hardy inequality*, Electron. J. Qual. Theory Differ. Equ. **2017**, no. 43. Zbl 06931274. MR3661719. DOI 10.14232/ejqtde.2017.1.43. 970
22. L. D. Ky, *New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators*, Integral Equations Operator Theory **78** (2014), no. 1, 115–150. Zbl 1284.42073. MR3147406. DOI 10.1007/s00020-013-2111-z. 969, 970, 972, 974
23. A. K. Lerner, *Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals*, Adv. Math. **226** (2011), no. 5, 3912–3926. Zbl 1226.42010. MR2770437. DOI 10.1016/j.aim.2010.11.009. 970
24. A. K. Lerner, *On sharp aperture-weighted estimates for square functions*, J. Fourier Anal. Appl. **20** (2014), no. 4, 784–800. Zbl 1310.42010. MR3232586. DOI 10.1007/s00041-014-9333-6. 970

25. Y. Liang, J. Huang, and D. Yang, *New real-variable characterizations of Musielak-Orlicz Hardy spaces*, J. Math. Anal. Appl. **395** (2012), no. 1, 413–428. Zbl 1256.42035. MR2943633. DOI 10.1016/j.jmaa.2012.05.049. 972, 974
26. Y. Liang and D. Yang, *Intrinsic square function characterizations of Musielak-Orlicz Hardy spaces*, Trans. Amer. Math. Soc. **367** (2015), no. 5, 3225–3256. Zbl 1318.42027. MR3314807. DOI 10.1090/S0002-9947-2014-06180-1. 970, 971, 974, 975, 976, 978, 985, 986
27. Y. Liang, D. Yang, and R. Jiang, *Weak Musielak-Orlicz Hardy spaces and applications*, Math. Nachr. **289** (2016), no. 5–6, 634–677. Zbl 1338.42023. MR3486149. DOI 10.1002/mana.201500152. 970, 971, 974, 977, 978, 979
28. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. **1034**, Springer, Berlin, 1983. Zbl 0557.46020. MR0724434. DOI 10.1007/BFb0072210. 969
29. W. Orlicz, *Über eine gewisse Klasse von Raumen vom Typus B*, Bull. Int. Acad. Pol. Ser. A **8** (1932), no. 8–9, 207–220. Zbl 0006.31503. 969
30. E. M. Stein, M. H. Taibleson, and G. Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Rend. Circ. Mat. Palermo (2) **1981**, suppl. 1, 81–97. Zbl 0503.42018. MR0639468. 973
31. H. Wang, *Boundedness of intrinsic square functions on the weighted weak Hardy spaces*, Integral Equations Operator Theory **75** (2013), no. 1, 135–149. Zbl 1276.42018. MR3016480. DOI 10.1007/s00020-012-2011-7. 970, 971, 976
32. H. Wang and H. Liu, *The intrinsic square function characterizations of weighted Hardy spaces*, Illinois J. Math. **56** (2012), no. 2, 367–381. Zbl 1284.42063. MR3161329. 970
33. M. Wilson, *The intrinsic square function*, Rev. Mat. Iberoam. **23** (2007), no. 3, 771–791. Zbl 1213.42072. MR2414491. DOI 10.4171/RMI/512. 970, 975, 978
34. M. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability*, Lecture Notes in Math. **1924**, Springer, Berlin, 2008. Zbl 1138.42011. MR2359017. 970
35. M. Wilson, *How fast and in what sense(s) does the Calderón reproducing formula converge?*, J. Fourier Anal. Appl. **16** (2010), no. 5, 768–785. Zbl 1204.42029. MR2673708. DOI 10.1007/s00041-009-9109-6. 970
36. M. Wilson, *Convergence and stability of the Calderón reproducing formula in H^1 and BMO*, J. Fourier Anal. Appl. **17** (2011), no. 5, 801–820. Zbl 1230.42028. MR2838108. DOI 10.1007/s00041-010-9165-y. 970
37. X. Yan, *Intrinsic square function characterizations of variable weak Hardy spaces*, to appear in Taiwanese J. Math., preprint, `https://projecteuclid.org/euclid.twjm/1555315214`. DOI 10.11650/tjm/190401. 970
38. D. Yang, Y. Liang, and L. D. Ky, *Real-Variable Theory of Musielak-Orlicz Hardy Spaces*, Lecture Notes in Math. **2182**, Springer, Cham, 2017. Zbl 1375.42038. MR3586020. DOI 10.1007/978-3-319-54361-1. 970
39. C. Zhuo, D. Yang, and Y. Liang, *Intrinsic square function characterizations of Hardy spaces with variable exponents*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 4, 1541–1577. Zbl 1356.42013. MR3549980. DOI 10.1007/s40840-015-0266-2. 970

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