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VARIABLE HARDY–LORENTZ SPACES ASSOCIATED TO OPERATORS SATISFYING DAVIES–GAFFNEY ESTIMATES

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Abstract. Let L be a one-to-one operator of type w with $w \in [0, \pi/2)$, which satisfies the Davies–Gaffney estimates and has a bounded holomorphic calculus, and let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with $0 < p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \leq \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty$. Under the assumption that $p(\cdot)$ satisfies the global log-Hölder condition, we introduce the variable Hardy–Lorentz space $H_L^{p(\cdot),q}(\mathbb{R}^n)$ for $0 < q < \infty$ and construct its molecular decomposition. Furthermore, we investigate the dual spaces of the variable Hardy–Lorentz space $H_L^{p(\cdot),q}(\mathbb{R}^n)$ with $0 < p_- \leq p_+ \leq 1$ and $0 < q < \infty$. These results are new even when $p(\cdot)$ is a constant.

1. Introduction

The real variable theory of classical Hardy spaces on the Euclidean space $\mathbb{R}^n$ was introduced by Stein and Weiss [31] and then developed by Fefferman and Stein [14]. It is well known that the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is a proper substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ when studying the boundedness of some operators (e.g., Riesz transforms). Nowadays, it plays an essential role in various fields of analysis such as harmonic analysis and partial differential equations.

The study of Hardy spaces associated with various differential operators has garnered considerable interest and has developed into an active research topic.

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In particular, let L be a linear operator on $L^2(\mathbb{R}^n)$, and generate an analytic semigroup $\{e^{-tL}\}_{t\geq 0}$ with heat kernels having pointwise upper bounds. Auscher, Duong, and McIntosh [3] investigated the Hardy space $H^1_L(\mathbb{R}^n)$ associated with the operator L. By introducing a new type of bounded mean oscillation (BMO) space $\mathrm{BMO}_L(\mathbb{R}^n)$ associated with L, Duong and Yan [11], [12] established the duality between the Hardy space $H^1_L(\mathbb{R}^n)$ and the space $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* is the adjoint operator of L. These results were further extended to $H^p_L(\mathbb{R}^n)$ for $p \leq 1$ by Yan in [32]. More generally, Jiang, Yang, and Zhou [21] studied the Orlicz–Hardy spaces associated with such operators. On the other hand, Auscher and Russ [4] treated the Hardy space $H^1_L(\mathbb{R}^n)$ on strongly Lipschitz domains associated with a divergence form elliptic operator L with heat kernels having Gaussian upper bounds and regularity. Under the assumption that L is a second-order divergence form elliptic operator with bounded complex coefficients, Hofmann and Mayboroda [18] studied the Hardy space $H^1_L(\mathbb{R}^n)$ and its dual space, which was generalized into the Orlicz–Hardy space by Jiang and Yang in [20]. It is also worth mentioning that, by subtly modifying some techniques created by Calderón in [6], Song and Yan [29] obtained the nontangential maximal characterization of $H^p_L(\mathbb{R}^n)$ with L being a nonnegative self-adjoint operator whose heat kernels satisfy Gaussian upper bound estimates.

Due to the rapid development of variable spaces and their wide applications in other areas, variable Hardy spaces associated with operators have also gained much attention in recent years. The literature on this subject is extensive and we briefly mention here several results closely related to the present paper. By means of tent spaces with variable exponents, Yang and Zhuo [36] established the molecular characterization of $H^{p(\cdot)}_L(\mathbb{R}^n)$ and obtained a duality theorem for $H^{p(\cdot)}_L(\mathbb{R}^n)$ with L being a linear operator on $L^2(\mathbb{R}^n)$ whose kernels have pointwise upper bounds. Let L be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ whose heat kernels satisfy the Gaussian upper bound estimates. Zhuo and Yang [38] constructed the atomic decompositions of variable Hardy spaces associated with such an operator L. Using the atomic decompositions, they obtained the nontangential maximal function characterization of $H^{p(\cdot)}_L(\mathbb{R}^n)$ which further induced the radial maximal function characterization of $H^{p(\cdot)}_L(\mathbb{R}^n)$. Yang, Zhang, and Zhuo [35] introduced the variable Hardy spaces associated to operators satisfying Davies–Gaffney estimates. Variable weak Hardy spaces associated to operators satisfying Davies–Gaffney estimates were also investigated by Zhuo and Yang in [39]. Recently, Yang and Zhang [34] considered the variable Hardy spaces associated to operators satisfying Davies-Gaffney estimates on metric measure spaces of homogeneous type.

As is well known, the family of Lorentz spaces is more extensive than that of Lebesgue spaces (see [7], [25]). The study of Hardy–Lorentz spaces has always been an interesting topic. For example, Fefferman, Rivière, and Sagher [13] considered the real interpolation of the Hardy–Lorentz space $H^{p,q}(\mathbb{R}^n)$; Fefferman and Soria [15] investigated the space $H^{1,\infty}(\mathbb{R}^n)$. Abu-Shammala and Torchinsky [1] established the atomic decomposition of $H^{p,q}(\mathbb{R}^n)$. They used atomic decompositions to obtain the dualities of $H^{p,q}(\mathbb{R}^n)$ for $0 < p \leq 1$ and $0 < q < \infty$.

Recently, the variable Lorentz space $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ was introduced by Kempka and Vybíral in [24]. It is a generalization of both variable Lebesgue space and Lorentz space. They provided some basic properties of variable Lorentz spaces and showed that these spaces arise through real interpolation between variable exponent spaces and $L^\infty(\mathbb{R}^n)$ when $q(\cdot) \equiv q$ is a constant. Very recently, Jiao, Zuo, Zhou, and Wu [23] introduced the variable Hardy–Lorentz spaces $H^{p(\cdot),q}(\mathbb{R}^n)$ and constructed the atomic decompositions on $H^{p(\cdot),q}(\mathbb{R}^n)$. As applications of the atomic decompositions, they developed a theory of real interpolation and formulated the dual space of the variable Hardy–Lorentz space with $0 < p_- \leq p_+ \leq 1$ and $0 < q < \infty$.

However, variable Hardy–Lorentz spaces associated with operators have not yet been investigated. This is exactly the main concern of the present paper. Actually, we will restrict our attention to the variable Hardy–Lorentz spaces associated with one-to-one operators of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$ with bounded holomorphic functional calculus and satisfying the Davies–Gaffney estimates, which are denoted by $H_L^{p(\cdot),q}(\mathbb{R}^n)$ in the remainder of this article. To be precise, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$S_L f(x) := \left(\iint_{\Gamma(x)} \left| (t^2 L) e^{-t^2 L} f(y) \right|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}}, \tag{1.1}$$

where, and in what follows, $\Gamma(x) := \{(y,t) \in \mathbb{R}_+^{n+1} : |y - x| < t\}$. The variable Hardy–Lorentz space associated with one-to-one operators of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$ with bounded holomorphic functional calculus and satisfying the Davies–Gaffney estimates is the completion of the set of all functions $f \in L^2(\mathbb{R}^n)$ such that $\|S_L(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty$. We establish its molecular characterizations via the atomic decompositions of the variable Lorentz tent space $T^{p(\cdot),q}(\mathbb{R}_+^{n+1})$. By using this molecular characterization, we interpret the dual spaces of the variable Hardy–Lorentz space $H_L^{p(\cdot),q}(\mathbb{R}^n)$. These results are new even in the case where $p(\cdot) \equiv p$ for some $0 < p \leq 1$.

This paper is organized as follows. Section 2 is devoted to describing the assumptions imposed on the operator L, recalling the definitions and some properties of the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the variable Lorentz spaces considered in this paper, and giving some basic properties which are useful for the proofs in the remainder of this article. In Section 3, we define the variable Lorentz tent space $T^{p(\cdot),q}(\mathbb{R}_+^{n+1})$ with $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < q < \infty$. Moreover, we characterize the atomic decompositions on $T^{p(\cdot),q}(\mathbb{R}_+^{n+1})$. In Section 4, we introduce the variable Hardy–Lorentz spaces associated to operators satisfying Davies–Gaffney estimates and prove that the operator $\pi_{L,M}$ is bounded from $T^{p(\cdot),q}(\mathbb{R}_+^{n+1})$ to $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Then we get the molecular decomposition of $H_L^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Furthermore, we deduce the molecular decomposition of $H_L^{p(\cdot),q}(\mathbb{R}^n)$.

Section 5 is focused on figuring out dual spaces of the variable Hardy–Lorentz space $H_L^{p(\cdot),q}(\mathbb{R}^n)$ for $0 < p_- \leq p_+ \leq 1$ and $0 < q < \infty$. We discuss the dualities according to the cases $0 < q \leq 1$ and $1 < q < \infty$ (see Theorems 5.6 and 5.9 below). The technical construction is inspired by [1] and the recent papers [22] and [23].

However, we use molecular decompositions instead of atomic decompositions to characterize dual spaces of variable Hardy–Lorentz spaces $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Precisely, we introduce two types of BMO spaces: $\mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)$ and $\mathrm{BMO}_L^{p(\cdot),q,M}(\mathbb{R}^n)$. Then, under the assumption that $p(\cdot)$ satisfies the log-Hölder condition, we prove that

$$\big(H_L^{p(\cdot),q}(\mathbb{R}^n)\big)^* = \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n), \quad 0 < q \le 1$$

and

$$\big(H_L^{p(\cdot),q}(\mathbb{R}^n)\big)^* = \mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n), \quad 1 < q < \infty$$

with equivalent norms.

Throughout this article, L denotes the one-to-one operator of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$ with bounded holomorphic functional calculus and satisfying the Davies–Gaffney estimates without any special explanation. We denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of nonnegative integers and the set of integers, respectively. And C denotes a positive constant independent of the parameters, which can vary from line to line. The symbols $A \lesssim B$ and $A \approx B$ are used for the inequalities $A \le CB$ and $A \lesssim B \lesssim A$, respectively. For a subset E of $\mathbb{R}^n$, we denote the set $\mathbb{R}^n \backslash E$ by $E^{\complement}$. Finally, for two spaces X and Y, if X is continuously embedded in Y, then we write $X \hookrightarrow Y$.

2. Preliminaries

In this section we introduce the assumptions imposed on the operator L and some useful definitions and preliminary information which are used in this paper.

2.1. Assumptions on the operator L. We begin by recalling some notions of bounded holomorphic calculi which was introduced by McIntosh in [26]. Let $0 \le \omega < \pi$. The closed sector in the complex plane $\mathbb{C}$ is defined by setting

$$S_\omega := \big\{z \in \mathbb{C} : |\arg z| \le \omega\big\} \cup \{0\};$$

S_ω^0 is its interior, namely,

$$S_\omega^0 := \big\{z \in \mathbb{C}\backslash\{0\} : |\arg z| \le \omega\big\}.$$

Let $H(S_\omega^0)$ denote the set of all holomorphic functions on S_ω^0, and let

$$H_\infty(S_\omega^0) := \big\{b \in H(S_\omega^0) : \|b\|_\infty < \infty\big\},$$

where $\|b\|_\infty = \sup\{b(z) : z \in S_\omega^0\}$. Define

$$\Psi(S_\omega^0) := \Big\{\psi \in H(S_\omega^0) : \exists\, \nu,\, C > 0 : \big|\psi(z)\big| \le \frac{c|z|^\nu}{1 + |z|^{2\nu}}\Big\}.$$

Let $\omega \in [0, \pi)$. A closed operator L on $L^2(\mathbb{R}^n)$ is said to be of type ω if $\sigma(L) \subset S_\omega$, where $\sigma(L)$ denotes the spectra of L, and for all $\upsilon \in (\omega, \pi)$, there exists a positive constant C_υ such that for all $\lambda \notin S_\upsilon$,

$$\big\|(L - \lambda I)^{-1}\big\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_\upsilon |z|^{-1},$$

where $\mathcal{L}(L^2(\mathbb{R}^n))$ is the set of all linear continuous operators from $L^2(\mathbb{R}^n)$ to itself, and for any operator $T \in \mathcal{L}(L^2(\mathbb{R}^n))$, $\|T\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ denotes the norm of T.

Let $\psi \in \Psi(S_\upsilon^0)$, and let L be an operator of type ω in $L^2(\mathbb{R}^n)$. The operator $\psi(L)$ is defined by

$$\psi(L) := \frac{1}{2\pi i}\int_\Theta \psi(\lambda)(\lambda I - L)^{-1}\, d\lambda, \tag{2.1}$$

where $\Theta = \{re^{+i\nu} : r \in (0,\infty)\} \cup \{re^{+i\nu} : r \in (0,\infty)\}$, $\nu \in (\omega, \upsilon)$ is the curve consisting of two rays parameterized counterclockwise. The integral in (2.1) is absolutely convergent in $L^2(\mathbb{R}^n)$ (see [16], [26]) and $\psi(L)$ does not depend on the choice of ν (see [2, Lecture 2]). The above holomorphic functional calculus on $\Psi(S_\upsilon^0)$ can be extended to $H_\infty(S_\upsilon^0)$ (see [26]). Let $\upsilon \in (0,\pi)$. An operator L is said to have a bounded $H_\infty(S_\upsilon^0)$-calculus in $L^2(\mathbb{R}^n)$ if there exists a positive constant C such that for all $\psi \in H_\infty(S_\upsilon^0)$,

$$\big\|\psi(L)\big\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C\|\psi\|_{L^\infty(S_\upsilon^0)}.$$

Let $\omega \in [0, \frac{\pi}{2})$. Then by [16, Proposition 7.1.1], it follows that if L is an operator of type ω in $L^2(\mathbb{R}^n)$, then L generates a bounded holomorphic semigroup $\{e^{-zL}\}_{z\in S^0_{\frac{\pi}{2}-\omega}}$ on the open sector $S^0_{\frac{\pi}{2}-\omega}$.

We now make two assumptions on the operator L.

Assumption (A). L is a one-to-one operator of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \frac{\pi}{2})$ and has a bounded holomorphic functional calculus.

Assumption (B). The semigroup $\{e^{-tL}\}_{t>0}$ generated by L satisfies the Davies–Gaffney estimates; that is, there exist positive constants c_1 and c_2 such that, for any function $f \in L^2(\mathbb{R}^n)$ and closed sets $E, F \subset \mathbb{R}^n$ with $\operatorname{supp} f \subset E$,

$$\big\|e^{-tL}(f)\big\|_{L^2(F)} \le c_1 e^{-c_2\frac{\operatorname{dist}(E,F)}{t}}\|f\|_{L^2(E)}.$$

Remark 2.1. Let L be an operator satisfying Assumption (A) and Assumption (B). Then,

(1) for any $i \in \mathbb{Z}_+$, the family of operators $\{(tL)^i e^{-tL}\}_{t>0}$ satisfies the Davies–Gaffney estimates (see [35, Remark 2.5(i)]);
(2) using Fubini's theorem and (4.1) in [17], one can prove that the operator S_L defined in (1.1) is bounded on $L^2(\mathbb{R}^n)$.

Example 2.2. One may check that the following operators satisfy Assumption (A) and Assumption (B):

(a) The one-to-one nonnegative self-adjoint operator L satisfying Gaussian upper bounds. Namely, there exist positive constants C and c such that, for any $x, y \in \mathbb{R}^n$ and $t \in (0,\infty)$,

$$\big|p_t(x,y)\big| \le \frac{C}{t^{\frac{n}{2}}}\exp\Big(-c\frac{|x-y|^2}{t}\Big).$$

(b) The second-order divergence form elliptic operators with complex bounded coefficients.
(c) The Schrödinger operator $-\triangle + V$ on $\mathbb{R}^n$ with the nonnegative potential $V \in L^1_{\rm loc}(\mathbb{R}^n)$ strictly greater than zero.

2.2. Variable exponent Lebesgue spaces. A variable exponent is a measurable function $p(\cdot):\mathbb{R}^n\to(0,\infty)$. For a variable exponent $p(\cdot)$, define

$$p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \qquad \text{and} \qquad p_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Denote by $\mathcal{P}(\mathbb{R}^n)$ the collection of variable exponents satisfying $0<p_-\le p_+<\infty$. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of measurable functions f such that

$$\int_{\mathbb{R}^n} \big|f(x)\big|^{p(x)}\,dx<\infty,$$

equipped with the quasinorm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\Big\{\lambda>0: \int_{\mathbb{R}^n}\Big[\frac{|f(x)|}{\lambda}\Big]^{p(x)}\,dx\le 1\Big\}.$$

Some elementary properties of variable Lebesgue spaces are collected below. The proofs can be found in [8, 21–22] and [27, p. 3671].

Remark 2.3. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $f,g\in L^{p(\cdot)}(\mathbb{R}^n)$.

(i) For $\lambda\in\mathbb{C}$, we have $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)}=|\lambda|\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.
(ii) For $s>0$, we have $\||f|^s\|_{L^{p(\cdot)}(\mathbb{R}^n)}=\|f\|^s_{L^{sp(\cdot)}(\mathbb{R}^n)}$.
(iii) If $p_-\in[1,\infty)$, then $\|f+g\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.
(iv) If $p_+\in(0,1)$, then $\|f+g\|_{L^{p(\cdot)}(\mathbb{R}^n)}\ge\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

The next lemma gives the embedding relationship between the classical and variable Lebesgue spaces, which can be found in [8, Corollary 2.27].

Lemma 2.4. *Given Ω and $p(\cdot)\in\mathcal{P}(\Omega)$ with $1<p_-<\infty$, suppose that $|\Omega|<\infty$. Then there exist constants $c_1,c_2>0$ such that*

$$c_1\|f\|_{L^{p_-}(\Omega)}\le\|f\|_{L^{p(\cdot)}(\Omega)}\le c_2\|f\|_{L^{p_+}(\Omega)}.$$

Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then $p(\cdot)$ is said to be globally log-Hölder continuous, denoted by $p(\cdot)\in C^{\log}(\mathbb{R}^n)$, if there exist constants $C_{p(\cdot)}$, C_∞, and p_∞ such that, for any $x,y\in\mathbb{R}^n$,

$$\big|p(x)-p(y)\big|\le\frac{C_{p(\cdot)}}{\log(e+1/|x-y|)}$$

and

$$\big|p(x)-p_\infty\big|\le\frac{C_\infty}{\log(e+|x|)}.$$

Recall that the Hardy–Littlewood maximal operator $\mathcal{M}$ is defined by setting, for all $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\mathcal{M}(f)(x):=\sup_{x\in B}\frac{1}{|B|}\int_B\big|f(y)\big|\,dy,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Lemma 2.5 ([10, Theorem 4.3.8])**.** *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\le p_+<\infty$. Then there exists a positive constant C such that for all $f\in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathcal{M}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.*

The following lemma (see [9, Corollary 2.1]) gives the vector-valued inequality for the Hardy–Littlewood maximal function on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.6. *Let $r \in (1,\infty)$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_- \in (1,\infty)$. Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions,*

$$\Big\|\Big(\sum_{j=1}^{\infty}|\mathcal{M}(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

From the preceding lemma and the fact that $\chi_{\beta B} \le \beta^{\frac{n}{r}}(\mathcal{M}(\chi_B))^{\frac{1}{r}}$, where $\beta \in [1,\infty)$, $r \in (0,\underline{p})$, and B is a ball in $\mathbb{R}^n$, we get the following result.

Corollary 2.7. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (0,\underline{p})$, and $\beta \in [1,\infty)$. Then for any sequence $\{B_j\}_{j\in\mathbb{N}}$, there exists a positive constant C such that*

$$\Big\|\sum_{j\in\mathbb{N}}\chi_{\beta B_j}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\beta^{\frac{n}{r}}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_j}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

2.3. Lorentz spaces with variable exponents. We now recall the definition of Lorentz spaces with variable exponents which we consider in this paper.

Definition 2.8. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < q \le \infty$. Then $L^{p(\cdot),q}(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that $\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} := \begin{cases} \big(\int_0^\infty \lambda^{q-1}\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\,d\lambda\big)^{\frac{1}{q}} & \text{if } 0<q<\infty,\\ \sup_{\lambda>0}\lambda\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} & \text{if } q=\infty.\end{cases}$$

An equivalent discrete characterization of the quasinorm $\|\cdot\|_{L^{p(\cdot),q}}$ is given below. For the proof, we refer to [24, Lemma 2.4].

Lemma 2.9. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < q \le \infty$. If $f \in L^{p(\cdot),q}(\mathbb{R}^n)$, then*

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \approx \begin{cases} \big(\sum_{i\in\mathbb{Z}} 2^{iq}\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^i\}}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\big)^{\frac{1}{q}} & \text{if } 0<q<\infty,\\ \sup_{i\in\mathbb{Z}} 2^i\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^i\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} & \text{if } q=\infty.\end{cases}$$

The following continuous embedding is just [24, Theorem 3.3(i)]. Indeed, it can also be easily seen from the lemma above.

Lemma 2.10. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and let $0 < q_1 \le q_2 \le \infty$. Then $L^{p(\cdot),q_1}(\mathbb{R}^n) \subset L^{p(\cdot),q_2}(\mathbb{R}^n)$. Moreover, we have $\|f\|_{L^{p(\cdot),q_2}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p(\cdot),q_1}(\mathbb{R}^n)}$.*

3. Tent spaces

In this section, we introduce the variable Lorentz tent space $T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$ for $0 < p_- \le p_+ < \infty$ and $0 < q < \infty$ and establish its atomic characterization. To this end, we begin with some notions and notation.

Let F be a closed set in $\mathbb{R}^n$, and let $O \equiv F^{\complement}$. We denote by $\widehat{O}$ the tent over O which is the set

$$\widehat{O} := \big\{(x,t) \in \mathbb{R}^{n+1}_+ : \mathrm{dist}(x,F) \ge t\big\}.$$

Assume that $|O| < \infty$. For any fixed $\gamma \in (0,1)$, if for all $r > 0$, we have

$$\frac{|B(x,r) \cap F|}{|B(x,r)|} \geq \gamma,$$

then x is said to have the global γ-density with respect to F. Define the set F^* by

$$F^* := \Big\{x \in \mathbb{R}^n : \text{ for all } r > 0, \frac{|B(x,r) \cap F|}{|B(x,r)|} \geq \gamma\Big\}.$$

It is obvious that F^* is closed and $F^* \subset F$. Let $O^* = (F^*)^{\complement}$. Then $O \subset O^*$. Also we have

$$O^* = \big\{x \in \mathbb{R}^n : \mathcal{M}(\chi_O)(x) > 1 - \gamma\big\},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator. Therefore, from the weak type $(1,1)$ of $\mathcal{M}$ it follows that $|O^*| < C_\gamma |O|$.

For all measurable functions f on $\mathbb{R}^{n+1}_+$ and for any $x \in \mathbb{R}^n$, define

$$\mathcal{A}(f)(x) := \Big(\iint_{\Gamma(x)} \big|f(y,t)\big|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2}.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The tent space $T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)$ is defined to be the space of all measurable functions f on $\mathbb{R}^{n+1}_+$ such that

$$\|f\|_{T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)} = \big\|\mathcal{A}(f)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

The variable Lorentz tent space $T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$ for $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < q < \infty$ is defined to be the space of all measurable functions f on $\mathbb{R}^{n+1}_+$ such that $\mathcal{A}(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $0 < q < \infty$. For $f \in T^{(\cdot),q}(\mathbb{R}^{n+1}_+)$, we define

$$\|f\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)} = \big\|\mathcal{A}(f)\big\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$

Remark 3.1. We have the following.

(i) If $p(\cdot) \equiv p$ for some $p \in (0,\infty)$, then $T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+) = T_2^{p}(\mathbb{R}^{n+1}_+)$ and $T^{p(\cdot),q}(\mathbb{R}^{n+1}_+) = T^{p,q}(\mathbb{R}^{n+1}_+)$. Moreover, if $p(\cdot) = p = q$, then $T^{p(\cdot),q}(\mathbb{R}^{n+1}_+) = T_2^{p}(\mathbb{R}^{n+1}_+)$.

(ii) If $f \in T_2^2(\mathbb{R}^{n+1}_+)$, then it is easy to show that

$$\|f\|_{T_2^2(\mathbb{R}^{n+1}_+)} = \Big(\iint_{\mathbb{R}^{n+1}_+} \big|f(x,t)\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}}.$$

Next, we want to establish the atomic characterization of the Lorentz tent space $T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$. Before going further, we recall the definition of $(p(\cdot),r)$-atoms.

Definition 3.2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $r \in (1,\infty)$. A function a on $\mathbb{R}^{n+1}_+$ is called a $(p(\cdot),\infty)$-atom if

(i) there exists a ball $B \subset \mathbb{R}^n$ such that $\operatorname{supp} a \subset \widehat{B}$,
(ii) $\|a\|_{T_2^r(\mathbb{R}^{n+1}_+)} \leq |B|^{1/r} \|\chi_B\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{C}$, and let $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^n$. Define

$$\begin{aligned}&\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big)\\&\quad:= \Big(\sum_{i\in\mathbb{Z}}\Big\|\Big\{\sum_{j\in\mathbb{N}}\Big[\frac{|\lambda_{i,j}|\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\Big]^{\underline{p}}\Big\}^{\frac{1}{\underline{p}}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q}},\end{aligned}$$

where and in what follows $\underline{p} = \min\{1, p_-\}$.

Theorem 3.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in (0,\infty)$. Then for $f \in T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$, there exist $(p(\cdot),\infty)$-atoms $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ associated with the balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that, for any $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}(x) \le A$, where A is a positive constant independent of i and x, numbers $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}} \subset \mathbb{C}$ such that, for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,*

$$f(x,t) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j}(x,t). \tag{3.1}$$

Moreover, there exists a positive constant C such that, for all $f \in T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$,

$$\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) \le C\|f\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)}. \tag{3.2}$$

Proof. Let $f \in T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$. For any $i \in \mathbb{Z}$, let

$$\Omega_i = \big\{x \in \mathbb{R}^n : \mathcal{A}(f)(x) > 2^i\big\}.$$

Since $f \in T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$, it is easy to check that Ω_i is a proper open set and that $|\Omega_i| < \infty$ for each $i \in \mathbb{Z}$. Indeed,

$$\begin{aligned}|\Omega_i| &= \int_{\mathbb{R}^n}\Big(\frac{\chi_{\Omega_i}(x)}{\|\chi_{\Omega_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\Big)^{p(x)}\|\chi_{\Omega_i}\|^{p(x)}_{L^{p(\cdot)}(\mathbb{R}^n)}\,dx\\&\lesssim \max\big\{\|\chi_{\Omega_i}\|^{p_-}_{L^{p(\cdot)}(\mathbb{R}^n)}, \|\chi_{\Omega_i}\|^{p_+}_{L^{p(\cdot)}(\mathbb{R}^n)}\big\}\\&\lesssim \max\{2^{-ip_-}, 2^{-ip_+}\}\big\|\mathcal{A}(f)\big\|_{L^{p(\cdot),\infty}(\mathbb{R}^n)}\\&\lesssim \max\{2^{-ip_-}, 2^{-ip_+}\}\|f\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1})},\end{aligned}$$

where the last inequality used Lemma 2.10.

By an argument similar to that used in the proof of [19, Theorem 3.2], we can prove that $\mathrm{supp} f \subset (\bigcup_{i\in\mathbb{Z}}\widehat{O_i^*}) \cup E$, where $E \subset \mathbb{R}^{n+1}_+$ satisfies $\int_E \frac{dy\,dt}{t} = 0$. Thus, for each $i \in \mathbb{Z}$, by applying the Whitney decomposition (see [30, p. 167]) to Ω_i^*, we get a sequence $\{Q_{i,j}\}_{j\in\mathbb{N}}$ of disjoint cubes such that

(1) $\bigcup_{j\in\mathbb{N}} Q_{i,j} = \Omega_i^*$ and $\{Q_{i,j}\}_{j\in\mathbb{N}}$ have disjoint interiors,
(2) for all $j \in \mathbb{N}$,

$$c_1\sqrt{n}l_{Q_{i,j}} \le \mathrm{dist}\big(Q_{i,j}, (\Omega_i^*)^\complement\big) \le c_2\sqrt{n}l_{Q_{i,j}}, \tag{3.3}$$

where $l_{Q_{i,j}}$ denotes the side length of the cube $Q_{i,j}$, $\mathrm{dist}(Q_{i,j}, (\Omega_i^*)^\complement) := \inf\{|x-y| : x \in Q_{i,j}, y \in (\Omega_i^*)^\complement\}$.

For each $j \in \mathbb{N}$, choose a ball $B_{i,j}$ with the same center with $Q_{i,j}$ and with radius $\frac{11}{2}\sqrt{n} l_{Q_{i,j}}$. Define

$$\begin{aligned} A_{i,j} &= \widehat{B}_{i,j} \cap \big(Q_{i,j} \times (0,\infty)\big) \cap (\widehat{\Omega_i^*} \setminus \widehat{\Omega_{i+1}^*}), \\ a_{i,j} &= 2^{-i} \|\chi_{B_{i,j}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} f \chi_{A_{i,j}}, \qquad \text{and} \\ \lambda_{i,j} &= 2^i \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.4}$$

Notice that $\{(Q_{i,j} \times (0,\infty)) \cap (\widehat{\Omega_i^*} \setminus \widehat{\Omega_{i+1}^*})\} \subset \widehat{B}_{i,j}$. It follows from the proof of [40, Theorem 2.16] that $a_{i,j}$ is a $(p(\cdot),\infty)$-atom associated to $B_{i,j}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. We obtain that $f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j}$ almost everywhere. It follows from [23, Remark 5.5] that we know the balls $B_{i,j}$ have the bounded overlapping property for any fixed $i \in \mathbb{Z}$. Namely,

$$\sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \lesssim \chi_{\Omega_i^*} \lesssim 1.$$

By the definition of $\lambda_{i,j}$, we get

$$\begin{aligned} \mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) &= \Bigg(\sum_{i\in\mathbb{Z}} \Big\| \Big(\sum_{j\in\mathbb{N}} (2^i \chi_{B_{i,j}})^{\underline{p}}\Big)^{\frac{1}{\underline{p}}} \Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}} \\ &\lesssim \Bigg(\sum_{i\in\mathbb{Z}} 2^{iq} \Big\| \Big(\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}\Big)^{\frac{1}{\underline{p}}} \Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}} \\ &\lesssim \Bigg(\sum_{i\in\mathbb{Z}} 2^{iq} \|\chi_{\Omega_i^*}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}} \\ &= \Bigg(\sum_{i\in\mathbb{Z}} 2^{iq} \|\chi_{\Omega_i^*}\|^{\frac{q}{r}}_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}, \end{aligned}$$

where $r \in (0,\underline{p})$. By Lemma 2.5, it follows that

$$\begin{aligned} \mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) &\lesssim \Bigg(\sum_{i\in\mathbb{Z}} 2^{iq} \Big\| \frac{\mathcal{M}(\chi_{\Omega_i})}{1-\gamma} \Big\|^{\frac{q}{r}}_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}} \\ &\lesssim \Bigg(\sum_{i\in\mathbb{Z}} 2^{iq} \|\chi_{\Omega_i}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}. \end{aligned}$$

Thus, we have $\mathcal{A}(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}) \lesssim \|\mathcal{A}(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} = \|f\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)}$. The proof is complete. □

Remark 3.4. We have the following.

(i) From the proof of Theorem 3.3 and the fact that the sequence of the balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ satisfies, for any $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \lesssim 1$, we have

$$\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) \approx \Bigg(\sum_{i\in\mathbb{Z}} 2^{iq} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}.$$

(ii) Let $\{a^k_{i,j}\}_{i,j}$ be $(p(\cdot),\infty)$-atoms, and let $\{\lambda^k_{i,j}\}_{i,j}$ be numbers in $\mathbb{C}$, where $k = 1, 2$. Then we have the following two cases:

(1) If $\underline{p} < q$, then

$$\left(\mathcal{A}\left(\{\lambda^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N},k}, \{B^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N},k}\right)\right)^{\underline{p}} \le \sum_{k=1}^{2}\left(\mathcal{A}\left(\{\lambda^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right)\right)^{\underline{p}}.$$

(2) If $\underline{p} \ge q$, then

$$\left(\mathcal{A}\left(\{\lambda^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N},k}, \{B^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N},k}\right)\right)^{q} \le \sum_{k=1}^{2}\left(\mathcal{A}\left(\{\lambda^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B^k_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right)\right)^{q}.$$

(iii) Let $r \in (0,\infty)$. From [20, Proposition 3.1], if $f \in T^r_2(\mathbb{R}^{n+1}_+)$, then (3.1) holds true in $T^r_2(\mathbb{R}^{n+1}_+)$.

Proposition 3.5. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$ and $q \in (0,\infty)$. Suppose that $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot),\infty)$-atoms associated with the balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that for $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \le A$, where A is a positive constant independent of x and i, and $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of numbers in $\mathbb{C}$ such that for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} \approx 2^i\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, satisfies*

$$\mathcal{A}\left(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right) < \infty.$$

If for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,

$$f(x,t) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j}(x,t),$$

then $f(x,t) \in T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$. Moreover,

$$\left\|\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j}(x,t)\right\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)} \lesssim \mathcal{A}\left(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right).$$

In order to prove the above proposition, we need the following technical lemma, which is a variant of [28, Theorem 4.1].

Lemma 3.6. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (0,\underline{p}]$, and $q \in [1,\infty) \cap (p_+,\infty)$. Then there exists a positive constant C such that, for all sequences $\{B_j\}_{j\in\mathbb{N}}$ of balls, numbers $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, and measurable functions $\{a_j\}_{j\in\mathbb{N}}$ satisfying that, for each $j \in \mathbb{N}$, $\operatorname{supp} a_j \subset B_j$ and $\|a_j\|_{L^q(\mathbb{R}^n)} \le |B_j|^{1/q}$, it holds true that*

$$\left\|\left(\sum_{j\in\mathbb{N}}|\lambda_j a_j|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|\left(\sum_{j\in\mathbb{N}}|\lambda_j \chi_{B_j}|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof of Proposition 3.5. Suppose that $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot),\infty)$-atoms and that $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ are numbers in $\mathbb{C}$ such that $\lambda_{i,j} \approx 2^i\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. For $i_0 \in \mathbb{Z}$, let

$$f = \sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j} + \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j} =: f_1 + f_2.$$

We get

$$\begin{aligned}\|\chi_{\{x\in\mathbb{R}^n:\mathcal{A}(f)(x)>2^{i_0}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\lesssim \|\chi_{\{x\in\mathbb{R}^n:\mathcal{A}(f_1)(x)>2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad+\|\chi_{\{x\in\mathbb{R}^n:\mathcal{A}(f_2)(x)>2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} =: I_1+I_2.\end{aligned}$$

We first estimate I_1. For any $b\in(0,\min\{p_-,q\})$ and $r\in(1,\infty)$, let $\widetilde{q}\in(1,\min\{r,\frac{1}{b}\})$ and $a\in(0,1-\frac{1}{\widetilde{q}})$. By the Hölder inequality, following the argument used in [33, pp. 2849–2850], we get

$$\begin{aligned}I_1 &\lesssim \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\lambda_{i,j}\mathcal{A}(a_{i,j})(x)>2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim 2^{-i_0\widetilde{q}(1-a)}\Big\|\sum_{i=-\infty}^{i_0-1}2^{-i_0a\widetilde{q}}\Big(\sum_{j\in\mathbb{N}}\lambda_{i,j}\mathcal{A}(a_{i,j})\Big)^{\widetilde{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim 2^{-i_0\widetilde{q}(1-a)}\Big\|\sum_{i=-\infty}^{i_0-1}2^{(1-a)i_0\widetilde{q}b}\Big(\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\mathcal{A}(a_{i,j})\Big)^{\widetilde{q}b}\Big\|^{\frac{1}{b}}_{L^{\frac{p(\cdot)}{b}}(\mathbb{R}^n)}.\end{aligned}$$

Since $\widetilde{q}b<1$, we have

$$\begin{aligned}I_1 &\lesssim 2^{-i_0\widetilde{q}(1-a)}\Big(\sum_{i=-\infty}^{i_0-1}2^{(1-a)i_0\widetilde{q}b}\Big\|\sum_{j\in\mathbb{N}}\big(\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\mathcal{A}(a_{i,j})\big)^{\widetilde{q}b}\Big\|_{L^{\frac{p(\cdot)}{b}}(\mathbb{R}^n)}\Big)^{\frac{1}{b}}\\ &\lesssim 2^{-i_0\widetilde{q}(1-a)}\Big(\sum_{i=-\infty}^{i_0-1}2^{(1-a)\widetilde{q}ib}\Big\|\Big(\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|^{\widetilde{q}b}_{L^{p(\cdot)}(\mathbb{R}^n)}\mathcal{A}(a_{i,j})^{\widetilde{q}b}\Big)^{\frac{1}{b}}\Big\|^{b}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{b}}.\end{aligned}$$

From the definition of $(p(\cdot),\infty)$-atoms, it follows that

$$\big\|\mathcal{A}(a_{i,j})\big\|_{L^r(\mathbb{R}^n)}=\|a_{i,j}\|_{T_2^r(\mathbb{R}^{n+1}_+)}\lesssim|B_{i,j}|^{\frac{1}{r}}\|\chi_{B_{i,j}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Let $\tilde{r}=\frac{r}{\tilde{q}}$. For all $i\in\mathbb{Z}$ and $j\in\mathbb{N}$, we know that

$$\|\chi_{B_{i,j}}\|^{\widetilde{q}}_{L^{p(\cdot)}(\mathbb{R}^n)}\big\|\mathcal{A}(a_{i,j})^{\widetilde{q}}\big\|_{L^{\tilde{r}}(\mathbb{R}^n)}\le\|\chi_{B_{i,j}}\|^{\widetilde{q}}_{L^{p(\cdot)}(\mathbb{R}^n)}\big\|\mathcal{A}(a_{i,j})\big\|^{\widetilde{q}}_{L^r(\mathbb{R}^n)}\lesssim|B_{i,j}|^{\frac{1}{\tilde{r}}}. \quad (3.5)$$

Let $1<\delta_1<(1-a)\widetilde{q}$. Using Lemma 3.6 and the Hölder inequality for $\frac{q-b}{q}+\frac{b}{q}=1$, we obtain

$$\begin{aligned}I_1 &\lesssim 2^{-i_0\widetilde{q}(1-a)}\Big(\sum_{i=-\infty}^{i_0-1}2^{[(1-a)\widetilde{q}-\delta_1]ib}2^{ib\delta_1}\Big\|\Big(\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big)^{\frac{1}{b}}\Big\|^{b}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{b}}\\ &\le 2^{-i_0\widetilde{q}(1-a)}\Big(\sum_{i=-\infty}^{i_0-1}2^{[(1-a)\widetilde{q}-\delta_1]ib\frac{q}{q-b}}\Big)^{\frac{q-b}{bq}}\\ &\quad\times\Big(\sum_{i=-\infty}^{i_0-1}2^{iq\delta_1}\Big\|\Big(\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big)^{\frac{1}{b}}\Big\|^{q}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q}}\\ &\lesssim 2^{-i_0\delta_1}\Big(\sum_{i=-\infty}^{i_0-1}2^{iq\delta_1}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^{q}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q}}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\sum_{i_0=-\infty}^{\infty}(2^{i_0}I_1)^q &\lesssim \sum_{i_0=-\infty}^{\infty}2^{i_0q}2^{-i_0\delta_1 q}\sum_{i=-\infty}^{i_0-1}2^{iq\delta_1}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &= \sum_{i=-\infty}^{\infty}2^{iq\delta_1}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i_0=i+1}^{\infty}2^{i_0q(1-\delta_1)}\\ &\lesssim \sum_{i\in\mathbb{Z}}2^{iq}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}. \qquad (3.6)\end{aligned}$$

We now deal with I_2. Let $r_1\in(0,1)$ and $b_1\in(0,\min\{p_-,q\})$. Then

$$\begin{aligned}I_2 &= \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\lambda_{i,j}\mathcal{A}(a_{i,j})(x)>2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim 2^{-i_0r_1}\Big\|\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\big(\lambda_{i,j}\mathcal{A}(a_{i,j})\big)^{r_1}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Using (ii) of Remark 2.3, we have

$$\begin{aligned}I_2 &\lesssim 2^{-i_0r_1}\Big\|\Big(\sum_{i=0}^{\infty}\sum_{j\in\mathbb{N}}\big(\lambda_{i,j}\mathcal{A}(a_{i,j})\big)^{r_1}\Big)^{b_1}\Big\|^{\frac{1}{b_1}}_{L^{\frac{p(\cdot)}{b_1}}(\mathbb{R}^n)}\\ &\lesssim 2^{-i_0r_1}\Big(\sum_{i=i_0}^{\infty}\Big\|\sum_{j\in\mathbb{N}}\big(\lambda_{i,j}\mathcal{A}(a_{i,j})\big)^{r_1b_1}\Big\|_{L^{\frac{p(\cdot)}{b_1}}(\mathbb{R}^n)}\Big)^{\frac{1}{b_1}}\\ &\lesssim 2^{-i_0r_1}\Big(\sum_{i=i_0}^{\infty}2^{ir_1b_1}\Big\|\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|^{r_1b_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\mathcal{A}(a_{i,j})^{r_1b_1}\Big\|_{L^{\frac{p(\cdot)}{b_1}}(\mathbb{R}^n)}\Big)^{\frac{1}{b_1}}\\ &= 2^{-i_0r_1}\Big(\sum_{i=i_0}^{\infty}2^{ir_1b_1}\Big\|\Big(\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|^{r_1b_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\mathcal{A}(a_{i,j})^{r_1b_1}\Big)^{\frac{1}{b_1}}\Big\|^{b_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{b_1}}.\end{aligned}$$

Let $r_1<\delta_2<1$. It follows from (3.5) with $\widetilde{q}$ replaced by r_1, Lemma 3.6, and the Hölder inequality for $\frac{q-b_1}{q}+\frac{b_1}{q}=1$ that

$$\begin{aligned}I_2 &\lesssim 2^{-i_0r_1}\Big(\sum_{i=i_0}^{\infty}2^{ib_1r_1-i\delta_2b_1}2^{i\delta b_1}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^{b_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{b_1}}\\ &\lesssim 2^{-i_0r_1}\Big(\sum_{i=i_0}^{\infty}2^{(ib_1r_1-i\delta_2b_1)\frac{q}{q-b_1}}\Big)^{\frac{q-b_1}{b_1q}}\\ &\quad\times\Big(\sum_{i=i_0}^{\infty}2^{i\delta q}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q}}\\ &\lesssim 2^{-i_0\delta_2}\Big(\sum_{i=i_0}^{\infty}2^{i\delta_2q}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i_0=-\infty}^{\infty}(2^{i_0}I_2)^q &\lesssim \sum_{i_0=-\infty}^{\infty} 2^{i_0 q}2^{-i_0\delta_2 q}\sum_{i=i_0}^{\infty}2^{iq\delta_2}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \sum_{i=-\infty}^{\infty}2^{iq\delta_2}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i_0=-\infty}^{i}2^{i_0 q(1-\delta_2)}\\ &\lesssim \sum_{i=-\infty}^{\infty}2^{iq}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}\tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\Big\|\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j}(x,t)\Big\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)} \lesssim \mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}},\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big).$$

The proof is complete. □

We denote by $T_c^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$ and $T_2^{2,c}(\mathbb{R}^{n+1}_+)$ the set of all functions in $T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$ and $T_2^2(\mathbb{R}^{n+1}_+)$ which have compact supports, respectively.

Lemma 3.7. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ and $q\in(0,\infty)$. Then $T_c^{p(\cdot),q}(\mathbb{R}^{n+1}_+)\subset T_c^2(\mathbb{R}^{n+1}_+)$ in the meaning of sets.*

Proof. By [21, Lemma 3.3], it is enough to prove that $T_c^{p(\cdot),q}(\mathbb{R}^{n+1}_+)\subset T_c^r(\mathbb{R}^{n+1}_+)$ for $r\in(0,\infty)$. Let $f\in T_c^{p(\cdot),q}(\mathbb{R}^{n+1}_+)$ be such that $\operatorname{supp} f\subset K$ ($K\subset\widehat{B}$ is a compact set), and let B be a ball in $\mathbb{R}^n$. Suppose that $\operatorname{supp}\mathcal{A}f\subset B$. Let $s\in(0,p_-)$ and $r\in(0,\infty)$ be such that $r<p_-$. We obtain

$$\begin{aligned}\|f\|^r_{T^r(\mathbb{R}^{n+1}_+)} &= \big\|\mathcal{A}(f)\big\|^r_{L^r(\mathbb{R}^n)} = r\int_0^\infty t^{r-1}\big|\{x\in B:\mathcal{A}f(x)>t\}\big|\,dt\\ &= r\Big(\int_0^1 t^{r-1}\big|\{x\in B:\mathcal{A}f(x)>t\}\big|\,dt\\ &\quad+\int_1^\infty t^{r-1}\big|\{x\in B:\mathcal{A}f(x)>t\}\big|\,dt\Big)\\ &\le |B| + r\int_1^\infty t^{r-p_--1}\big(t\big|\{x\in B:\mathcal{A}f(x)>t\}\big|^{\frac{1}{p_-}}\big)^{p_-}\,dt.\end{aligned}$$

It follows from (ii) of Remark 2.3, Lemma 2.4, and Lemma 2.10 that

$$\begin{aligned}\|f\|^r_{T^r(\mathbb{R}^{n+1}_+)} &= |B| + r\int_1^\infty t^{r-p_--1}\big(t\|\chi_{\{x\in B:\mathcal{A}f(x)>t\}}\|^{\frac{1}{s}}_{L^{\frac{p_-}{s}}(B)}\big)^{p_-}\,dt\\ &\lesssim |B| + r\int_1^\infty t^{r-p_--1}\big(t\|\chi_{\{x\in B:\mathcal{A}f(x)>t\}}\|_{L^{p(\cdot)}(B)}\big)^{p_-}\,dt\\ &\lesssim |B| + r\int_1^\infty t^{r-p_--1}\|\mathcal{A}f\|^{p_-}_{L^{p(\cdot),\infty}(\mathbb{R}^n)}\,dt\\ &\lesssim |B| + \frac{r}{p_- - r}\|\mathcal{A}f\|^{p_-}_{L^{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned}$$

The proof is complete. □

4. Variable Hardy–Lorentz spaces associated to operators satisfying Davies–Gaffney estimates

In this section, we introduce the variable Hardy–Lorentz spaces associated to operators satisfying Davies–Gaffney estimates, which we denote by $H_L^{p(\cdot),q}(\mathbb{R}^n)$. We begin with a definition.

Definition 4.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (0,\infty)$. Define the space $\widetilde{H}_L^{p(\cdot),q}(\mathbb{R}^n)$ by

$$\widetilde{H}_L^{p(\cdot),q}(\mathbb{R}^n) := \big\{ f \in L^2 : S_L(f) \in L^{p(\cdot),q}(\mathbb{R}^n) \big\}.$$

Moreover, define

$$\|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)} := \big\|S_L(f)\big\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$

The variable Hardy–Lorentz space $H_L^{p(\cdot),q}$ is the completion of the space $\widetilde{H}_L^{p(\cdot),q}(\mathbb{R}^n)$ in the quasinorm $\|\cdot\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}$.

Remark 4.2. (i) From the theorem of Yosida [37, p. 65], we have that $\widetilde{H}_L^{p(\cdot),q}(\mathbb{R}^n)$ is dense in $H_L^{p(\cdot),q}(\mathbb{R}^n)$. (ii) When $L = -\triangle$, the space $H_L^{p(\cdot),q}(\mathbb{R}^n)$ comes back to the Hardy–Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$.

To introduce the molecular variable Hardy–Lorentz space $H_{L,M,\epsilon}^{p(\cdot),q}(\mathbb{R}^n)$, we give the definition of a $(p(\cdot), M, \epsilon)_L$-molecule.

Definition 4.3. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $M \in \mathbb{N}$ and $\epsilon \in (0,\infty)$. A function $\alpha \in L^2(\mathbb{R}^n)$ is called a $(p(\cdot), M, \epsilon)_L$*-molecule* adapted to the ball B if $\alpha \in R(L^M)$ ($R(L^M)$ denotes the range of L^M) and there exists a ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B > 0$ such that, for every $k = 0, \dots, M$ and $j \in \mathbb{Z}_+$,

$$\big\|(r_B^{-2}L^{-1})^k \alpha\big\|_{L^2(U_j(B))} \le 2^{-j\epsilon} |2^j B|^{\frac{1}{2}} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1},$$

where for $j \in \mathbb{Z}_+$,

$$U_j(B) = B(x_B, 2^j r_B) \backslash B(x_B, 2^{j-1} r_B).$$

Definition 4.4. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (0,\infty)$, $M \in \mathbb{N}$, and $\epsilon \in (0,\infty)$. Let L be an operator satisfying Assumption (A) and Assumption (B). The space $\widetilde{H}_{L,M,\epsilon}^{p(\cdot),q}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^2(\mathbb{R}^n)$ which can be decomposed as

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} \alpha_{i,j} \quad \text{in } L^2(\mathbb{R}^n), \tag{4.1}$$

where $\{\alpha_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot), M, \epsilon)_L$-molecules, associated to balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that, for any $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \lesssim 1$. For any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} \approx 2^i \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and $\mathcal{A}(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}) \le C$. The molecular variable Hardy–Lorentz space $H_{L,M,\epsilon}^{p(\cdot),q}(\mathbb{R}^n)$ is the completion of $\widetilde{H}_{L,M,\epsilon}^{p(\cdot),q}(\mathbb{R}^n)$ with respect to the quasinorm

$$\|f\|_{H_{L,M,\epsilon}^{p(\cdot),q}(\mathbb{R}^n)} := \inf\big\{\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big)\big\},$$

where the infimum is taken over all decompositions of f as (4.1).

The following proposition is the first main result of this section.

Proposition 4.5. *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$*, with* $p_+ \in (0,1]$*,* $q \in (0,\infty)$*, let* L *satisfy Assumption (A) and Assumption (B), let* $M \in (\frac{n}{2}[\frac{1}{\min\{p_-,q\}} - \frac{1}{2}], \infty) \cap \mathbb{N}$*, and let* $\epsilon \in (\frac{n}{\min\{p_-,q\}}, \infty)$*. Then there exists a positive constant* C *such that, for any* $f \in \widetilde{H}^{p(\cdot),q}_{L,M,\epsilon}(\mathbb{R}^n)$*,*

$$\|f\|_{H^{p(\cdot),q}_L(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot),q}_{L,M,\epsilon}(\mathbb{R}^n)}.$$

Proof. Let $f \in \widetilde{H}^{p(\cdot),q}_{L,M,\epsilon}(\mathbb{R}^n)$. There exist a sequence of $(p(\cdot),M,\epsilon)_L$-molecules $\{\alpha_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ associated to balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ and a sequence of numbers $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ in $\mathbb{C}$ such that

$$f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}\alpha_{i,j} \quad \text{in } L^2(\mathbb{R}^n)$$

and $\mathcal{A}(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}) \lesssim \|f\|_{H^{p(\cdot),q}_{L,M,\epsilon}(\mathbb{R}^n)}$. Since the operator S_L is bounded on $L^2(\mathbb{R}^n)$, we have that $\|S_L(f) - S_L(\sum_{|i|+j<N} \lambda_{i,j}\alpha_{i,j})\|_{L^2(\mathbb{R}^n)}$ tends to zero as $N \to 0$. Hence, there exists a subsequence $\{S_L(\sum_{|i|+j<N_k} \lambda_{i,j}\alpha_{i,j})\}_{k\in\mathbb{N}}$, which converges almost every $x \in \mathbb{R}^n$ to $S_L(f)$, that is,

$$S_L(f)(x) = \lim_{k\to\infty} S_L\Big(\sum_{|i|+j<N_k} \lambda_{i,j}\alpha_{i,j}\Big)(x) \quad \text{a.e.}$$

In what follows, without loss of generality, we will use the same notation as the original sequence. Thus, for almost every $x \in \mathbb{R}^n$, we have

$$S_L(f)(x) \le \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} |\lambda_{i,j}| S_L(\alpha_{i,j})(x) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\sum_{k=0}^{\infty} |\lambda_{i,j}| S_L(\alpha_{i,j})(x)\chi_{U_k(B_{i,j})}(x).$$

For $i_0 \in \mathbb{Z}$, we obtain

$$\begin{aligned} S_L(f)(x) \le & \sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\sum_{k=0}^{\infty} \lambda_{i,j} S_L(\alpha_{i,j})(x)\chi_{U_k(B_{i,j})}(x) \\ & + \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\sum_{k=0}^{\infty} \lambda_{i,j} S_L(\alpha_{i,j})(x)\chi_{U_k(B_{i,j})}(x) \\ =: & \, S_L(f_1)(x) + S_L(f_2)(x). \end{aligned}$$

We get

$$\begin{aligned} \|\chi_{\{x\in\mathbb{R}^n: S_L(f)(x)>2^{i_0}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim & \|\chi_{\{x\in\mathbb{R}^n: S_L(f_1)(x)>2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & + \|\chi_{\{x\in\mathbb{R}^n: S_L(f_2))(x)>2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ =: & \, I_1 + I_2. \end{aligned}$$

Now we start to estimate I_1. Let $b \in (0, \min\{p_-,q\})$ be such that $\epsilon > \frac{n}{b}$. From [35, (3.12)], we know that

$$\big\|S_L(\alpha_{i,j})\chi_{U_k(B_{i,j})}\big\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-k\epsilon}|2^k B_{i,j}|^{\frac{1}{2}}\|\chi_{B_{i,j}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Let $\widetilde{q}\in(1,\min\{2,\frac{1}{b}\})$, and let $\tilde{r}=\frac{2}{\widetilde{q}}$. For all $i\in\mathbb{Z}$ and $j\in\mathbb{N}$, we know that

$$\begin{aligned}&\left\|2^{k\epsilon\widetilde{q}}\left(\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}S_L(\alpha_{i,j})\chi_{U_k(B_{i,j})}\right)^{\widetilde{q}}\right\|_{L^{\tilde{r}}(\mathbb{R}^n)}\\&\quad\le 2^{k\widetilde{q}\epsilon}\|\chi_{B_{i,j}}\|^{\widetilde{q}}_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|S_L(\alpha_{i,j})\chi_{U_k(B_{i,j})}\right\|^{\widetilde{q}}_{L^2(\mathbb{R}^n)}\lesssim|2^kB_{i,j}|^{\frac{1}{\tilde{r}}}.\end{aligned}\tag{4.2}$$

Let $a\in(0,1-\frac{1}{\widetilde{q}})$, $1<\delta_3<(1-a)\widetilde{q}$, and $s\in(\frac{n}{\epsilon\widetilde{q}b},1)$. By an argument similar to the one used in Proposition 3.5 with $\mathcal{A}$ replaced by S_L, we can show that

$$\begin{aligned}I_1&\lesssim\sum_{k=0}^{\infty}2^{-k(\epsilon\widetilde{q}-\frac{n}{sb})}2^{-i_0\delta_3}\Bigg(\sum_{i=-\infty}^{i_0-1}2^{iq\delta_3}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}\\&\lesssim 2^{-i_0\delta_3}\Bigg(\sum_{i=-\infty}^{i_0-1}2^{iq\delta_3}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}.\end{aligned}$$

Then we get

$$\begin{aligned}\sum_{i_0=-\infty}^{\infty}(2^{i_0}I_1)^q&\lesssim\sum_{i_0=-\infty}^{\infty}2^{i_0q}2^{-i_0\delta_3q}\sum_{i=-\infty}^{i_0-1}2^{iq\delta_3}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\lesssim\left(\mathcal{A}\left(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}},\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right)\right)^q.\end{aligned}\tag{4.3}$$

And for I_2, let $\widetilde{b}\in(0,\min\{p_-,q\})$ be such that $\epsilon>\frac{n}{\widetilde{b}}$ and $r\in(\frac{n}{\epsilon\widetilde{b}},1)$. Then, by an argument similar to that used in Proposition 3.5 with $\mathcal{A}$ and b_1 replaced by S_L and $\widetilde{b}$, respectively, choosing $r<\delta_4<1$ and $t\in(\frac{n}{\epsilon r\widetilde{b}},1)$, we obtain

$$\begin{aligned}I_2&\lesssim\sum_{k=0}^{\infty}2^{-k(\epsilon r-\frac{n}{t\widetilde{b}})}2^{-i_0\delta_4}\Bigg(\sum_{i=i_0}^{\infty}2^{i\delta_4q}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}\\&\lesssim 2^{-i_0\delta_4}\Bigg(\sum_{i=i_0}^{\infty}2^{i\delta_4q}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i_0=-\infty}^{\infty}(2^{i_0}I_2)^q&\lesssim\sum_{i_0=-\infty}^{\infty}2^{i_0q}2^{-i_0\delta_4q}\sum_{i=i_0}^{\infty}2^{iq\delta_4}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\lesssim\left(\mathcal{A}\left(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}},\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right)\right)^q.\end{aligned}\tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\Big\|\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{i,j}\alpha_{i,j}\Big\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}\lesssim\mathcal{A}\left(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}},\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\right).$$

The proof is complete. □

For all functions $F\in L^2(\mathbb{R}^{n+1}_+)$ with compact supports, define the operator $\pi_{L,M}$, which was introduced in [11], by

$$\pi_{L,M}(F):=C_M\int_0^{\infty}(t^2L)^{M+1}e^{-t^2L}F(\cdot,t)\,\frac{dt}{t},\tag{4.5}$$

where C_M is such that

$$C_M \int_0^\infty t^{2(M+2)} e^{-2t^2} \frac{dt}{t} = 1.$$

Remark 4.6. The operator $\pi_{L,M}$ is bounded from $T_c^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ (see [5, Proposition 4.5(i)]).

Proposition 4.7. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $q \in (0,\infty)$, $M \in \mathbb{N}$, and $\pi_{L,M}$ be as in (4.5). Then the operator $\pi_{L,M}$, initially defined on $T_c^{p(\cdot),q}(\mathbb{R}_+^{n+1})$, extends to a bounded linear operator from $T^{p(\cdot),q}(\mathbb{R}_+^{n+1})$ to $H_L^{p(\cdot),q}(\mathbb{R}^n)$.*

Proof. The proof is similar to that of Proposition 3.5. Assume that $f \in T_c^{p(\cdot),q}(\mathbb{R}_+^{n+1})$. From Lemma 3.7, Theorem 3.3, and Remark 4.6, it follows that

$$\pi_{L,M}(f) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} \pi_{L,M}(a_{i,j}) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} \alpha_{i,j}$$

in $L^2(\mathbb{R}^n)$, where $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot),\infty)$-atoms and $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of numbers in $\mathbb{C}$ such that $f(x) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j}(x)$ for almost every $x \in \mathbb{R}_+^{n+1}$. By an argument similar to the one used for [35, Lemma 3.11], we can prove that for any fixed ϵ, $\alpha_{i,j} = \pi_{L,M}(a_{i,j})$ is a multiple of a $(p(\cdot), M, \epsilon)_L$-molecule adapted to $B_{i,j}$. For $i_0 \in \mathbb{Z}$, we write

$$f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} =: f_1 + f_2.$$

We get

$$\begin{aligned}\|\chi_{\{x\in\mathbb{R}^n: S_L(\pi_{L,M}(f))(x) > 2^{i_0}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\lesssim \|\chi_{\{x\in\mathbb{R}^n: S_L(\pi_{L,M}(f_1))(x) > 2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\chi_{\{x\in\mathbb{R}^n: S_L(\pi_{L,M}(f_2))(x) > 2^{i_0-1}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &=: I_1 + I_2.\end{aligned}$$

By substituting the operator $\mathcal{A}$ by S_L in the proof of Proposition 3.5 and combining Remark 2.1(2) and Remark 4.6, we have

$$\begin{aligned}\left\|\mathcal{S}_L(\alpha_{i,j})\right\|_{L^2(\mathbb{R}^n)} &\lesssim \|\alpha_{i,j}\|_{L^2(\mathbb{R}^n)} = \left\|\pi_{L,M}(a_{i,j})\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|a_{i,j}\|_{T_2^2(\mathbb{R}_+^{n+1})} \lesssim |B_{i,j}|^{\frac{1}{2}} \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}.\end{aligned}$$

Let $\widetilde{q}$ be as in the proof of Proposition 3.5, and let $\tilde{r} = \frac{2}{\tilde{q}}$. For all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, we have

$$\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\widetilde{q}} \left\|S_L(\alpha_{i,j})^{\widetilde{q}}\right\|_{L^{\tilde{r}}(\mathbb{R}^n)} \leq \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\widetilde{q}} \left\|S_L(\alpha_{i,j})\right\|_{L^2(\mathbb{R}^n)}^{\widetilde{q}} \lesssim |B_{i,j}|^{\frac{1}{\tilde{r}}}.$$

Following the argument used in the proof of Proposition 3.5, we conclude that

$$\left\|\pi_{L,M}(f)\right\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|f\|_{T^{p(\cdot),q}(\mathbb{R}_+^{n+1})}.$$

The proof is complete. □

Proposition 4.8. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $q \in (0,\infty)$, $\epsilon > 0$, and $M \in \mathbb{N}$. If $f \in H_L^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then there exist $(p(\cdot), M, \epsilon)_L$-molecules $\{\alpha_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ and numbers $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}} \subset \mathbb{C}$ such that for almost every $x \in \mathbb{R}^n$,*

$$f(x) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j}\alpha_{i,j}(x) \tag{4.6}$$

in both $H_L^{p(\cdot),q}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in H_L^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) \le C\|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{4.7}$$

Proof. Let $f \in H_L^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then by Definition 4.1 and [2, Theorem F], we can deduce that $t^2Le^{-t^2L}f \in T^{p(\cdot),q}(\mathbb{R}^{n+1}_+) \cap T_2^2(\mathbb{R}^{n+1}_+)$. Hence, by Theorem 3.3, we have

$$t^2Le^{-t^2L}f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j}$$

almost everywhere on $\mathbb{R}^{n+1}$, and

$$\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) \lesssim \|t^2Le^{-t^2L}f\|_{T^{p(\cdot),q}(\mathbb{R}^{n+1}_+)} \approx \|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}, \tag{4.8}$$

where $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ and $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ are as in Theorem 3.3. By the H_∞-calculi of L,

$$f = C_M \int_0^\infty (t^2L)^{M+1}e^{-t^2L}(t^2Le^{-t^2L}f)\,\frac{dt}{t} = \pi_{L,M}(t^2Le^{-t^2L}f) \quad \text{in } L^2(\mathbb{R}^n),$$

where C_M is such that

$$C_M \int_0^\infty t^{2(M+2)}e^{-2t^2}\,\frac{dt}{t} = 1.$$

From Remark 3.4(iii), Proposition 3.5, Remark 4.6, and Proposition 4.7, we have

$$f = C_M\pi_{L,M}\Big(\sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j}\Big) = C_M \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j}\pi_{L,M}(a_{i,j})$$

in $L^2(\mathbb{R}^n)$ and $H_L^{p(\cdot),q}(\mathbb{R}^n)$. From the proof of Proposition 4.7, we know that $\pi_{L,M}(a_{i,j})$ is a $(p(\cdot), M, \epsilon)_L$-molecule up to a harmless constant, which implies the desired result. □

Corollary 4.9. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $0 < q < \infty$, $M \in \mathbb{N}$, and $\epsilon \in (0,\infty)$. Then for every $f \in H_L^{p(\cdot),q}(\mathbb{R}^n)$, there exist $(p(\cdot), M, \epsilon)_L$-molecules $\{\alpha_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ and numbers $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}} \subset \mathbb{C}$ such that for almost every $x \in \mathbb{R}^n$, $f(x) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j}\alpha_{i,j}(x)$ in $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover,*

$$\mathcal{A}\big(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}\big) \le C\|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)},$$

where C is a positive constant independent of f.

Proof. Let $f \in H_L^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then the desired result follows immediately from Proposition 4.8. For $f \in H_L^{p(\cdot),q}(\mathbb{R}^n)$, there exists $\{f_k\}_{k\in\mathbb{N}} \subset H_L^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that for all $k \in \mathbb{N}$, we have $\|f - f_k\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)} \leq 2^{-k}\|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}$. Let $f_0 = 0$. Then $f = \sum_{k\geq 1} f_k - f_{k-1}$ in $H_L^{p(\cdot),q}(\mathbb{R}^n)$. By the preceding argument for all $k \in \mathbb{N}$, there exist a sequence $\{\alpha_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ of $(p(\cdot), M, \epsilon)_L$-molecules and numbers $\{\lambda_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N}} \subset \mathbb{C}$ such that $\lambda_{i,j}^k \approx 2^i\|\chi_{B_{i,j}^k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, satisfies

$$f_k - f_{k-1} = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}^k \alpha_{i,j}^k$$

and

$$\mathcal{A}\big(\{\lambda_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}, \{B_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}\big) \lesssim \|f_k - f_{k-1}\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}.$$

By Remark 3.4(ii), we have that if $\underline{p} < q$, then

$$\begin{aligned}
&\big(\mathcal{A}\big(\{\lambda_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}^k\big)\big)^{\underline{p}} \\
&\quad\leq \sum_{k\geq 1}\big(\mathcal{A}\big(\{\lambda_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}, \{B_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}\big)\big)^{\underline{p}} \\
&\quad\lesssim \sum_{k\geq 1}\|f_k - f_{k-1}\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}^{\underline{p}} \lesssim \|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}^{\underline{p}}.
\end{aligned}$$

If $\underline{p} \geq q$, then

$$\begin{aligned}
&\big(\mathcal{A}\big(\{\lambda_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}, \{B_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}\big)\big)^{q} \\
&\quad\leq \sum_{k\geq 1}\big(\mathcal{A}\big(\{\lambda_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}, \{B_{i,j}^k\}_{i\in\mathbb{Z},j\in\mathbb{N},k\in\mathbb{N}}\big)\big)^{q} \\
&\quad\lesssim \sum_{k\geq 1}\|f_k - f_{k-1}\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}^{q} \lesssim \|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}^{q}.
\end{aligned}$$

The proof is complete. □

Let $H_{\mathrm{mole,fin}}^{p(\cdot),q,M,\epsilon}(\mathbb{R}^n)$ denote the set of all finite combinations of $(p(\cdot), r, M, \epsilon)$-molecules. We get the following density result.

Proposition 4.10. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (0,\infty)$, $M \in \mathbb{N}$, and $\epsilon > 0$. Then $H_{\mathrm{mole,fin}}^{p(\cdot),q,M,\epsilon}(\mathbb{R}^n)$ is dense in $H_L^{p(\cdot),q}(\mathbb{R}^n)$.*

The following theorem establishes the molecular characterization of the variable Hardy–Lorentz spaces, which is a result of combining Propositions 4.5 and 4.8.

Theorem 4.11. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, with $p_+ \in (0,1]$, $q \in (0,\infty)$, let L satisfy Assumption (A) and Assumption (B), let $M \in (\frac{n}{2}[\frac{1}{\min\{p_-,q\}} - \frac{1}{2}],\infty) \cap \mathbb{N}$, and let $\epsilon \in (\frac{n}{\min\{p_-,q\}},\infty)$. Then $H_L^{p(\cdot),q}(\mathbb{R}^n)$ and $H_{L,M,\epsilon}^{p(\cdot),q}(\mathbb{R}^n)$ coincide with equivalent quasinorms.*

5. Duality results

In this section, we discuss the dualities of $H_L^{p(\cdot),q}(\mathbb{R}^n)$ with $p_+ \in (0,1]$ and $0 < q < \infty$. Let us first recall some basic notions and definitions. Let $\epsilon > 0$ and $M \in \mathbb{N}$. We have

$$\mathcal{M}^{p(\cdot),M,\epsilon}(L) := \left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{\mathcal{M}^{p(\cdot),M,\epsilon}(L)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{M}^{p(\cdot),M,\epsilon}(L)} := \sup_{j\geq 0} 2^{j(\epsilon-n/2)} \|\chi_{B(0,1)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=0}^{M} \|L^{-k} f\|_{L^2(U_j(B(0,1)))}.$$

Remark 5.1. If α is a $(p(\cdot), M, \epsilon)_L$-molecule adapted to a ball B in $\mathbb{R}^n$, then $\alpha \in \mathcal{M}_L^{p(\cdot),M,\epsilon}$ (see [35, Proposition 4.3]).

For any $M \in \mathbb{N}$, we define $\mathcal{M}_{L^*}^{p(\cdot),M} := \bigcap_{\epsilon>0} (\mathcal{M}^{p(\cdot),M,\epsilon}(L))^*$.

Definition 5.2. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$ and $M \in \mathbb{N}$. We say that $f \in \mathcal{M}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$ is in $\mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left(\frac{1}{|B|} \int_B \left|(I - e^{-r_B^2 L})^M f(x)\right|^2 dx \right)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$.

The following three estimates play important roles in the proofs of our main theorems in this section. Their proofs are similar to those of [20, Lemma 4.3] and [18, Lemmas 8.1 and 8.3]. We leave the details to the reader.

Lemma 5.3. *Let $p(\cdot)$ and M be as in Definition 5.2, let $\epsilon, \epsilon_1 > 0$, and let $\widetilde{M} > M + \epsilon_1 + n/4$. If $f \in \mathcal{M}_{L^*}^{p(\cdot),M}$ satisfies*

$$\int_{\mathbb{R}^n} \frac{|(I - (I + L^*)^{-1})^M f(x)|^2}{1 + |x|^{n+\epsilon_1}} \, dx < \infty, \tag{5.1}$$

then for every $(p(\cdot), \widetilde{M}, \epsilon)_L$-molecule α,

$$\langle f, \alpha \rangle_M = C_M \iint_{\mathbb{R}^{n+1}} (t^2 L^*)^M e^{-t^2 L^*} f(x) \overline{t^2 L e^{-t^2 L} \alpha(x)} \, \frac{dx\, dt}{t},$$

where $\langle f, \alpha \rangle_M$ denotes the duality between $\mathcal{M}_{L^*}^{p(\cdot),M}$ and $(\mathcal{M}_{L^*}^{p(\cdot),M})^*$, and C_M is a positive constant depending on M.

Lemma 5.4. *Let $p(\cdot)$ and M be as in Definition 5.2. Then $f \in \mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)$ if and only if*

$$\|f\|_{\mathrm{BMO}_{L,\mathrm{res}}^{p(\cdot),M}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{|B|^{\frac{1}{2}}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \int_B \left|\left(I - (I + r_B^2 L)^{-1}\right)^M f(x)\right|^2 dx \right\}^{\frac{1}{2}} < \infty.$$

Moreover, $\|f\|_{\mathrm{BMO}_{L,\mathrm{res}}^{p(\cdot),M}(\mathbb{R}^n)}$ is equivalent to $\|f\|_{\mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)}$.

Lemma 5.5. *Let $p(\cdot)$ and M be as in Definition 5.2. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)$,*

$$\sup_{B\subset\mathbb{R}^n} \frac{|B|^{1/2}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \Big(\iint_{\widehat{B}} \big|(t^2L)^M e^{-t^2L} f(x)\big|^2 \,\frac{dx\,dt}{t}\Big)^{\frac{1}{2}} \le C\|f\|_{\mathrm{BMO}_L^{p(\cdot),M}(\mathbb{R}^n)},$$

where the supremum is taken over all balls of $\mathbb{R}^n$.

The first main result is to deal with the case where $p_+ \in (0,1)$ and $0 < q \le 1$.

Theorem 5.6. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1)$, $0 < q \le 1$, let $\epsilon \in (\frac{n}{\min\{p_-,q\}}, \infty)$, let $M \in \mathbb{N} \cap (\frac{n}{2}[\frac{1}{p_-} - \frac{1}{2}], \infty)$, and let $\widetilde{M} > M + n(1 + \frac{2}{\min\{p_-,q\}} - \frac{2}{p_+}) + n/4$. Then we have*

$$\big(H_L^{p(\cdot),q}(\mathbb{R}^n)\big)^* = \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$$

with equivalent norms. More precisely, we have the following.

(i) *Let $g \in \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$. Then the functional $\ell_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx$, defined initially on $H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n)$, can be extended to a bounded functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, $\|\ell_g\| \le C\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)}$.*

(ii) *Conversely, let ℓ be a bounded linear functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Then there exists $g \in \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$ such that $\ell = \ell_g$ and $\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)} \le C\|\ell\|$.*

Proof. We first show (i). Let $g \in \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$. For $f \in H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n)$, we define

$$\ell_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

Since $f \in H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n) \subset H_L^{p(\cdot),q}(\mathbb{R}^n)$, we have that $t^2Le^{-t^2L}f \in T^{p(\cdot),q}(\mathbb{R}^{n+1})$. Then by Theorem 3.3, we can assume that $t^2Le^{-t^2L}f = \sum_{|i|\le M}\sum_{j\le N}\lambda_{i,j}a_{i,j}$ for some $M, N \in \mathbb{N}$, where $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot),\infty)$-atoms supported on $\{\widehat{B}_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$. By Lemma 5.4, we know that g satisfies the inequality (5.1) for $\epsilon_1 > n(1+\frac{2}{p_-} - \frac{2}{p_+})$ (see also [35, Remark 4.6]). Thus, it follows from Lemma 5.3, the Hölder inequality, and Lemma 5.5 that

$$\begin{aligned}
\big|\ell_g(f)\big| &= \Big|C_M \iint_{\mathbb{R}^{n+1}_+} (t^2L^*)^M e^{-t^2L^*} g(x)\overline{t^2Le^{-t^2L}f(x)}\,\frac{dx\,dt}{t}\Big| \\
&\lesssim \sum_{i=-M}^{M}\sum_{j=1}^{N} |\lambda_{i,j}| \iint_{\mathbb{R}^{n+1}_+} (t^2L^*)^M e^{-t^2L^*} g(x)\overline{a_{i,j}(x,t)}\,\frac{dx\,dt}{t} \\
&\lesssim \sum_{i=-M}^{M}\sum_{j=1}^{N} |\lambda_{i,j}|\|a_{i,j}\|_{T_2^2(\mathbb{R}^{n+1}_+)} \Big(\iint_{\widehat{B}_{i,j}} \big|(t^2L^*)^M e^{-t^2L^*} g(x)\big|^2\,\frac{dx\,dt}{t}\Big)^{1/2} \\
&\lesssim \sum_{i=-M}^{M}\sum_{j=1}^{N} |\lambda_{i,j}|\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)}.
\end{aligned}$$

Since $0 < q \le 1$ and $p_+ \in (0,1]$, we have

$$\begin{aligned}|\ell_g(f)| &\lesssim \sum_{i\in\mathbb{Z}} 2^i \Big\|\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)} \\ &\lesssim \Big(\sum_{i\in\mathbb{Z}} 2^{iq} \Big\|\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q}} \|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)} \|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, by Proposition 4.10, ℓ_g can be extended to a bounded linear functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, $\|\ell_g\| \le C\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)}$.

Now we turn to prove (ii). Let $\ell \in (H_L^{p(\cdot),q}(\mathbb{R}^n))^*$. By Proposition 4.5, it follows that, for any $(p(\cdot), M, \epsilon)_L$-molecule, we have $\|\alpha\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)} \lesssim 1$. Thus, we get $|\ell(\alpha)| \lesssim \|\ell\|$. Combining this with Remark 5.1, we can deduce that $\ell \in \bigcap_{\epsilon>0}(\mathcal{M}^{p(\cdot),M,\epsilon}(L))^*$. This implies that $\ell \in \mathcal{M}_{L^*}^{p(\cdot),M}$. To complete the proof we need to show that $\ell \in \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$. Take a ball $B \subset \mathbb{R}^n$, and let $h \in L^2(B)$ with $\|h\|_{L^2(B)} \le |B|^{1/2}\|\chi_B\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}$ and $\widetilde{\alpha} = (I - e^{-r_B^2 L})^M h$. Following the argument used in [35, (4.14)], we know that $\widetilde{\alpha}$ is a $(p(\cdot), M, \epsilon)_L$-molecule and

$$\int_B (I - e^{-r_B^2 L^*})^M \ell(x) h(x)\, dx \lesssim \|\ell\|_{(H_L^{p(\cdot),q}(\mathbb{R}^n))^*}.$$

Therefore,

$$\begin{aligned}&\frac{|B|^{1/2}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\Big(\int_B \big|(I - e^{-r_B^2 L^*})^M \ell(x)\big|^2\, dx\Big)^{1/2} \\ &\quad = \sup_{\|h\|_{L^2(B)}\le 1} \int_B (I - e^{-r_B^2 L^*})^M \ell(x) h(x) \frac{|B|^{1/2}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\, dx \\ &\quad \lesssim \|\ell\|_{(H_L^{p(\cdot),q}(\mathbb{R}^n))^*}.\end{aligned}$$

Taking the supremum over all balls B in $\mathbb{R}^n$, we get the desired result. □

We now investigate the dual space of $H_L^{p(\cdot),q}(\mathbb{R}^n)$ with $p_+ \in (0,1]$ and $1 < q < \infty$. Inspired by [1], [22], and [23], we introduce the following new kind of BMO spaces.

Definition 5.7. Let $1 \le r < \infty$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, and let Ω be an open set with $|\Omega| < \infty$. Define

$$O_L^r(g, \Omega) := \sup \sum_j |B_j| \Big(\frac{1}{|B_j|}\int_{B_j} \big|(I - e^{-r_{B_j}^2 L})^M g(x)\big|^r\, dx\Big)^{\frac{1}{r}},$$

where the supremum is taken over all collections of balls in Ω, $\{B_j\}_{j\in\mathbb{N}}$ such that $\sum_j \chi_{B_j} \le C$. Then the BMO space $\mathrm{BMO}_L^{p(\cdot),q,r,M}(\mathbb{R}^n)$ is defined by

$$\mathrm{BMO}_L^{p(\cdot),q,r,M}(\mathbb{R}^n) := \big\{g \in L^r_{\mathrm{loc}} : \|g\|_{\mathrm{BMO}_L^{p(\cdot),q,r,M}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|g\|_{\mathrm{BMO}_L^{p(\cdot),q,r,M}(\mathbb{R}^n)} := \sup \frac{\sum_{i\in\mathbb{Z}} 2^i O_L^r(g,\Omega_i)}{(\sum_{i\in\mathbb{Z}} 2^{iq}\|\chi_{\Omega_i}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)})^{\frac{1}{q}}}$$

with the supremum taken over all collections of open sets $\{\Omega_i\}_{i\in\mathbb{Z}}$ such that

$$\sum_{i\in\mathbb{Z}} 2^{iq}\|\chi_{\Omega_i}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

In the following, we simply denote $O_L^2(g,\Omega)$ and $\mathrm{BMO}_L^{p(\cdot),q,2,M}(\mathbb{R}^n)$ by $O_L(g,\Omega)$ and $\mathrm{BMO}_L^{p(\cdot),q,M}(\mathbb{R}^n)$, respectively.

The lemma below should be compared with Lemma 5.5. The proof is similar to that for [18, Lemma 8.3].

Lemma 5.8. *Let $p(\cdot)$ and M be as in Definition 5.2. Then there exists a positive constant C such that for all $g \in \mathrm{BMO}_L^{p(\cdot),q,M}(\mathbb{R}^n)$,*

$$\sup \sum_{j\in\mathbb{N}} |B_j|^{1/2}\Big(\iint_{\widehat{B_j}} \big|(t^2L)^M e^{-t^2L} g(x)\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}} \le C O_L(g,\Omega),$$

where the supremum is taken over all collections of balls in Ω, $\{B_j\}_{j\in\mathbb{N}}$ such that $\sum_j \chi_{B_j} \le C$.

Now we give another main result of this section. It states that the above new BMO space can be used to characterize the duality of $H_L^{p(\cdot),q}(\mathbb{R}^n)$ for $1 < q < \infty$.

Theorem 5.9. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $1 < q < \infty$, let $\epsilon \in (\frac{n}{\min\{p_-,q\}}, \infty)$, let $M \in \mathbb{N} \cap (\frac{n}{2}(\frac{1}{\min\{p_-,q\}} - \frac{1}{2}), \infty)$, and let $\widetilde{M} > M + n(1 + \frac{2}{p_-} - \frac{2}{p_+}) + n/4$. Then*

$$\big(H_L^{p(\cdot),q}(\mathbb{R}^n)\big)^* = \mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)$$

with equivalent norms. More precisely, we have the following.

(i) *Let $g \in \mathrm{BMO}_L^{p(\cdot),q,M}(\mathbb{R}^n)$. Then the functional $\ell_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx$, defined initially on $H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n)$, can be extended to a bounded functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, $\|\ell_g\| \le C\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)}$.*

(ii) *Conversely, let ℓ be a bounded linear functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$. Then there exists $g \in \mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)$ such that $\ell = \ell_g$ and $\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)} \le C\|\ell\|$.*

Proof. We first show (i). Let $g \in \mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)$. For $f \in H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n)$, we define

$$\ell_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

Since $p_+ < q$, we have $H_{\mathrm{mole,fin}}^{p(\cdot),p_+,\widetilde{M},\epsilon}(\mathbb{R}^n) \hookrightarrow H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n)$ by Lemma 2.10. Then ℓ_g is also a bounded functional on $H_{\mathrm{mole,fin}}^{p(\cdot),p_+,\widetilde{M},\epsilon}(\mathbb{R}^n)$. Moreover, by Theorem 5.6, we know that $g \in \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$. It follows from $f \in H_{\mathrm{mole,fin}}^{p(\cdot),q,\widetilde{M},\epsilon}(\mathbb{R}^n) \subset H_L^{p(\cdot),q}(\mathbb{R}^n)$ that $t^2Le^{-t^2L}f \in T^{p(\cdot),q}(\mathbb{R}_+^{n+1})$ and

$$t^2Le^{-t^2L}f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j},$$

where $\{a_{i,j}\}_{i,j}$ is a sequence of $(p(\cdot),\infty)$-atoms associated with balls $\{B_{i,j}\}$ and $\lambda_{i,j} \approx 2^i\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Then by Lemma 5.3, the Hölder inequality, and Lemma 5.8, we have

$$\begin{aligned}\big|\ell_g(f)\big| &= \Big|C_M\iint_{\mathbb{R}^{n+1}}(t^2L^*)^Me^{-t^2L^*}g(x)\overline{t^2Le^{-t^2L}f(x)}\,\frac{dx\,dt}{t}\Big|\\ &\lesssim \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}2^i\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|a_{i,j}\|_{T_2^2(\mathbb{R}^{n+1})}\\ &\quad\cdot\Big(\iint_{\widehat{B}_{i,j}}\big|(t^2L^*)^Me^{-t^2L^*}g(x)\big|^2\,\frac{dx\,dt}{t}\Big)^{1/2}\\ &\lesssim \sum_{i\in\mathbb{Z}}2^iO_L(g,\Omega_i) \lesssim \|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)}\Big(\sum_{i\in\mathbb{Z}}2^{iq}\|\chi_{\Omega_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\Big)^{\frac{1}{q}}\\ &\lesssim \|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)}\|f\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, by Proposition 4.10, ℓ_g can be extended to a bounded linear functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$ and $\|\ell_g\| \le C\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)}$.

Now we prove (ii). Let ℓ be a bounded linear functional on $H_L^{p(\cdot),q}(\mathbb{R}^n)$. From the fact that $1 < q < \infty$ and $0 < p_+ \le 1$, we know that $H_L^{p(\cdot),p_+}(\mathbb{R}^n) \hookrightarrow H_L^{p(\cdot),q}(\mathbb{R}^n)$. Then we deduce that ℓ is a bounded functional on $H_L^{p(\cdot),p_+}(\mathbb{R}^n)$. By Theorem 5.6, there exists $g \in \mathrm{BMO}_{L^*}^{p(\cdot),M}(\mathbb{R}^n)$ such that

$$\ell(f) = \int_{\mathbb{R}^n}f(x)g(x)\,dx.$$

It remains to show that $g \in \mathrm{BMO}_{L^*}^{p(\cdot),q,M}(\mathbb{R}^n)$. For any $i \in \mathbb{Z}$, let

$$\Omega_i = \big\{x \in \mathbb{R}^n : S_L(f)(x) > 2^i\big\}.$$

Then it is easy to show that $\{\Omega_i\}_{i\in\mathbb{Z}}$ is a collection of open sets since $f \in H_L^{p(\cdot),q}(\mathbb{R}^n)$ and $\sum_{i\in\mathbb{Z}}2^{iq}\|\chi_{\Omega_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q < \infty$. We take $\{B_{i,j}\}_{j\in\mathbb{N}}$ a collection of balls in Ω_i satisfying $\sum_{j\in\mathbb{N}}\chi_{B_{i,j}} \le C$ and

$$O_L(g,\Omega_i) \lesssim \sum_{j\in\mathbb{N}}|B_{i,j}|\Big(\frac{1}{|B_{i,j}|}\int_{B_{i,j}}\big|(I-e^{-(r_{B_{i,j}})^2L})^Mg(x)\big|^2\,dx\Big)^{\frac{1}{2}}.$$

Let $h_{i,j} \in L^2(B_{i,j})$ with $\|h_{i,j}\|_{L^2(B_{i,j})} = 1$ such that

$$\Big(\int_{B_{i,j}}\big|(I-e^{-(r_{B_{i,j}})^2L})^Mg(x)\big|^2\,dx\Big)^{\frac{1}{2}} = \int_{B_{i,j}}\big((I-e^{-(r_{B_{i,j}})^2L})^Mg(x)\big)h_{i,j}(x)\,dx.$$

Set

$$\alpha_{i,j}(x) = |B_{i,j}|^{1/2} \|\chi_{B_{i,j}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} (I - e^{-(r_{B_{i,j}})^2 L^*})^M (h_{i,j})(x).$$

Then $\alpha_{i,j}$ is a $(p(\cdot), M, \epsilon)_L$-molecule up to a harmless constant. Indeed, by the definition of $h_{i,j}$, we know that

$$\big\| |B_{i,j}|^{1/2} \|\chi_{B_{i,j}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} h_{i,j} \|_{L^2(B_{i,j})} = |B_{i,j}|^{1/2} \|\chi_{B_{i,j}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

From the same argument used in the second part of Theorem 5.6, we know that $\alpha_{i,j}$ is a $(p(\cdot), M, \epsilon)_L$-molecule up to a harmless constant. Moreover, by Proposition 4.5, we get

$$\sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} 2^i \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \alpha_{i,j} \in H_L^{p(\cdot),q}(\mathbb{R}^n)$$

and

$$\Big\| \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} 2^i \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \alpha_{i,j} \Big\|_{H_L^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \Big(\sum_{i\in\mathbb{Z}} 2^{iq} \|\chi_{\Omega_i}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{q}}. \tag{5.2}$$

Therefore, by the definition of $h_{i,j}$, we have

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} 2^i O_L(g, \Omega_i) &\lesssim \sum_{i\in\mathbb{Z}} 2^i \sum_{j\in\mathbb{N}} |B_{i,j}|^{1/2} \Big(\int_{B_{i,j}} \big|(I - e^{-(r_{B_{i,j}})^2 L})^M g(x)\big|^2 \, dx \Big)^{\frac{1}{2}} \\
&\lesssim \sum_{i\in\mathbb{Z}} 2^i \sum_{j\in\mathbb{N}} |B_{i,j}|^{1/2} \int_{B_{i,j}} (I - e^{-(r_{B_{i,j}})^2 L})^M g(x) h_{i,j}(x) \, dx \\
&\lesssim \sum_{i\in\mathbb{Z}} 2^i \sum_{j\in\mathbb{N}} |B_{i,j}|^{1/2} \int_{B_{i,j}} g(x) (I - e^{-(r_{B_{i,j}})^2 L^*})^M h_{i,j}(x) \, dx.
\end{aligned}$$

Using (5.2) and Definition 5.7, we obtain

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} 2^i O_L(g, \Omega_i) &\lesssim \sum_{i\in\mathbb{Z}} 2^i \sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \int_{B_{i,j}} g(x) \alpha_{i,j}(x) \, dx \\
&= \ell\Big(\sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} 2^i \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \alpha_{i,j} \Big) \\
&\lesssim \|\ell\| \Big(\sum_{i\in\mathbb{Z}} 2^{iq} \|\chi_{\Omega_i}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{q}}.
\end{aligned}$$

Hence, $g \in \mathrm{BMO}_{L^*}^{p(\cdot),q,r,M}(\mathbb{R}^n)$ and $\|g\|_{\mathrm{BMO}_{L^*}^{p(\cdot),q,r,M}(\mathbb{R}^n)} \lesssim \|\ell\|$. □

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