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DISJOINT HYPERCYCLIC WEIGHTED PSEUDOSHIFT OPERATORS GENERATED BY DIFFERENT SHIFTS

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Abstract. Let I be a countably infinite index set, and let X be a Banach sequence space over I. In this article, we characterize the disjoint hypercyclic and supercyclic weighted pseudoshift operators on X in terms of the weights, the OP-basis, and the shift mappings on I. Also, the shifts on weighted L^p spaces of a directed tree and the operator weighted shifts on $\ell^2(\mathbb{Z}, \mathcal{K})$ are investigated as special cases.

1. Introduction and preliminaries

The study of disjointness in hypercyclicity was initiated in 2007 by Bernal-González [2] and Bès and Peris [6]. Since then, disjoint hypercyclicity has been investigated in many works, including [3]–[5], [19], [20], [22], and [24].

Two new notions, *disjoint hypercyclic operator* and *disjoint supercyclic operator*, are derived from the much older notions of *hypercyclic operator* and *supercyclic operator* in linear dynamics. Let X be a separable infinite-dimensional complex Banach space. We denote by $L(X)$ the set of all continuous and linear operators on X. An operator $T \in L(X)$ is said to be *hypercyclic* if there is some vector $x \in X$ such that the *orbit* $\mathrm{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ (where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$) is dense in X. In such a case, x is called a *hypercyclic vector* for T. Similarly, T is said to be *supercyclic* if there exists an $x \in X$ such that $\mathbb{C} \cdot \mathrm{Orb}(T, x) = \{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in X. (For background

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on hypercyclicity and supercyclicity, see the excellent monographs by Bayart and Matheron [1] and by Grosse-Erdmann and Peris Manguillot [10].)

We say that $N \geq 2$ hypercyclic (resp., supercyclic) operators $T_1, T_2, \ldots, T_N$ acting on the same space X are *disjoint hypercyclic* or *d-hypercyclic* (resp., *d-supercyclic*) if their direct sum $\bigoplus_{m=1}^{N} T_m$ has a hypercyclic (resp., supercyclic) vector of the form $(x, x, \ldots, x)$ in X^N. We call x a *d-hypercyclic* (resp., *d-supercyclic*) vector for $T_1, T_2, \ldots, T_N$. If the set of d-hypercyclic (resp., d-supercyclic) vectors is dense in X, then we say that $T_1, T_2, \ldots, T_N$ are *densely d-hypercyclic* (resp., *densely d-supercyclic*).

In the study of linear dynamics, one large source of examples is the class of *weighted shifts*. In [17] and [18], Salas characterized the hypercyclic and supercyclic weighted shifts on $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$, respectively. The characterizations for weighted shifts on $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ to be disjoint hypercyclic and disjoint supercyclic were provided in [6], [13], and [15]. As generalizations of characterizations for weighted shifts in [17] and [18], Grosse-Erdmann [9] studied the hypercyclicity of weighted pseudoshifts on F-sequence spaces, Hazarika and Arora considered the hypercyclic operator weighted shifts on $\ell^2(\mathbb{Z}, \mathcal{K})$ in [11], and the equivalent conditions for the weighted pseudoshifts and operator weighted shifts to be supercyclic were obtained in [14]. Inspired by above results, in [21] we characterized the disjoint hypercyclic powers of weighted pseudoshifts. And in this article, we will continue our research on the disjointness of these weighted pseudoshifts. Before we proceed further, let us recall some terminology about the sequence spaces and weighted pseudoshifts. (For a comprehensive survey, we recommend Grosse-Erdmann's work [9].)

Definition 1.1 (Sequence space). If we allow an arbitrary, countably infinite set I as an index set, then a *sequence space over* I is a subspace of the space $\omega(I) = \mathbb{C}^I$ of all scalar families $(x_i)_{i \in I}$. The space $\omega(I)$ is endowed with its natural product topology. A *topological sequence space* X *over* I is a sequence space over I that is endowed with a linear topology in such a way that the inclusion mapping $X \hookrightarrow \omega(I)$ is continuous or, equivalently, that every *coordinate functional* $f_i : X \to \mathbb{C}$, $(x_k)_{k \in I} \mapsto x_i (i \in I)$ is continuous. A *Banach* (*Hilbert*, *F-*) *sequence space over* I is a topological sequence space over I that is a Banach (Hilbert, F-) space.

Definition 1.2 (OP-basis). By $(e_i)_{i \in I}$ we denote the canonical unit vectors $e_i = (\delta_{ik})_{k \in I}$ in a topological sequence space X over I. We say that $(e_i)_{i \in I}$ is an *OP-basis* (*Ovsepian–Pełczyński basis*) if $\operatorname{span}\{e_i : i \in I\}$ is a dense subspace of X and the family of *coordinate projections* $x \mapsto x_i e_i$ $(i \in I)$ on X is equicontinuous. Note that in a Banach sequence space over I, the family of coordinate projections is equicontinuous if and only if $\sup_{i \in I} \|e_i\| \|f_i\| < \infty$.

Definition 1.3 (Pseudoshift operator). Let X be a Banach sequence space over I. Then a continuous linear operator $T : X \to X$ is considered a weighted pseudoshift if there is a sequence $b = (b_i)_{i \in I}$ of nonzero scalars and an injective mapping $\varphi : I \to I$ such that

$$T(x_i)_{i \in I} = (b_i x_{\varphi(i)})_{i \in I}$$

for $(x_i) \in X$. We then write $T = T_{b,\varphi}$, and $(b_i)_{i\in I}$ is called the *weight sequence*. The weighted pseudoshift can also be described with the sequence $(e_i)_{i\in I}$. We consider the inverse

$$\psi = \varphi^{-1} : \varphi(I) \to I$$

of the mapping φ, and we set

$$b_{\psi(i)} = 0 \qquad \text{and} \qquad e_{\psi(i)} = 0 \quad \text{if } i \in I \setminus \varphi(I),$$

since the map ψ is defined on $\varphi(I)$ and $\psi(i)$ is undefined when $i \in I \setminus \varphi(I)$. Then for all $i \in I$,

$$T_{b,\varphi} e_i = b_{\psi(i)} e_{\psi(i)}.$$

By the definition of weighted pseudoshift, it is easy to verify that if $T = T_{b,\varphi} : X \to X$ is a weighted pseudoshift, then for each integer $n \geq 1$, T^n is a weighted pseudoshift

$$T^n (x_i)_{i\in I} = (b_{n,i} x_{\varphi^n(i)})_{i\in I},$$

where

$$\varphi^n(i) = (\varphi \circ \varphi \circ \cdots \circ \varphi)(i) \quad (n\text{-fold}),$$

$$b_{n,i} = b_i b_{\varphi(i)} \cdots b_{\varphi^{n-1}(i)} = \prod_{v=0}^{n-1} b_{\varphi^v(i)}.$$

Similarly, for any integer $n \geq 1$, we denote $\psi^n = \psi \circ \psi \circ \cdots \circ \psi$ (n-fold), and we set $b_{\psi^n(i)} = 0$ and $e_{\psi^n(i)} = 0$ when $\psi^n(i)$ is "undefined." Then for each $n \geq 1$,

$$T^n_{b,\varphi} e_i = \prod_{v=1}^{n} b_{\psi^v(i)} e_{\psi^n(i)} \quad \text{for all } i \in I.$$

Definition 1.4 (Runaway sequence). Let $\varphi : I \to I$ be a map on I, and let $(\varphi^n)_n$ be the sequence of iterates of the mapping φ. We call $(\varphi^n)_n$ a *runaway sequence* if for any finite subset $I_0 \subset I$, there exists an $n_0 \in \mathbb{N}$ such that $\varphi^n(I_0) \cap I_0 = \emptyset$ for every $n \geq n_0$. Let $\varphi_1 : I \to I$ and $\varphi_2 : I \to I$ be two maps on I. We call the sequence $(\varphi_1^n)_n$ *runaway* with $(\varphi_2^n)_n$ if for any finite subset $I_0 \subset I$, there exists an $n_0 \in \mathbb{N}$ such that $\varphi_1^n(I_0) \cap \varphi_2^n(I_0) = \emptyset$ for every $n \geq n_0$. We note that this property is symmetric since $\varphi_1^n(I_0) \cap \varphi_2^n(I_0) = \emptyset$ and $\varphi_2^n(I_0) \cap \varphi_1^n(I_0) = \emptyset$ are equivalent.

The following two criteria provide sufficient conditions for densely d-hypercyclicity and d-supercyclicity, which also play an important role in our main results in this article. The first criterion is due to Bès and Peris [6] and the second one is due to Martin [15].

Definition 1.5 (Disjoint hypercyclicity criterion). Let $(n_k)_k$ be a strictly increasing sequence of positive integers, and let $N \geq 2$. We say that $T_1, T_2, \ldots, T_N \in L(X)$ satisfy the *d-hypercyclicity criterion* with respect to $(n_k)_k$ provided that there exist dense subsets $X_0, X_1, \ldots, X_N$ of X and mappings $S_{l,k} : X_l \to X$ ($1 \leq l \leq N$, $k \in \mathbb{N}$) satisfying

$$T_l^{n_k} \xrightarrow[k\to\infty]{} 0 \quad \text{pointwise on } X_0,$$
$$S_{l,k} \xrightarrow[k\to\infty]{} 0 \quad \text{pointwise on } X_l, \text{ and}$$
$$(T_l^{n_k} S_{i,k} - \delta_{i,l} Id_{X_i}) \xrightarrow[k\to\infty]{} 0 \quad \text{pointwise on } X_i \quad (1 \le i \le N).$$

In general,we say that $T_1, T_2, \ldots, T_N$ satisfy the d-hypercyclicity criterion if there exists some sequence $(n_k)_k$ for which the conditions in Definition 1.5 are satisfied.

If $T_1, T_2, \ldots, T_N$ satisfy the d-hypercyclicity criterion with respect to a sequence $(n_k)_k$, then $T_1, T_2, \ldots, T_N$ are densely d-hypercyclic.

Definition 1.6 (Disjoint supercyclicity criterion). Let X be a Banach space, let $(n_k)_k$ be a strictly increasing sequence of positive integers, and let $N \ge 2$. We say that $T_1, T_2, \ldots, T_N \in L(X)$ satisfy the *d-supercyclicity criterion* with respect to $(n_k)_k$ provided that there exist dense subsets $X_0, X_1, \ldots, X_N$ of X and mappings $S_{l,k} : X_l \to X$ $(1 \le l \le N,\ k \in \mathbb{N})$ such that for $1 \le l \le N$,

(i) $(T_l^{n_k} S_{i,k} - \delta_{i,l} Id_{X_i}) \xrightarrow[k\to\infty]{} 0$ pointwise on X_i $(1 \le i \le N)$,

(ii) $\lim_{k\to\infty} \|T_l^{n_k} x\| \cdot \|\sum_{j=1}^N S_{j,k} y_j\| = 0$ for $x \in X_0$, $y_j \in X_j$.

If $T_1, T_2, \ldots, T_N$ satisfy the d-supercyclicity criterion with respect to a sequence $(n_k)_k$, then $T_1, T_2, \ldots, T_N$ are densely d-supercyclic. For a better understanding, we present the main result in [21] as follows.

Theorem 1.7 ([21, Theorem 2.1])**.** *Let I be a countably infinite index set, and let X be a Banach sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $\varphi : I \to I$ be an injective map. Let $N \ge 2$ be any integer, and for each integer $1 \le l \le N$, $T_l = T_{b^{(l)},\varphi} : X \to X$ is a weighted pseudoshift generated by φ and weight sequence $b^{(l)} = (b_i^{(l)})_{i\in I}$. Then for any integers $1 \le r_1 < r_2 < \cdots < r_N$, the following are equivalent.*

(1) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are densely d-hypercyclic.*

(2) (α) *The mapping $\varphi : I \to I$ has no periodic points.*

(β) *There exists an increasing sequence $(n_k)_{k\ge1}$ of positive integers such that for every $i \in I$, we have:*

(H1) *If $1 \le l \le N$, then*

$$\begin{cases} (\prod_{v=0}^{r_l n_k - 1} b_{\varphi^v(i)}^{(l)})^{-1} e_{\varphi^{r_l n_k}(i)} \to 0, \\ (\prod_{v=1}^{r_l n_k} b_{\psi^v(i)}^{(l)}) e_{\psi^{r_l n_k}(i)} \to 0 \end{cases} \quad \textit{in } X \textit{ as } k \to \infty.$$

(H2) *If $1 \le s < l \le N$, then*

$$\begin{cases} (\prod_{v=0}^{r_s n_k - 1} b_{\varphi^v(i)}^{(s)})^{-1} (\prod_{v=1}^{r_l n_k} b_{\psi^v(\varphi^{r_s n_k}(i))}^{(l)}) e_{\psi^{(r_l - r_s) n_k}(i)} \to 0, \\ (\prod_{v=0}^{r_l n_k - 1} b_{\varphi^v(i)}^{(l)})^{-1} (\prod_{v=1}^{r_s n_k} b_{\psi^v(\varphi^{r_l n_k}(i))}^{(s)}) e_{\varphi^{(r_l - r_s) n_k}(i)} \to 0 \end{cases} \quad \textit{in } X, \textit{ as } k \to \infty.$$

(3) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ satisfy the d-hypercyclicity criterion.*

Remark 1.8. In [21], Theorem 1.7 holds for weighted pseudoshifts on an arbitrary F-sequence space.

Note that the weighted pseudoshifts $T_{b^{(1)},\varphi}, T_{b^{(1)},\varphi}, \ldots, T_{b^{(N)},\varphi}$ in the above theorem are determined by different weights $b^{(l)}$ $(1 \le l \le N)$ and the same injective mapping φ. In this article, we will consider the disjoint hypercyclicity and disjoint supercyclicity of finite weighted pseudoshifts which are generated by different weights and different injective mappings.

2. Disjoint hypercyclic powers of weighted pseudoshifts

In this section, let X be a Banach sequence space over I. We will characterize the disjoint hypercyclic weighted pseudoshifts on X generated by different weights and different injective maps, which is a generalization of Theorem 1.7. In the following main result in this section, it is important to note that the proof of the following theorem follows the same basic ideas presented by Bès and Peris in [6, Theorem 4.7].

Theorem 2.1. *Let X be a Banach sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $N \ge 2$, and let integers $1 \le r_1 < r_2 < \cdots < r_N$ be given. For each integer $1 \le l \le N$, let $T_l = T_{b^{(l)},\varphi_l} : X \to X$ be a weighted pseudoshift with the weight sequence $b^{(l)} = (b_i^{(l)})_{i\in I}$ and the injective mapping $\varphi_l : I \to I$. For each $1 \le l \le N$, let ψ_l be the inverse of φ_l. If for any $1 \le s < l \le N$ the sequence $((\varphi_s^{r_s})^n)_n$ is runaway with $((\varphi_l^{r_l})^n)_n$, then the following are equivalent.*

(1) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are densely d-hypercyclic.*

(2) (α) *For each $1 \le l \le N$, the mapping $\varphi_l : I \to I$ has no periodic points.*

(β) *There exists an increasing sequence $(n_k)_{k\ge1}$ of positive integers such that for every $i \in I$, we have:*

(H1) *If $1 \le l \le N$, then*

$$\begin{cases} \lim_{k\to\infty} \|(\prod_{v=0}^{r_l n_k - 1} b^{(l)}_{\varphi_l^v(i)})^{-1} e_{\varphi_l^{r_l n_k}(i)}\| = 0, \\ \lim_{k\to\infty} \|(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}) e_{\psi_l^{r_l n_k}(i)}\| = 0. \end{cases}$$

(H2) *If $1 \le s < l \le N$, then*

$$\begin{cases} \lim_{k\to\infty} \|(\prod_{v=0}^{r_s n_k - 1} b^{(s)}_{\varphi_s^v(i)})^{-1} (\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\| = 0, \\ \lim_{k\to\infty} \|(\prod_{v=0}^{r_l n_k - 1} b^{(l)}_{\varphi_l^v(i)})^{-1} (\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}) e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\| = 0. \end{cases}$$

(3) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ satisfy the d-hypercyclicity criterion.*

Proof. (1) $\Rightarrow$ (2). Since $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are d-hypercyclic, T_l is hypercyclic for each $1 \le l \le N$. In [9, Theorem 5], Grosse-Erdmann proved that if the weighted pseudoshift $T_{b^{(l)},\varphi_l}$ is hypercyclic, then the mapping $\varphi_l : I \to I$ has no periodic points, which implies that, for every finite subset I_0 of I and any $i \in I_0$, there is an $n_0 \in \mathbb{N}$ such that $\varphi_l^n(i) \notin I_0$ for $n \ge n_0$. This shows that $(\varphi_l^n)_n$ is a runaway sequence.

By assumption, I is a countably infinite set. We fix

$$I := \{i_1, i_2, \ldots, i_n, \ldots\},$$

and we set $I_k := \{i_1, i_2, \ldots, i_k\}$ for each integer k with $k \geq 1$. To complete the proof of (β), it suffices to show that for any fixed integers $k \geq 1$ and $N_0 \in \mathbb{N}$, there is an integer $n_k > N_0$ such that for every $i \in I_k$, if $1 \leq l \leq N$, then

$$\begin{cases} \text{(i)} \quad \|(\prod_{v=0}^{r_l n_k - 1} b^{(l)}_{\varphi_l^v(i)})^{-1} e_{\varphi_l^{r_l n_k}(i)}\| < \frac{1}{k}, \\ \text{(ii)} \quad \|(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}) e_{\psi_l^{r_l n_k}(i)}\| < \frac{1}{k}, \end{cases} \tag{2.1}$$

and if $1 \leq s < l \leq N$, then

$$\begin{cases} \text{(i)} \quad \|(\prod_{v=0}^{r_s n_k - 1} b^{(s)}_{\varphi_s^v(i)})^{-1} (\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\| < \frac{1}{k}, \\ \text{(ii)} \quad \|(\prod_{v=0}^{r_l n_k - 1} b^{(l)}_{\varphi_l^v(i)})^{-1} (\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}) e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\| < \frac{1}{k}. \end{cases} \tag{2.2}$$

To see this, we take $k = 1, 2, 3, \ldots$ in the above argument, and inductively define an increasing sequence $(n_k)_{k\geq 1}$ of positive integers by letting n_k be a positive integer satisfying (2.1) and (2.2) for $N_0 = n_{k-1}$ (where we set $N_0 = 0$ when $k = 1$). It is clear that the sequence $(n_k)_{k\geq 1}$ satisfies (H1) and (H2), since for any fixed $i_0 \in I$ there exists an integer $m_0 \in \mathbb{N}$ such that $i_0 \in I_k$ for all $k \geq m_0$. Thus, (2.1) and (2.2) are satisfied for i_0 and all integers k with $k \geq m_0$. This shows that $(n_k)_{k\geq 1}$ satisfies (H1) and (H2) for i_0. By the arbitrariness of i_0, condition (β) holds.

We therefore have to prove (2.1) and (2.2) under the assumption of (1). Let integers $k \geq 1$ and $N_0 \in \mathbb{N}$ be given. Note that $(e_i)_{i\in I}$ is an OP-basis. By the equicontinuity of the coordinate projections in X, there is some $\delta_k > 0$ so that for any $h = (h_i)_{i\in I} \in X$,

$$\|h_i e_i\| < \frac{1}{2k} \quad \text{for all } i \in I \text{ if } \|h\| < \delta_k. \tag{2.3}$$

Since for each $1 \leq l \leq N$ the sequence $(\varphi_l^n)_n$ is runaway, and since for $1 \leq s < l \leq N$ the sequence $((\varphi_s^{r_s})^n)_n$ is runaway with $((\varphi_l^{r_l})^n)_n$, there exists an integer $\widetilde{N_0} \in \mathbb{N}$ such that

$$\begin{cases} \text{(i)} \quad \varphi_l^{r_l n}(I_k) \cap I_k = \emptyset & (1 \leq l \leq N), \\ \text{(ii)} \quad \varphi_s^{r_s n}(I_k) \cap \varphi_l^{r_l n}(I_k) = \emptyset & (1 \leq s < l \leq N) \end{cases} \tag{2.4}$$

for all $n \geq \widetilde{N_0}$.

By the fact that each ψ_l $(1 \leq l \leq N)$ is the inverse of φ_l, we have that, for any positive integer n, $\psi_l^{r_l n}$ $(1 \leq l \leq N)$ is injective and defined on $\varphi_l^{r_l n}(I)$. Hence, (ii) of (2.4) implies that

$$\psi_l^{r_l n}\big(\varphi_s^{r_s n}(I_k) \cap \varphi_l^{r_l n}(I)\big) \cap I_k = \emptyset \quad (1 \leq s < l \leq N) \tag{2.5}$$

for every $n \geq \widetilde{N_0}$.

By the density of d-hypercyclic vectors in X and the continuous inclusion of X into $\mathbb{K}^I$, we can find a d-hypercyclic vector $x = (x_i)_{i\in I} \in X$ and an integer $n_k > \max\{N_0, \widetilde{N_0}\}$ such that

$$\begin{cases} \text{(i)} \quad \|x - \sum_{i\in I_k} e_i\| < \delta_k, \\ \text{(ii)} \quad \sup_{i\in I_k} |x_i - 1| \leq \frac{1}{2} \end{cases} \tag{2.6}$$

and for each $1 \leq l \leq N$,

$$\begin{cases} \text{(i)} \quad \|y^{(l)} - \sum_{i\in I_k} e_i\| < \delta_k, \\ \text{(ii)} \quad \sup_{i\in I_k} |y_i^{(l)} - 1| \leq \frac{1}{2}, \end{cases} \tag{2.7}$$

where $y^{(l)} := T_l^{r_l n_k} x = ((\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}) x_{\varphi_l^{r_l n_k}(i)})_{i\in I} = (y_i^{(l)})_{i\in I}$.

By (2.3), the first inequality in (2.6) implies that

$$\|x_i e_i\| < \frac{1}{2k} \quad \text{if } i \notin I_k; \tag{2.8}$$

thus by the first set equality in (2.4), for each $1 \leq l \leq N$, we have that

$$\|x_{\varphi_l^{r_l n_k}(i)} e_{\varphi_l^{r_l n_k}(i)}\| < \frac{1}{2k} \quad \text{for } i \in I_k. \tag{2.9}$$

By the second inequality in (2.7), for each $i \in I_k$ and $1 \leq l \leq N$,

$$\Big|\Big(\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}\Big) x_{\varphi_l^{r_l n_k}(i)} - 1\Big| \leq \frac{1}{2},$$

which means that

$$\begin{cases} \text{(i)} \quad x_{\varphi_l^{r_l n_k}(i)} \neq 0, \\ \text{(ii)} \quad |((\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}) x_{\varphi_l^{r_l n_k}(i)})^{-1}| \leq 2. \end{cases} \tag{2.10}$$

For each $i \in I_k$ and $1 \leq l \leq N$, by (2.9) and (2.10), it follows that

$$\begin{aligned} \Big\|\Big(\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\| &= \Big\|\frac{1}{(\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}) x_{\varphi_l^{r_l n_k}(i)}} x_{\varphi^{r_l n_k}(i)} e_{\varphi_l^{r_l n_k}(i)}\Big\| \\ &\leq 2\|x_{\varphi_l^{r_l n_k}(i)} e_{\varphi_l^{r_l n_k}(i)}\| \\ &< \frac{1}{k}. \end{aligned}$$

Similar with (2.5), we deduce from (2.4) that

$$\begin{cases} \psi_l^{r_l n_k}(I_k \cap \varphi_l^{r_l n_k}(I)) \cap I_k = \emptyset & \text{for } 1 \leq l \leq N, \\ \psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(I_k) \cap \varphi_s^{r_s n_k}(I)) \cap I_k = \emptyset & \text{for } 1 \leq s < l \leq N. \end{cases} \tag{2.11}$$

By (2.3), the first inequality in (2.7) implies that

$$\Big\|\Big(\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}\Big) x_{\varphi_l^{r_l n_k}(i)} e_i\Big\| < \frac{1}{2k} \quad \text{for } i \notin I_k \text{ and } 1 \leq l \leq N. \tag{2.12}$$

Note that for each $1 \leq l \leq N$,

$$e_{\psi_l^{r_l n_k}(i)} = 0 \quad \text{if } i \in I_k \backslash \varphi_l^{r_l n_k}(I),$$

and for $1 \leq s < l \leq N$,

$$\begin{cases} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))} = 0 & \text{if } i \in I_k \text{ and } \varphi_s^{r_s n_k}(i) \notin \varphi_l^{r_l n_k}(I), \\ e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))} = 0 & \text{if } i \in I_k \text{ and } \varphi_l^{r_l n_k}(i) \notin \varphi_s^{r_s n_k}(I). \end{cases}$$

So by (2.12), (2.11), and (2.5) we obtain that, for each $i \in I_k$ and $1 \le l \le N$:

$$\begin{aligned}
&\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(\psi_l^{r_l n_k}(i))}\Big) x_{\varphi_l^{r_l n_k}(\psi_l^{r_l n_k}(i))} e_{\psi_l^{r_l n_k}(i)}\Big\| \\
&\quad = \Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) x_i e_{\psi_l^{r_l n_k}(i)}\Big\| < \frac{1}{2k}. \qquad (2.13)
\end{aligned}$$

For each $i \in I_k$ and $1 \le s < l \le N$,

$$\begin{aligned}
&\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i)))}\Big) x_{\varphi_l^{r_l n_k}(\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i)))} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| \\
&\quad = \Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n}(i))}\Big) x_{\varphi_s^{r_s n_k}(i)} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| < \frac{1}{2k} \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\Big(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i)))}\Big) x_{\varphi_s^{r_s n_k}(\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i)))} e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\| \\
&\quad = \Big\|\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}\Big) x_{\varphi_l^{r_l n_k}(i)} e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\| < \frac{1}{2k}. \qquad (2.15)
\end{aligned}$$

By the second inequality in (2.6),

$$0 < \frac{1}{|x_i|} \le 2 \quad \text{for } i \in I_k. \qquad (2.16)$$

Now by (2.13) and (2.16), we get that for each $i \in I_k$ and $1 \le l \le N$,

$$\begin{aligned}
\Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) e_{\psi_l^{r_l n_k}(i)}\Big\| &= \Big\|\frac{1}{x_i}\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) x_i e_{\psi_l^{r_l n_k}(i)}\Big\| \\
&\le 2\Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) x_i e_{\psi_l^{r_l n_k}(i)}\Big\| \\
&< \frac{1}{k}.
\end{aligned}$$

Similarly, (2.10), (2.14), and (2.15) give that for each $i \in I_k$ and $1 \le s < l \le N$,

$$\begin{aligned}
&\Big\|\Big(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}\Big) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| \\
&\quad = \Big\|\Big(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(i)} x_{\varphi_s^{r_s n_k}(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}\Big) x_{\varphi_s^{r_s n_k}(i)} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| \\
&\quad \le 2\Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}\Big) x_{\varphi_s^{r_s n_k}(i)} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| < \frac{1}{k}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}\Big)e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\| \\
&\quad = \Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)} x_{\varphi_l^{r_l n_k}(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}\Big) x_{\varphi_l^{r_l n_k}(i)} e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\| \\
&\quad < \frac{1}{k}.
\end{aligned}$$

(2) $\Rightarrow$ (3). Suppose that (2) holds; let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers for (H1), (H2). Set $X_0 = X_1 = \cdots = X_N = \operatorname{span}\{e_i : i \in I\}$, which are dense in X. For each $1 \leq l \leq N$ and integer $n \geq 1$, we consider the linear mapping $S_{l,n} : X_l \to X$ given by

$$S_{l,n}(e_i) = \Big(\prod_{v=0}^{r_l n-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n}(i)} \quad (i \in I).$$

Since $T_l^{r_l n} e_i = (\prod_{v=1}^{r_l n} b^{(l)}_{\psi_l^v(i)}) e_{\psi_l^{r_l n}(i)}$ for any integer $n \geq 1$, we have

$$T_l^{r_l n} S_{l,n}(e_i) = e_i \quad \text{for all } i \in I \text{ and } n \geq 1.$$

By (H1), for any $i \in I$ and $1 \leq l \leq N$,

$$\lim_{k\to\infty} \|S_{l,n_k} e_i\| = \lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\| = 0$$

and

$$\lim_{k\to\infty} \|T_l^{r_l n_k} e_i\| = \lim_{k\to\infty}\Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) e_{\psi_l^{r_l n_k}(i)}\Big\| = 0.$$

An easy calculation gives that, for any $i \in I$ and $1 \leq s < l \leq N$,

$$\begin{cases}
T_l^{r_l n_k} S_{s,n_k}(e_i) = (\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(i)})^{-1}(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}, \\
T_s^{r_s n_k} S_{l,n_k}(e_i) = (\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)})^{-1}(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}) e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))},
\end{cases}$$

and therefore by (H2), $T_l^{r_l n_k} S_{s,n_k}(e_i)$ and $T_s^{r_s n_k} S_{l,n_k}(e_i)$ both tend to zero for any $i \in I$. We conclude by using linearity.

(3) $\Rightarrow$ (1). This implication is obvious. □

Remark 2.2. For each $1 \leq l \leq N$, the fact that the mapping $\varphi_l : I \to I$ has no periodic points does not imply that $(\varphi_s^{r_s n})_n$ is runaway with $(\varphi_l^{r_l n})_n$ for $1 \leq s < l \leq N$. For example, if we let $I = \mathbb{Z}$, then define $\varphi_1(i) = i + 2$ and $\varphi_2(i) = i + 1$ for $i \in \mathbb{Z}$. Let $r_1 = 1$, $r_2 = 2$. Clearly, for $l = 1, 2$, φ_l has no periodic points. But for any positive integer $n \geq 1$, $\varphi_1^{r_1 n} = \varphi_2^{r_2 n}$.

Now we will illustrate Theorem 2.1 with the following example. We refer to [12] for the relevant definitions and notation of directed trees and directed graphs. The definitions about shifts on weighted L^p spaces of directed trees are borrowed from Martínez-Avendaño [16].

Example 2.3. Let $T = (V, E)$ be an infinite, unrooted, directed tree such that T has no vertices of outdegree larger than 1. Let $1 \leq p < \infty$, and let $\lambda = \{\lambda_v\}_{v\in V}$ be a sequence of positive numbers such that $\sup_{u\in V, v=\mathrm{chi}(u)} \frac{\lambda_v}{\lambda_u} < \infty$ and $\sup_{u\in V, v=\mathrm{chi}^2(u)} \frac{\lambda_v}{\lambda_u} < \infty$. We denote by $L^p(T, \lambda)$ the space of complex-valued functions $f : V \to \mathbb{C}$ such that

$$\sum_{v\in V} \big|f(v)\big|^p \lambda_v < \infty.$$

The space $L^p(T, \lambda)$ is a Banach space with the norm

$$\|f\|_p = \Big(\sum_{v\in V} \big|f(v)\big|^p \lambda_v\Big)^{\frac{1}{p}}.$$

Now we define the shifts S_1 and S_2 on $L^p(T, \lambda)$ by

$$(S_1 f)(v) = f\big(\mathrm{par}(v)\big) \quad \text{for } f \in L^p(T, \lambda)$$

and

$$(S_2 f)(v) = f\big(\mathrm{par}^2(v)\big) \quad \text{for } f \in L^p(T, \lambda).$$

The weight assumption ensures that S_1 and S_2 are bounded on $L^p(T, \lambda)$ (see [16, Proposition 3.2]). Let f be any vector in $L^p(T, \lambda)$. If we identify f by $(f(v))_{v\in V}$, then $L^p(T, \lambda)$ can be seen as a Banach sequence space over $I := V$. Let $u \in V$, and denote by χ_u the characteristic function of vertex u. Define $e_u := \chi_u$. Clearly, $(e_u)_{u\in V}$ is an OP-basis of $L^p(T, \lambda)$. In this interpretation, for $l = 1, 2$, S_l is a weighted pseudoshift $T_{b^{(l)},\varphi_l}$ with

$$b^{(l)}_v = 1 \qquad \text{and} \qquad \varphi_l(v) = \mathrm{par}^l(v) \quad (v \in V)$$

(since T has no vertices of outdegree larger than 1, for each $l = 1, 2$, φ_l is an injective mapping). Thus for $l = 1, 2$, the inverse $\psi_l = \varphi_l^{-1} : \mathrm{par}^l(V) \to V$ is given by

$$\psi_l(v) = \mathrm{chi}^l(v) \quad \text{for } v \in \mathrm{par}^l(V).$$

If we set

$$b^{(l)}_{\mathrm{chi}^l(v)} = 0 \qquad \text{and} \qquad \lambda_{\mathrm{chi}^l(v)} = 0 \quad \text{if } v \in V \backslash \mathrm{par}^l(V),$$

then by Theorem 2.1, S_1, S_2^2 are densely d-hypercyclic if and only if there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that for every $v \in V$ and $l = 1, 2$,

$$\lim_{k\to\infty} \Big\| \Big(\prod_{t=0}^{ln_k-1} b^{(l)}_{\varphi_l^t(v)} \Big)^{-1} e_{\varphi_l^{ln_k}(v)} \Big\| = \lim_{k\to\infty} \|\chi_{\mathrm{par}^{l^2 n_k}(v)}\| = \lim_{k\to\infty} (\lambda_{\mathrm{par}^{l^2 n_k}(v)})^{\frac{1}{p}} = 0,$$

$$\lim_{k\to\infty} \Big\| \Big(\prod_{t=1}^{ln_k} b^{(l)}_{\psi_l^t(v)} \Big) e_{\psi_l^{ln_k}(v)} \Big\| = \lim_{k\to\infty} \|\chi_{\mathrm{chi}^{l^2 n_k}(v)}\| = \lim_{k\to\infty} (\lambda_{\mathrm{chi}^{l^2 n_k}(v)})^{\frac{1}{p}} = 0$$

and for $s = 1$, $l = 2$,

$$\lim_{k\to\infty}\Big\|\Big(\prod_{t=0}^{sn_k-1} b^{(s)}_{\varphi_s^t(v)}\Big)^{-1}\Big(\prod_{t=1}^{ln_k} b^{(l)}_{\psi_l^t(\varphi_s^{sn_k}(v))}\Big)e_{\psi_l^{ln_k}(\varphi_s^{sn_k}(v))}\Big\|$$
$$= \lim_{k\to\infty}\|\chi_{\mathrm{chi}^{(l^2-s^2)n_k}(v)}\| = \lim_{k\to\infty}(\lambda_{\mathrm{chi}^{3n_k}(v)})^{\frac{1}{p}} = 0,$$
$$\lim_{k\to\infty}\Big\|\Big(\prod_{t=0}^{ln_k-1} b^{(l)}_{\varphi_l^t(v)}\Big)^{-1}\Big(\prod_{t=1}^{sn_k} b^{(s)}_{\psi_s^t(\varphi_l^{ln_k}(v))}\Big)e_{\psi_s^{sn_k}(\varphi_l^{ln_k}(v))}\Big\|$$
$$= \lim_{k\to\infty}\|\chi_{\mathrm{par}^{(l^2-s^2)n_k}(v)}\| = \lim_{k\to\infty}(\lambda_{\mathrm{par}^{3n_k}(v)})^{\frac{1}{p}} = 0.$$

We let $s_1, s_2 \in \mathbb{R}$ with $1 < s_1 \le s_2$. Select an arbitrary fixed vertex and call it ω^*. For each $u \in V$, set

$$\lambda_u = \begin{cases} \frac{1}{s_1^d} & \text{if } \omega^* \text{ is a descendant of } u,\\ \frac{1}{s_2^d} & \text{if } \omega^* \text{ is an ancestor of } u,\end{cases}$$

where $d = \mathrm{dist}(u, \omega^*)$ ($\mathrm{dist}(u, \omega^*)$ denotes the length of the shortest path in the undirected graph joining u and ω^*). Fix $u \in V$ with $d = \mathrm{dist}(u, \omega^*)$. Also, let $n > d$. We then have that

$$\lambda_{\mathrm{par}^n(u)} = \Big(\frac{1}{s_1}\Big)^{n\pm d},$$

where the plus sign corresponds to the case where ω^* is a descendant of u and the minus sign corresponds to the case where ω^* is an ancestor of u. Also,

$$\lambda_{\mathrm{chi}^n(u)} = \begin{cases} \frac{1}{s_2^{n-d}} & \text{if } \omega^* \text{ is a descendant of } u \text{ and } u \in \mathrm{par}^n(V),\\ \frac{1}{s_2^{n+d}} & \text{if } \omega^* \text{ is an ancestor of } u \text{ and } u \in \mathrm{par}^n(V),\\ 0 & \text{if } u \in V\backslash \mathrm{par}^n(V).\end{cases}$$

Therefore, $\lim_{n\to\infty}\lambda_{\mathrm{par}^{l^2n}(u)} = \lim_{n\to\infty}\lambda_{\mathrm{chi}^{l^2n}(u)} = 0$ for $l = 1, 2$ and $\lim_{n\to\infty}\lambda_{\mathrm{par}^{3n}(u)} = \lim_{n\to\infty}\lambda_{\mathrm{chi}^{3n}(u)} = 0$.

3. Disjoint supercyclic powers of weighted pseudoshifts

In this section, we will extend the characterizations in Theorem 2.1 from d-hypercyclicity to d-supercyclicity and present some corollaries.

Theorem 3.1. *Let X be a Banach sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $N \ge 2$, and let integers $1 \le r_1 < r_2 < \cdots < r_N$ be given. For each integer $1 \le l \le N$, let $T_l = T_{b^{(l)},\varphi_l} : X \to X$ be a weighted pseudoshift with weight sequence $b^{(l)} = (b_i^{(l)})_{i\in I}$ and injective mapping $\varphi_l : I \to I$. For each $1 \le l \le N$, let ψ_l be the inverse of φ_l. If, for each integer $1 \le l \le N$, $(\varphi_l^n)_n$ is a runaway sequence, and if for any $1 \le s < l \le N$ the sequence $(\varphi_s^{r_sn})_n$ is runaway with $(\varphi_l^{r_ln})_n$, then the following statements are equivalent.*

(1) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ have a dense set of d-supercyclic vectors.*
(2) *There exists an increasing sequence $(n_k)_{k\ge1}$ of positive integers such that:*

(H1) *For any* $i, j \in I$ *and* $1 \le l,\ s \le N$,

$$\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\|\Big\|\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(j)}\Big) e_{\psi_s^{r_s n_k}(j)}\Big\| = 0.$$

(H2) *For any* $i \in I$ *and* $1 \le s < l \le N$,

$$\begin{cases} \text{(i)} & \lim_{k\to\infty}\|(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(i)})^{-1}(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\| = 0, \\ \text{(ii)} & \lim_{k\to\infty}\|(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)})^{-1}(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}) e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\| = 0. \end{cases}$$

(3) $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ *satisfy the d-supercyclicity criterion.*

Proof. (1) ⇒ (2). Fix

$$I := \{i_1, i_2, \ldots, i_n, \ldots\},$$

and for each $k \in \mathbb{N}$ with $k \ge 1$ set $I_k := \{i_1, i_2, \ldots, i_k\}$. To prove (2), it is enough to verify that for any positive integer $k \ge 1$ and any $N_0 \in \mathbb{N}$, there is an integer $n_k > N_0$ such that:

For any $i, j \in I_k$ and $1 \le l,\ s \le N$,

$$\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\|\Big\|\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(j)}\Big) e_{\psi_s^{r_s n_k}(j)}\Big\| < \frac{1}{k}.$$

For every $i \in I_k$ and $1 \le s < l \le N$,

$$\begin{cases} \text{(i)} & \|(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(i)})^{-1}(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\| < \frac{1}{k}, \\ \text{(ii)} & \|(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)})^{-1}(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}) e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\| < \frac{1}{k}. \end{cases}$$

Let integers $k \ge 1$ and $N_0 \in \mathbb{N}$ be given. By assumption, there is some $\delta_k > 0$ such that for any $h = (h_i)_{i\in I} \in X$,

$$\|h_i e_i\| < \frac{1}{2k} \quad \text{for } i \in I \text{ if } \|h\| < \delta_k.$$

Let $\widetilde{N_0} \in \mathbb{N}$ be the integer such that

$$\begin{cases} \text{(i)} & \varphi_l^{r_l n}(I_k) \cap I_k = \emptyset \qquad (1 \le l \le N), \\ \text{(ii)} & \psi_l^{r_l n}(\varphi_s^{r_s n}(I_k) \cap \varphi_l^{r_l n}(I)) \cap I_k = \emptyset \quad (1 \le s < l \le N) \end{cases}$$

for all $n \ge \widetilde{N_0}$.

Since the d-supercyclic vectors are dense in X, there exists a d-supercyclic vector $x = (x_i)_{i\in I} \in X$ such that

$$\Big\|x - \sum_{i\in I_k} e_i\Big\| < \delta_k.$$

Now, let $A = \{\alpha(T_1^{r_1 n}x, T_2^{r_2 n}x, \ldots, T_N^{r_N n}x) : \alpha \in \mathbb{C}, n \in \mathbb{N}\}$. Obviously, A is dense in X^N. For each integer $p \in \mathbb{N}$, let $B_p = \{\alpha(T_1^{r_1 n}x, T_2^{r_2 n}x, \ldots, T_N^{r_N n}x) : \alpha \in \mathbb{C}, n \in \mathbb{N}, n \le p\}$. Since X is an infinite-dimensional Banach space, $A \setminus B_p$ remains dense in X^N for any $p \in \mathbb{N}$. Thus, we can find a complex number $\alpha \ne 0$ and an integer $n_k > \max\{N_0, \widetilde{N_0}\}$ such that for each $1 \le l \le N$,

$$\Big\|\alpha y^{(l)} - \sum_{i\in I_k} e_i\Big\| < \delta_k,$$

where $y^{(l)} := T_l^{r_l n_k} x = ((\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}) x_{\varphi_l^{r_l n_k}(i)})_{i\in I} = (y_i^{(l)})_{i\in I}$.

By the continuous inclusion of X into $\mathbb{K}^I$, we can in addition assume that

$$\begin{cases} \text{(i)} \quad \sup_{i\in I_k} |x_i - 1| \le \frac{1}{2}, \\ \text{(ii)} \quad \sup_{i\in I_k} |\alpha y_i^{(l)} - 1| \le \frac{1}{2} \quad \text{for } 1 \le l \le N. \end{cases}$$

It follows that for any $i \in I_k$,

$$\begin{cases} \text{(i)} \quad 0 < \frac{1}{|x_i|} \le 2, \\ \text{(ii)} \quad x_{\varphi_l^{r_l n_k}(i)} \ne 0 \quad (1 \le l \le N), \\ \text{(iii)} \quad 0 < |(\alpha(\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}) x_{\varphi_l^{r_l n_k}(i)})^{-1}| \le 2 \quad (1 \le l \le N). \end{cases} \tag{3.1}$$

By repeating a similar argument as in the proof of Theorem 2.1, one can obtain that, for each $i \in I_k$ and $1 \le l \le N$,

$$\Big\|\Big(\alpha \prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\| < \frac{1}{k} \tag{3.2}$$

and

$$\Big\|\Big(\alpha \prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) e_{\psi_l^{r_l n_k}(i)}\Big\| < \frac{1}{k}. \tag{3.3}$$

For each $i \in I_k$ and $1 \le s < l \le N$,

$$\Big\|\alpha\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n}(i))}\Big) x_{\varphi_s^{r_s n_k}(i)} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| < \frac{1}{2k} \tag{3.4}$$

and

$$\Big\|\alpha\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}\Big) x_{\varphi_l^{r_l n_k}(i)} e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\| < \frac{1}{2k}. \tag{3.5}$$

Hence by (3.2) and (3.3), for any $i, j \in I_k$ and $1 \le l,\ s \le N$,

$$\begin{aligned} &\Big\|\Big(\prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\| \Big\|\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(j)}\Big) e_{\psi_s^{r_s n_k}(j)}\Big\| \\ &\quad = \Big\|\Big(\alpha \prod_{v=0}^{r_l n_k -1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n_k}(i)}\Big\| \Big\|\Big(\alpha \prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(j)}\Big) e_{\psi_s^{r_s n_k}(j)}\Big\| < \frac{1}{k^2} \le \frac{1}{k}. \end{aligned}$$

By (3.1), (3.4), and (3.5), for each $i \in I_k$ and $1 \le s < l \le N$,

$$\begin{aligned} &\Big\|\Big(\prod_{v=0}^{r_s n_k -1} b^{(s)}_{\varphi_s^v(i)}\Big)^{-1} \Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}\Big) e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| \\ &\quad = \Big\|\Big(\alpha\Big(\prod_{v=0}^{r_s n_k -1} b^{(s)}_{\varphi_s^v(i)}\Big) x_{\varphi_s^{r_s n_k}(i)}\Big)^{-1} \alpha\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}\Big) x_{\varphi_s^{r_s n_k}(i)} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| \end{aligned}$$

$$\leq 2\Big\|\alpha\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(\varphi_s^{r_s n_k}(i))}\Big) x_{\varphi_s^{r_s n_k}(i)} e_{\psi_l^{r_l n_k}(\varphi_s^{r_s n_k}(i))}\Big\| < \frac{1}{k} \tag{3.6}$$

and

$$\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}\Big) e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\|$$
$$= \Big\|\Big(\alpha\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi_l^v(i)}\Big) x_{\varphi_l^{r_l n_k}(i)}\Big)^{-1}\Big(\alpha\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi_s^v(\varphi_l^{r_l n_k}(i))}\Big) x_{\varphi_l^{r_l n_k}(i)} e_{\psi_s^{r_s n_k}(\varphi_l^{r_l n_k}(i))}\Big\|$$
$$< \frac{1}{k}.$$

This completes the proof.

(2) $\Rightarrow$ (3). Suppose that (2) holds, and let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying (H1) and (H2). Let us show that $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ satisfy the d-supercyclicity criterion. Set $X_0 = X_1 = \cdots = X_N = \operatorname{span}\{e_i : i \in I\}$. For each $1 \leq l \leq N$ and $n \in \mathbb{N}$ with $n \geq 1$, we consider the linear mappings $S_{l,n} : X_l \to X$ given by

$$S_{l,n}(e_i) = \Big(\prod_{v=0}^{r_l n-1} b^{(l)}_{\varphi_l^v(i)}\Big)^{-1} e_{\varphi_l^{r_l n}(i)} \quad (i \in I).$$

The same argument as used in the proof of (2) $\Rightarrow$ (3) in Theorem 2.1 yields that Definition 1.6(i) is satisfied. So we just need to check that $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ satisfy Definition 1.6(ii) with respect to $(n_k)_{k\geq 1}$. Let $y_0, y_1, \ldots, y_N \in \operatorname{span}\{e_i : i \in I\}$. There exists a $k_0 \in \mathbb{N}$ such that

$$y_i = \sum_{j\in I_{k_0}} y_{i,j} e_j \quad (0 \leq i \leq N).$$

Set $C := \max\{|y_{i,j}| : 0 \leq i \leq N, j \in I_{k_0}\}$. By (H1), for any $i, j \in I$ and $1 \leq l$, $s \leq N$,

$$\lim_{k\to\infty} \|T_l^{r_l n_k} e_i\|\|S_{s,n_k} e_j\|$$
$$= \lim_{k\to\infty}\Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(i)}\Big) e_{\psi_l^{r_l n_k}(i)}\Big\|\Big\|\Big(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(j)}\Big)^{-1} e_{\varphi_s^{r_s n_k}(j)}\Big\| = 0.$$

It follows that

$$\|T_l^{r_l n_k} y_0\|\Big\|\sum_{s=1}^{N} S_{s,n_k} y_s\Big\|$$
$$= \Big\|\sum_{j\in I_{k_0}} y_{0,j}\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(j)}\Big) e_{\psi_l^{r_l n_k}(j)}\Big\|\Big\|\sum_{s=1}^{N}\sum_{j\in I_{k_0}} y_{s,j}\Big(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(j)}\Big)^{-1} e_{\varphi_s^{r_s n_k}(j)}\Big\|$$

$$\leq C^2\Big(\sum_{j\in I_{k_0}}\Big\|\Big(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi_l^v(j)}\Big)e_{\psi_l^{r_l n_k}(j)}\Big\|\Big)\Big(\sum_{s=1}^{N}\sum_{j\in I_{k_0}}\Big\|\Big(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi_s^v(j)}\Big)^{-1}e_{\varphi_s^{r_s n_k}(j)}\Big\|\Big)$$
$$\xrightarrow[k\to\infty]{} 0.$$

(3) $\Rightarrow$ (1). This implication is obvious. □

Next, we consider the special case $\varphi_1 = \varphi_2 = \cdots = \varphi_N$ in Theorem 3.1.

Corollary 3.2. *Let X be a Banach sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $N \geq 2$, for each $1 \leq l \leq N$, and let $T_l = T_{b^{(l)},\varphi} : X \to X$ be a weighted pseudoshift with weight sequence $b^{(l)} = (b_i^{(l)})_{i\in I}$ and injective mapping φ. Then for any integers $1 \leq r_1 < r_2 < \cdots < r_N$, the following are equivalent.*

(1) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ have a dense set of d-supercyclic vectors.*

(2) (α) *The mapping $\varphi : I \to I$ has no periodic points.*

(β) *There exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that:*

(H1) *For any $i, j \in I$ and $1 \leq l,\ s \leq N$ we have*

$$\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi^v(i)}\Big)^{-1}e_{\varphi^{r_l n_k}(i)}\Big\|\Big\|\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi^v(j)}\Big)e_{\psi^{r_s n_k}(j)}\Big\| = 0.$$

(H2) *For every $i \in I$ and any $1 \leq s < l \leq N$,*

$$\begin{cases}\text{(i)} & \lim_{k\to\infty}\|(\prod_{v=0}^{r_s n_k-1} b^{(s)}_{\varphi^v(i)})^{-1}(\prod_{v=1}^{r_l n_k} b^{(l)}_{\psi^v(\varphi^{r_s n_k}(i))})e_{\psi^{(r_l-r_s)n_k}(i)}\| = 0,\\ \text{(ii)} & \lim_{k\to\infty}\|(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi^v(i)})^{-1}(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi^v(\varphi^{r_l n_k}(i))})e_{\varphi^{(r_l-r_s)n_k}(i)}\| = 0.\end{cases}$$

(3) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ satisfy the d-supercyclicity criterion.*

Proof. By Theorem 3.1, we just need to prove that (1) implies (α). Suppose on the contrary that φ has a periodic point. Then there exist an $i \in I$ and an integer $M \geq 1$ such that $\varphi^M(i) = i$. For each $1 \leq l \leq N$ and any $x \in X$, the entry of $T_l^{r_l n}x$ at position i is $(\prod_{v=0}^{r_l n-1} b^{(l)}_{\varphi^v(i)})x_{\varphi^{r_l n}(i)}$. Since $\varphi^M(i) = i$, both $(b^{(l)}_{\varphi^v(i)})_v$ and $(x_{\varphi^{r_l n}(i)})_n$ are periodic sequences, which implies that $\{\frac{(\prod_{v=0}^{r_1 n-1} b^{(1)}_{\varphi^v(i)})x_{\varphi^{r_1 n}(i)}}{(\prod_{v=0}^{r_2 n-1} b^{(2)}_{\varphi^v(i)})x_{\varphi^{r_2 n}(i)}}\}_{n\in\mathbb{N}}$ cannot be dense in $\mathbb{K}$. It follows that the set

$$\Big\{\Big(\alpha\Big(\prod_{v=0}^{r_1 n-1} b^{(1)}_{\varphi^v(i)}\Big)x_{\varphi^{r_1 n}(i)}, \alpha\Big(\prod_{v=0}^{r_2 n-1} b^{(2)}_{\varphi^v(i)}\Big)x_{\varphi^{r_2 n}(i)}\Big) : \alpha \in \mathbb{C}, n \in \mathbb{N}\Big\}$$

cannot be dense in $\mathbb{K}^2$. By continuous inclusion of X into $\mathbb{K}^I$, the set

$$\{\alpha(T_1^{r_1 n}x, T_2^{r_2 n}x, \ldots, T_N^{r_N n}x) : \alpha \in \mathbb{C}, n \in \mathbb{N}\}$$

cannot be dense in X^N, which is contrary to condition (1). Hence φ has no periodic points. □

If the mapping φ satisfies that each $i \in I$ lies outside $\varphi^n(I)$ for all sufficiently large n, then this implies in particular that the sequence $(\varphi^n)_n$ is runaway. In this case, for every $i \in I$, $e_{\psi^n(i)} = 0$ is eventually zero when n is large enough by the definition of ψ^n. Thus, (H1) and (i) of (H2) in Corollary 3.2 are automatically satisfied. Therefore, the following result holds.

Corollary 3.3. *Let X be a Banach sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $N \geq 2$, and let integers $1 \leq r_1 < r_2 < \cdots < r_N$ be given. For each $1 \leq l \leq N$, let $T_l = T_{b^{(l)},\varphi} : X \to X$ be a weighted pseudoshift with weight sequence $b^{(l)} = (b_i^{(l)})_{i\in I}$ and injective mapping φ. If the mapping $\varphi : I \to I$ satisfies that each $i \in I$ lies outside $\varphi^n(I)$ for all sufficiently large n, then the following assertions are equivalent:*

(1) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are densely d-supercyclic;*
(2) *there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that for every $i \in I$, we have*

$$\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi^v(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi^v(\varphi^{r_l n_k}(i))}\Big)e_{\varphi^{(r_l-r_s)n_k}(i)}\Big\| = 0 \quad (1 \leq s < l \leq N);$$

(3) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ satisfy the d-supercyclicity criterion.*

Example 3.4. Let $X = \ell^2(\mathbb{N})$, and let integers $1 \leq r_1 < r_2 < \cdots < r_N$ $(N \geq 2)$ be given. For each $1 \leq l \leq N$, let $(a_{l,n})_{n=1}^{\infty}$ be a bounded sequence of nonzero scalars, and let T_l be the unilateral backward weighted shift on X defined by

$$T_l e_0 = 0 \qquad \text{and} \qquad T_l e_j = a_{l,j} e_{j-1} \quad \text{for } j \geq 1,$$

where $(e_j)_{j\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$. Clearly, in this case, X is a Banach sequence space over $I = \mathbb{N}$ with OP-basis $(e_j)_{j\in\mathbb{N}}$. Each T_l is the pseudoshift $T_{b^{(l)},\varphi}$ with

$$b_i^{(l)} = a_{l,i+1} \qquad \text{and} \qquad \varphi(i) = i+1 \quad \text{for any } i \in \mathbb{N}.$$

By Corollary 3.3, $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are densely d-supercyclic if and only if there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that for every $i \in I$ and $1 \leq s < l \leq N$,

$$\begin{aligned}&\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_l n_k-1} b^{(l)}_{\varphi^v(i)}\Big)^{-1}\Big(\prod_{v=1}^{r_s n_k} b^{(s)}_{\psi^v(\varphi^{r_l n_k}(i))}\Big)e_{\varphi^{(r_l-r_s)n_k}(i)}\Big\| \\ &= \lim_{k\to\infty}\Big|\Big(\prod_{v=i+1}^{i+r_l n_k} a_{l,v}\Big)^{-1}\Big(\prod_{v=i+(r_l-r_s)n_k+1}^{i+r_l n_k} a_{s,v}\Big)\Big| = 0.\end{aligned}$$

Remark 3.5. Recently, disjoint hypercyclic and disjoint supercyclic weighted translations on a locally compact group G were studied in [7] and [23]. It is easy to see that when G is discrete, these weighted translations are special cases of pseudoshifts. Therefore by Theorem 1.7 and Corollary 3.2, we can also get the equivalent conditions for weighted translations on locally compact discrete groups to be disjoint hypercyclic and disjoint supercyclic.

4. Disjoint supercyclic operator weighted shifts on $\ell^2(\mathbb{Z},\mathcal{K})$

The bilateral operator weighted shifts on the space $\ell^2(\mathbb{Z},\mathcal{K})$ were studied by Hazarika and Arora in [11]. In [21, Theorem 3.1], we proved that the bilateral operator weighted shifts are special cases of weighted pseudoshifts. In this section, we will use Corollary 3.2 to characterize the d-supercyclicity of bilateral operator weighted shifts on $\ell^2(\mathbb{Z},\mathcal{K})$. First of all, let us recall some terminology.

Let $\mathcal{K}$ be a separable complex Hilbert space with an orthonormal basis $\{f_k\}_{k=0}^{\infty}$. Define a separable Hilbert space

$$\ell^2(\mathbb{Z},\mathcal{K}) := \Big\{x = (\ldots, x_{-1}, [x_0], x_1, \ldots) : x_i \in \mathcal{K} \text{ and } \sum_{i\in\mathbb{Z}} \|x_i\|^2 < \infty\Big\}$$

under the inner product $\langle x, y\rangle = \sum_{i\in\mathbb{Z}}\langle x_i, y_i\rangle_{\mathcal{K}}$.

Let $\{A_n\}_{n=-\infty}^{\infty}$ be a uniformly bounded sequence of invertible positive operators on $\mathcal{K}$, where the operators $\{A_n\}_{n=-\infty}^{\infty}$ are all diagonal with respect to the basis $\{f_k\}_{k=0}^{\infty}$. That is, there exists a uniformly bounded sequence of positive sequences $(\ldots, (a_{i,-1})_{i\in\mathbb{N}}, (a_{i,0})_{i\in\mathbb{N}}, (a_{i,1})_{i\in\mathbb{N}}, \ldots)$ such that for each $n \in \mathbb{Z}$,

$$A_n f_i = a_{i,n} f_i \qquad \text{and} \qquad (A_n)^{-1} f_i = (a_{i,n})^{-1} f_i \quad \text{for } i \in \mathbb{N}.$$

Then for any $g = \sum_{i=0}^{\infty} g_i f_i \in \mathcal{K}$, $A_n g = \sum_{i=0}^{\infty} g_i a_{i,n} f_i$ and $A_n^{-1} g = \sum_{i=0}^{\infty} g_i (a_{i,n})^{-1} f_i$ $(n \in \mathbb{Z})$. Also, for each $n \in \mathbb{Z}$, $\|A_n\| = \sup_{i\in\mathbb{N}} a_{i,n}$.

The bilateral forward and backward operator weighted shifts on $\ell^2(\mathbb{Z},\mathcal{K})$ are defined as follows:

(i) The bilateral forward operator weighted shift T on $\ell^2(\mathbb{Z},\mathcal{K})$ is defined by

$$T\big(\ldots, x_{-1}, [x_0], x_1, \ldots\big) = \big(\ldots, A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0 x_0, \ldots\big).$$

That is,

$$T\big(\ldots, x_{-1}, [x_0], x_1, \ldots\big) = \big(\ldots, z_{-1}, [z_0], z_1, \ldots\big),$$

where $z_j = A_{j-1}x_{j-1}$. Since $\{A_n\}_{n=-\infty}^{\infty}$ is uniformly bounded, T is bounded and $\|T\| = \sup_{j\in\mathbb{Z}}\|A_j\| < \infty$. For $n > 0$,

$$T^n\big(\ldots, x_{-1}, [x_0], x_1, \ldots\big) = \big(\ldots, y_{-1}, [y_0], y_1, \ldots\big),$$

where $y_j = \prod_{s=0}^{n-1} A_{j+s-n} x_{j-n}$.

(ii) The bilateral backward operator weighted shift T on $\ell^2(\mathbb{Z},\mathcal{K})$ is defined by

$$T\big(\ldots, x_{-1}, [x_0], x_1, \ldots\big) = \big(\ldots, A_0 x_0, [A_1 x_1], A_2 x_2, \ldots\big).$$

That is,

$$T\big(\ldots, x_{-1}, [x_0], x_1, \ldots\big) = \big(\ldots, z_{-1}, [z_0], z_1, \ldots\big),$$

where $z_j = A_{j+1}x_{j+1}$. For $n > 0$,

$$T^n\big(\ldots, x_{-1}, [x_0], x_1, \ldots\big) = \big(\ldots, y_{-1}, [y_0], y_1, \ldots\big),$$

where $y_j = \prod_{s=1}^{n} A_{j+s} x_{j+n}$. Since each A_n is an invertible diagonal operator on $\mathcal{K}$, we conclude that

$$\|A_n\| = \sup_k \|A_n f_k\| \qquad \text{and} \qquad \|A_n^{-1}\| = \sup_k \|A_n^{-1} f_k\|.$$

Now we are ready to state the main result in this section.

Theorem 4.1. *Let $N \geq 2$, for each $1 \leq l \leq N$, and let T_l be a bilateral forward operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence $\{A_n^{(l)}\}_{n=-\infty}^{\infty}$, where $\{A_n^{(l)}\}_{n=-\infty}^{\infty}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then for any integers $1 \leq r_1 < r_2 < \cdots < r_N$, the following statements are equivalent.*

(1) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are densely d-supercyclic.*

(2) *There exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that: For every $i_1, i_2 \in \mathbb{N}$, $j_1, j_2 \in \mathbb{Z}$ and $1 \leq l,\ s \leq N$,*

$$\lim_{k\to\infty}\Big\|\Big(\prod_{v=j_1-r_ln_k}^{j_1-1}(A_v^{(l)})^{-1}\Big)f_{i_1}\Big\|\Big\|\Big(\prod_{v=j_2}^{j_2+r_sn_k-1}A_v^{(s)}\Big)f_{i_2}\Big\| = 0.$$

For every $i \in \mathbb{N}$, $j \in \mathbb{Z}$ and $1 \leq s < l \leq N$,

$$\begin{cases}\lim_{k\to\infty}\|(\prod_{v=j-r_sn_k}^{j-1}(A_v^{(s)})^{-1})(\prod_{v=j-r_sn_k}^{j+(r_l-r_s)n_k-1}A_v^{(l)})f_i\| = 0,\\ \lim_{k\to\infty}\|(\prod_{v=j-r_ln_k}^{j-1}(A_v^{(l)})^{-1})(\prod_{v=j-r_ln_k}^{j-(r_l-r_s)n_k-1}A_v^{(s)})f_i\| = 0.\end{cases}$$

Proof. In [21], we proved that $\ell^2(\mathbb{Z}, \mathcal{K})$ is a Hilbert sequence space over $I := \mathbb{N}\times\mathbb{Z}$ with an OP-basis $(e_{i,j})_{(i,j)\in I}$, where for $(i,j) \in I$, $e_{i,j} := (\ldots, z_{-1}, [z_0], z_1, \ldots) \in \ell^2(\mathbb{Z}, \mathcal{K})$ is defined by

$$z_k = \begin{cases} f_i & \text{if } k = j,\\ 0 & \text{if } k \neq j.\end{cases}$$

By assumption, for each $1 \leq l \leq N$, $\{A_n^{(l)}\}_{n\in\mathbb{Z}}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. That is, there exists a uniformly bounded sequence of positive sequences $\{(a_{i,n}^{(l)})_{i\in\mathbb{N}}\}_{n\in\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$,

$$A_n^{(l)}f_i = a_{i,n}^{(l)}f_i \qquad \text{and} \qquad (A_n^{(l)})^{-1}f_i = (a_{i,n}^{(l)})^{-1}f_i \quad \text{for } i \in \mathbb{N}.$$

In this interpretation, each T_l is a weighted pseudoshift $T_{b^{(l)},\varphi}$ on $\ell^2(\mathbb{Z}, \mathcal{K})$ with

$$b_{i,j}^{(l)} = a_{i,j-1}^{(l)} \qquad \text{and} \qquad \varphi(i,j) = (i, j-1) \quad \text{for } (i,j) \in I.$$

It follows from Corollary 3.2 that $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are densely d-supercyclic if and only if there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that:

For any $(i_1, j_1), (i_2, j_2) \in I$ and $1 \leq l,\ s \leq N$,

$$\begin{aligned}&\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_ln_k-1}b_{\varphi^v(i_1,j_1)}^{(l)}\Big)^{-1}e_{\varphi^{r_ln_k}(i_1,j_1)}\Big\|\Big\|\Big(\prod_{v=1}^{r_sn_k}b_{\psi^v(i_2,j_2)}^{(s)}\Big)e_{\psi^{r_sn_k}(i_2,j_2)}\Big\|\\ &= \lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_ln_k-1}a_{(i_1,j_1-v-1)}^{(l)}\Big)^{-1}e_{(i_1,j_1-r_ln_k)}\Big\|\Big\|\Big(\prod_{v=1}^{r_sn_k}a_{(i_2,j_2+v-1)}^{(s)}\Big)e_{(i_2,j_2+r_sn_k)}\Big\|\\ &= \lim_{k\to\infty}\Big\|\Big(\prod_{v=j_1-r_ln_k}^{j_1-1}(A_v^{(l)})^{-1}\Big)f_{i_1}\Big\|\Big\|\Big(\prod_{v=j_2}^{j_2+r_sn_k-1}A_v^{(s)}\Big)f_{i_2}\Big\| = 0.\end{aligned}$$

For every $(i,j) \in I$ and $1 \le s < l \le N$,

$$\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_sn_k-1} b^{(s)}_{\varphi^v(i,j)}\Big)^{-1}\Big(\prod_{v=1}^{r_ln_k} b^{(l)}_{\psi^v(\varphi^{r_sn_k}(i,j))}\Big)e_{\psi^{(r_l-r_s)n_k}(i,j)}\Big\|$$
$$=\lim_{k\to\infty}\Big\|\Big(\prod_{v=j-r_sn_k}^{j-1}(A_v^{(s)})^{-1}\Big)\Big(\prod_{v=j-r_sn_k}^{j+(r_l-r_s)n_k-1}A_v^{(l)}\Big)f_i\Big\| = 0$$

and

$$\lim_{k\to\infty}\Big\|\Big(\prod_{v=0}^{r_ln_k-1} b^{(l)}_{\varphi^v(i,j)}\Big)^{-1}\Big(\prod_{v=1}^{r_sn_k} b^{(s)}_{\psi^v(\varphi^{r_ln_k}(i,j))}\Big)e_{\varphi^{(r_l-r_s)n_k}(i,j)}\Big\| \tag{4.1}$$
$$=\lim_{k\to\infty}\Big\|\Big(\prod_{v=j-r_ln_k}^{j-1}(A_v^{(l)})^{-1}\Big)\Big(\prod_{v=j-r_ln_k}^{j-(r_l-r_s)n_k-1}A_v^{(s)}\Big)f_i\Big\| = 0. \tag{4.2}$$

This completes the proof. □

In [8], Feldman considered the hypercyclicity of bilateral weighted shifts on $\ell^2(\mathbb{Z})$ that are invertible. Motivated by Feldman's work, in [21] we showed that if the weight sequence $\{A_n\}_{n=-\infty}^{\infty}$ is assumed to satisfy that there exists some $m > 0$ such that $\|A_n^{-1}\| \le m$ for all $n < 0$ (or for all $n > 0$), then the characterizing conditions for d-hypercyclicity simplify. Now, we establish the d-supercyclic conditions for this special case. The proof is similar to that in [21, Theorem 3.3], so we omit it here.

Corollary 4.2. *Let T be a bilateral forward operator weighted shift on $\ell^2(\mathbb{Z},\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}_{n=-\infty}^{\infty}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$, and assume that there exists $m > 0$ such that $\|A_n^{-1}\| \le m$ for all $n < 0$ (or for all $n > 0$). Then for any integer $N \ge 2$, the following are equivalent.*

(1) *$T, T^2, \ldots, T^N$ are densely d-supercyclic.*

(2) *There exists an increasing sequence $(n_k)_{k\ge1}$ of positive integers such that: For every $i_1, i_2 \in \mathbb{N}$ and $1 \le l \le N$,*

$$\begin{cases}\lim_{k\to\infty}\|(\prod_{v=1}^{ln_k}(A_{-v})^{-1})f_{i_1}\|\|(\prod_{v=1}^{Nn_k}A_v)f_{i_2}\| = 0,\\ \lim_{k\to\infty}\|(\prod_{v=1}^{Nn_k}(A_{-v})^{-1})f_{i_1}\|\|(\prod_{v=1}^{ln_k}A_v)f_{i_2}\| = 0.\end{cases}$$

For every $i \in \mathbb{N}$ and $1 \le l \le N-1$,

$$\begin{cases}\lim_{k\to\infty}\|(\prod_{v=1}^{ln_k}(A_{-v})^{-1})f_i\| = 0,\\ \lim_{k\to\infty}\|(\prod_{v=1}^{ln_k}A_v)f_i\| = 0.\end{cases}$$

Example 4.3. For each $s \in \mathbb{N}$, let

$$\mathcal{C}_s = \big\{2^{2s+1} - (2s+1), \ldots, 2^{2s+1} - 1\big\},$$
$$\mathcal{D}_s = \big\{2^{2s+1}, \ldots, 2^{2s+1} + (2s+1) - 1\big\},$$

and let

$$\mathcal{C} = \bigcup_{s=0}^{\infty} \mathcal{C}_s, \qquad \mathcal{D} = \bigcup_{s=0}^{\infty} \mathcal{D}_s, \qquad \mathcal{E} = \bigcup_{s=0}^{\infty} \{-2^{2s+1}\}.$$

Let $\{A_n\}_{n=-\infty}^{\infty}$ be a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$, defined as follows.

$$\text{If } n \in \mathcal{C}, \text{then} A_n(f_k) = \begin{cases} \frac{1}{2} f_k & 0 \le k \le n, \\ f_k & k > n. \end{cases}$$

$$\text{If } n \in \mathcal{D}, \text{ then } A_n(f_k) = \begin{cases} 2 f_k & 0 \le k \le n, \\ f_k & k > n. \end{cases}$$

$$\text{If } n \in \mathcal{E}, \text{ then } A_n(f_k) = \begin{cases} 2 f_k & 0 \le k \le -n, \\ f_k & k > -n. \end{cases}$$

$$\text{If } n \in \mathbb{Z} \backslash (\mathcal{C} \cup \mathcal{D} \cup \mathcal{E}), \text{ then } A_n(f_k) = f_k \text{ for all } k \ge 0.$$

Let T be the bilateral forward operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$. Then T, T^2 are d-supercyclic.

Proof. Note that for any $n \in \mathbb{Z}$, $\frac{1}{2} \le \|A_n\| \le 2$, we use Corollary 4.2 to give the proof. Let $(n_k)_{k\ge1} = (2^{2k+1})_{k\ge1}$. Then for each $i \in \mathbb{N}$,

$$\Big\| \Big(\prod_{v=1}^{n_k} (A_{-v})^{-1} \Big) f_i \Big\| \le \Big(\frac{1}{2}\Big)^{k-i+1} \to 0 \quad \text{as } k \to \infty$$

and

$$\Big\| \Big(\prod_{v=1}^{n_k} A_v \Big) f_i \Big\| \le \Big(\frac{1}{2}\Big)^{2k-i} \to 0 \quad \text{as } k \to \infty.$$

Since $2^{2k+1} + (2k+1) - 1 < 2 \cdot 2^{2k+1} < 2^{2k+3} - (2k+3)$,

$$\Big(\frac{1}{2}\Big)^{i} \le \Big\| \Big(\prod_{v=1}^{2n_k} A_v \Big) f_i \Big\| \le \Big(\frac{1}{2}\Big)^{2k-i} \le 2^i, \tag{4.3}$$

hence for any $i_1, i_2 \in I$,

$$\lim_{k\to\infty} \Big\| \Big(\prod_{v=1}^{n_k} (A_{-v})^{-1} \Big) f_{i_1} \Big\| \Big\| \Big(\prod_{v=1}^{2n_k} A_v \Big) f_{i_2} \Big\| = 0;$$

also by the fact that $2^{2k+1} < 2 \cdot 2^{2k+1} < 2^{2k+3}$, it is easy to see that

$$\lim_{k\to\infty} \Big\| \Big(\prod_{v=1}^{2n_k} (A_{-v})^{-1} \Big) f_{i_1} \Big\| \Big\| \Big(\prod_{v=1}^{n_k} A_v \Big) f_{i_2} \Big\| = 0$$

and

$$\lim_{k\to\infty} \Big\| \Big(\prod_{v=1}^{2n_k} (A_{-v})^{-1} \Big) f_{i_1} \Big\| \Big\| \Big(\prod_{v=1}^{2n_k} A_v \Big) f_{i_2} \Big\| = 0.$$

It follows from Corollary 4.2 that T, T^2 are d-supercyclic. But it follows from (4.3) that $\|(\prod_{v=1}^{2n_k} A_v)f_i\| \nrightarrow 0$ as $k \to \infty$. Thus by Theorem 1.7, T, T^2 are not d-hypercyclic. □

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