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ATOMIC CHARACTERIZATIONS OF WEAK MARTINGALE MUSIELAK–ORLICZ HARDY SPACES AND THEIR APPLICATIONS

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Abstract. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ be a Musielak–Orlicz function. In this article, we establish the atomic characterizations of weak martingale Musielak–Orlicz Hardy spaces $WH^s_\varphi(\Omega)$, $WH^M_\varphi(\Omega)$, $WH^S_\varphi(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$. We then use these atomic characterizations to obtain the boundedness of σ-sublinear operators from weak martingale Musielak–Orlicz Hardy spaces to weak Musielak–Orlicz spaces, as well as some martingale inequalities which further clarify the relationships among these weak martingale Musielak–Orlicz Hardy spaces. All these results improve and generalize the corresponding results on weak martingale Orlicz–Hardy spaces. Moreover, we improve all the known results on weak martingale Musielak–Orlicz Hardy spaces. In particular, both the boundedness of σ-sublinear operators and the martingale inequalities, for weak weighted martingale Hardy spaces as well as for weak weighted martingale Orlicz–Hardy spaces, are new.

1. Introduction

The weak Hardy space $WH^1(\mathbb{R}^n)$ was originally introduced by Fefferman and Soria [9] to find out the biggest space from which the Hilbert transform is bounded to the weak Lebesgue space $WL^1(\mathbb{R}^n)$. The ∞-atomic decomposition of $WH^1(\mathbb{R}^n)$

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and the boundedness of some Calderón–Zygmund operators from $WH^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$ were also established in [9]. As is well known, for any $p \in (0, 1]$, the weak Hardy spaces $WH^p(\mathbb{R}^n)$ naturally appear and prove a good substitute of Hardy spaces $H^p(\mathbb{R}^n)$ when studying the boundedness of operators in the critical case (see, e.g., [24], [3], [2]). It should also be pointed out that Fefferman, Rivière, and Sagher [8] proved that the weak Hardy spaces are the intermediate spaces of Hardy spaces in the real interpolation method.

Recently, various martingale Hardy spaces have been investigated (see, e.g., Weisz [33], [34], [36], Ho [13], [11], Nakai, Miyamoto, Sadasue, and Sawano in [29], [30], and [31], Sadasue [32], Jiao [18], and Xie, Jiao, and Yang in [37] for several different martingale Hardy spaces and their applications). Observe that weak martingale Hardy spaces naturally appear when studying the interpolation spaces between martingale Hardy spaces (see [34, Chapter 5] for more details). Moreover, the theory of weak martingale Hardy spaces has also been developed rapidly. Weak Hardy spaces consisting of Vilenkin martingales were originally studied by Weisz in [35] and then later fully generalized by Hou and Ren in [14]. Inspired by this earlier work, Jiao [16], Jiao, Wu, and Peng [17], and Liu, along with Zhou and Peng in [26] and [25], investigated weak martingale Orlicz–Hardy spaces associated with concave functions. In [41], Zhou, Wu, and Jiao introduced weak martingale Orlicz–Karamata–Hardy spaces associated with concave functions and established their atomic characterizations.

On the other hand, as a generalization of the Orlicz space and the weighted Lebesgue space, the Musielak–Orlicz space has proved very useful in partial differential equations and image filtering (see, e.g., [1], [19], and references therein). As a suitable substitute of the Musielak–Orlicz space in dealing with some problems of analysis such as the boundedness of operators, Ky [21] introduced the Musielak–Orlicz Hardy space, which plays a key role in establishing sharp endpoint estimates for the div-curl lemma and the boundedness of commutators generated by Calderón–Zygmund operators and $\mathrm{BMO}(\mathbb{R}^n)$ functions (see, e.g., [5], [20], [40]). Very recently, Ho [12] and Fu and Yang [10] established the intrinsic atomic and molecular characterizations, as well as the wavelet characterizations of Musielak–Orlicz Hardy spaces, respectively. Moreover, Liang, Yang, and Jiang [23] introduced and studied the weak Musielak–Orlicz Hardy space, which has proved useful in establishing the endpoint boundedness of Calderón–Zygmund operators (see [23], [40]) and parametric Marcinkiewicz integrals with rough kernels (see [22]). We also refer the reader to the monograph [40] for a complete survey of recent progress made on the real-variable theory of the Musielak–Orlicz Hardy space on $\mathbb{R}^n$.

The martingale Musielak–Orlicz Hardy space was also investigated in [37] and [38]. In [39], Yang introduced weak martingale Musielak–Orlicz Hardy spaces which are a generalization of weak martingale Orlicz–Hardy spaces (see, e.g., [17]). Moreover, Yang also established in [39] the atomic characterizations of weak martingale Musielak–Orlicz Hardy spaces and the boundedness of σ-sublinear operators from weak martingale Musielak–Orlicz Hardy spaces to weak Musielak–Orlicz spaces.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A function $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ is called a *Musielak–Orlicz function* if the function $\varphi(\cdot, t)$ is a measurable function for any given $t \in [0, \infty)$, and the function $\varphi(x, \cdot) : [0, \infty) \to [0, \infty)$ is an Orlicz function for almost every given $x \in \Omega$; namely, $\varphi(x, \cdot)$ is nondecreasing, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t\to\infty} \varphi(x, t) = \infty$. For any $p \in (0, \infty)$, a Musielak–Orlicz function φ is said to be of *uniformly lower* (resp., *upper*) *type* p if there exists a positive constant $C_{(p)}$, depending on p, such that

$$\varphi(x, st) \le C_{(p)} s^p \varphi(x, t) \tag{1.1}$$

for any $x \in \Omega$ and $t \in [0, \infty)$, $s \in (0, 1)$ (resp., $s \in [1, \infty)$) (see [40] for more details).

Recall that the following assumption is needed through [39].

Assumption 1.A. Let φ be a Musielak–Orlicz function, and let φ be of uniformly lower type $p \in (0, 1]$ and of uniformly upper type 1.

Observe that Assumption 1.A is quite restrictive. Indeed, for any given $p \in (1, \infty)$, if $\varphi(x, t) := t^p$ for any $x \in \Omega$ and $t \in (0, \infty)$, then φ is of uniformly lower type p and also of uniformly upper type p. However, in this case, φ is not of uniformly upper type 1. Thus, under Assumption 1.A, all the results in [39] cannot cover the corresponding results on weak Lebesgue spaces $WL_p(\Omega)$ with any given $p \in (1, \infty)$ in [35] and [14].

On the other hand, Jiao, Wu, and Peng [17] studied weak martingale Orlicz–Hardy spaces under the following assumption. For any $\ell \in (0, 1]$, let $\mathcal{G}_\ell$ be the set of all Orlicz functions Φ satisfying that Φ is of lower type ℓ and of upper type 1 (see, e.g., [17], [29]). Let Φ be a concave function, and let Φ' be its derivative function. Its lower index and its upper index of Φ are defined, respectively, by setting

$$p_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)} \qquad \text{and} \qquad q_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}. \tag{1.2}$$

All the results in [17] need the assumptions that $\Phi \in \mathcal{G}_\ell$ for some $\ell \in (0, 1]$ and $q_{\Phi^{-1}} \in (0, \infty)$, where Φ^{-1} denotes the inverse function of Φ. Observe that, when $\varphi(x, t) := \Phi(t)$ for any $x \in \Omega$ and $t \in (0, \infty)$, φ satisfies Assumption 1.A if and only if $\Phi \in \mathcal{G}_\ell$ for some $\ell \in (0, 1]$.

The first goal of this article is to weaken Assumption 1.A of [39] and to remove the unnecessary assumption $q_{\Phi^{-1}} \in (0, \infty)$ of [17]. Indeed, instead of Assumption 1.A, in this article, we *always make* the following assumption.

Assumption 1.1. Let φ be a Musielak–Orlicz function, and let φ be of uniformly lower type p_φ^- for some $p_\varphi^- \in (0, \infty)$ and of uniformly upper type p_φ^+ for some $p_\varphi^+ \in (0, \infty)$.

In this article, under Assumption 1.1, we first establish the atomic characterizations of weak martingale Musielak–Orlicz Hardy spaces $WH_\varphi^s(\Omega)$, $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$. Using these atomic characterizations, we then obtain the boundedness of σ-sublinear operators from weak martingale Musielak–Orlicz Hardy spaces to weak Musielak–Orlicz spaces, as well as some martingale inequalities which further clarify the relationships among $WH_\varphi^s(\Omega)$, $WH_\varphi^M(\Omega)$,

$WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$. All these results improve and generalize the corresponding results on weak martingale Orlicz–Hardy spaces (see [17]). Moreover, we also improve all the results on weak martingale Musielak–Orlicz Hardy spaces in [39]. In particular, both the boundedness of σ-sublinear operators and the martingale inequalities, for weak weighted martingale Hardy spaces as well as for weak weighted martingale Orlicz–Hardy spaces, are new.

To be precise, this article is organized as follows. In Section 2, we first recall some notation and notions on Musielak–Orlicz functions, weak Musielak–Orlicz spaces, and weak martingale Musielak–Orlicz Hardy spaces. Then we introduce various weak atomic martingale Musielak–Orlicz Hardy spaces.

Section 3 is devoted to establishing the atomic characterizations of spaces $WH_\varphi^s(\Omega)$, $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$ (see Theorems 3.1, 3.2, and 3.5 below). The above five weak martingale Musielak–Orlicz Hardy spaces contain weak weighted martingale Hardy spaces, weak martingale Orlicz–Hardy spaces as in [17], and weak variable martingale Hardy spaces as special cases (see Remark 2.6 below for more details). Recall that, even for weak martingale Hardy spaces in [35] and [14], only the ∞-atomic characterizations are known. Nevertheless, in this article we establish the q-atomic characterizations for any $q \in (\max\{p_\varphi^+, 1\}, \infty]$, where p_φ^+ denotes the uniformly upper type index of φ. Moreover, in [17] for weak martingale Orlicz–Hardy spaces and [39] for weak martingale Musielak–Orlicz Hardy spaces, the results of atomic characterizations need the index $p_\varphi^+ = 1$. Differently from [17] and [39], we allow $p_\varphi^+ \in (0, \infty)$ in Theorems 3.1, 3.2, and 3.5 below. So, the classical argument used in the proofs of [14, Theorem 1] and [17, Theorem 2.1] does not work here anymore. We overcome this difficulty by using some ideas from the proof of [23, Theorem 3.5] and constructing some appropriate atoms (see the proofs of Theorems 3.1 and 3.5). Moreover, our atomic characterizations of weak martingale Musielak–Orlicz Hardy spaces cover weak variable martingale Hardy spaces, weak weighted martingale Hardy spaces, and weak weighted martingale Orlicz–Hardy spaces, which are also new (see Remarks 3.3 and 3.6 below).

In Section 4, we study the boundedness of σ-sublinear operators on weak martingale Musielak–Orlicz Hardy spaces. Recall that, for a martingale space X and a measurable function space Y, an operator $T : X \to Y$ is called a *σ-sublinear operator* if, for any $\{f_k\}_{k\in\mathbb{N}} \subset X$ and $\lambda \in \mathbb{C}$,

$$\Big|T\Big(\sum_{k\in\mathbb{N}} f_k\Big)\Big| \le \sum_{k\in\mathbb{N}} \big|T(f_k)\big| \qquad \text{and} \qquad \big|T(\lambda f)\big| \le |\lambda|\big|T(f)\big|.$$

The boundedness of σ-sublinear operators from weak martingale Hardy spaces to weak Lebesgue spaces was studied in [14] and [35], and from weak martingale Orlicz–Hardy spaces to weak Orlicz spaces in [17]. All these results need the assumption that σ-sublinear operators T are bounded on $L^q(\Omega)$ for some $q \in [1, 2]$ or some $q \in [1, \infty)$. In particular, Yang [39, Theorem 4.2] also gave some sufficient conditions for a σ-sublinear operator T to be bounded from weak martingale Musielak–Orlicz Hardy spaces to weak Musielak–Orlicz spaces. In what follows, for any measurable set $E \subseteq \Omega$ and $t \in [0, \infty)$, let

$$\varphi(E, t) := \int_E \varphi(x, t)\, d\mathbb{P}.$$

The following assumption on φ is needed in Yang [39, Theorems 4.2–4.5].

Assumption 1.B.

(i) Let T be a σ-sublinear operator bounded on $L^2(\Omega)$.

(ii) Let φ be a Musielak–Orlicz function satisfying Assumption 1.A, and suppose that there exist two positive constants B and D such that, for any measurable subset $E \subseteq \Omega$, $x \in \Omega$ and $t \in (0, \infty)$,

$$B\varphi(x, t)\mathbb{P}(E) \le \varphi(E, t) \le D\varphi(x, t)\mathbb{P}(E). \tag{1.3}$$

Observe that (1.3) is also quite restrictive. Indeed, using (1.3) with $E = \Omega$, we find that, for any $x \in \Omega$ and $t \in (0, \infty)$,

$$\frac{1}{D}\varphi(\Omega, t) \le \varphi(x, t) \le \frac{1}{B}\varphi(\Omega, t).$$

Thus, Assumption 1.B(ii) requires φ to be essentially an Orlicz function. Moreover, [39, Theorems 4.2–4.5] do not cover the very important case, namely, the weighted case.

Note that all these assumptions for the boundedness of σ-sublinear operators used in [14], [17], [35], and [39] ensure that T is bounded from some martingale Hardy spaces to some Lebesgue spaces, which, together with the fact that Musielak–Orlicz functions unify Orlicz functions and weights, motivates us to introduce the following assumption.

Assumption 1.2. Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. Let T be a σ-sublinear operator satisfying one of the following:

(i) For some given $q \in (p_\varphi^+, \infty)$, there exists a positive constant C such that, for any $(\varphi, \infty)_s$-atom a and any $t \in (0, \infty)$,

$$\big\|T(a)\big\|_{L^q(\Omega, \varphi(\cdot, t)\, d\mathbb{P})} \le C\|a\|_{H_q^s(\Omega, \varphi(\cdot, t)\, d\mathbb{P})}.$$

(ii) For some given $q \in (p_\varphi^+, \infty)$, there exists a positive constant C such that, for any $(\varphi, \infty)_S$-atom a and any $t \in (0, \infty)$,

$$\big\|T(a)\big\|_{L^q(\Omega, \varphi(\cdot, t)\, d\mathbb{P})} \le C\|a\|_{H_q^S(\Omega, \varphi(\cdot, t)\, d\mathbb{P})}.$$

(iii) For some given $q \in (p_\varphi^+, \infty)$, there exists a positive constant C such that, for any $(\varphi, \infty)_M$-atom a and any $t \in (0, \infty)$,

$$\big\|T(a)\big\|_{L^q(\Omega, \varphi(\cdot, t)\, d\mathbb{P})} \le C\|a\|_{H_q^M(\Omega, \varphi(\cdot, t)\, d\mathbb{P})}.$$

(See Section 2 for the definitions of these spaces and atoms.)

In Section 4 of this article, under Assumption 1.2, we obtain the boundedness of σ-sublinear operators from $WH_\varphi^s(\Omega)$ (resp., $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, or

$WQ_\varphi(\Omega)$) to $WL_\varphi(\Omega)$ (see Theorems 4.1, 4.2, and 4.3 below). In particular, we obtain the same results as in [39] via replacing Assumption 1.B by Assumption 1.2.

Observe that Assumption 1.2 is much weaker than Assumption 1.B. Indeed, Assumption 1.1 is weaker than Assumption 1.A, and (1.3) in Assumption 1.B is not needed in Assumption 1.2. Moreover, under Assumption 1.B, we find that $p_\varphi^+ = 1$ and the weighted Hardy space $H_2^s(\Omega, \varphi(\cdot, t)\, d\mathbb{P})$ (resp., $H_q^S(\Omega, \varphi(\cdot, t)\, d\mathbb{P})$ or $H_q^M(\Omega, \varphi(\cdot, t)\, d\mathbb{P})$) becomes the martingale Hardy space $H_2^s(\Omega)$ (resp., $H_2^S(\Omega)$ or $H_2^M(\Omega)$), which, together with the boundedness of T on $L^2(\Omega)$ and the boundedness of the operator s (resp., S or M) on $L^2(\Omega)$ (see, e.g., [34, Proposition 2.6, Theorems 2.11–2.12]), further implies that Assumption 1.2 holds true. Thus, compared with Assumption 1.B, Assumption 1.2 is much weaker. In particular, Theorems 4.1, 4.2, and 4.3 of this article indeed improve [14, Theorems 4–6], [17, Theorem 3.1, Remark 3.2], and [39, Theorems 4.2–4.4], respectively (see Remark 4.4 below for more details).

Also, in this section, using Theorems 4.1 and 4.2, we obtain some martingale inequalities among the spaces $WH_\varphi^s(\Omega)$, $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$, which further clarify the relations among these spaces, in Theorem 4.6 below. Moreover, Theorem 4.6 generalizes and improves the corresponding results on weak martingale Orlicz–Hardy spaces in [17, Theorem 3.3] (see Remark 4.7 below for details).

In Section 5, the last section of this article, we obtain some bounded convergence theorems and dominated convergence theorems on weak Musielak–Orlicz spaces $WL_\varphi(\Omega)$ (see Theorems 5.8 and 5.9 below), which are of independent interest.

Finally, we describe some conventions on notation used throughout this article. We always let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and we let C denote a positive constant, which may vary from line to line. We use the symbol $f \lesssim g$ to denote that there exists a positive constant C such that $f \le Cg$. The symbol $f \sim g$ is used as an abbreviation of $f \lesssim g \lesssim f$. We also use the following convention. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. For any subset E of Ω, denote by $\mathbf{1}_E$ its *characteristic function*. For any $p \in [1, \infty]$, let p' denote the conjugate number of p, namely, $1/p + 1/p' = 1$.

2. Preliminaries

In this section, we first recall some notation and notions on Musielak–Orlicz functions, weak Musielak–Orlicz spaces, and weak martingale Musielak–Orlicz Hardy spaces, and then we introduce various weak atomic martingale Musielak–Orlicz Hardy spaces.

Let $L^0(\Omega)$ denote the set of all measurable functions f on Ω. Now we introduce the notion of the weak Musielak–Orlicz space.

Definition 2.1. Let φ be a Musielak–Orlicz function. The *weak Musielak–Orlicz space* $WL_\varphi(\Omega)$ is defined to be the set of all $f \in L^0(\Omega)$ such that

$$\|f\|_{WL_\varphi(\Omega)} := \inf\left\{\lambda \in (0,\infty) : \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \Omega : \left|f(x)\right| > \alpha\right\}, \frac{\alpha}{\lambda}\right) \le 1\right\}$$

is finite.

Let $p \in (0,\infty)$, and let Φ be an Orlicz function. If $\varphi(x,t) := t^p$ or $\Phi(t)$ for any $x \in \Omega$ and $t \in (0,\infty)$, then $WL_\varphi(\Omega)$ becomes the weak Lebesgue space $WL_p(\Omega)$ (see, e.g., [35]) or the weak Orlicz space $WL_\Phi(\Omega)$ (see, e.g., [17]). Here and thereafter, $WL_p(\Omega)$ denotes the set of all $f \in L^0(\Omega)$ such that

$$\|f\|_{WL_p(\Omega)} := \sup_{\alpha\in(0,\infty)} \alpha\left[\mathbb{P}\left(\left\{x \in \Omega : \left|f(x)\right| > \alpha\right\}\right)\right]^{\frac{1}{p}}$$

is finite, and $WL_\Phi(\Omega)$ denotes the set of all $f \in L^0(\Omega)$ such that

$$\|f\|_{WL_\Phi(\Omega)} := \inf\left\{\lambda \in (0,\infty) : \sup_{\alpha\in(0,\infty)} \Phi\left(\frac{\alpha}{\lambda}\right)\mathbb{P}\left(\left\{x \in \Omega : \left|f(x)\right| > \alpha\right\}\right) \le 1\right\}$$

is finite.

Remark 2.2.

(i) If a Musielak–Orlicz function φ is of uniformly upper type p_φ^+ for some $p_\varphi^+ \in (0,\infty)$, then there exists a positive constant C such that, for any measurable functions f and g,

$$\|f+g\|_{WL_\varphi(\Omega)} \le C\left[\|f\|_{WL_\varphi(\Omega)} + \|g\|_{WL_\varphi(\Omega)}\right].$$

Indeed, by the uniformly upper type p_φ^+ property of φ, we find that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \Omega : \left|f(x)+g(x)\right| > \alpha\right\}, \frac{\alpha}{\lambda}\right)\\
&\quad\lesssim \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \Omega : \left|f(x)\right| > \frac{\alpha}{2}\right\}, \frac{\alpha}{2\lambda}\right)\\
&\qquad + \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \Omega : \left|g(x)\right| > \frac{\alpha}{2}\right\}, \frac{\alpha}{2\lambda}\right)\\
&\quad\sim \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \Omega : \left|f(x)\right| > \alpha\right\}, \frac{\alpha}{\lambda}\right)\\
&\qquad + \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \Omega : \left|g(x)\right| > \alpha\right\}, \frac{\alpha}{\lambda}\right).
\end{aligned}$$

Then the above claim follows immediately.

(ii) Obviously, if φ is both of uniformly lower type p_1 and of uniformly upper type p_2, then $p_1 \le p_2$. Moreover, if φ is of uniformly lower (resp., upper) type p, then it is also of uniformly lower (resp., upper) type $\widetilde{p}$ for any $\widetilde{p} \in (0,p)$ (resp., $\widetilde{p} \in (p,\infty)$).

(iii) Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. If there exist an Orlicz function Φ and two positive constants B and D such that, for any $x \in \Omega$ and $t \in (0,\infty)$,

$$B\Phi(t) \le \varphi(x,t) \le D\Phi(t),$$

then, for any $f \in WL_\varphi(\Omega)$, $\|f\|_{WL_\varphi(\Omega)} \sim \|f\|_{WL_\Phi(\Omega)}$ with the positive equivalence constants independent of f.

Let $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$, and let $\{\mathbb{E}_n\}_{n\in\mathbb{Z}_+}$ be the associated conditional expectations. Let w be a strictly positive function satisfying $\int_\Omega w(x)\,d\mathbb{P} < \infty$. The weight we consider in this article is a *special weight* $\{\mathbb{E}_n(w)\}_{n\in\mathbb{Z}_+}$, the martingale generated by w, with respect to $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_n\}_{n\in\mathbb{Z}_+})$. In what follows, with an abuse of notation, we denote this weight simply by $w := \{\mathbb{E}_n(w)\}_{n\in\mathbb{Z}_+}$. In particular, if φ is a Musielak–Orlicz function which is strictly positive and satisfies $\sup_{t\in(0,\infty)} \int_\Omega \varphi(x,t)\,d\mathbb{P} < \infty$, we then write $\varphi(\cdot,t) := \{\varphi_n(\cdot,t)\}_{n\in\mathbb{Z}_+} := \{\mathbb{E}_n(\varphi(\cdot,t))\}_{n\in\mathbb{Z}_+}$ for any $t \in (0,\infty)$.

The following weighted condition is due to Izumisawa and Kazamaki [15, p. 115].

Definition 2.3. Let $q \in [1,\infty)$. A positive Musielak–Orlicz function $\varphi : \Omega \times [0,\infty) \to [0,\infty)$ is said to satisfy the *uniformly* $A_q(\Omega)$ *condition*, denoted by $\varphi \in \mathbb{A}_q(\Omega)$, if there exists a positive constant K such that, when $q \in (1,\infty)$,

$$\sup_{n\in\mathbb{Z}_+} \sup_{t\in(0,\infty)} \mathbb{E}_n(\varphi)(\cdot,t)\left[\mathbb{E}_n(\varphi^{-\frac{1}{q-1}})(\cdot,t)\right]^{q-1} \le K \quad \mathbb{P}\text{-a.e.}$$

and, when $q = 1$,

$$\sup_{n\in\mathbb{Z}_+} \sup_{t\in(0,\infty)} \mathbb{E}_n(\varphi)(\cdot,t)\frac{1}{\varphi(\cdot,t)} \le K \quad \mathbb{P}\text{-a.e.}$$

A positive Musielak–Orlicz function φ is said to belong to $\mathbb{A}_\infty(\Omega)$ if $\varphi \in \mathbb{A}_q(\Omega)$ for some $q \in [1,\infty)$.

The following $\mathbb{S}$ condition arises naturally when dealing with weighted martingale inequalities. We refer to Doléans-Dade and Meyer [7] and Bonami and Lépingle [6] for more details.

Definition 2.4. Let $t \in [0,\infty)$. The martingale $\varphi(\cdot,t) := \{\varphi_n(\cdot,t)\}_{n\in\mathbb{Z}_+}$ is said to satisfy the *uniformly* $\mathbb{S}$ *condition*, denoted by $\varphi \in \mathbb{S}$, if there exists a positive constant K such that, for any $n \in \mathbb{N}$, $t \in (0,\infty)$ and almost every $x \in \Omega$,

$$\frac{1}{K}\varphi_{n-1}(x,t) \le \varphi_n(x,t) \le K\varphi_{n-1}(x,t). \tag{2.1}$$

The *conditions* $\mathbb{S}^-$ and $\mathbb{S}^+$ denote two parts of $\mathbb{S}$ satisfying only the left-hand or the right-hand sides of the preceding inequalities, respectively.

Let w be a special weight on Ω, and let $\varphi(x,t) := w(x)$ for any $x \in \Omega$ and $t \in (0,\infty)$. Then Definitions 2.3 and 2.4 go back to the original weighted definition (see, e.g., [7], [15]).

Denote by $\mathcal{M}$ the set of all martingales $f := (f_n)_{n\in\mathbb{Z}_+}$ related to $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ such that $f_0 = 0$. For any $f \in \mathcal{M}$, denote its *martingale difference sequence* by $\{d_n f\}_{n\in\mathbb{N}}$, where $d_n f := f_n - f_{n-1}$ for any $n \in \mathbb{N}$. Then the *maximal functions* $M_n(f)$ and $M(f)$, the *quadratic variations* $S_n(f)$ and $S(f)$, and the *conditional quadratic variations* $s_n(f)$ and $s(f)$ of the martingale f are defined, respectively, by setting

$$M_n(f) := \sup_{0\le i\le n} |f_i|, \qquad M(f) := \sup_{n\in\mathbb{Z}_+} |f_n|,$$
$$S_n(f) := \Big(\sum_{i=1}^{n} |d_i f|^2\Big)^{\frac12}, \qquad S(f) := \Big(\sum_{i=1}^{\infty} |d_i f|^2\Big)^{\frac12},$$
$$s_n(f) := \Big(\sum_{i=1}^{n} \mathbb{E}_{i-1}|d_i f|^2\Big)^{\frac12} \quad \text{and} \quad s(f) := \Big(\sum_{i=1}^{\infty} \mathbb{E}_{i-1}|d_i f|^2\Big)^{\frac12}.$$

Definition 2.5. Let φ be a Musielak–Orlicz function. The *weak martingale Musielak–Orlicz Hardy spaces* $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, and $WH_\varphi^s(\Omega)$ are defined, respectively, as follows:

$$WH_\varphi^M(\Omega) := \big\{f \in \mathcal{M} : \|f\|_{WH_\varphi^M(\Omega)} := \big\|M(f)\big\|_{WL_\varphi(\Omega)} < \infty\big\},$$
$$WH_\varphi^S(\Omega) := \big\{f \in \mathcal{M} : \|f\|_{H_\varphi^S} := \big\|S(f)\big\|_{WL_\varphi(\Omega)} < \infty\big\},$$

and

$$WH_\varphi^s(\Omega) := \big\{f \in \mathcal{M} : \|f\|_{H_\varphi^s} := \big\|s(f)\big\|_{WL_\varphi(\Omega)} < \infty\big\}.$$

Let Λ be the collection of all sequences $(\lambda_n)_{n\in\mathbb{Z}_+}$ of nondecreasing, nonnegative and *adapted functions* (namely, for any $n \in \mathbb{Z}_+$, λ_n is $\mathcal{F}_n$ measurable). Let $\lambda_\infty := \lim_{n\to\infty} \lambda_n$. For any $f \in \mathcal{M}$, let

$$\Lambda[WP_\varphi](f) := \big\{(\lambda_n)_{n\in\mathbb{Z}_+} \in \Lambda : |f_n| \le \lambda_{n-1} \ (n \in \mathbb{N}), \lambda_\infty \in WL_\varphi(\Omega)\big\}$$

and

$$\Lambda[WQ_\varphi](f) := \big\{(\lambda_n)_{n\in\mathbb{Z}_+} \in \Lambda : S_n(f) \le \lambda_{n-1} \ (n \in \mathbb{N}), \lambda_\infty \in WL_\varphi(\Omega)\big\}.$$

The *weak martingale Musielak–Orlicz Hardy spaces* $WP_\varphi(\Omega)$ and $WQ_\varphi(\Omega)$ are defined, respectively, as follows:

$$WP_\varphi(\Omega) := \Big\{f \in \mathcal{M} : \|f\|_{WP_\varphi(\Omega)} := \inf_{(\lambda_n)_{n\in\mathbb{Z}_+}\in\Lambda[WP_\varphi(\Omega)]} \|\lambda_\infty\|_{WL_\varphi(\Omega)} < \infty\Big\}$$

and

$$WQ_\varphi(\Omega) := \Big\{f \in \mathcal{M} : \|f\|_{WQ_\varphi(\Omega)} := \inf_{(\lambda_n)_{n\in\mathbb{Z}_+}\in\Lambda[WQ_\varphi(\Omega)]} \|\lambda_\infty\|_{WL_\varphi(\Omega)} < \infty\Big\}.$$

Remark 2.6. Several known weak martingale Hardy spaces can be regarded as special cases of the above five weak martingale Musielak–Orlicz Hardy spaces. For example, let $p \in (0,\infty)$, let Φ be an Orlicz function on $(0,\infty)$, let w be a weight, and let $p(\cdot) : \Omega \to [1,\infty]$ be a measurable function. If $\varphi(x,t) := t^p$, $\Phi(t)$, $t^{p(x)}$ or $w(x)\Phi(t)$ for any $x \in \Omega$ and $t \in (0,\infty)$, then the corresponding weak martingale Musielak–Orlicz Hardy space becomes, respectively, the weak martingale Hardy space (see [14], [35]), the weak martingale Orlicz–Hardy space (see [17]), the weak variable martingale Hardy space, or the weak weighted martingale Orlicz–Hardy space.

In what follows, for any $q \in [1,\infty]$, any measurable set $B \subseteq \Omega$, and any measurable function f on Ω, let

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup_{t\in(0,\infty)}[\frac{1}{\varphi(B,t)}\int_\Omega |f(x)|^q\varphi(x,t)\,d\mathbb{P}(x)]^{1/q} & \text{when } q\in[1,\infty), \\ \|f\|_{L^\infty(\Omega)} & \text{when } q=\infty. \end{cases}$$

Let $\mathcal{T}$ be the set of all stopping times related to $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$. For any $\nu \in \mathcal{T}$, let

$$B_\nu := \big\{x\in\Omega : \nu(x) < \infty\big\}.$$

Now we introduce the notion of atoms associated with a Musielak–Orlicz function.

Definition 2.7. Let $q \in (1,\infty]$, and let φ be a Musielak–Orlicz function. A measurable function a is called a $(\varphi,q)_s$-*atom* if there exists a stopping time ν relative to $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ (ν is called the *stopping time associated with* a) such that

(i) $a_n := \mathbb{E}_n a = 0$ a.e. on $\{x\in\Omega : \nu(x)\ge n\}$,
(ii) $\|s(a)\|_{L^q_\varphi(B_\nu)} \le \|\mathbf{1}_{B_\nu}\|^{-1}_{L^\varphi(\Omega)}$.

Similarly, $(\varphi,q)_S$-*atom* and $(\varphi,q)_M$-*atom* are defined via replacing (ii) in the above definition, respectively, by

$$\big\|S(a)\big\|_{L^q_\varphi(B_\nu)} \le \|\mathbf{1}_{B_\nu}\|^{-1}_{L^\varphi(\Omega)}$$

and

$$\big\|M(a)\big\|_{L^q_\varphi(B_\nu)} \le \|\mathbf{1}_{B_\nu}\|^{-1}_{L^\varphi(\Omega)}.$$

Via $(\varphi,q)_s$-atoms, $(\varphi,q)_S$-atoms, and $(\varphi,q)_M$-atoms, we now introduce three weak atomic martingale Musielak–Orlicz Hardy spaces $WH^{\varphi,q,s}_{\mathrm{at}}(\Omega)$, $WH^{\varphi,q,S}_{\mathrm{at}}(\Omega)$, and $WH^{\varphi,q,M}_{\mathrm{at}}(\Omega)$, respectively, as follows.

Definition 2.8. Let $q \in (1,\infty]$, and let φ be a Musielak–Orlicz function. The *weak atomic martingale Musielak–Orlicz Hardy space* $WH^{\varphi,q,s}_{\mathrm{at}}(\Omega)$ (resp., $WH^{\varphi,q,S}_{\mathrm{at}}(\Omega)$ or $WH^{\varphi,q,M}_{\mathrm{at}}(\Omega)$) is defined to be the space of all $f \in \mathcal{M}$ satisfying that there exist a sequence of $(\varphi,q)_s$-atoms (resp., $(\varphi,q)_S$-atoms or $(\varphi,q)_M$-atoms) $\{a^k\}_{k\in\mathbb{Z}}$, related to stopping times $\{\nu^k\}_{k\in\mathbb{Z}}$, and a positive constant $\widetilde{C}$, independent of f, such that, for any $n\in\mathbb{Z}_+$,

$$\sum_{k\in\mathbb{Z}} \mu^k a^k_n = f_n \quad \mathbb{P}\text{-a.e.},$$

where $\mu^k := \widetilde{C}2^k\|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}$ for any $k\in\mathbb{Z}$, and

$$\begin{aligned} &\|f\|_{WH^{\varphi,q,s}_{\mathrm{at}}(\Omega)} \quad \big(\text{resp., } \|f\|_{WH^{\varphi,q,S}_{\mathrm{at}}(\Omega)} \text{ or } \|f\|_{WH^{\varphi,q,M}_{\mathrm{at}}(\Omega)}\big) \\ &\quad := \inf\Big\{\inf\Big[\lambda\in(0,\infty) : \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big) \le 1\Big]\Big\} < \infty, \end{aligned}$$

where the first infimum is taken over all decompositions of f as above.

Let $p \in (0,\infty)$, and let w be a special weight. Recall that the *weighted Lebesgue space* $L^p(\Omega, w\,d\mathbb{P})$ is defined to be the set of all measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,w\,d\mathbb{P})} := \left[\int_\Omega \left|f(x)\right|^p w(x)\,d\mathbb{P}(x)\right]^{\frac{1}{p}} < \infty.$$

Moreover, the *weighted martingale Hardy spaces* $H_p^s(\Omega, w\,d\mathbb{P})$*,* $H_p^S(\Omega, w\,d\mathbb{P})$*, and* $H_p^M(\Omega, w\,d\mathbb{P})$ are defined, respectively, as follows:

$$H_p^s(\Omega, w\,d\mathbb{P}) := \left\{f \in \mathcal{M} : \left\|s(f)\right\|_{L^p(\Omega,w\,d\mathbb{P})} < \infty\right\},$$
$$H_p^S(\Omega, w\,d\mathbb{P}) := \left\{f \in \mathcal{M} : \left\|S(f)\right\|_{L^p(\Omega,w\,d\mathbb{P})} < \infty\right\},$$

and

$$H_p^M(\Omega, w\,d\mathbb{P}) := \left\{f \in \mathcal{M} : \left\|M(f)\right\|_{L^p(\Omega,w\,d\mathbb{P})} < \infty\right\}.$$

If $w \equiv 1$, then the weighted Hardy space $H_p^s(\Omega, w\,d\mathbb{P})$ (resp., $H_p^S(\Omega, w\,d\mathbb{P})$ or $H_p^M(\Omega, w\,d\mathbb{P})$) becomes the classical martingale Hardy space $H_p^s(\Omega)$ (resp., $H_p^S(\Omega)$ or $H_p^M(\Omega)$) (see, e.g., [34, p. 6]).

3. Atomic characterizations

In this section, we establish atomic characterizations of the weak martingale Musielak–Orlicz Hardy spaces $WH_\varphi^s(\Omega)$, $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$. We begin with the atomic characterization of $WH_\varphi^s(\Omega)$.

Theorem 3.1. *Let* $q \in (0,\infty)$*, and let* φ *be a Musielak–Orlicz function satisfying Assumption 1.1. If* $q \in (\max\{p_\varphi^+, 1\}, \infty]$*, then* $WH_\varphi^s(\Omega) = WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)$ *with equivalent quasinorms.*

Proof. We prove this theorem in two steps.

Step 1: Prove $WH_{\mathrm{at}}^{\varphi,q,s}(\Omega) \subseteq WH_\varphi^s(\Omega)$. To this end, let $f \in WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)$. Then, by Definition 2.8, we know that there exists a sequence of $(\varphi, q)_s$-atoms, $\{a^k\}_{k\in\mathbb{Z}}$, related to stopping times $\{\nu^k\}_{k\in\mathbb{Z}}$ such that, for any $n \in \mathbb{Z}_+$,

$$f_n = \sum_{k\in\mathbb{Z}} \mu^k a_n^k \quad \mathbb{P}\text{-a.e.},$$

where $\mu^k := \widetilde{C} 2^k \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}$ for any $k \in \mathbb{Z}$ and $\widetilde{C}$ is a positive constant independent of f. To show the desired conclusion, by the definitions of $WH_\varphi^s(\Omega)$ and $WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)$, it suffices to prove that, for any α, $\lambda \in (0,\infty)$,

$$\varphi\Big(\big\{x \in \Omega : s(f)(x) > \alpha\big\}, \frac{\alpha}{\lambda}\Big) \lesssim \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big). \tag{3.1}$$

To this end, for any fixed $\alpha \in (0,\infty)$, let $k_0 \in \mathbb{Z}$ be such that

$$2^{k_0} \le \alpha < 2^{k_0+1}.$$

Combining this and the subadditivity of the operator s, we conclude that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\varphi\Big(\Big\{x\in\Omega: s(f)(x)>\alpha\Big\},\frac{\alpha}{\lambda}\Big)\\
&\quad\le \varphi\Big(\Big\{x\in\Omega: \sum_{k\in\mathbb{Z}}\mu^k s(a^k)(x)>\alpha\Big\},\frac{\alpha}{\lambda}\Big)\\
&\quad\le \varphi\Big(\Big\{x\in\Omega: \sum_{k=-\infty}^{k_0-1}\mu^k s(a^k)(x)>2^{k_0-1}\Big\},\frac{2^{k_0+1}}{\lambda}\Big)\\
&\qquad+\varphi\Big(\Big\{x\in\Omega: \sum_{k=k_0}^{\infty}\mu^k s(a^k)(x)>2^{k_0-1}\Big\},\frac{2^{k_0+1}}{\lambda}\Big)\\
&\quad=: \mathrm{I}_{\alpha,1}+\mathrm{I}_{\alpha,2}.
\end{aligned}$$

Thus, in order to show (3.1), we only need to estimate $\mathrm{I}_{\alpha,1}$ and $\mathrm{I}_{\alpha,2}$.

We first estimate $\mathrm{I}_{\alpha,1}$. For any $r\in(\max\{p_\varphi^+,1\},\infty)$ and $\ell\in(0,1-\frac{\max\{p_\varphi^+,1\}}{r})$, by the Hölder inequality, we know that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}
\mathrm{I}_{\alpha,1}&\le \frac{1}{2^{(k_0-1)r}}\int_\Omega\Big[\sum_{k=-\infty}^{k_0-1}\mu^k s(a^k)(x)\Big]^r\varphi\Big(x,\frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}\\
&\le \frac{1}{2^{(k_0-1)r}}\Big(\sum_{k=-\infty}^{k_0-1}2^{k\ell r'}\Big)^{\frac{r}{r'}}\\
&\quad\times\int_\Omega\Big\{\sum_{k=-\infty}^{k_0-1}2^{-k\ell r}\big[\mu^k s(a^k)(x)\big]^r\Big\}\varphi\Big(x,\frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}\\
&\le 2^{-r(k_0-1)(1-\ell)}(1-2^{-\ell r'})^{-r/r'}\\
&\quad\times\sum_{k=-\infty}^{k_0-1}2^{-k\ell r}(\mu^k)^r\big\|s(a^k)\big\|^r_{L^r_\varphi(\Omega)}\int_{B_{\nu^k}}\varphi\Big(x,\frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P},
\end{aligned}\tag{3.2}$$

where, in the last inequality, we used the fact that

$$\sum_{k=-\infty}^{k_0-1}2^{k\ell r'}=2^{(k_0-1)\ell r'}(1-2^{-\ell r'})^{-1},$$

the monotone convergence theorem, and the definition of $L^r_\varphi(\Omega)$. For the case $q\in(\max\{p_\varphi^+,1\},\infty)$, let $r:=q$. From (3.2), the uniformly upper type p_φ^+ property of φ, and the fact that a^k is a $(\varphi,q)_s$-atom for any $k\in\mathbb{Z}$, we deduce that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}
\mathrm{I}_{\alpha,1}&\le 2^{-q(k_0-1)(1-\ell)}(1-2^{-\ell q'})^{-q/q'}\\
&\quad\times\sum_{k=-\infty}^{k_0-1}2^{-k\ell q}(\mu^k)^q\big\|s(a^k)\big\|^q_{L^q_\varphi(\Omega)}\int_{B_{\nu^k}}\varphi\Big(x,\frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}
\end{aligned}$$

$$\leq 2^{-q(k_0-1)(1-\ell)}(1-2^{-\ell q'})^{-q/q'}$$
$$\times \sum_{k=-\infty}^{k_0-1} 2^{-k\ell q}(\widetilde{C}2^k)^q 2^{(k_0+1-k)p_\varphi^+} \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big),$$

which, together with $(1-\ell)q > p_\varphi^+$, implies that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned} \mathrm{I}_{\alpha,1} &\leq (\widetilde{C})^q 2^{-q(k_0-1)(1-\ell)}(1-2^{-\ell q'})^{-q/q'} 2^{(k_0+1)p_\varphi^+} \\ &\quad \times \sum_{k=-\infty}^{k_0-1} 2^{k[(1-\ell)q-p_\varphi^+]} \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big) \\ &\leq (\widetilde{C})^q (1-2^{-\ell q'})^{-q/q'}[1-2^{p_\varphi^+-(1-\ell)q}] \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big). \end{aligned} \tag{3.3}$$

Letting $\ell := \frac{1}{2}(1-\max\{p_\varphi^+/q, 1/q\})$ in (3.3), we conclude that, for any given $q \in (\max\{p_\varphi^+, 1\}, \infty)$ and any $\lambda \in (0,\infty)$,

$$\mathrm{I}_{\alpha,1} \lesssim \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big). \tag{3.4}$$

For the case $q = \infty$, notice that, for any $k \in \mathbb{Z}$, $\|s(a^k)\|_{L_\varphi^r(\Omega)} \leq \|s(a^k)\|_{L^\infty(\Omega)}$. Combining this and (3.2), similarly to the estimation of (3.3), we know that, for any $r \in (\max\{p_\varphi^+, 1\}, \infty)$, $\ell \in (0, 1-\frac{\max\{p_\varphi^+,1\}}{r})$ and $\lambda \in (0,\infty)$,

$$\mathrm{I}_{\alpha,1} \lesssim (1-2^{-\ell r'})^{-r/r'}[1-2^{p_\varphi^+-(1-\ell)r}] \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big).$$

Letting $r := p_\varphi^+ + 1$ and $\ell := \frac{1}{2}(1-\max\{\frac{p_\varphi^+}{p_\varphi^+ +1}, \frac{1}{p_\varphi^+ +1}\})$ in the above inequality, we finally find that, for any $\lambda \in (0,\infty)$,

$$\mathrm{I}_{\alpha,1} \lesssim \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big). \tag{3.5}$$

Now we estimate $\mathrm{I}_{\alpha,2}$. For any $k \in \mathbb{Z}$, by the definition of a^k, we have

$$\big\{x \in \Omega : s(a^k)(x) \neq 0\big\} \subseteq B_{\upsilon^k}.$$

From this, it follows that

$$0 \leq s(f_{\alpha,2}) \leq \sum_{k=k_0}^{\infty} \mu^k s(a^k) = \sum_{k=k_0}^{\infty} \mu^k s(a^k) \mathbf{1}_{B_{\nu^k}},$$

which implies that

$$\big\{x \in \Omega : s(f_{\alpha,2})(x) \neq 0\big\} \subseteq \bigcup_{k=k_0}^{\infty} B_{\upsilon^k}.$$

Combining this and the fact that φ is of uniformly lower type p_φ^- and of uniformly upper type p_φ^+, we obtain, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}\mathrm{I}_{\alpha,2} &\le \sum_{k=k_0}^{\infty} \varphi\Big(B_{\nu^k}, \frac{2^{k_0+1}}{\lambda}\Big) \lesssim \sum_{k=k_0}^{\infty} 2^{p_\varphi^+} \varphi\Big(B_{\nu^k}, \frac{2^{k_0}}{\lambda}\Big) \\ &\lesssim 2^{p_\varphi^+} \sum_{k=k_0}^{\infty} 2^{(k_0-k)p_\varphi^-} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big) \lesssim \sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big). \end{aligned} \tag{3.6}$$

From (3.4), (3.5), and (3.6), it follows that, for any α, $\lambda \in (0,\infty)$, (3.1) holds true, which further implies that $\|f\|_{WH_\varphi^s(\Omega)} \lesssim \|f\|_{WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)}$. Thus, we have $WH_{\mathrm{at}}^{\varphi,q,s}(\Omega) \subseteq WH_\varphi^s(\Omega)$. This finishes the proof of step 1.

Step 2: Prove $WH_\varphi^s(\Omega) \subseteq WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)$. To this end, let $f \in WH_\varphi^s(\Omega)$. For any $k \in \mathbb{Z}$ and $x \in \Omega$, let

$$\nu^k(x) := \inf\big\{n \in \mathbb{N} : s_{n+1}(f)(x) > 2^k\big\} \qquad \text{and} \qquad \mu^k := 2^{k+1}\|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}.$$

Then $(\nu^k)_{k\in\mathbb{Z}}$ is a sequence of nondecreasing stopping times. Moreover, for any $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, let

$$a_n^k := \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\mu^k} \quad \text{if } \mu^k \neq 0;$$

otherwise, let $a_n^k := 0$. Then we have

$$f_n = \sum_{k\in\mathbb{Z}} \mu^k a_n^k \quad \mathbb{P}\text{-a.e.}$$

Now we claim that, for any $k \in \mathbb{Z}$, $a^k := (a_n^k)_{n\in\mathbb{Z}_+}$ is a $(\varphi, q)_s$-atom. Indeed, for any $k \in \mathbb{Z}$, it is clear that a^k is a martingale. When $\nu^k \ge n$, we can easily see that $a_n^k = 0$. Thus, a^k satisfies Definition 2.7(i). Similarly to the proof of [37, Theorem 1.4], we know that, for any $k \in \mathbb{Z}$.

$$\big\|s(a^k)\big\|_{L^\infty(\Omega)} \le \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}^{-1}.$$

This implies that a^k is an $L^2(\Omega)$-bounded martingale and hence $(a_n^k)_{n\in\mathbb{Z}_+}$ converges in $L^2(\Omega)$ as $n \to \infty$. Denoting this limit still by a^k, we have that $\mathbb{E}_n(a^k) = a_n^k$ for any $n \in \mathbb{Z}_+$. Moreover, for any given $q \in (0,\infty]$ and any $k \in \mathbb{Z}$,

$$\big\|s(a^k)\big\|_{L_\varphi^q(B_{\nu^k})} \le \big\|s(a^k)\big\|_{L^\infty(\Omega)} \le \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}^{-1}.$$

Thus, a^k satisfies Definition 2.7(ii) and hence a^k is a $(\varphi, q)_s$-atom. This proves the above claim. On the other hand, for any $k \in \mathbb{Z}$, we have $\{x \in \Omega : s(f)(x) > 2^k\} = B_{\nu^k}$. From this, it follows that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}\sup_{k\in\mathbb{Z}} \varphi\Big(B_{\nu^k}, \frac{2^k}{\lambda}\Big) &= \sup_{k\in\mathbb{Z}} \varphi\Big(\big\{x \in \Omega : s(f)(x) > 2^k\big\}, \frac{2^k}{\lambda}\Big) \\ &\le \sup_{\alpha\in(0,\infty)} \varphi\Big(\big\{x \in \Omega : s(f)(x) > \alpha\big\}, \frac{\alpha}{\lambda}\Big),\end{aligned}$$

which implies that $f \in WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)$ and $\|f\|_{WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)} \le \|f\|_{WH_\varphi^s(\Omega)}$. This finishes the proof of step 2 and hence of Theorem 3.1. $\square$

Theorem 3.2. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. Then $WP_\varphi(\Omega) = WH_{\rm at}^{\varphi,\infty,M}(\Omega)$ and $WQ_\varphi(\Omega) = WH_{\rm at}^{\varphi,\infty,S}(\Omega)$ with equivalent quasinorms.*

Proof. This proof is just a slight modification of the one given for Theorem 3.1. Details are provided for the reader's convenience. We only give the proof for $WP_\varphi(\Omega)$ because the proof for $WQ_\varphi(\Omega)$ is similar.

We first prove $WP_\varphi(\Omega) \subseteq WH_{\rm at}^{\varphi,\infty,M}(\Omega)$. To this end, let $f \in WP_\varphi(\Omega)$. For any $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $x \in \Omega$, let

$$\nu^k(x) := \{n \in \mathbb{Z}_+ : \lambda_n(x) > 2^k\}, \qquad \mu^k := 3 \cdot 2^k \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)},$$

and

$$a_n^k := \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\mu^k}$$

if $\mu^k \neq 0$; otherwise, let $a_n^k := 0$, where $(\lambda_n)_{n\in\mathbb{Z}_+} \in \Lambda[WP_\varphi](f)$. Then, using the same method as that used in the proof of Theorem 3.1, we can prove that, for any $k \in \mathbb{Z}$, $\|M(a^k)\|_{L^\infty(\Omega)} \le \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}^{-1}$, a^k is a $(\varphi,\infty)_M$-atom and $\|f\|_{WH_{\rm at}^{\varphi,\infty,M}(\Omega)} \le \|f\|_{WP_\varphi(\Omega)}$.

Conversely, let $f \in WH_{\rm at}^{\varphi,\infty,M}(\Omega)$. Then there exist a sequence of $(\varphi,\infty)_M$-atoms, $\{a^k\}_{k\in\mathbb{Z}}$, related to stopping times $\{\nu^k\}_{k\in\mathbb{Z}}$ and a positive constant $\widetilde{C}$, independent of f, such that, for any $n \in \mathbb{Z}_+$,

$$f_n = \sum_{k\in\mathbb{Z}} \widetilde{C} 2^k \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)} a_n^k \quad \mathbb{P}\text{-a.e.}$$

For any $n \in \mathbb{Z}_+$, let $\lambda_n := \sum_{k\in\mathbb{Z}} \widetilde{C} 2^k \mathbf{1}_{\{x\in\Omega:\nu^k(x)\le n\}}$. Then, by the definition of a^k, we know that $(\lambda_n)_{n\in\mathbb{Z}_+}$ is a nonnegative adapted sequence and, for any $n \in \mathbb{N}$ and almost every $x \in \Omega$,

$$|f_n(x)| \le \sum_{k\in\mathbb{Z}} \widetilde{C} 2^k \|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)} \|a_n^k\|_{L^\infty(\Omega)} \mathbf{1}_{\{x\in\Omega:\nu^k(x)\le n-1\}}(x) \le \lambda_{n-1}(x).$$

Now we show that $\|\lambda_\infty\|_{WL_\varphi(\Omega)} \lesssim \|f\|_{WH_{\rm at}^{\varphi,\infty,M}(\Omega)}$. For any fixed $\alpha \in (0,\infty)$, let $k_0 \in \mathbb{Z}$ be such that $2^{k_0} \le \alpha < 2^{k_0+1}$. Similarly to the estimations of (3.5) and (3.6) via replacing $\mu^k s(a^k)$ by $\widetilde{C} 2^k \mathbf{1}_{B_{\nu^k}}$, we conclude that, for any $\gamma \in (0,\infty)$,

$$\begin{aligned}
&\varphi\left(\{x \in \Omega : \lambda_\infty(x) > \alpha\}, \frac{\alpha}{\gamma}\right) \\
&\quad \le \varphi\left(\left\{x \in \Omega : \sum_{k=-\infty}^{k_0-1} \widetilde{C} 2^k \mathbf{1}_{B_{\nu^k}}(x) > 2^{k_0-1}\right\}, \frac{2^{k_0+1}}{\gamma}\right) \\
&\qquad + \varphi\left(\left\{x \in \Omega : \sum_{k=k_0}^{\infty} \widetilde{C} 2^k \mathbf{1}_{B_{\nu^k}}(x) > 2^{k_0-1}\right\}, \frac{2^{k_0+1}}{\gamma}\right) \\
&\quad \lesssim \sup_{k\in\mathbb{Z}} \varphi\left(B_{\nu^k}, \frac{2^k}{\gamma}\right).
\end{aligned}$$

This implies that $f \in WP_\varphi(\Omega)$ and $\|f\|_{WP_\varphi(\Omega)} \le \|\lambda_\infty\|_{WL_\varphi(\Omega)} \lesssim \|f\|_{WH_{\rm at}^{\varphi,\infty,M}(\Omega)}$, which completes the proof of step 2 and hence of Theorem 3.2. □

Remark 3.3.

(i) For any given $p \in (0,\infty)$, when $\varphi(x,t) := t^p$ for any $x \in \Omega$ and $t \in (0,\infty)$, in this case, Theorem 3.1 with $q = \infty$ for Vilenkin martingales was investigated by Weisz [35, Theorem 1], and then Theorem 3.1 with $q = \infty$ and Theorem 3.2 were obtained by Hou and Ren [14, Theorems 1, 2, and 3]. Observing that Theorem 3.1 includes the q-atomic characterization of $WH_\varphi^s(\Omega)$ for any $q \in (\max\{p,1\},\infty)$, Theorem 3.1 generalizes and improves [35, Theorem 1] and [14, Theorem 1]. Moreover, since φ is of wide generality, we know that Theorem 3.2 generalizes [14, Theorems 2 and 3].

(ii) Let Φ be an Orlicz function. Theorem 3.1 with $q = \infty$ and Theorem 3.2 when $\varphi(x,t) := \Phi(t)$ for any $x \in \Omega$ and $t \in (0,\infty)$ were obtained by Jiao, Wu, and Peng [17, Theorems 2.1 and 2.4] under some slightly stronger assumptions. Indeed, [17, Theorems 2.1 and 2.4] require that $\Phi \in \mathcal{G}_\ell$ for some $\ell \in (0,1]$ and the upper index $q_{\Phi^{-1}} \in (0,\infty)$ (see (1.2) for its definition); however, Theorems 3.1 and 3.2 in this case only need $\Phi \in \mathcal{G}_\ell$ for some $\ell \in (0,\infty)$. Moreover, in Theorem 3.1, $q \in (\max\{p_\Phi^+,1\},\infty]$ is much more difficult and important than the endpoint case $q = \infty$. Therefore, Theorems 3.1 and 3.2 generalize and improve [17, Theorems 2.1 and 2.4].

(iii) Theorem 3.1 with $q = \infty$ and Theorem 3.2 were first proved by Yang [39, Theorems 3.1, 3.2, and 3.3] under Assumption 1.A. However, Theorems 3.1 and 3.2 only need Assumption 1.1 which is much weaker than Assumption 1.A. Thus, Theorems 3.1 and 3.2 indeed essentially improve [39, Theorem 3.1] and [39, Theorems 3.2 and 3.3], respectively.

(iv) Let $p(\cdot)$ be a measurable function on Ω satisfying

$$0 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{x\in\Omega} p(x) < \infty.$$

Let $\varphi(x,t) := t^{p(x)}$ for any $x \in \Omega$ and $t \in (0,\infty)$. Observe that, in this case, φ is of uniformly lower type p^- and of uniformly upper type p^+. From this and Remark 2.6, it follows that Theorems 3.1 and 3.2 give the atomic characterizations of weak variable martingale Hardy spaces, which are also new.

Now we establish the atomic characterizations of $WH_\varphi^S(\Omega)$ and $WH_\varphi^M(\Omega)$. To do this, we need an additional notion. The stochastic basis $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is said to be *regular* if there exists a positive constant R such that, for any $n \in \mathbb{N}$,

$$f_n \le R f_{n-1} \tag{3.7}$$

holds true for any nonnegative martingale $(f_n)_{n\in\mathbb{Z}_+}$. Clearly, if $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular, then any special weight automatically satisfies the condition $\mathbb{S}^+$. Moreover, it was proved in [28, Proposition 6.3.7] that, if $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular, then any special weight from $\mathbb{A}_\infty(\Omega)$ satisfies the condition $\mathbb{S}$.

The following technical lemma was proved in [38, Lemma 4.7].

Lemma 3.4. *Let $w := (w_n)_{n\in\mathbb{Z}_+} \in \mathbb{S}^-$ be a special weight. If the stochastic basis $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular, then, for any nonnegative adapted process $\gamma = (\gamma_n)_{n\in\mathbb{Z}_+}$ and any $\lambda \in (\|\gamma_0\|_{L^\infty(\Omega)}, \infty)$, there exists a stopping time τ_λ such that, for any $n \in \mathbb{Z}_+$,*

$$\sup_{n\le\tau_\lambda(x)} \gamma_n(x) =: M_{\tau_\lambda}\gamma(x) \le \lambda, \quad \forall\, x \in \Omega,$$

$$\{x \in \Omega : M\gamma(x) > \lambda\} \subseteq \{x \in \Omega : \tau_\lambda(x) < \infty\},$$

and

$$w(\{x \in \Omega : \tau_\lambda(x) < \infty\}) \le KRw(\{x \in \Omega : M\gamma(x) > \lambda\}),$$

where K and R are the same as in (2.1) and (3.7), respectively. Moreover, for any $\lambda_1,\ \lambda_2 \in (0,\infty)$ with $\lambda_1 < \lambda_2$, $\tau_{\lambda_1} \le \tau_{\lambda_2}$.

Theorem 3.5. *Let $q \in (0,\infty)$, and let $\varphi \in \mathbb{S}^-$ be a Musielak–Orlicz function satisfying Assumption 1.1. If $q \in (\max\{p_\varphi^+, 1\}, \infty]$ and the stochastic basis $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular, then $WH_\varphi^S(\Omega) = WH_{\rm at}^{\varphi,q,S}(\Omega)$ and $WH_\varphi^M(\Omega) = WH_{\rm at}^{\varphi,q,M}(\Omega)$ with equivalent quasinorms.*

Proof. We only prove this theorem for $WH_\varphi^S(\Omega)$, because the proof for $WH_\varphi^M(\Omega)$ only needs a slight modification. We proceed in two steps.

Step 1: Prove $WH_\varphi^S(\Omega) \subseteq WH_{\rm at}^{\varphi,q,S}(\Omega)$. To this end, let $f \in WH_\varphi^S(\Omega)$. For any $k \in \mathbb{Z}$ and for the nonnegative adapted sequence $\{S_n(f)\}_{n\in\mathbb{Z}_+}$, by Lemma 3.4, we know that there exists a stopping time $\nu^k \in \mathcal{T}$ such that

$$\begin{aligned}\{x \in \Omega : S(f)(x) > 2^k\} &\subseteq \{x \in \Omega : \nu^k(x) < \infty\},\\ S_{\nu^k}(f)(x) &\le 2^k, \quad \forall x \in \Omega,\end{aligned} \tag{3.8}$$

and, for any $t \in (0,\infty)$,

$$\varphi(\{x \in \Omega : \nu^k(x) < \infty\}, t) \le KR\varphi(\{x \in \Omega : S(f)(x) > 2^k\}, t), \tag{3.9}$$

where K and R are the same as in (2.1) and (3.7), respectively. Moreover, for any $k \in \mathbb{Z}$, $\nu^k \le \nu^{k+1}$ and $\nu^k \to \infty$ as $k \to \infty$. For any $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, let

$$\mu^k := 2^{k+1}\|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)} \qquad \text{and} \qquad a_n^k := \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\mu^k}$$

if $\mu^k \ne 0$; otherwise, let $a_n^k := 0$. Then, for any $n \in \mathbb{N}$, $f_n(x) = \sum_{k\in\mathbb{Z}} \mu^k a_n^k(x)$ for almost every $x \in \Omega$. Now we claim that, for any fixed $k \in \mathbb{Z}$, $a^k := (a_n^k)_{n\in\mathbb{Z}_+}$ is a $(\varphi, q)_S$-atom. Indeed, it is clear that $(a_n^k)_{n\in\mathbb{Z}_+}$ is a martingale. Moreover, by (3.8), we know that

$$\begin{aligned}\left[S(a^k)\right]^2 &= \sum_{n\in\mathbb{N}} |d_n a^k|^2 = \frac{1}{(\mu^k)^2} \sum_{n\in\mathbb{N}} |d_n f^{\nu^{k+1}} - d_n f^{\nu^k}|^2\\ &= \frac{1}{(\mu^k)^2} \sum_{n\in\mathbb{N}} |d_n f \mathbf{1}_{\{x\in\Omega:\nu^k(x)<n\le\nu^{k+1}(x)\}}|^2\\ &\le \frac{1}{(\mu^k)^2}\left[S_{\nu^{k+1}}(f)\right]^2 \le \left(\frac{2^{k+1}}{\mu^k}\right)^2.\end{aligned} \tag{3.10}$$

From this, it follows that a^k is an $L^2(\Omega)$-bounded martingale and hence $(a_n^k)_{n\in\mathbb{Z}_+}$ converges in $L^2(\Omega)$ as $n\to\infty$. Denoting its limit still by a^k, then $\mathbb{E}_n(a^k)=a_n^k$. For any $n\in\mathbb{Z}_+$ and $x\in\{x\in\Omega:\nu^k(x)\ge n\}$, by the definition of $f_n^{\nu^k}$, we know that $a_n^k(x)=0$. Thus, a^k satisfies Definition 2.7(i). From (3.10), it follows that

$$\left\|S(a^k)\right\|_{L^q_\varphi(B_{\nu^k})}\le\left\|S(a^k)\right\|_{L^\infty(\Omega)}\le\|\mathbf{1}_{B_{\nu^k}}\|^{-1}_{L^\varphi(\Omega)},$$

which implies that a^k satisfies Definition 2.7(ii) and hence a^k is a $(\varphi,q)_S$-atom. This proves the above claim.

Now we show that $f\in WH^{\varphi,q,S}_{\rm at}(\Omega)$. From (3.9), we deduce that, for any $k\in\mathbb{Z}$ and $\lambda\in(0,\infty)$,

$$\begin{aligned}\varphi\left(B_{\nu^k},\frac{2^k}{\lambda}\right)&\le RK\varphi\left(\left\{x\in\Omega:S(f)(x)>2^k\right\},\frac{2^k}{\lambda}\right)\\&\le RK\sup_{\alpha\in(0,\infty)}\varphi\left(\left\{x\in\Omega:S(f)(x)>\alpha\right\},\frac{\alpha}{\lambda}\right).\end{aligned}$$

This implies that

$$\|f\|_{WH^{\varphi,q,S}_{\rm at}(\Omega)}\lesssim\|f\|_{WH^S_\varphi(\Omega)},$$

which completes the proof of step 1.

Step 2: Prove $WH^{\varphi,q,S}_{\rm at}(\Omega)\subseteq WH^S_\varphi(\Omega)$. To prove this, let $f\in WH^{\varphi,q,S}_{\rm at}(\Omega)$. Then there exists a sequence of triples, $\{\mu^k,a^k,\nu^k\}_{k\in\mathbb{Z}}$, such that

$$f=\sum_{k\in\mathbb{Z}}\mu^k a^k\quad\text{a.e.},$$

where $\{a^k\}_{k\in\mathbb{Z}}$ are $(\varphi,q)_S$-atoms, $\{\nu^k\}_{k\in\mathbb{Z}}$ are the stopping times associated with $\{a^k\}_{k\in\mathbb{Z}}$, $\mu^k:=\widetilde{C}2^k\|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}$ for any $k\in\mathbb{Z}$, and $\widetilde{C}$ is a positive constant independent of f.

Now we prove that $f\in WH^S_\varphi(\Omega)$. For any fixed $\alpha\in(0,\infty)$, let $k_0\in\mathbb{Z}$ be such that $2^{k_0}\le\alpha<2^{k_0+1}$. Then by arguments similar to those used in the estimations of (3.4), (3.5), and (3.6), we find that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}&\varphi\left(\left\{x\in\Omega:S(f)(x)>\alpha\right\},\frac{\alpha}{\lambda}\right)\\&\quad\le\varphi\left(\left\{x\in\Omega:\sum_{k=-\infty}^{k_0-1}\mu^kS(a^k)(x)>2^{k_0-1}\right\},\frac{2^{k_0+1}}{\lambda}\right)\\&\qquad+\varphi\left(\left\{x\in\Omega:\sum_{k=k_0}^{\infty}\mu^kS(a^k)(x)>2^{k_0-1}\right\},\frac{2^{k_0+1}}{\lambda}\right)\\&\quad\lesssim\sup_{k\in\mathbb{Z}}\varphi\left(B_{\nu^k},\frac{2^k}{\lambda}\right),\end{aligned}$$

which implies that $\|f\|_{WH^S_\varphi(\Omega)}\lesssim\|f\|_{WH^{\varphi,q,S}_{\rm at}(\Omega)}$ and hence $f\in WH^S_\varphi(\Omega)$. This finishes the proof of Theorem 3.5. □

Remark 3.6.

(i) Let Φ be an Orlicz function. Theorem 3.5 with $q=\infty$, when $\varphi(x,t):=\Phi(t)$ for any $x\in\Omega$ and $t\in(0,\infty)$, was proved by Jiao, Wu, and Peng in [17, Theorem 2.3] under the regularity assumption and the assumptions that $\Phi\in\mathcal{G}_\ell$ for some $\ell\in(0,1]$ and the upper index $q_{\Phi^{-1}}\in(0,\infty)$. However, Theorem 3.5, in this case, only needs $\Phi\in\mathcal{G}_\ell$ for some $\ell\in(0,\infty)$ and the regularity condition. Moreover, Theorem 3.5 includes the q-atomic characterizations for any $q\in(\max\{p_\varphi^+,1\},\infty]$. Thus, Theorem 3.5 generalizes and improves [17, Theorem 2.3].

(ii) Let $p\in(0,\infty)$, and let w be a special weight. If $\varphi(x,t):=w(x)t^p$ for any $x\in\Omega$ and $t\in(0,\infty)$, then Theorems 3.1, 3.2, and 3.5 give the atomic characterizations of weak weighted martingale Hardy spaces, which are also new.

4. Boundedness of σ-sublinear operators

In this section, we first obtain the boundedness of σ-sublinear operators from $WH_\varphi^s(\Omega)$ (resp., $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WP_\varphi(\Omega)$, or $WQ_\varphi(\Omega)$) to $WL_\varphi(\Omega)$, and then clarify relations among these weak martingale Musielak–Orlicz Hardy spaces.

Theorem 4.1. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1, and let $T:WH_\varphi^s(\Omega)\to L^0(\Omega)$ be a σ-sublinear operator satisfying Assumption 1.2(i). If there exists a positive constant C such that, for any $(\varphi,\infty)_s$-atom a and any $t\in(0,\infty)$,*

$$\varphi\big(\{x\in\Omega:\big|T(a)(x)\big|>0\},t\big)\le C\varphi(B_\nu,t),\tag{4.1}$$

where ν is the stopping time associated with a, then there exists a positive constant C such that, for any $f\in WH_\varphi^s(\Omega)$,

$$\|Tf\|_{WL_\varphi(\Omega)}\le C\|f\|_{WH_\varphi^s(\Omega)}.\tag{4.2}$$

Proof. Let $f\in WH_\varphi^s(\Omega)$. By step 2 of the proof of Theorem 3.1, we know that there exists a sequence of $(\varphi,\infty)_s$-atoms $\{a^k\}_{k\in\mathbb{Z}}$, related to stopping times $\{\nu^k\}_{k\in\mathbb{Z}}$, such that, for any $\lambda\in(0,\infty)$,

$$f=\sum_{k\in\mathbb{Z}}\mu^k a^k\quad\text{a.e.}$$

and

$$\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big)\le\sup_{\alpha\in(0,\infty)}\varphi\Big(\{x\in\Omega:s(f)(x)>\alpha\},\frac{\alpha}{\lambda}\Big),$$

where $\mu^k:=\widetilde{C}2^k\|\mathbf{1}_{B_{\nu^k}}\|_{L^\varphi(\Omega)}$ for any $k\in\mathbb{Z}$ and $\widetilde{C}$ is a positive constant independent of f. Thus, in order to prove (4.2), we only need to prove that, for any $\alpha\in(0,\infty)$ and $\lambda\in(0,\infty)$,

$$\varphi\Big(\{x\in\Omega:\big|T(f)(x)\big|>\alpha\},\frac{\alpha}{\lambda}\Big)\lesssim\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big).\tag{4.3}$$

For any fixed $\alpha \in (0,\infty)$, let $k_0 \in \mathbb{Z}$ be such that $2^{k_0} \le \alpha < 2^{k_0+1}$. Then, from the definition of T, it follows that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\varphi\Big(\big\{x\in\Omega : \big|T(f)(x)\big| > \alpha\big\}, \frac{\alpha}{\lambda}\Big)\\
&\quad\lesssim \varphi\Big(\Big\{x\in\Omega : \sum_{k=-\infty}^{k_0-1} \mu^k \big|T(a^k)(x)\big| > \frac{\alpha}{2}\Big\}, \frac{\alpha}{\lambda}\Big)\\
&\qquad + \varphi\Big(\Big\{x\in\Omega : \sum_{k=k_0}^{\infty} \mu^k \big|T(a^k)(x)\big| > \frac{\alpha}{2}\Big\}, \frac{\alpha}{\lambda}\Big)\\
&\quad\lesssim \varphi\Big(\Big\{x\in\Omega : \sum_{k=-\infty}^{k_0-1} \mu^k \big|T(a^k)(x)\big| > 2^{k_0-1}\Big\}, \frac{2^{k_0+1}}{\lambda}\Big)\\
&\qquad + \varphi\Big(\Big\{x\in\Omega : \sum_{k=k_0}^{\infty} \mu^k \big|T(a^k)(x)\big| > 2^{k_0-1}\Big\}, \frac{2^{k_0+1}}{\lambda}\Big)\\
&\quad=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Thus, to show (4.3), we only need to estimate I_1 and I_2, respectively.

To estimate I_1, we consider two cases.

Case 1: $q \in (1,\infty) \cap (p_\varphi^+,\infty)$. In this case, for any $\ell \in (0, 1-\frac{p_\varphi^+}{q})$, by the Hölder inequality, the fact that $\sum_{k=-\infty}^{k_0-1} 2^{k\ell q'} = 2^{(k_0-1)\ell q'}(1-2^{-\ell q'})^{-1}$, the monotone convergence theorem, and the boundedness of T, we know that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \frac{1}{2^{(k_0-1)q}} \int_\Omega \Big[\sum_{k=-\infty}^{k_0-1} \mu^k \big|T(a^k)(x)\big|\Big]^q \varphi\Big(x, \frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}\\
&\lesssim \frac{1}{2^{(k_0-1)q}} \Big(\sum_{k=-\infty}^{k_0-1} 2^{k\ell q'}\Big)^{\frac{q}{q'}}\\
&\quad\times \int_\Omega \sum_{k=-\infty}^{k_0-1} 2^{-k\ell q}(\mu^k)^q \big|T(a^k)(x)\big|^q \varphi\Big(x, \frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}\\
&\lesssim 2^{-q(k_0-1)(1-\ell)}(1-2^{-\ell q'})^{-q/q'} \sum_{k=-\infty}^{k_0-1} 2^{-k\ell q}(\mu^k)^q\\
&\quad\times \int_\Omega \big|s(a^k)(x)\big|^q \varphi\Big(x, \frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}.
\end{aligned}$$

From this, $q(1-\ell) > p_\varphi^+$ and the fact that, for any $k \in \mathbb{Z}$, a^k is a $(\varphi,\infty)_s$-atom, we deduce that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{I}_1 &\lesssim 2^{-q(k_0-1)(1-\ell)}(1-2^{-\ell q'})^{-q/q'}\\
&\quad\times \sum_{k=-\infty}^{k_0-1} 2^{-k\ell q}(\mu^k)^q \big\|s(a^k)\big\|_{L^\infty(B_{\nu^k})}^q \varphi\Big(B_{\nu^k}, \frac{2^{k_0+1}}{\lambda}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{-q(k_0-1)(1-\ell)}(1-2^{-\ell q'})^{-q/q'}\\
&\quad\times\sum_{k=-\infty}^{k_0-1}2^{-k\ell q}2^{kq}2^{(k_0+1-k)p_\varphi^+}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big)\\
&\lesssim (1-2^{-\ell q'})^{-q/q'}[1-2^{p_\varphi^+-(1-\ell)q}]\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big).
\end{aligned}
$$

Letting $\ell := \frac{1}{2}(1-\frac{p_\varphi^+}{q})$ in the above inequality, we conclude that, for any $\lambda\in(0,\infty)$,

$$
\mathrm{I}_1\lesssim\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big). \tag{4.4}
$$

Case 2: $q\in(0,1]\cap(p_\varphi^+,\infty)$. From the boundedness of T and the uniformly upper type p_φ^+ property of φ, it follows that, for any $\lambda\in(0,\infty)$,

$$
\begin{aligned}
\mathrm{I}_1&\lesssim\frac{1}{2^{(k_0-1)q}}\int_\Omega\Big[\sum_{k=-\infty}^{k_0-1}\mu^k\big|T(a^k)(x)\big|\Big]^q\varphi\Big(x,\frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}\\
&\lesssim\frac{1}{2^{(k_0-1)q}}\sum_{k=-\infty}^{k_0-1}(\mu^k)^q\int_\Omega\big|T(a^k)(x)\big|^q\varphi\Big(x,\frac{2^{k_0+1}}{\lambda}\Big)\,d\mathbb{P}\\
&\lesssim\frac{1}{2^{(k_0-1)q}}\sum_{k=-\infty}^{k_0-1}(\mu^k)^q\big\|s(a^k)\big\|_{L^\infty(\Omega)}^q\varphi\Big(B_{\nu^k},\frac{2^{k_0+1}}{\lambda}\Big)\\
&\lesssim\frac{1}{2^{(k_0-1)q}}\sum_{k=-\infty}^{k_0-1}2^{kq}2^{(k_0+1-k)p_\varphi^+}\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big)\\
&\sim\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big).
\end{aligned}\tag{4.5}
$$

Now we estimate I_2. Clearly,

$$
\begin{aligned}
&\Big\{x\in\Omega:\sum_{k=k_0}^{\infty}\mu^k\big|T(a^k)(x)\big|>2^{k_0-1}\Big\}\\
&\qquad\subseteq\bigcup_{k=k_0}^{\infty}\big\{x\in\Omega:\big|T(a^k)(x)\big|>0\big\}.
\end{aligned}
$$

Combining this, (4.1) and the fact that φ is of uniformly lower type p_φ^- and of uniformly upper type p_φ^+, we find that, for any $\lambda\in(0,\infty)$,

$$
\begin{aligned}
\mathrm{I}_2&\lesssim\sum_{k=k_0}^{\infty}\varphi\Big(\big\{x\in\Omega:\big|T(a^k)(x)\big|>0\big\},\frac{2^{k_0+1}}{\lambda}\Big)\\
&\lesssim 2^{p_\varphi^+}\sum_{k=k_0}^{\infty}\varphi\Big(B_{\nu^k},\frac{2^{k_0}}{\lambda}\Big)\lesssim\sum_{k=k_0}^{\infty}2^{(k_0-k)p_\varphi^-}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big)\lesssim\sup_{k\in\mathbb{Z}}\varphi\Big(B_{\nu^k},\frac{2^k}{\lambda}\Big),
\end{aligned}
$$

which, together with (4.4) and (4.5), further implies that (4.3) holds true. This finishes the proof of Theorem 4.1. □

Using Theorems 3.2 and 3.5, we can also show that the σ-sublinear operator T is bounded from $WP_\varphi(\Omega)$ (resp., $WQ_\varphi(\Omega)$, $WH^S_\varphi(\Omega)$, or $WH^M_\varphi(\Omega)$) to $WL_\varphi(\Omega)$. The proofs are similar to that of Theorem 4.1. We omit the details.

Theorem 4.2. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1, and let $T : WQ_\varphi(\Omega) \to L^0(\Omega)$ (resp., $T : WP_\varphi(\Omega) \to L^0(\Omega)$) be a σ-sublinear operator satisfying Assumption 1.2(ii) (resp., Assumption 1.2(iii)). If there exists a positive constant C such that, for any $(\varphi,\infty)_S$-atom (resp., $(\varphi,\infty)_M$-atom) a and any $t \in (0,\infty)$,*

$$\varphi\big(\big\{x \in \Omega : \big|T(a)(x)\big| > 0\big\}, t\big) \le C\varphi(B_\nu, t), \tag{4.6}$$

where ν is the stopping time associated with a, then there exists a positive constant C such that, for any $f \in WQ_\varphi(\Omega)$ (resp., $f \in WP_\varphi(\Omega)$),

$$\|Tf\|_{WL_\varphi(\Omega)} \le C\|f\|_{WQ_\varphi(\Omega)} \quad \big[\text{resp., } \|Tf\|_{WL_\varphi(\Omega)} \le C\|f\|_{WP_\varphi(\Omega)}\big].$$

Theorem 4.3. *Let $\varphi \in \mathbb{S}^-$ be a Musielak–Orlicz function satisfying Assumption 1.1, and let $T : WH^M_\varphi(\Omega) \to L^0(\Omega)$ (resp., $T : WH^S_\varphi(\Omega) \to L^0(\Omega)$) be a σ-sublinear operator satisfying Assumption 1.2(ii) (resp., Assumption 1.2(iii)). If the stochastic basis $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular and there exists a positive constant C such that, for any $(\varphi,\infty)_S$-atom (resp., $(\varphi,\infty)_M$-atom) a and any $t \in (0,\infty)$,*

$$\varphi\big(\big\{x \in \Omega : \big|T(a)(x)\big| > 0\big\}, t\big) \le C\varphi(B_\nu, t),$$

where ν is the stopping time associated with a, then there exists a positive constant C such that, for any $f \in WH^M_\varphi(\Omega)$ (resp., $f \in WH^S_\varphi(\Omega)$),

$$\|Tf\|_{WL_\varphi(\Omega)} \le C\|f\|_{WH^M_\varphi(\Omega)} \quad \big[\text{resp., } \|Tf\|_{WL_\varphi(\Omega)} \le C\|f\|_{WH^S_\varphi(\Omega)}\big].$$

Remark 4.4.

(i) For any given $p \in (0,\infty)$, when $\varphi(x,t) := t^p$ for any $x \in \Omega$ and $t \in (0,\infty)$, Theorem 4.1 for Vilenkin martingales was originally obtained by Weisz [35, Theorem 2]. Then Theorems 4.1 and 4.2, in this case, were proved by Hou and Ren [14, Theorems 4, 5, and 6]. Observe that the assumptions of Theorem 4.1 in this case are weaker than those of [14, Theorem 4]. Indeed, the assumptions of [14, Theorem 4] require that T be bounded on $L^q(\Omega)$ for some $q \in [1,2] \cap (p,\infty)$ and that (4.1) holds true in this case. Therefore, to prove our claim, we only need to show that the boundedness of T on $L^q(\Omega)$ for some $q \in [1,2] \cap (p,\infty)$ implies the boundedness of T from $H^s_q(\Omega)$ to $L^q(\Omega)$. This follows immediately from the well-known fact that, for any $f \in L^q(\Omega)$ with $q \in (0,2]$ (see [34, Theorem 2.11(i)]),

$$\|f\|_{L^q(\Omega)} \le C\|f\|_{H^s_q(\Omega)},$$

where C is a positive constant independent of f. Similarly, we can also deduce that the assumptions of Theorem 4.2 in this case are weaker than those of [14, Theorems 5 and 6]. Thus, Theorems 4.1 and 4.2 generalize and improve [14, Theorem 4] and [14, Theorems 5 and 6], respectively.

(ii) Let Φ be an Orlicz function. Theorems 4.1 and 4.2 when $\varphi(x,t) := \Phi(t)$ for any $x \in \Omega$ and $t \in (0,\infty)$ were proved by Jiao, Wu, and Peng in [17, Theorem 3.1 and Remark 3.2] under the assumptions that $\Phi \in \mathcal{G}_\ell$ for some $\ell \in (0,1]$, $p_{\Phi^{-1}} \in (1,\infty)$ and $q_{\Phi^{-1}} \in (0,\infty)$ (see (1.2) for the definitions of $p_{\Phi^{-1}}$ and $q_{\Phi^{-1}}$). However, on the assumption on Φ, Theorems 4.1 and 4.2 in this case only need that $\Phi \in \mathcal{G}_\ell$ for some $\ell \in (0,\infty)$. Thus, in this sense, Theorems 4.1 and 4.2 essentially improve and generalize [17, Theorem 3.1 and Remark 3.2], respectively.

(iii) Replacing Assumption 1.2 by Assumption 1.B, Yang [39, Theorems 4.2, 4.3, and 4.4] also proved Theorems 4.1 and 4.2. Clearly, Assumption 1.2 is quite weaker than Assumption 1.B. Thus, Theorems 4.1 and 4.2 essentially improve [39, Theorem 4.2] and [39, Theorems 4.3 and 4.4], respectively. In particular, Theorem 4.3 is new.

(iv) Recall that, in [35, Theorem 2] and [14, Theorems 4, 5, and 6], it was proved that the σ-sublinear operator T, originally defined on $L^p(\Omega)$ for some $p \in [1,2]$, is bounded from $WH_p^s(\Omega)$ (or $WP_p(\Omega)$ or $WQ_p(\Omega)$) to $WL^p(\Omega)$. These results might be *problematic* because $L^p(\Omega)$ is not dense in $WL^p(\Omega)$ (or $WP_p(\Omega)$ or $WQ_p(\Omega)$) and hence T cannot automatically extend to the whole $WL^p(\Omega)$ (or $WP_p(\Omega)$ or $WQ_p(\Omega)$). The same gap appears in very recent articles [17, Theorem 3.1 and Remark 3.2], [39, Theorems 4.2, 4.3, and 4.4], and [41, Theorems 4.1 and 4.3]. (We thank the referee for reminding us of this gap.)

(v) Let $p \in (0,\infty)$, and let w be a special weight. If, for any $x \in \Omega$ and $t \in (0,\infty)$,

$$\varphi(x,t) := w(x)t^p,$$

then Theorems 4.1, 4.2, and 4.3 give the boundedness of σ-sublinear operators from weak weighted martingale Hardy spaces to weak weighted Lebesgue spaces, which are also new.

The following weighted martingale inequalities come from Bonami and Lépingle [6, Thèoréme 1] and Long [28, Remark 6.6.12, Theorems 6.6.11 and 6.6.12].

Theorem 4.5. *Let w be a special weight.*

(i) *If $w \in \mathbb{A}_\infty(\Omega) \cap \mathbb{S}(\Omega)$ and $p \in [1,\infty)$, then there exists a positive constant C such that, for any $f \in H_p^M(\Omega, w\, d\mathbb{P})$,*

$$\frac{1}{C}\|f\|_{H_p^M(\Omega,w\,d\mathbb{P})} \le \|f\|_{H_p^S(\Omega,w\,d\mathbb{P})} \le C\|f\|_{H_p^M(\Omega,w\,d\mathbb{P})}. \tag{4.7}$$

(ii) *If $w \in \mathbb{S}^-(\Omega)$ and $p \in [2,\infty)$, then there exists a positive constant C such that, for any $f \in H_p^S(\Omega, w\, d\mathbb{P})$,*

$$\|f\|_{H_p^s(\Omega,w\,d\mathbb{P})} \le C\|f\|_{H_p^S(\Omega,w\,d\mathbb{P})}. \tag{4.8}$$

(iii) *If $w \in \mathbb{S}^+(\Omega)$ and $p \in (0,2]$, then there exists a positive constant C such that, for any $f \in H_p^s(\Omega, w\, d\mathbb{P})$,*

$$\|f\|_{H_p^S(\Omega,w\,d\mathbb{P})} \le C\|f\|_{H_p^s(\Omega,w\,d\mathbb{P})}. \tag{4.9}$$

(iv) *If $w \in \mathbb{A}_\infty(\Omega) \cap \mathbb{S}(\Omega)$ and $p \in (0,2]$, then there exists a positive constant C such that, for any $f \in H_p^s(\Omega, w\,d\mathbb{P})$,*

$$\|f\|_{H_p^M(\Omega,w\,d\mathbb{P})} \le C\|f\|_{H_p^s(\Omega,w\,d\mathbb{P})}. \tag{4.10}$$

Theorem 4.6. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1.*

(i) *If $\varphi \in \mathbb{S}^+(\Omega)$ and $p_\varphi^+ \in (0,2)$, then there exists a positive constant C such that, for any $f \in WH_\varphi^s(\Omega)$,*

$$\|f\|_{WH_\varphi^S(\Omega)} \le C\|f\|_{WH_\varphi^s(\Omega)}. \tag{4.11}$$

(ii) *If $\varphi \in \mathbb{A}_\infty(\Omega) \cap \mathbb{S}(\Omega)$ and $p_\varphi^+ \in (0,2)$, then there exists a positive constant C such that, for any $f \in WH_\varphi^s(\Omega)$,*

$$\|f\|_{WH_\varphi^M(\Omega)} \le C\|f\|_{WH_\varphi^s(\Omega)}. \tag{4.12}$$

(iii) *There exists a positive constant C such that, for any $f \in WP_\varphi(\Omega)$ (resp., $f \in WQ_\varphi(\Omega)$),*

$$\|f\|_{WH_\varphi^M(\Omega)} \le C\|f\|_{WP_\varphi(\Omega)} \quad \big(\textit{resp.}, \ \|f\|_{WH_\varphi^S(\Omega)} \le C\|f\|_{WQ_\varphi(\Omega)}\big). \tag{4.13}$$

(iv) *If $\varphi \in \mathbb{A}_\infty(\Omega) \cap \mathbb{S}(\Omega)$, then there exists a positive constant C such that, for any $f \in WP_\varphi(\Omega)$ (resp., $f \in WQ_\varphi(\Omega)$),*

$$\|f\|_{WH_\varphi^S(\Omega)} \le C\|f\|_{WP_\varphi(\Omega)} \qquad \textit{and} \qquad \|f\|_{WH_\varphi^s(\Omega)} \le C\|f\|_{WP_\varphi(\Omega)}$$
$$\big(\textit{resp.}, \ \|f\|_{WH_\varphi^M(\Omega)} \le C\|f\|_{WQ_\varphi(\Omega)}\big). \tag{4.14}$$

(v) *If $\varphi \in \mathbb{S}^-(\Omega)$, then there exists a positive constant C such that, for any $f \in WQ_\varphi(\Omega)$,*

$$\|f\|_{WH_\varphi^s(\Omega)} \le C\|f\|_{WQ_\varphi(\Omega)}. \tag{4.15}$$

(vi) *If $\varphi \in \mathbb{A}_\infty(\Omega) \cap \mathbb{S}(\Omega)$ and $p_\varphi^+ \in (0,2)$, then there exists a positive constant C such that, for any $f \in WQ_\varphi(\Omega)$,*

$$\frac{1}{C}\|f\|_{WQ_\varphi(\Omega)} \le \|f\|_{WP_\varphi(\Omega)} \le C\|f\|_{WQ_\varphi(\Omega)}. \tag{4.16}$$

Moreover, if $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular and $\varphi \in \mathbb{A}_\infty(\Omega)$, then

$$WH_\varphi^s(\Omega) = WH_\varphi^M(\Omega) = WH_\varphi^S(\Omega) = WP_\varphi(\Omega) = WQ_\varphi(\Omega).$$

Proof. In order to prove (4.11) and (4.12), we use Theorem 4.1 with the operator $T := S$ or M. From Definition 2.7(i), it follows that, for any $(\varphi, q)_s$-atom a,

$$\begin{aligned} 0 &\le \mathbf{1}_{\{x\in\Omega:\nu(x)=\infty\}}\big[S(a)\big]^2 = \mathbf{1}_{\{x\in\Omega:\nu(x)=\infty\}}\sum_{n\in\mathbb{N}}|d_n a|^2 \\ &\le \sum_{n\in\mathbb{N}}\mathbf{1}_{\{x\in\Omega:\nu(x)\ge n\}}|d_n a|^2 = 0, \end{aligned}$$

which implies that $\{x \in \Omega : S(f)(x) > 0\} \subseteq B_\nu$ and hence the operator S satisfies (4.1). Clearly, the Doob maximal operator M also satisfies (4.1). By this, (4.9), (4.10), and Theorem 4.1, we obtain (4.11) and (4.12).

Inequalities (4.13) follow immediately from the definitions of $WP_\varphi(\Omega)$ and $WQ_\varphi(\Omega)$. To prove inequalities (4.14) and (4.15), we apply Theorem 4.2, respectively, to the operator $T = S$, M or s. Observe that operators M, S, and s all satisfy the condition (4.6). From (4.7) and (4.8), it follows that, for any $q \in [2,\infty)$ and $t \in (0,\infty)$,

$$s : H_q^M\big(\Omega, \varphi(\cdot,t)\, d\mathbb{P}\big) \to L^q\big(\Omega, \varphi(\cdot,t)\big)$$

is bounded. Combining this, (4.7), (4.8), and Theorem 4.2, we obtain (4.14) and (4.15).

To show inequalities (4.16), let $f \in WQ_\varphi(\Omega)$. For any $\varepsilon \in (0,\infty)$, there exists an adapted process $\{\lambda_n^{(1)}\}_{n\in\mathbb{Z}_+} \in \Lambda[WQ_\varphi](f)$ such that, for any $n \in \mathbb{N}$,

$$S_n(f) \le \lambda_{n-1}^{(1)} \qquad \text{and} \qquad \|\lambda_\infty^{(1)}\|_{WL_\varphi(\Omega)} \le \|f\|_{WQ_\varphi(\Omega)} + \varepsilon.$$

By this, we find that, for any $n \in \mathbb{N}$,

$$|f_n| \le M_{n-1}(f) + |d_n f| \le M_{n-1}(f) + S_n(f) \le M_{n-1}(f) + \lambda_{n-1}^{(1)}.$$

Combining this and (4.14), we know that

$$\|f\|_{WP_\varphi(\Omega)} \lesssim \|f\|_{WH_\varphi^M(\Omega)} + \|\lambda_\infty^{(1)}\|_{WL_\varphi(\Omega)} \lesssim \|f\|_{WQ_\varphi(\Omega)} + \varepsilon,$$

which, together with letting $\varepsilon \to 0$, implies that $\|f\|_{WP_\varphi(\Omega)} \lesssim \|f\|_{WQ_\varphi(\Omega)}$ and $f \in WP_\varphi(\Omega)$. Moreover, for any $\varepsilon \in (0,\infty)$, there exists an adapted process $\{\lambda_n^{(2)}\}_{n\in\mathbb{Z}_+} \in \Lambda[WP_\varphi](f)$ such that, for any $n \in \mathbb{N}$,

$$|f_n| \le \lambda_{n-1}^{(2)} \qquad \text{and} \qquad \|\lambda_\infty^{(2)}\|_{WL_\varphi(\Omega)} \lesssim \|f\|_{WP_\varphi(\Omega)} + \varepsilon,$$

which implies that, for any $n \in \mathbb{N}$,

$$S_n(f) \le S_{n-1}(f) + |d_n f| \le S_{n-1}(f) + 2\lambda_{n-1}^{(2)}.$$

From this and (4.14), it follows that

$$\|f\|_{WQ_\varphi(\Omega)} \lesssim \|f\|_{WH_\varphi^S(\Omega)} + \|\lambda_\infty^{(1)}\|_{WL_\varphi(\Omega)} \lesssim \|f\|_{WP_\varphi(\Omega)} + \varepsilon$$

and hence, by letting $\varepsilon \to 0$, $\|f\|_{WQ_\varphi(\Omega)} \lesssim \|f\|_{WP_\varphi(\Omega)}$. Thus, we conclude that the inequalities (4.16) hold true.

Finally, assume that $\{\mathcal{F}_n\}_{n\in\mathbb{Z}_+}$ is regular. From this and $\varphi \in \mathbb{A}_\infty(\Omega)$, it follows that $\varphi \in \mathbb{S}$ (see [28, Proposition 6.3.7]). Then, by Theorems 3.2 and 3.5, we have

$$WQ_\varphi(\Omega) = WH_\varphi^S(\Omega) \qquad \text{and} \qquad WP_\varphi(\Omega) = WH_\varphi^M(\Omega).$$

Combining this and (4.14), we know that

$$WQ_\varphi(\Omega) = WH_\varphi^S(\Omega) = WH_\varphi^M(\Omega) = WP_\varphi(\Omega) \subseteq WH_\varphi^s(\Omega).$$

Thus, to complete the proof of this theorem, we only need to show $WH_\varphi^s(\Omega) \subseteq WH_\varphi^S(\Omega)$. By the regularity and [34, Lemma 2.18], we have $|d_n f|^2 \lesssim \mathbb{E}_{n-1}(|d_n f|^2)$ for any $n \in \mathbb{N}$. From this, it follows that $S(f) \lesssim s(f)$ and hence

$$\|f\|_{WH_\varphi^S(\Omega)} \lesssim \|f\|_{WH_\varphi^s(\Omega)}.$$

Thus, $WH_\varphi^s(\Omega) \subseteq WH_\varphi^S(\Omega)$, which completes the proof of Theorem 4.6. $\square$

Remark 4.7.

(i) For any given $p \in (0,\infty)$, if $\varphi(x,t) := t^p$ for any $x \in \Omega$ and $t \in (0,\infty)$, then Theorem 4.6 in this case coincides with [14, Theorem 7].

(ii) Let Φ be an Orlicz function. Theorem 4.6 when $\varphi(x,t) := \Phi(t)$ for any $x \in \Omega$ and $t \in (0,\infty)$ was proved by Jiao, Wu, and Peng in [17, Theorem 3.3] under some slightly stronger assumptions. Indeed, [17, Theorem 3.3] needs the condition that Φ be of lower type p_Φ^- for some $p_\Phi^- \in (0,1]$ and of upper type $p_\Phi^+ := 1$, and $q_{\Phi^{-1}} \in (0,\infty)$. However, the conclusions (4.11), (4.12), and (4.16) of Theorem 4.6 only need $p_\Phi^+ \in (0,2)$. Thus, Theorem 4.6 essentially generalizes and improves [17, Theorem 3.3].

(iii) Under Assumption 1.B, Yang [39, Theorem 4.5] also proved the martingale inequalities among spaces $WH_\varphi^M(\Omega)$, $WH_\varphi^S(\Omega)$, $WH_\varphi^s(\Omega)$, $WP_\varphi(\Omega)$, and $WQ_\varphi(\Omega)$. By Assumption 1.B and Remark 2.2(iii), we know that Theorem 4.6 essentially improves [39, Theorem 4.5].

(iv) Similarly to the discussion of Remark 4.4(iv), Theorem 4.6 is also new on weak weighted martingale (Orlicz) Hardy spaces.

5. Convergence theorems

In this section, we obtain bounded convergence theorems and dominated convergence theorems on weak Musielak–Orlicz spaces $WL_\varphi(\Omega)$. We begin with the following notion of the absolute continuity of the weak Musielak–Orlicz norm (see also the monograph [4] of Bennett and Sharpley for more details on the absolute continuity of Banach function norms).

Definition 5.1. Let φ be a Musielak–Orlicz function. A function f in $WL_\varphi(\Omega)$ is said to have *absolutely continuous quasinorm* if

$$\lim_{n\to\infty} \|f\mathbf{1}_{\{x\in\Omega:|f(x)|>n\}}\|_{WL_\varphi(\Omega)} = 0.$$

But, not every function in $WL_\varphi(\Omega)$ has absolutely continuous quasinorm even when φ satisfies Assumption 1.1. For example, let $\Omega := (0,1]$, and let $\mathbb{P}$ be the Lebesgue measure. For any given $p \in (0,\infty)$ and any $x \in (0,1]$ and $t \in (0,\infty)$, let $\varphi(x,t) := t^p$ and $f(x) := x^{-\frac{1}{p}}$ (see, e.g., [27, Example 2.5]). Via a simple calculation, we know that $\|f\|_{WL_\varphi(\Omega)} = 1$ and that φ is of uniformly lower type p and of uniformly upper type p. However, for any $n \in \mathbb{N}$, we have $\|f\mathbf{1}_{\{x\in\Omega:|f(x)|>n\}}\|_{WL_\varphi(\Omega)} = 1$. Thus, f has no absolutely continuous quasinorm.

Definition 5.2. Let φ be a Musielak–Orlicz function. The *absolutely continuous part of the weak Musielak–Orlicz space* $W\mathcal{L}_\varphi(\Omega)$ is defined as follows:

$$W\mathcal{L}_\varphi(\Omega) := \left\{f \in WL_\varphi(\Omega) : \lim_{n\to\infty} \|f\mathbf{1}_{\{x\in\Omega:|f(x)|>n\}}\|_{WL_\varphi(\Omega)} = 0\right\}.$$

The proof of the following lemma is similar to that of [4, Theorem 3.8]. For the convenience of the reader, we give some details here.

Lemma 5.3. *Let φ be a Musielak–Orlicz function with uniformly upper type p_φ^+ for some $p_\varphi^+ \in (0,\infty)$.*

(i) *For any measurable functions* $g \in WL_\varphi(\Omega)$ *and* $h \in W\mathcal{L}_\varphi(\Omega)$*, if* $|g|$ *is pointwise* $\mathbb{P}$*-a.e. bounded by* $|h|$*, then* $g \in W\mathcal{L}_\varphi(\Omega)$*.*

(ii) *If* $g,\ h \in W\mathcal{L}_\varphi(\Omega)$*, then, for any complex numbers* c_1 *and* c_2*,* $c_1 g + c_2 h \in W\mathcal{L}_\varphi(\Omega)$*.*

(iii) *If* $\{g_n\}_{n\in\mathbb{N}} \subset W\mathcal{L}_\varphi(\Omega)$ *and there exists a measurable function* g *such that* $\lim_{n\to\infty} \|g_n - g\|_{WL_\varphi(\Omega)} = 0$*, then* $g \in W\mathcal{L}_\varphi(\Omega)$*.*

Proof. It is clear that (i) and (ii) hold true. Now we prove (iii). For any fixed $\varepsilon \in (0,\infty)$, by the condition that $\lim_{n\to\infty} \|g_n - g\|_{WL_\varphi(\Omega)} = 0$, we know that there exists a positive integer N_0 such that, for any $n \in \mathbb{N} \cap (N_0,\infty)$,

$$\|g_n - g\|_{WL_\varphi(\Omega)} < \varepsilon. \tag{5.1}$$

Moreover, for any fixed $n_0 \in \mathbb{N}\cap(N_0,\infty)$, since $g_{n_0} \in W\mathcal{L}_\varphi(\Omega)$, we find that there exists a positive integer k_0 such that

$$\|g_{n_0} \mathbf{1}_{\{x\in\Omega:|g_{n_0}(x)|>k_0\}}\|_{WL_\varphi(\Omega)} < \varepsilon. \tag{5.2}$$

Combining this and the definition of $WL_\varphi(\Omega)$, we conclude that

$$\sup_{\alpha\in(0,\infty)} \int_{\{x\in\Omega:|g_{n_0}(x)|>\alpha\}\cap\{x\in\Omega:|g_{n_0}(x)|>k_0\}} \varphi\Big(x, \frac{\alpha}{\varepsilon}\Big)\, d\mathbb{P} \le 1.$$

From this, it follows that

$$\sup_{\alpha\in(k_0,\infty)} \int_{\{x\in\Omega:|g_{n_0}(x)|>\alpha\}} \varphi\Big(x, \frac{\alpha}{\varepsilon}\Big)\, d\mathbb{P} \le 1. \tag{5.3}$$

On the other hand, since $n_0 \in \mathbb{N} \cap (N_0,\infty)$, from (5.1), it follows that

$$\begin{aligned}
&\int_{\{x\in\Omega:|g_{n_0}(x)-g(x)|>k_0\}} \varphi\Big(x, \frac{k_0}{\varepsilon}\Big)\, d\mathbb{P} \\
&\quad \le \sup_{\alpha\in(0,\infty)} \int_{\{x\in\Omega:|g_{n_0}(x)-g(x)|>\alpha\}} \varphi\Big(x, \frac{\alpha}{\varepsilon}\Big)\, d\mathbb{P} \le 1,
\end{aligned}$$

which, together with (5.3), implies that, for any $k \in \mathbb{N} \cap (2k_0,\infty)$,

$$\begin{aligned}
&\sup_{\alpha\in(0,\infty)} \int_{\{x\in\Omega:|g_{n_0}(x)|>\alpha\}\cap\{x\in\Omega:|g_{n_0}(x)-g(x)|>k/2\}} \varphi\Big(x, \frac{\alpha}{\varepsilon}\Big)\, d\mathbb{P} \\
&\quad \le \max\Big\{ \sup_{\alpha\in(0,k_0]} \int_{\{x\in\Omega:|g_{n_0}(x)-g(x)|>k/2\}} \varphi\Big(x, \frac{\alpha}{\varepsilon}\Big)\, d\mathbb{P}, \\
&\qquad \sup_{\alpha\in(k_0,\infty)} \int_{\{x\in\Omega:|g_{n_0}(x)|>\alpha\}} \varphi\Big(x, \frac{\alpha}{\varepsilon}\Big)\, d\mathbb{P} \Big\} \\
&\quad \le \max\Big\{ \int_{\{x\in\Omega:|g_{n_0}(x)-g(x)|>k_0\}} \varphi\Big(x, \frac{k_0}{\varepsilon}\Big)\, d\mathbb{P}, 1 \Big\} \le 1.
\end{aligned}$$

By this and the definition of $WL_\varphi(\Omega)$, we find that, for any $k \in \mathbb{N} \cap (2k_0,\infty)$,

$$\|g_{n_0} \mathbf{1}_{\{x\in\Omega:|g_{n_0}(x)-g(x)|>k/2\}}\|_{WL_\varphi(\Omega)} < \varepsilon.$$

Combining this, Remark 2.2(i), (5.1), and (5.2), we conclude that, for any $k \in \mathbb{N} \cap (2k_0, \infty)$,

$$\begin{aligned}\|g\mathbf{1}_{\{x\in\Omega:|g(x)|>k\}}\|_{WL_\varphi(\Omega)} &\lesssim \|g_{n_0} - g\|_{WL_\varphi(\Omega)} + \|g_{n_0}\mathbf{1}_{\{x\in\Omega:|g_{n_0}(x)|>k/2\}}\|_{WL_\varphi(\Omega)} \\ &\quad + \|g_{n_0}\mathbf{1}_{\{x\in\Omega:|g_{n_0}(x)-g(x)|>k/2\}}\|_{WL_\varphi(\Omega)} \\ &\lesssim \varepsilon.\end{aligned}$$

Thus, we have $\lim_{k\to\infty} \|g\mathbf{1}_{\{x\in\Omega:|g(x)|>k\}}\|_{WL_\varphi(\Omega)} = 0$, which completes the proof of (iii) and hence of Lemma 5.3. □

Remark 5.4. Let φ be a Musielak–Orlicz function with uniformly upper type p_φ^+ for some $p_\varphi^+ \in (0, \infty)$. From Lemma 5.3, we deduce that $W\mathcal{L}_\varphi(\Omega)$ is a closed subspace of $WL_\varphi(\Omega)$.

The following lemma is just [23, Lemma 3.3(ii)].

Lemma 5.5. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. Then, for any $f \in WL_\varphi(\Omega)$ satisfying $\|f\|_{WL_\varphi(\Omega)} \neq 0$,*

$$\sup_{\alpha\in(0,\infty)} \varphi\Big(\big\{x \in \Omega : \big|f(x)\big| > \alpha\big\}, \frac{\alpha}{\|f\|_{WL_\varphi(\Omega)}}\Big) = 1.$$

For any measurable function f, let

$$\rho_\varphi(f) := \sup_{\alpha\in(0,\infty)} \varphi\big(\big\{x \in \Omega : \big|f(x)\big| > \alpha\big\}, \alpha\big).$$

Lemma 5.6. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. Then, for any measurable functions $\{h_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \|h_n\|_{WL_\varphi(\Omega)} = 0$ if and only if $\lim_{n\to\infty} \rho_\varphi(h_n) = 0$.*

Proof. If $\lim_{n\to\infty} \|h_n\|_{WL_\varphi(\Omega)} = 0$, then, for any fixed $\varepsilon \in (0, 1)$, there exists a positive integer $N_0 \in \mathbb{N}$ such that, for any $n \in \mathbb{N} \cap (N_0, \infty)$, $\|h_n\|_{WL_\varphi(\Omega)} < \varepsilon$. From this, Lemma 5.5 and the fact that φ is of uniformly lower type p_φ^-, we deduce that, for any $n \in \mathbb{N} \cap (N_0, \infty)$,

$$\rho_\varphi(h_n) \lesssim \big[\|h_n\|_{WL_\varphi(\Omega)}\big]^{p_\varphi^-} \sup_{\alpha\in(0,\infty)} \int_{\{x\in\Omega:|h_n(x)|>\alpha\}} \varphi\Big(x, \frac{\alpha}{\|h_n\|_{WL_\varphi(\Omega)}}\Big)\, d\mathbb{P} \lesssim \varepsilon^{p_\varphi^-}.$$

This implies that $\lim_{n\to\infty} \rho_\varphi(h_n) = 0$.

Conversely, if $\lim_{n\to\infty} \|h_n\|_{WL_\varphi(\Omega)} = 0$ is not true, then there exist a constant $\varepsilon_0 \in (0, 1)$ and a sequence $\{h_{n_k}\}_{k\in\mathbb{N}}$ of measurable functions such that, for any $k \in \mathbb{N}$, $\|h_{n_k}\|_{WL_\varphi(\Omega)} \geq \varepsilon_0$. Combining this, Lemma 5.5 and the uniformly upper type p_φ^+ property of φ, we find that, for any $k \in \mathbb{N}$,

$$\begin{aligned}1 &\leq \sup_{\alpha\in(0,\infty)} \varphi\Big(\big\{x \in \Omega : \big|h_{n_k}(x)\big| > \alpha\big\}, \frac{\alpha}{\varepsilon_0}\Big) \\ &\lesssim \varepsilon_0^{-p_\varphi^+} \sup_{\alpha\in(0,\infty)} \varphi\big(\big\{x \in \Omega : \big|h_{n_k}(x)\big| > \alpha\big\}, \alpha\big),\end{aligned}$$

which implies that, for any $k \in \mathbb{N}$, $\rho_\varphi(h_{n_k}) \gtrsim \varepsilon_0^{p_\varphi^+}$. This contradicts

$$\lim_{n\to\infty} \rho_\varphi(h_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} \|h_n\|_{WL_\varphi(\Omega)} = 0,$$

which completes the proof of Lemma 5.6. □

Remark 5.7. Let φ be a Musielak–Orlicz function. Since $\sup_{t\in(0,\infty)} \int_\Omega \varphi(x,t)\,d\mathbb{P} < \infty$, it follows that, for any $t \in (0,\infty)$, $d\widehat{\mathbb{P}}_t := \varphi(\cdot,t)\,d\mathbb{P}$ is a finite measure on $(\Omega,\mathcal{F},\mathbb{P})$. Now we claim that, for any $F \in \mathcal{F}$ and $t \in (0,\infty)$,

$$\widehat{\mathbb{P}}_t(F) = 0 \Longleftrightarrow \mathbb{P}(F) = 0.$$

To show this, it suffices to prove that, for any $t \in (0,\infty)$, $\widehat{\mathbb{P}}_t(F) = 0$ for some $F \in \mathcal{F}$ implies that $\mathbb{P}(F) = 0$. Indeed, for any $t \in (0,\infty)$, $0 = \widehat{\mathbb{P}}_t(F) = \int_F \varphi(\cdot,t)\,d\mathbb{P}$. From this and the fact that $\varphi(\cdot,t)$ is strictly positive, we deduce that $\mathbb{P}(F) = 0$. This proves the above claim.

We now state the following bounded convergence theorem.

Theorem 5.8. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. Let h be a measurable function on Ω, and let $\{h_n\}_{n\in\mathbb{N}} \subset WL_\varphi(\Omega)$ be a sequence of measurable functions such that h_n converges to h almost everywhere on Ω as $n \to \infty$. If there exists a positive constant M such that, for any $n \in \mathbb{N}$, $|h_n(x)| \le M$ for almost every $x \in \Omega$, then*

$$\lim_{n\to\infty} \|h_n - h\|_{WL_\varphi(\Omega)} = 0.$$

Proof. For any fixed $\varepsilon \in (0,\infty)$, let

$$\delta := \min\left\{\left[\frac{\varepsilon}{2C_{(p_\varphi^-)}\|\varphi(\cdot,1)\|_{L^1(\Omega)}}\right]^{1/p_\varphi^-}, \frac{1}{2}\right\};$$

here and thereafter, $C_{(p_\varphi^-)}$ is the positive constant same as in (1.1). For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho_\varphi(h_n - h) &= \sup_{\alpha\in(0,\infty)} \int_{\{x\in\Omega:|h_n(x)-h(x)|>\alpha\}} \varphi(x,\alpha)\,d\mathbb{P} \\
&= \max\left\{ \sup_{\alpha\in(0,\delta]} \int_{\{x\in\Omega:|h_n(x)-h(x)|>\alpha\}} \varphi(x,\alpha)\,d\mathbb{P}, \right. \\
&\qquad \left. \sup_{\alpha\in(\delta,\infty)} \int_{\{x\in\Omega:|h_n(x)-h(x)|>\alpha\}} \varphi(x,\alpha)\,d\mathbb{P} \right\} \\
&=: \max\{\mathrm{J}_{n,1}, \mathrm{J}_{n,2}\}.
\end{aligned}$$

We first estimate $\mathrm{J}_{n,1}$. By the uniformly lower type p_φ^- property of φ, we know that, for any $n \in \mathbb{N}$,

$$\mathrm{J}_{n,1} \le \int_\Omega \varphi(x,\delta)\,d\mathbb{P} \le C_{(p_\varphi^-)}\delta^{p_\varphi^-} \int_\Omega \varphi(x,1)\,d\mathbb{P} < \varepsilon. \tag{5.4}$$

Now we estimate $\mathrm{J}_{n,2}$. Since, for any $n \in \mathbb{N}$, $|h_n|$ is pointwise $\mathbb{P}$-a.e. bounded by M and h_n converges $\mathbb{P}$-a.e. to h as $n \to \infty$, we know that $|h|$ is pointwise $\mathbb{P}$-a.e.

bounded by M. From this, we deduce that, for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathrm{J}_{n,2} &\le \sup_{\alpha\in(\delta,\infty)} \int_{\{x\in\Omega:|h_n(x)-h(x)|>\alpha\}} \varphi\big(x, \big|h_n(x)-h(x)\big|\big)\, d\mathbb{P} \\ &\le \varphi\big(\big\{x\in\Omega : \big|h_n(x)-h(x)\big| > \delta\big\}, 2M\big). \end{aligned} \tag{5.5}$$

Moreover, there exists a measurable set $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 0$ and $h_n \to h$ on E as $n \to \infty$. From this and Remark 5.7, it follows that $\widehat{\mathbb{P}_{2M}}(E) = 0$ and $\widehat{\mathbb{P}_{2M}}(\Omega) < \infty$. Then we have that h_n converges to h in measure $\widehat{\mathbb{P}_{2M}}$; that is, for every $\sigma \in (0,\infty)$,

$$\lim_{n\to\infty} \varphi\big(\big\{x\in\Omega : \big|h_n(x)-h(x)\big| > \sigma\big\}, 2M\big) = 0.$$

Combining this and (5.5), we find that there exists a positive integer N_0 such that, for any $n \in \mathbb{N}\cap(N_0,\infty)$, $\mathrm{J}_{n,2} < \varepsilon$, which, together with (5.4), implies that, for any $n \in \mathbb{N}\cap(N_0,\infty)$, $\rho_\varphi(h_n - h) < \varepsilon$. By this and the arbitrariness of ε, we find that

$$\lim_{n\to\infty} \rho_\varphi(h_n - h) = 0.$$

From this and Lemma 5.6, it follows that $\lim_{n\to\infty} \|h_n - h\|_{WL_\varphi(\Omega)} = 0$, which completes the proof of Theorem 5.8. □

Finally, we establish the following dominated convergence theorem.

Theorem 5.9. *Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions that converges $\mathbb{P}$-a.e. to a measurable function h. Suppose that there exists a measurable function $g \in W\mathcal{L}_\varphi(\Omega)$ such that $|h_n|$ is pointwise $\mathbb{P}$-a.e. bounded by g for any $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \|h_n - h\|_{WL_\varphi(\Omega)} = 0.$$

Proof. For any $\varepsilon \in (0,\infty)$, since $g \in W\mathcal{L}_\varphi(\Omega)$, we deduce that there exists a positive integer N_0 such that

$$\|g\mathbf{1}_{\{x\in\Omega:|g(x)|>N_0\}}\|_{WL_\varphi(\Omega)} < \varepsilon.$$

Combining this, Remark 2.2(i) and the fact that $\{h_n\}_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.e. to h as $n \to \infty$, we obtain

$$\big\|(h_n - h)\mathbf{1}_{\{x\in\Omega:|g(x)|>N_0\}}\big\|_{WL_\varphi(\Omega)} \le \|2g\mathbf{1}_{\{x\in\Omega:|g(x)|>N_0\}}\|_{WL_\varphi(\Omega)} \lesssim \varepsilon. \tag{5.6}$$

On the other hand, notice that $|h_n(x)| \le N_0$ for $\mathbb{P}$-almost every $x \in \{x \in \Omega : |g(x)| \le N_0\}$. Then, by Remark 2.2(i) and Theorem 5.8, we know that there exists a positive integer N such that, for any $n \in \mathbb{N}\cap(N,\infty)$,

$$\big\|(h_n - h)\mathbf{1}_{\{x\in\Omega:|g(x)|\le N_0\}}\big\|_{WL_\varphi(\Omega)} < \varepsilon.$$

From this, (5.6) and Remark 2.2(i), it follows that, for any $n \in \mathbb{N} \cap (N, \infty)$,

$$\begin{aligned}\|h_n - h\|_{WL_\varphi(\Omega)} &\lesssim \left\|(h_n - h)\mathbf{1}_{\{x\in\Omega:|g(x)|>N_0\}}\right\|_{WL_\varphi(\Omega)} \\ &\quad + \left\|(h_n - h)\mathbf{1}_{\{x\in\Omega:|g(x)|\le N_0\}}\right\|_{WL_\varphi(\Omega)} \\ &\lesssim \varepsilon.\end{aligned}$$

By this and the arbitrariness of ε, we have $\lim_{n\to\infty} \|h_n - h\|_{WL_\varphi(\Omega)} = 0$. This finishes the proof of Theorem 5.9. □

Remark 5.10.

(i) Let φ be a Musielak–Orlicz function satisfying Assumption 1.1. We then let

$$W\mathcal{H}_\varphi^s(\Omega) := \left\{f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M} : s(f) \in W\mathcal{L}_\varphi(\Omega)\right\},$$
$$W\mathcal{H}_\varphi^S(\Omega) := \left\{f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M} : S(f) \in W\mathcal{L}_\varphi(\Omega)\right\},$$

and

$$W\mathcal{H}_\varphi^M(\Omega) := \left\{f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M} : M(f) \in W\mathcal{L}_\varphi(\Omega)\right\}.$$

From Remark 5.4 and the sublinearity of the operator s, we deduce that $W\mathcal{H}_\varphi^s(\Omega)$ is a closed subspace of $WH_\varphi^s(\Omega)$. Similarly, $W\mathcal{H}_\varphi^S(\Omega)$ and $W\mathcal{H}_\varphi^M(\Omega)$ are the closed subspaces of $WH_\varphi^S(\Omega)$ and $WH_\varphi^M(\Omega)$, respectively.

If $f \in W\mathcal{H}_\varphi^s(\Omega) \subset WH_\varphi^s(\Omega)$, then, by Theorem 3.1, we have $f \in WH_{\mathrm{at}}^{\varphi,q,s}(\Omega)$. Thus, there exists a sequence of triples, $\{\mu^k, a^k, \nu^k\}_{k\in\mathbb{Z}}$, such that $f = \sum_{k\in\mathbb{Z}} \mu^k a^k$ $\mathbb{P}$-a.e. Now we claim that the sum $\sum_{k=m}^{\ell} \mu^k a^k$ converges to f in $WH_\varphi^s(\Omega)$ as $m \to -\infty$ and $\ell \to \infty$. Indeed, for any $m, \ell \in \mathbb{Z}$ with $m < \ell$, we have

$$f - \sum_{k=m}^{\ell} \mu^k a^k = (f - f^{\nu^{\ell+1}}) + f^{\nu^m}$$

and

$$\left[s(f - f^{\nu^{\ell+1}})\right]^2 = \left[s(f)\right]^2 - \left[s(f^{\nu^{\ell+1}})\right]^2. \tag{5.7}$$

Thus, we obtain $s(f - f^{\nu^{\ell+1}}) \le s(f)$ and $s(f^{\nu^m}) \le s(f)$. From this, (5.7), the fact that, for $\mathbb{P}$-almost every $x \in \Omega$,

$$\lim_{\ell\to\infty} s(f - f^{\nu^{\ell+1}})(x) = 0, \qquad \lim_{m\to-\infty} s(f^{\nu^m})(x) = 0$$

and Theorem 5.9, we deduce that

$$\lim_{\ell\to\infty} \left\|s(f - f^{\nu^{\ell+1}})\right\|_{WL_\varphi(\Omega)} = 0 \quad \text{and} \quad \lim_{m\to-\infty} \left\|s(f^{\nu^m})\right\|_{WL_\varphi(\Omega)} = 0.$$

Combining this, Remark 2.2(i) and the sublinearity of the operator s, we complete the proof of the claim.

(ii) Let φ be as in Theorem 3.5, and let $f \in W\mathcal{H}_\varphi^S(\Omega)$ (resp., $W\mathcal{H}_\varphi^M(\Omega)$). Analogously to (i) of this remark, from Theorem 5.9, we deduce that, in step 1 of the proof of Theorem 3.5, the sum $\sum_{k=m}^{\ell} \mu^k a^k$ converges to f in $WH_\varphi^M(\Omega)$ (resp., $WH_\varphi^S(\Omega)$) as $m \to -\infty$ and $\ell \to \infty$, which may be of independent interest.

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