Banach J. Math. Anal. 13 (2019), no. 4, 798–814
https://doi.org/10.1215/17358787-2018-0026
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

INTERPOLATION OF HAAGERUP NONCOMMUTATIVE HARDY SPACES

TURDEBEK N. BEKJAN[1*] and MADI RAIKHAN[2]

Communicated by Q. Xu

ABSTRACT. Let $\mathcal{M}$ be a σ-finite von Neumann algebra equipped with a normal faithful state φ, and let $\mathcal{A}$ be a maximal subdiagonal algebra of $\mathcal{M}$. We prove a Stein–Weiss-type interpolation theorem of Haagerup noncommutative H^p-spaces associated with $\mathcal{A}$.

1. Introduction

In [1], for a von Neumann algebra $\mathcal{M}$ with a faithful, normal finite trace, Arveson introduced the notion of finite, maximal subdiagonal algebras $\mathcal{A}$ of $\mathcal{M}$ as noncommutative analogues of weak* Dirichlet algebras. In [19], Marsalli and West defined the noncommutative H^p-space by the closure of $\mathcal{A}$ in the noncommutative L^p-space $L^p(\mathcal{M})$ and extended some results of the classical Hardy spaces on the torus to this noncommutative setting. Using some of Arveson's ideas, Labuschagne [17] showed that in the context of finite von Neumann algebras, these maximal subdiagonal algebras satisfy a Szegö formula. As a result of this breakthrough, the theory of (noncommutative) H^p-spaces associated with such algebras has been rapidly developing. Many classical results of H^p-spaces associated with weak* Dirichlet algebras have been transferred to the noncommutative setting (see [6]).

Received May 28, 2018; Accepted Aug. 22, 2018.
First published online Aug. 30, 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L52; Secondary 47L05.
Keywords. subdiagonal algebras, Haagerup noncommutative H^p-space, interpolation.

In [10] and [11], Ji studied Haagerup noncommutative H^p-spaces based on Haagerup's noncommutative L^p-space $L^p(\mathcal{M})$ associated with a σ-finite von Neumann algebra $\mathcal{M}$ and a maximal subdiagonal algebra $\mathcal{A}$ of $\mathcal{M}$. Ji extended some results in [19] to the Haagerup noncommutative H^p-space case (see also [13]). Labuschagne [18] showed that a Beurling-type theory of invariant subspaces of Haagerup noncommutative H^2 spaces holds. In [3, Theorem 3.4] the first author proved a Szegö-type factorization theorem for Haagerup noncommutative H^p-spaces.

Kosaki [16, Theorem 11.1] proved a Haagerup noncommutative L^p-spaces analogue of the classical Stein–Weiss interpolation theorem. In general, the real interpolation theorem of classical L^p-spaces is no longer valid for Haagerup noncommutative L^p-spaces (see Example 3.3 in [21]).

Pisier and Xu [21] obtained a noncommutative version of Jones's theorem for noncommutative Hardy spaces associated with a finite subdiagonal algebra. It is stated in [21] without proof (see the remark following Lemma 8.5 there). In [2], the first-named author extended the results of the real interpolation method to the semifinite case; results of the complex interpolation method were extended to the semifinite case by the first-named author and Ospanov in [4]. The main objective of this paper is to prove a Stein–Weiss-type interpolation theorem of Haagerup noncommutative H^p-spaces.

The organization of our paper is as follows. In Section 2, we give some definitions and related results of Haagerup noncommutative H^p-spaces. The Stein–Weiss-type interpolation theorems of Haagerup noncommutative H^p-space are presented in Section 3.

2. Preliminaries

We use standard notation in operator algebras. We refer to [20] and [22] for modular theory, [8] for Haagerup noncommutative L^p-spaces, and [10] and [11] for Haagerup noncommutative H^p-spaces. Let us recall some basic facts about these spaces and fix the relevant notation used throughout this paper. Let $\mathcal{M}$ be a σ-finite von Neumann algebra on a complex Hilbert space $\mathcal{H}$ equipped with a distinguished normal faithful state φ. Recall briefly the definition of Haagerup noncommutative L^p spaces associated with $\mathcal{M}$. Let $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ be the 1-parameter modular automorphism group of $\mathcal{M}$ associated with φ, and let $\mathcal{N}$ denote the crossed product $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ of $\mathcal{M}$ by $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$. Then $\mathcal{N}$ is the von Neumann algebra acting on the Hilbert space $L^2(\mathbb{R},\mathcal{H})$ generated by

$$\bigl\{\pi(x) : x\in\mathcal{M}\bigr\}\cup\bigl\{\lambda(s) : s\in\mathbb{R}\bigr\},$$

where the operator $\pi(x)$ is defined by

$$\bigl(\pi(x)\xi\bigr)(t)=\sigma_{-t}^\varphi(x)\xi(t),\quad \forall\xi\in L^2(\mathbb{R},\mathcal{H}),\forall t\in\mathbb{R},$$

and the operator $\lambda(s)$ is defined by

$$\bigl(\lambda(s)\xi\bigr)(t)=\xi(t-s),\quad \forall\xi\in L^2(\mathbb{R},\mathcal{H}),\forall t\in\mathbb{R}.$$

We identify $\mathcal{M}$ with its image $\pi(\mathcal{M})$ in $\mathcal{N}$. The operators $\pi(x)$ and $\lambda(t)$ satisfy the following commutation relation:

$$\lambda(t)\pi(x)\lambda(t)^* = \pi\big(\sigma_t^{\varphi}(x)\big), \quad \forall t \in \mathbb{R}, \forall x \in \mathcal{M}.$$

Then the modular automorphism group $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ is given by

$$\sigma_t^{\varphi}(x) = \lambda(t)x\lambda^*(t), \quad x \in \mathcal{M}, t \in \mathbb{R}.$$

We denote by $\{\hat{\sigma}_t\}_{t\in\mathbb{R}}$ the dual action of $\mathbb{R}$ on $\mathcal{N}$. Then the dual action $\hat{\sigma}_t$ is uniquely determined by the following conditions: for any $x \in \mathcal{M}$ and $s \in \mathbb{R}$,

$$\hat{\sigma}_t(x) = x \qquad \text{and} \qquad \hat{\sigma}_t\big(\lambda(s)\big) = e^{-ist}\lambda(s), \quad \forall t \in \mathbb{R}.$$

Hence

$$\mathcal{M} = \big\{x \in \mathcal{N} : \hat{\sigma}_t(x) = x, \forall t \in \mathbb{R}\big\}.$$

Since $\mathcal{N}$ is semifinite (cf. [20]), there exists a normal semifinite faithful trace τ on $\mathcal{N}$ satisfying the equation

$$\tau \circ \hat{\sigma}_t = e^{-t}\tau, \quad \forall t \in \mathbb{R}.$$

Also recall that any normal semifinite faithful weight ψ on $\mathcal{M}$ induces a dual normal semifinite weight $\hat{\psi}$ on $\mathcal{N}$. Then $\hat{\psi}$ admits a Radon–Nikodym derivative with respect to τ (cf. [20]). In particular, the dual weight $\hat{\varphi}$ of our distinguished state φ has the Radon–Nikodym derivative D with respect to τ. Then

$$\hat{\varphi}(x) = \tau(Dx), \quad x \in \mathcal{N}_+.$$

Recall that D is an invertible positive self-adjoint operator on $L^2(\mathbb{R}, \mathcal{H})$, affiliated with $\mathcal{N}$, and that the regular representation $\lambda(t)$ above is given by

$$\lambda(t) = D^{it}, \quad \forall t \in \mathbb{R}.$$

Now we define Haagerup noncommutative L^p-spaces. Recall that $L^0(\mathcal{N}, \tau)$ denotes the topological $*$-algebra of all operators on $L^2(\mathbb{R}, \mathcal{H})$ measurable with respect to $(\mathcal{N}, \tau)$. Then the Haagerup noncommutative L^p-spaces, $0 < p \leq \infty$, are defined by

$$L^p(\mathcal{M}, \varphi) = \big\{x \in L^0(\mathcal{N}, \tau) : \hat{\sigma}_t(x) = e^{-\frac{t}{p}}x, \forall t \in \mathbb{R}\big\}.$$

It is clear that $L^p(\mathcal{M}, \varphi)$ is a vector subspace of $L^0(\mathcal{N}, \tau)$, invariant under the $*$-operation. The algebraic structure of $L^p(\mathcal{M}, \varphi)$ is inherited from that of $L^0(\mathcal{N}, \tau)$. Let $x \in L^p(\mathcal{M}, \varphi)$, and let $x = u|x|$ be its polar decomposition, where

$$|x| = (x^*x)^{\frac{1}{2}}$$

is the modulus of x. Then

$$u \in \mathcal{M} \qquad \text{and} \qquad |x| \in L^p(\mathcal{M}, \varphi).$$

Recall that

$$L^\infty(\mathcal{M}, \varphi) = \mathcal{M}.$$

As mentioned previously, for any $\psi \in \mathcal{M}_*^+$, the dual weight $\hat{\psi}$ has a Radon–Nikodym derivative with respect to τ denoted by h_ψ:

$$\hat{\psi}(x) = \tau(h_\psi x), \quad x \in \mathcal{N}_+.$$

Then

$$h_\psi \in L^0(\mathcal{N}, \tau)$$

and

$$\hat{\sigma}_t(h_\psi) = e^{-t} h_\psi, \quad \forall t \in \mathbb{R}.$$

So

$$h_\psi \in L^1(\mathcal{M}, \varphi)_+.$$

Hence, this correspondence between $\mathcal{M}_*^+$ and $L^1(\mathcal{M}, \varphi)_+$ extends to a bijection between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \varphi)$. So that for $\psi \in \mathcal{M}_*$, if $\psi = u|\psi|$ is its polar decomposition, then the corresponding $h_\psi \in L^1(\mathcal{N}, \tau)$ admits the polar decomposition

$$h_\psi = u|h_\psi| = u h_{|\psi|}.$$

Then the norm on $L^1(\mathcal{M}, \varphi)$ is defined as

$$\|h_\psi\|_1 = |\psi|(1) = \|\psi\|_*, \quad \forall \psi \in \mathcal{M}_*.$$

In this way,

$$L^1(\mathcal{M}, \varphi) = \mathcal{M}_*$$

isometrically. For $0 < p < \infty$, we define

$$\|x\|_p = \||x|^p\|_1^{\frac{1}{p}}, \quad \forall x \in L^p(\mathcal{M}, \varphi),$$

since $x \in L^p(\mathcal{M}, \varphi)$ if and only if $|x| \in L^p(\mathcal{M}, \varphi)$. Then for $1 \leq p < \infty$ (resp., $0 < p < 1$),

$$\big(L^p(\mathcal{M}, \varphi), \|\cdot\|_p\big)$$

is a Banach space (resp., a quasi-Banach space), and the norm satisfies the following equations:

$$\|x\|_p = \|x^*\|_p = \||x|\|_p, \quad \forall x \in L^p(\mathcal{M}, \varphi).$$

The usual Hölder inequality also holds for the $L^p(\mathcal{M}, \varphi)$ spaces. Let

$$0 < p, q, r \leq \infty, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

If $x \in L^p(\mathcal{M}, \varphi)$ and $y \in L^q(\mathcal{M}, \varphi)$, then

$$xy \in L^r(\mathcal{M}, \varphi) \qquad \text{and} \qquad \|xy\|_r \leq \|x\|_p \|x\|_q.$$

It is well known that $L^p(\mathcal{M}, \varphi)$ is independent of φ up to isometry. So, following Haagerup, we will use the notation $L^p(\mathcal{M})$ for the abstract Haagerup noncommutative L^p-space $L^p(\mathcal{M}, \varphi)$. To describe duality of Haagerup noncommutative L^p-space, we use the distinguished linear function on $L^1(\mathcal{M})$ which is defined by

$$\mathrm{tr}(x) = \psi_x(1), \quad \forall x \in L^1(\mathcal{M}),$$

where $\psi_x \in \mathcal{M}_*$ is the unique normal functional associated with x by the above identification between $\mathcal{M}_*$ and $L^1(\mathcal{M})$. Then tr is a continuous functional on $L^1(\mathcal{M})$ satisfying

$$\big|\mathrm{tr}(x)\big| \leq \mathrm{tr}\big(|x|\big) = \|x\|_1, \quad \forall x \in L^1(\mathcal{M}).$$

Let
$$1 \le p < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$
Then tr has the following property:
$$\mathrm{tr}(xy) = \mathrm{tr}(yx), \quad \forall x \in L^p(\mathcal{M}), \forall y \in L^q(\mathcal{M}).$$
The bilinear form $(x, y) \mapsto \mathrm{tr}(xy)$ defines a duality bracket between $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$, for which
$$\big(L^p(\mathcal{M})\big)' = L^q(\mathcal{M})$$
isometrically for all $1 \le p < \infty$. Moreover, our distinguished state φ can be recovered from tr (recalling that D is the Radon–Nikodym derivative of $\hat{\varphi}$ with respect to τ); that is,
$$\varphi(x) = \mathrm{tr}(Dx), \quad \forall x \in \mathcal{M}.$$

Let $0 < p \le \infty$, $K \subset L^p(\mathcal{M})$. We denote by $[K]_p$ the closed linear span of K in $L^p(\mathcal{M})$ (relative to the w*-topology for $p = \infty$), and we set
$$J(K) = \{x^* : x \in K\}.$$
For $0 < p < \infty$, $0 \le \eta \le 1$, we have
$$L^p(\mathcal{M}) = [D^{\frac{1-\eta}{p}} \mathcal{M} D^{\frac{\eta}{p}}]_p.$$

Let $\mathcal{D}$ be a von Neumann subalgebra of $\mathcal{M}$. Let $\mathcal{E}$ be the (unique) normal faithful conditional expectation of $\mathcal{M}$ with respect to $\mathcal{D}$ which leaves φ invariant.

Definition 2.1. A w*-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a *subdiagonal algebra* of $\mathcal{M}$ with respect to $\mathcal{E}$ (or to $\mathcal{D}$) if

(1) $\mathcal{A} + J(\mathcal{A})$ is w*-dense in $\mathcal{M}$,
(2) $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$, $\forall x, y \in \mathcal{A}$,
(3) $\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}$.

The algebra $\mathcal{D}$ is called the *diagonal* of $\mathcal{A}$.

We say that $\mathcal{A}$ is a *maximal subdiagonal algebra* in $\mathcal{M}$ with respect to $\mathcal{E}$ in the case that $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is a subdiagonal with respect to $\mathcal{E}$. Let
$$\mathcal{A}_0 = \big\{x \in \mathcal{A} : \mathcal{E}(x) = 0\big\}$$
and
$$\mathcal{A}_m = \big\{x \in \mathcal{M} : \mathcal{E}(yxz) = \mathcal{E}(yxz) = 0, \forall y \in \mathcal{A}, \forall z \in \mathcal{A}_0\big\}.$$
By Theorem 2.2.1 in [1], $\mathcal{A}_m$ is a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$ containing $\mathcal{A}$.

It follows from Theorem 2.4 in [12] and Theorem 1.1 in [24] that a subdiagonal algebra $\mathcal{A}$ of $\mathcal{M}$ with respect to $\mathcal{D}$ is maximal if and only if
$$\sigma_t^{\varphi}(\mathcal{A}) = \mathcal{A}, \quad \forall t \in \mathbb{R}. \tag{2.1}$$
In this paper, $\mathcal{A}$ always denotes a maximal subdiagonal algebra in $\mathcal{M}$ with respect to $\mathcal{E}$.

Definition 2.2. For $0 < p < \infty$, we define the *Haagerup noncommutative H^p-space* by

$$H^p(\mathcal{A}) = [\mathcal{A}D^{\frac{1}{p}}]_p, \qquad H_0^p(\mathcal{A}) = [\mathcal{A}_0 D^{\frac{1}{p}}]_p.$$

From Proposition 2.1 in [11], we know that for $1 \le p < \infty$, $0 \le \eta \le 1$,

$$H^p(\mathcal{A}) = [D^{\frac{1-\eta}{p}}\mathcal{A}D^{\frac{\eta}{p}}]_p, \qquad H_0^p(\mathcal{A}) = [D^{\frac{1-\eta}{p}}\mathcal{A}_0 D^{\frac{\eta}{p}}]_p. \tag{2.2}$$

It is known that

$$L^p(\mathcal{D}) = [D^{\frac{1-\eta}{p}}\mathcal{D}D^{\frac{\eta}{p}}]_p, \quad \forall p \in [1,\infty), \forall \eta \in [1,0],$$

and the conditional expectation $\mathcal{E}$ extends to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{D})$. The extension will be denoted still by $\mathcal{E}$. Let

$$1 \le r, p, q \le \infty, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Then

$$\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y), \quad \forall x \in H^p(\mathcal{A}), \forall y \in H^q(\mathcal{A}).$$

Indeed, let $a, b \in \mathcal{A}$. By Proposition 2.3 in [15], we have

$$\begin{aligned}\mathcal{E}(D^{\frac{1}{p}}abD^{\frac{1}{q}}) &= D^{\frac{1}{p}}\mathcal{E}(ab)D^{\frac{1}{q}} \\ &= D^{\frac{1}{p}}\mathcal{E}(a)\mathcal{E}(b)D^{\frac{1}{q}} \\ &= \mathcal{E}(D^{\frac{1}{p}}a)\mathcal{E}(bD^{\frac{1}{q}}).\end{aligned}$$

Hence, by (2.2), we obtain the desired result. We will repeatedly use the following fact:

$$\mathrm{tr}\big(\mathcal{E}(x)\big) = \mathrm{tr}(x), \quad x \in L^1(\mathcal{M}). \tag{2.3}$$

Let $\mathcal{M}_1$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\mathcal{M}_1$ is invariant under σ^φ and $\mathcal{E}$; that is,

$$\sigma_t^\varphi(\mathcal{M}_1) \subset \mathcal{M}_1, \quad \forall t \in \mathbb{R} \qquad \text{and} \qquad \mathcal{E}(\mathcal{M}_1) \subset \mathcal{M}_1.$$

Since $\mathcal{M}_1$ is σ^φ-invariant, it is well known (see [22])) that there is a (unique) normal conditional expectation Ψ such that

$$\varphi \circ \Psi = \varphi.$$

This conditional expectation $\Psi : \mathcal{M} \to \mathcal{M}_1$ commutes with $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ (see [7]); that is,

$$\Psi \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \Psi, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

Now let $\phi = \varphi|_{\mathcal{M}_1}$ be the restriction of φ to $\mathcal{M}_1$. Using the fact that $\mathcal{M}_1$ is σ^φ-invariant, we obtain that the modular automorphism group associated with ϕ is $\sigma^\varphi|_{\mathcal{M}_1}$, that is,

$$\sigma_t^\phi = \sigma_t^\varphi|_{\mathcal{M}_1}, \quad \forall t \in \mathbb{R}. \tag{2.5}$$

Hence, the crossed product

$$\mathcal{N}_1 = \mathcal{M}_1 \rtimes_{\sigma^\phi} \mathbb{R}$$

is a von Neumann subalgebra of

$$\mathcal{N} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}.$$

Let υ be the canonical normal semifinite faithful trace on $\mathcal{N}_1$. Then υ is equal to the restriction of τ to $\mathcal{N}_1$ (recalling that τ is the canonical trace on $\mathcal{N}$). Observe that the conditional expectation Ψ extends to a normal faithful conditional expectation $\widehat{\Psi}$ from $\mathcal{N}$ onto $\mathcal{N}_1$, satisfying $\tau \circ \widehat{\Psi} = \tau$ (i.e., $\upsilon \circ \widehat{\Psi} = \tau$). Let $\widehat{\varphi}$ and $\widehat{\phi}$ be the dual weights of φ and ϕ, respectively. Then $\widehat{\phi} \circ \widehat{\Psi} = \widehat{\varphi}$, and by [7], the Radon–Nikodym derivative of $\widehat{\phi}$ with respect to υ is equal to D, the Radon–Nikodym derivative of $\widehat{\varphi}$ with respect to τ.

Now let $\mathcal{M}$, φ, σ_t^φ, $\mathcal{E}$, $\mathcal{D}$, $\mathcal{A}$, $\mathcal{M}_1$, Ψ be fixed as in the above.

Proposition 2.3. *Let $\mathcal{A}$ be Ψ-invariant; that is, $\Psi(\mathcal{A}) \subset \mathcal{A}$. Set $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{M}_1$. Then $\mathcal{A}_1$ is a maximal subdiagonal algebra of $\mathcal{M}_1$ with respect to $\mathcal{E}|_{\mathcal{M}_1}$ and $H^p(\mathcal{A}_1)$ coincides isometrically with a subspace of $H^p(\mathcal{A})$ for every $0 < p < \infty$.*

Proof. It is clear that $\mathcal{A}_1$ is a w*-closed subalgebra of $\mathcal{M}$ and $\mathcal{E}|_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{D}_1$, where $\mathcal{D}_1 = \mathcal{D} \cap \mathcal{M}_1$. Since $\mathcal{E}$ is multiplicative on $\mathcal{A}$, $\mathcal{E}|_{\mathcal{M}_1}$ is multiplicative on $\mathcal{A}_1$. On the other hand,

$$\mathcal{A}_1 \cap J(\mathcal{A}_1) = \mathcal{A} \cap J(A) \cap \mathcal{M}_1 = \mathcal{D} \cap \mathcal{M}_1 = \mathcal{D}_1.$$

Thus it remains to show the σ-weak density of $\mathcal{A}_1 + J(\mathcal{A}_1)$ in $\mathcal{M}_1$. Let $x \in \mathcal{M}_1$. Since $\mathcal{A} + J(\mathcal{A})$ is σ-weakly dense in $\mathcal{M}$, there are $a_i, b_i \in \mathcal{A}$ such that

$$x = \lim_i (a_i + b_i^*) \quad \sigma\text{-weakly}.$$

Then by the normality of Ψ, we have

$$x = \Psi(x) = \lim_i \big(\Psi(a_i) + \Psi(b_i)^*\big) \quad \sigma\text{-weakly}.$$

Since $\mathcal{A}$ is Ψ-invariant, $\Psi(a_i), \Psi(b_i) \in \mathcal{A} \cap \mathcal{M}_1 = \mathcal{A}_1$. It follows that $\mathcal{A}_1 + J(\mathcal{A}_1)$ is σ-weakly dense in $\mathcal{M}_1$. Thus, A_1 is a subdiagonal algebra with respect to $\mathcal{E}|_{\mathcal{M}_1}$.

From

$$\mathcal{A}_1 = \Psi(\mathcal{A}_1) \subset \Psi(\mathcal{A}) \subset \mathcal{A} \cap \mathcal{M}_1 = \mathcal{A}_1$$

it follows that $\mathcal{A}_1 = \Psi(\mathcal{A})$. Hence, by (2.1), (2.4), and (2.5), we have

$$\sigma_t^\phi(a) = \sigma_t^\varphi(a) = \sigma_t^\varphi\big(\Psi(a)\big) = \Psi\big(\sigma_t^\varphi(a)\big) \in \mathcal{A}_1, \quad \forall a \in \mathcal{A}_1, \forall t \in \mathbb{R}.$$

Using Theorem 1.1 in [24] we obtain that $\mathcal{A}_1$ is a maximal subdiagonal algebra of $\mathcal{M}_1$ with respect to $\mathcal{E}|_{\mathcal{M}_1}$.

Since $\mathcal{A}_1 \subset \mathcal{A}$, we obtain that for every $0 < p < \infty$,

$$H^p(\mathcal{A}_1) = [\mathcal{A}_1 D^{\frac{1}{p}}]_p$$

coincides isometrically with a subspace of

$$H^p(\mathcal{A}) = [\mathcal{A} D^{\frac{1}{p}}]_p. \qquad \square$$

Remark 2.4. For $1 \leq p < \infty$, $H^p(\mathcal{A})$ is independent of φ (see Theorem 2.5 in [10]).

We also need to give a brief description of Haagerup's reduction theorem (cf. [9], [24]). Let

$$G = \bigcup_{n \geq 1} 2^{-n}\mathbb{Z}.$$

Then it is a discrete subgroup of $\mathbb{R}$. We consider the discrete crossed product

$$\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} G \tag{2.6}$$

with respect to $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ as above by replacing $\mathbb{R}$ by G and $L_2(\mathbb{R}, H)$ by $\ell_2(G, H)$. Recall that $\mathcal{R}$ is a von Neumann algebra on $\ell_2(G, H)$ generated by the operators $\pi(x), x \in \mathcal{M}$ and $\lambda(t), s \in G$, which are defined by

$$\big(\pi(x)\xi\big)(t) = \sigma_{-t}(x)\xi(t), \qquad \big(\lambda(s)\xi\big)(t) = \xi(t-s), \quad \xi \in \ell_2(G, H), \forall t \in G.$$

Then π is a normal faithful representation of $\mathcal{M}$ on $\ell_2(G, H)$ and we identify $\pi(\mathcal{M})$ with $\mathcal{M}$. The operators $\pi(x)$ and $\lambda(t)$ satisfy the following commutation relation:

$$\lambda(t)\pi(x)\lambda(t)^* = \pi\big(\sigma_t^\varphi(x)\big), \quad \forall t \in G, \forall x \in \mathcal{M}. \tag{2.7}$$

Let $\widehat{\varphi}$ be the dual weight of φ on $\mathcal{R}$. Then $\widehat{\varphi}$ is again a faithful normal state on $\mathcal{R}$ whose restriction on $\mathcal{M}$ is φ. The modular automorphism group of $\widehat{\varphi}$ is uniquely determined by

$$\sigma_t^{\widehat{\varphi}}(x) = \sigma_t^\varphi(x), \qquad \sigma_t^{\widehat{\varphi}}\big(\lambda_s(x)\big) = \lambda_s(x), \quad x \in \mathcal{M}, s, t \in G.$$

Consequently, $\sigma_t^{\widehat{\varphi}}|_{\mathcal{M}} = \sigma_t^\varphi$, and so $\sigma_t^{\widehat{\varphi}}(\mathcal{M}) = \mathcal{M}$ for all $t \in \mathbb{R}$. We recall that there is a unique normal faithful conditional expectation Φ from $\mathcal{R}$ onto $\mathcal{M}$ determined by

$$\Phi\big(\lambda(t)x\big) = \begin{cases} x & \text{if } t = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad \forall x \in \mathcal{M}, \forall t \in G. \tag{2.8}$$

It satisfies

$$\widehat{\varphi} \circ \Phi = \widehat{\varphi}, \qquad \sigma_t^{\widehat{\varphi}} \circ \Phi = \Phi \circ \sigma_t^{\widehat{\varphi}}, \quad \forall t \in \mathbb{R}.$$

We denote by $\mathcal{R}_{\widehat{\varphi}}$ the centralizer of $\widehat{\varphi}$ in $\mathcal{R}$. Then

$$\mathcal{R}_{\widehat{\varphi}} = \big\{x \in \mathcal{R} : \sigma_t^{\widehat{\varphi}}(x) = x, \forall t \in \mathbb{R}\big\}.$$

For each $n \in \mathbb{N}$, there exists a unique $b_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ such that

$$0 \leq b_n \leq 2\pi \qquad \text{and} \qquad e^{ib_n} = \lambda(2^{-n}),$$

where $\mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ is the center of $\mathcal{R}_{\widehat{\varphi}}$. Let $a_n = 2^n b_n$, and define normal faithful positive functionals on $\mathcal{R}$ by

$$\varphi_n(x) = \widehat{\varphi}(e^{-a_n}x), \quad \forall x \in \mathcal{R}, \forall n \in \mathbb{N}.$$

Then $\sigma_t^{\varphi_n}$ is 2^{-n}-periodic and

$$\sigma_t^{\varphi_n}(x) = e^{-ita_n}\sigma_t^{\widehat{\varphi}}(x)e^{ita_n}, \quad \forall x \in \mathcal{R}, \forall t \in \mathbb{R}, \forall n \in \mathbb{N}. \tag{2.9}$$

Let $\mathcal{R}_n = \mathcal{R}_{\varphi_n}$. Then $\mathcal{R}_n$ is a finite von Neumann algebra equipped with the normal faithful tracial state $\tau_n = \varphi_n|_{\mathcal{R}_n}$.

Define

$$\Phi_n(x) = 2^n \int_0^{2^{-n}} \sigma_t^{\varphi_n}(x)\, dt, \quad \forall x \in \mathcal{R}.$$

By the 2^{-n}-periodicity of $\sigma_t^{\varphi_n}$, we have

$$\Phi_n(x) = \int_0^1 \sigma_t^{\varphi_n}(x)\, dt, \quad \forall x \in \mathcal{R}. \tag{2.10}$$

Haagerup's reduction theorem (Theorem 2.1 and Lemma 2.7 in [9]) asserts that $\{\mathcal{R}_n\}_{n\geq 1}$ is an increasing sequence of von Neumann subalgebras of $\mathcal{R}$ with the following properties:

(1) each $\mathcal{R}_n$ is finite;

(2) Φ_n is a faithful normal conditional expectation from $\mathcal{R}$ onto $\mathcal{R}_n$ such that

$$\hat{\varphi} \circ \Phi_n = \hat{\varphi}, \qquad \sigma_t^{\hat{\varphi}} \circ \Phi_n = \Phi_n \circ \sigma_t^{\hat{\varphi}}, \qquad \Phi_n \circ \Phi_{n+1} = \Phi_n, \quad \forall t \in \mathbb{R}, \forall n \in \mathbb{N};$$

(3) for any $x \in \mathcal{R}$, $\Phi_n(x)$ converges to x σ-strongly as $n \to \infty$;

(4) $\bigcup_{n\geq 1} \mathcal{R}_n$ is σ-weakly dense in $\mathcal{R}$.

Since $\mathcal{D}$ is σ_t^{φ}-invariant and $\sigma_t^{\varphi}|_{\mathcal{D}}$ is exactly the modular automorphism group of $\varphi|_{\mathcal{D}}$, we do not need to distinguish φ and $\varphi|_{\mathcal{D}}$, σ_t^{φ} and $\sigma_t^{\varphi}|_{\mathcal{D}}$, respectively. Now let

$$\widehat{\mathcal{D}} = \mathcal{D} \rtimes_{\sigma^{\varphi}} G.$$

Then $\widehat{\mathcal{D}}$ is naturally identified as a von Neumann subalgebra of $\mathcal{R}$ generated by all operators $\pi(x), x \in \mathcal{D}$ and $\lambda(t), t \in G$. We can extend $\mathcal{E}$ to a normal faithful conditional expectation $\widehat{\mathcal{E}}$ from $\mathcal{R}$ onto $\widehat{\mathcal{D}}$, which is uniquely determined by

$$\widehat{\mathcal{E}}\big(\lambda(t)x\big) = \lambda(t)\mathcal{E}(x), \quad \forall x \in \mathcal{M}, \forall t \in G. \tag{2.11}$$

This conditional expectation satisfies

$$\widehat{\mathcal{E}} \circ \Phi_n = \Phi_n \circ \widehat{\mathcal{E}}, \quad \forall n \in \mathbb{N}.$$

Now let $n \in \mathbb{N}$ and $\mathcal{D}_n = \mathcal{R}_n \cap \widehat{\mathcal{D}}$. Note that $\Phi_n|_{\widehat{\mathcal{D}}}$ and $\widehat{\mathcal{E}}|_{\mathcal{R}_n}$ are normal faithful conditional expectations from $\widehat{\mathcal{D}}$ onto $\mathcal{D}_n$ (resp., from $\mathcal{R}_n$ onto $\mathcal{D}_n$).

Since $\mathcal{A}$ is σ_t^{φ}-invariant, by (2.7), the family of all linear combinations on $\lambda(t)\pi(x)$, $t \in G$, $x \in \mathcal{A}$, is a *-subalgebra of $\mathcal{R}$. Let $\widehat{\mathcal{A}}$ be its σ-weakly closure in $\mathcal{R}$, that is,

$$\widehat{\mathcal{A}} = \overline{\operatorname{span}\big\{\lambda(t)\pi(x) : t \in G, x \in \mathcal{A}\big\}}^{\sigma\text{-weakly}}, \tag{2.12}$$

and let $\mathcal{A}_n = \widehat{\mathcal{A}} \cap \mathcal{R}_n$. By Lemmas 3.1–3.3 in [24], $\widehat{\mathcal{A}}$ (resp., $\mathcal{A}_n$) is a maximal subdiagonal algebra of $\mathcal{R}$ (resp., $\mathcal{R}_n$) with respect to $\widehat{\mathcal{E}}$ (resp., $\widehat{\mathcal{E}}|_{\mathcal{R}_n}$).

The next result is known. For easy reference, we give its proof (see [10], [24]).

Lemma 2.5. *Let $\mathcal{M}$, $\mathcal{A}$, $\mathcal{E}$, $\mathcal{R}$, $\widehat{\mathcal{A}}$, $\widehat{\mathcal{E}}$, Φ, $\mathcal{A}_n$, Φ_n ($\forall n \in \mathbb{N}$) be fixed as in the above. Then*

(1) $\Phi(\widehat{\mathcal{A}}) = \mathcal{A}$, $\Phi_n(\widehat{\mathcal{A}}) = \mathcal{A}_n$, $\forall n \in \mathbb{N}$;

(2) $\bigcup_{n\geq 1} \mathcal{A}_n$ is σ-weakly dense in $\widehat{\mathcal{A}}$;

(3) $\widehat{\mathcal{E}}(\mathcal{M}) \subset \mathcal{M}$.

Proof. (1) By (2.8), we know that $\Phi(\widehat{\mathcal{A}}) = \mathcal{A}$. Since $\mathcal{A}$ is σ_t^φ-invariant, by (2.9) and (2.10), we conclude that $\widehat{\mathcal{A}}$ is Φ_n-invariant for all $n \in \mathbb{N}$. It is clear that

$$\mathcal{A}_n = \Phi_n(\mathcal{A}_n) \subset \Phi_n(\widehat{\mathcal{A}}) \subset \widehat{\mathcal{A}} \cap \mathcal{R}_n = \mathcal{A}_n,$$

so it follows that $\mathcal{A}_n = \Phi_n(\widehat{\mathcal{A}})$.

(2) For any $x \in \widehat{\mathcal{A}}$, $\Phi_n(x)$ converges to x σ-strongly as $n \to \infty$. Hence, (2) follows from (1).

(3) This follows from (2.11). □

As an extension of Theorem 3.1 in [9], we have the following.

Theorem 2.6. *Let $\mathcal{M}$, $\mathcal{A}$, $\mathcal{E}$, $\mathcal{R}$, $\widehat{\mathcal{A}}$, $\widehat{\mathcal{E}}$, Φ, $\mathcal{A}_n$, Φ_n ($\forall n \in \mathbb{N}$) be fixed as in the above, and let $0 < p < \infty$. Then*

1. *$\{H^p(\mathcal{A}_n)\}_{n\geq 1}$ is an increasing sequence of subspaces of $H^p(\widehat{\mathcal{A}})$;*
2. *$\bigcup_{n\geq 1} H^p(\mathcal{A}_n)$ is dense in $H^p(\widehat{\mathcal{A}})$;*
3. *for each n, $H^p(\mathcal{A}_n)$ is isometric to the usual noncommutative H^p-space associated with a finite subdiagonal algebra;*
4. *$H^p(\mathcal{A})$ and all $H^p(\mathcal{A}_n)$ are 1-complemented in $H^p(\widehat{\mathcal{A}})$ for $1 \leq p < \infty$.*

Proof. (1) By Proposition 2.3, $H^p(\mathcal{A})$ and all $H^p(\mathcal{A}_n)$ are naturally isometrically identified as subspaces of $H^p(\widehat{\mathcal{A}})$ for $0 < p < \infty$, and the sequence $\{H^p(\mathcal{A}_n)\}_{n\geq 1}$ is increasing.

(2) By Lemma 2.2 in [14], $\bigcup_{n\geq 1} H^p(\mathcal{A}_n)$ is dense in $H^p(\widehat{\mathcal{A}})$ for $0 < p < \infty$.

(3) Since $\mathcal{R}_n$ is a finite von Neumann algebra, the result holds (see Remark 2.4).

(4) Using Lemma 2.5(1) and Lemma 2.2 in [15], we obtain the desired results. □

Let

$$1 \leq p \leq \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

If $x \in L^p(\mathcal{M})$ and $\mathrm{tr}(xa) = 0$, $\forall a \in H^q(\mathcal{A})$, then we write $x \perp J(H^q(\mathcal{A}))$. Similarly, we define $x \perp J(H_0^q(\mathcal{A}))$.

Proposition 2.7. *Let $\mathcal{A}$ be a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$. Then we have*

(i) $\mathcal{A} = \{x \in \mathcal{M} : x \perp J(H_0^1(\mathcal{A}))\}$, $\mathcal{A}_0 = \{x \in \mathcal{M} : x \perp J(H^1(\mathcal{A}))\}$,
(ii) $H^1(\mathcal{A}) = \{x \in L^1(\mathcal{M}) : x \perp J(\mathcal{A}_0)\}$, $H_0^1(\mathcal{A}) = \{x \in L^1(\mathcal{M}) : x \perp J(\mathcal{A})\}$.

Proof. (i) Assume that $x \in \mathcal{A}$. If $y^* \in J(H_0^1(\mathcal{A}))$, then there exists $\{a_n^*\} \subset \mathcal{A}_0$ such that $y = \lim_{n\to\infty} a_n D$. Hence, by (2.3),

$$\mathrm{tr}(xy) = \lim_{n\to\infty} \mathrm{tr}(xa_n D) = \lim_{n\to\infty} \mathrm{tr}\big(\mathcal{E}(xa_n)D\big) = 0,$$

whence $\mathcal{A} \subset \{x \in \mathcal{M}, x \perp J(H_0^1(\mathcal{A}))\}$.

Conversely, we take $x \in \mathcal{M}, x \perp J(H_0^1(\mathcal{A}))$. By (2.2),

$$\mathrm{tr}(xD^{\frac{1}{2}}aD^{\frac{1}{2}}) = 0, \quad \forall a \in \mathcal{A}_0.$$

Hence

$$\operatorname{tr}(xD^{\frac{1}{2}}abD^{\frac{1}{2}}) = 0, \quad \forall b \in \mathcal{A}_0, \forall a \in \mathcal{A}.$$

Using (2.2), we obtain that $xH^2(\mathcal{A}) \perp J(H_0^2(\mathcal{A}))$. Since

$$L^2(\mathcal{M}) = H^2(\mathcal{A}) \oplus J\big(H_0^2(\mathcal{A})\big), \quad xH^2(\mathcal{A}) \subseteq H^2(\mathcal{A}).$$

By Theorem 2.2 in [13], $x \in \mathcal{A}$. Thus, we have proved the first equality. The proof of second equality is similar to that of the first equality.

(ii) It is clear that $H^1(\mathcal{A}) \subset \{x \in L^1(\mathcal{M}) : x \perp J(\mathcal{A}_0)\}$. Conversely, if $x \in L^1(\mathcal{M})$, $x \perp J(\mathcal{A}_0)$ and $x \notin H^1(\mathcal{A})$, then there exists $y \in \mathcal{M}$ such that $\operatorname{tr}(y^*x) = 1$ and $y \perp H^1(\mathcal{A})$. By (i), $y^* \in \mathcal{A}_0$. Therefore, $\operatorname{tr}(y^*x) = 0$. This is a contradiction, so the first equality holds. The proof of the second equality is similar. □

By Theorem 3.3 in [11], we know that

$$L^p(\mathcal{M})) = H^p(\mathcal{A}) \oplus J\big(H_0^p(\mathcal{A})\big)$$

for all $1 < p < \infty$. Hence, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$H^p(\mathcal{A}) = \big\{x \in L^p(\mathcal{M}) : x \perp J\big(H_0^q(\mathcal{A})\big)\big\} \tag{2.13}$$

and

$$H_0^p(\mathcal{A}) = \big\{x \in L^p(\mathcal{M}) : x \perp J\big(H^q(\mathcal{A})\big)\big\}.$$

Proposition 2.8. *Let* $1 \le r, p, s \le \infty$, $\frac{1}{p} + \frac{1}{s} = \frac{1}{r}$, *and* $0 \le \eta \le 1$. *Then*

(i) $H^p(\mathcal{A}) = \{a \in L^p(\mathcal{M}) : D^{\frac{\eta}{s}} a D^{\frac{1-\eta}{s}} \in H^r(\mathcal{A})\}$,
(ii) $H_0^p(\mathcal{A}) = \{a \in L^p(\mathcal{M}) : D^{\frac{\eta}{s}} a D^{\frac{1-\eta}{s}} \in H_0^r(\mathcal{A})\}$.

Proof. We only prove (i). The proof of (ii) is similar. It is clear that

$$H^p(\mathcal{A}) \subseteq \big\{a \in L^p(\mathcal{M}) : D^{\frac{\eta}{s}} a D^{\frac{1-\eta}{s}} \in H^r(\mathcal{A})\big\}.$$

If $x \in \{a \in L^p(\mathcal{M}) : D^{\frac{\eta}{s}} a D^{\frac{1-\eta}{s}} \in H^r(\mathcal{A})\}$, then

$$\operatorname{tr}(D^{\frac{\eta}{s}} x D^{\frac{1-\eta}{s}} b) = 0, \quad \forall b \in H_0^{r'}(\mathcal{A}),$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Therefore, by (2.2),

$$\operatorname{tr}(xb) = 0, \quad \forall b \in H_0^q(\mathcal{A}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Using (2.13) or Lemma 2.7, we get $x \in H^p(\mathcal{A})$. Hence the desired result holds. □

3. Complex interpolation theorem of the Haagerup H^p-spaces

First, we briefly recall the complex interpolation method we will use in this section. Our main reference is [5]. Let $X = (X_0, X_1)$ be a pair of two compatible Banach spaces with norms $\|\cdot\|_{X_0} = \|\cdot\|_0$ and $\|\cdot\|_{X_1} = \|\cdot\|_1$, respectively. The algebraic sum $\Sigma(X_0, X_1) = X_0 + X_1$ is a Banach space under the norm

$$\|x\|_\Sigma = \inf\big\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1, x_j \in X_j, j = 0, 1\big\}.$$

One defines a space of certain $\Sigma(X_0, X_1)$-valued functions on the strip $0 \le \operatorname{Re} z \le 1$ as $F(X_0, X_1)$, which consists of functions $f : 0 \le \operatorname{Re} z \le 1 \mapsto \sum(X_0, X_1)$ satisfying the following:

(1) f is bounded and continuous, and analytic in the interior (with respect to $\|\cdot\|_\Sigma$);
(2) $f(j + it) \in X_j$, $t \in \mathbb{R}$, $j = 0, 1$;
(3) for $j = 0, 1$, the map $t \in \mathbb{R} \mapsto f(j + it) \in X_j$ is $\|\cdot\|_j$-continuous and $\lim_{t\to\pm\infty} \|f(j + it)\|_j = 0$.

It follows from the Phragmén–Lindelöf theorem [5, Lemma 1.1.2] that the space $F(X_0, X_1)$ is Banach space under the norm

$$|||f||| = \max\Big\{\sup_{t\in\mathbb{R}}\big\|f(it)\big\|_0, \sup_{t\in\mathbb{R}}\big\|f(1 + it)\big\|_1\Big\}.$$

Definition 3.1. For each $0 < \theta < 1$, the complex interpolation space $C_\theta(X_0, X_1)$ is the set of all $f(\theta)$, $f \in F(X_0, X_1)$, equipped with the complex interpolation norm

$$\|x\|_{C_\theta(X_0,X_1)} = \inf\big\{|||f||| : f \in F(X_0, X_1), f(\theta) = x\big\}.$$

As in [16], for each $0 \le \eta \le 1$, we define the embedding $i^\eta_\infty : \mathcal{M} \hookrightarrow L^{p_0}(\mathcal{M})$ by

$$i^\eta_\infty(x) = D^\eta x D^{1-\eta}, \quad \forall x \in \mathcal{M}.$$

Let $\mathcal{M}^\eta = i^\eta_\infty(\mathcal{M}) = D^\eta \mathcal{M} D^{1-\eta}$. Then $\mathcal{M}^\eta \subseteq L^1(\mathcal{M})$. We write (unless there is a chance of confusion)

$$\big\|i^\eta_\infty(x)\big\|^\eta_\infty = \|D^\eta x D^{1-\eta}\|^\eta_\infty = \|x\|_\infty, \qquad \|x\|^\eta_1 = \|D^\eta x D^{1-\eta}\|_1, \quad \forall x \in \mathcal{M}.$$

In what follows, we never consider a power of $\|\cdot\|_\infty$, $\|\cdot\|_1$, so that we avoid any chance of confusion in the above notation. Since

$$\big\|i^\eta_\infty(x)\big\|^\eta_1 = \|D^\eta x D^{1-\eta}\|_1 \le \|x\|_\infty = \|D^\eta x D^{1-\eta}\|^\eta_\infty = \big\|i^\eta_\infty(x)\big\|^\eta_\infty$$

for all $x \in \mathcal{M}$, $\|\cdot\|^\eta_1 \le \|\cdot\|^\eta_\infty$ on the subspace $\mathcal{M}^\eta$ of $L^1(\mathcal{M})$. Therefore, the pair $(\mathcal{M}^\eta, L^1(\mathcal{M}))$ is compatible and satisfies

$$\Sigma\big(\mathcal{M}^\eta, L^1(\mathcal{M})\big) = L^1(\mathcal{M}), \qquad \mathcal{M}^\eta \cap L^1(\mathcal{M}) = \mathcal{M}^\eta, \qquad \|\cdot\|_\Sigma = \|\cdot\|_1.$$

If $1 < p < \infty$, then for each $0 \le \eta \le 1$, we embed $L^p(\mathcal{M})$ into $L^1(\mathcal{M})$ via

$$i^\eta_p : x \in L^p(\mathcal{M}) \to D^{\frac{\eta}{q}} x D^{\frac{1-\eta}{q}} \in L^1(\mathcal{M}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. We have $i^\eta_p(L^p(\mathcal{M})) = D^{\frac{\eta}{q}} L^p(\mathcal{M}) D^{\frac{1-\eta}{q}} \subseteq L^1(\mathcal{M})$. We also write (unless there is a chance of confusion)

$$\big\|i^\eta_p(x)\big\|^\eta_p = \|D^{\frac{\eta}{q}} x D^{\frac{1-\eta}{q}}\|^\eta_p = \|x\|_p, \quad \forall x \in L^p(\mathcal{M}).$$

By Theorem 9.1 in [16], we have

$$C_\theta\big(\mathcal{M}^\eta, L^1(\mathcal{M})\big) = i_p^\eta\big(L^p(\mathcal{M})\big) \tag{3.1}$$

with equal norms, where $\theta = \frac{1}{p}$.

As in the above, for each $0 \le \eta \le 1$, we can embed $\mathcal{A}$ into $H^1(\mathcal{A})$ via

$$i_\infty^\eta : a \in \mathcal{A} \to D^\eta a D^{1-\eta} \in H^1(\mathcal{A}).$$

Let $\mathcal{A}^\eta = i_\infty^\eta(\mathcal{A}) = D^\eta \mathcal{A} D^{1-\eta}$. We write

$$\big\|i_\infty^\eta(a)\big\|_\infty^\eta = \|D^\eta a D^{1-\eta}\|_\infty^\eta = \|a\|_\infty, \qquad \|a\|_1^\eta = \|D^\eta a D^{1-\eta}\|_1, \quad \forall a \in \mathcal{A}.$$

Hence $(\mathcal{A}^\eta, H^1(\mathcal{A}))$ is compatible.

For the Haagerup noncommutative H^p-space, we obtain the following result.

Lemma 3.2. *Let $\mathcal{R}$ and $\widehat{\mathcal{A}}$ be as in (2.6) and (2.12), respectively. Then for $1 < p < \infty$, we have*

$$C_\theta\big(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})\big) = i_p^\eta\big(H^p(\widehat{\mathcal{A}})\big)$$

with equivalent norms, where $\theta = \frac{1}{p}$.

Proof. By (3.1), $C_\theta(\mathcal{R}^\eta, L^1(\mathcal{R})) = i_p^\eta(L^p(\mathcal{R}))$ with equal norms. We have

$$C_\theta\big(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})\big) \subset C_\theta\big(\mathcal{R}^\eta, L^1(\mathcal{R})\big) = i_p^\eta\big(L^p(\mathcal{R})\big).$$

On the other hand, $C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})) \subset H^1(\widehat{\mathcal{A}})$. By Proposition 2.8,

$$C_\theta\big(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})\big) \subset i_p^\eta\big(H^p(\widehat{\mathcal{A}})\big),$$

whence

$$\|x\|_p \le \|x\|_{C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))}, \quad \forall x \in C_\theta\big(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})\big). \tag{3.2}$$

Let $\{H^p(\mathcal{A}_n)\}_{n\ge1}$ be the sequence in Theorem 2.6. Since $H^p(\mathcal{A}_n)$ is isometric to the usual noncommutative H^p-space associated with a finite subdiagonal algebra,

$$C_\theta\big(\mathcal{A}_n, H^1(\mathcal{A}_n)\big) = H^p(\mathcal{A}_n)$$

with equivalent norms, for all $n \in \mathbb{N}$ (see the remark following Lemma 8.5 in [21] or Theorem 1.1 in [4]). Hence, by Lemma 2.5(1) and Theorem 2.6(4),

$$H^p(\mathcal{A}_n) = C_\theta\big(\mathcal{A}_n, H^1(\mathcal{A}_n)\big) \subset C_\theta\big(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})\big)$$

and

$$\|a\|_{C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))} \le C\|a\|_p, \quad \forall a \in H^p(\mathcal{A}_n), \tag{3.3}$$

where C depends only on θ. Let $x \in H^p(\widehat{\mathcal{A}})$. Using Theorem 2.6(2), we obtain a sequence $\{a_n\}_{n\ge1}$ in $\bigcup_{n\ge1} H^p(\mathcal{A}_n)$ such that

$$\lim_{n\to\infty} \|a_n - x\|_p = 0.$$

Since $\{H^p(\mathcal{A}_n)\}_{n\ge1}$ is increasing, using (3.3) we obtain that

$$\|a_n - a_m\|_{C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))} \le C\|a_n - a_m\|_p, \quad n, m \ge 1.$$

Hence $\{a_n\}_{n\geq 1}$ is a Cauchy sequence in $C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))$. So, there exists y in $C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))$ such that

$$\lim_{n\to\infty} \|a_n - y\|_{C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))} = 0.$$

By (3.2), $\lim_{n\to\infty} \|a_n - y\|_p = 0$. Therefore, $y = x$. We deduce that

$$\|x\|_{C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))} \leq C\|x\|_p, \quad \forall x \in H^p(\widehat{\mathcal{A}}).$$

Thus, $C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}})) = i_p^\eta(H^p(\widehat{\mathcal{A}}))$. ☐

Theorem 3.3. *Let $1 < p < \infty$. Then*

$$C_\theta\big(\mathcal{A}^\eta, H^1(\mathcal{A})\big) = i_p^\eta\big(H^p(\mathcal{A})\big)$$

with equivalent norms, where $\theta = \frac{1}{p}$.

Proof. Using (3.1), we obtain that $C_\theta(\mathcal{M}^\eta, L^1(\mathcal{M})) = i_p^\eta(L^p(\mathcal{M}))$ with equal norms. It is clear that

$$C_\theta\big(\mathcal{A}^\eta, H^1(\mathcal{A})\big) \subset C_\theta\big(\mathcal{M}^\eta, L^1(\mathcal{M})\big) = i_p^\eta\big(L^p(\mathcal{M})\big).$$

Since $C_\theta(\mathcal{A}^\eta, H^1(\mathcal{A})) \subset H^1(\mathcal{A})$, by Proposition 2.8, we have

$$C_\theta\big(\mathcal{A}, H^1(\mathcal{A})\big) \subset i_p^\eta\big(H^p(\mathcal{A})\big).$$

On the other hand, by Lemma 2.5(1) and Theorem 2.6(4), we have that $\mathcal{A}^\eta$, $H^1(\mathcal{A})$, $H^p(\mathcal{A})$ are 1-completed in $\widehat{\mathcal{A}}^\eta$, $H^1(\widehat{\mathcal{A}})$, $H^p(\widehat{\mathcal{A}})$, respectively. Hence,

$$\|x\|_{C_\theta(\mathcal{A}, H^1(\mathcal{A}))} = \|x\|_{C_\theta(\widehat{\mathcal{A}}^\eta, H^1(\widehat{\mathcal{A}}))}, \quad \forall x \in C_\theta\big(\mathcal{A}^\eta, H^1(\mathcal{A})\big)$$

and

$$\|x\|_{H^p(\widehat{\mathcal{A}})} = \|x\|_{H^p(\mathcal{A})}, \quad \forall x \in H^p(\mathcal{A}).$$

Using Lemma 3.2, we get

$$\|x\|_{C_\theta(\mathcal{A}, H^1(\mathcal{A}))} \leq C\|x\|_p, \quad \forall x \in C_\theta\big(\mathcal{A}^\eta, H^1(\mathcal{A})\big). \tag{3.4}$$

Let $a \in \mathcal{A}$. For $\delta > 0$, we define

$$f_a(z) = e^{\delta(z^2-\theta^2)} D^{\eta(1-z)} a D^z D^{(1-\eta)(1-z)}, \quad 0 \leq \operatorname{Re} z \leq 1.$$

Obviously,

$$f_a(it) \in \mathcal{A}^\eta, \quad \big\|f_a(it)\big\|_\infty^\eta \leq \|a\|_\infty,$$
$$f_a(1+it) \in H^1(\mathcal{A}), \quad \big\|f_a(1+it)\big\|_1 \leq e^{\delta(1-\theta^2)}\|aD\|_1 \leq e^{\delta(1-\theta^2)}\|a\|_\infty,$$
$$f_a(z) \in F\big(\mathcal{A}^\eta, H^1(\mathcal{A})\big), \quad |||f_a||| \leq e^{\delta(1-\theta^2)}\|a\|_\infty, \qquad f_a(\theta) = D^{\frac{\eta}{q}} a D^{\frac{1}{p}} D^{\frac{1-\eta}{q}}.$$

Thus $i_p^\eta(aD^{\frac{1}{p}}) \in C_\theta(\mathcal{A}^\eta, H^1(\mathcal{A}))$. Therefore, applying (3.4) we deduce that

$$\|aD^{\frac{1}{p}}\|_{C_\theta(\mathcal{A}, H^1(\mathcal{A}))} \leq C\|aD^{\frac{1}{p}}\|_p, \quad \forall a \in \mathcal{A}. \tag{3.5}$$

The proof of $C_\theta(\mathcal{A}^\eta, H^1(\mathcal{A})) = i_p^\eta(H^p(\mathcal{A}))$ is similar to that of Lemma 3.2 by using (3.5) and Definition 2.2. We omit the details. ☐

Let $H^p(\mathcal{A})_L = C_\theta(\mathcal{A}^0, H^1(\mathcal{A}))$ and $H^p(\mathcal{A})_R = C_\theta(\mathcal{A}^1, H^1(\mathcal{A}))$. We call $H^p(\mathcal{A})_L$ and $H^p(\mathcal{A})_R$ the left and the right H^p-spaces with respect to D, respectively. By Theorem 3.3, we have

$$H^p(\mathcal{A})_L = H^p(\mathcal{A})D^{\frac{1}{q}}, \qquad H^p(\mathcal{A})_R = D^{\frac{1}{q}}H^p(\mathcal{A}).$$

We have the following Stein–Weiss-type interpolation theorem.

Theorem 3.4. *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 \le \eta \le 1$. *Then*

$$C_\eta\big(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A})\big) = D^{\frac{\eta}{q}}H^p(\mathcal{A})D^{\frac{1-\eta}{q}}.$$

Consequently,

$$C_\eta\big(H^p(\mathcal{A})_L, H^p(\mathcal{A})_R\big) = C_\theta\big(\mathcal{A}^\eta, H^1(\mathcal{A})\big)|_{\theta=\frac{1}{p}}.$$

Proof. By Theorem 11.2 of [16], we see that

$$C_\eta\big(L^p(\mathcal{M})D^{\frac{1}{q}}, D^{\frac{1}{q}}L^p(\mathcal{M})\big) = D^{\frac{\eta}{q}}L^p(\mathcal{M})D^{\frac{1-\eta}{q}}.$$

Thus,

$$C_\eta\big(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A})\big) \subset D^{\frac{\eta}{q}}L^p(\mathcal{M})D^{\frac{1-\eta}{q}}.$$

It is trivial that $C_\eta(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A})) \subset H^1(\mathcal{A})$. By Proposition 2.8, we deduce that

$$C_\eta\big(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A})\big) \subset D^{\frac{\eta}{q}}H^p(\mathcal{A})D^{\frac{1-\eta}{q}}. \tag{3.6}$$

Conversely, assume that $x \in H^p(\mathcal{A})$. Let $\delta > 0$. Define for $0 \le \operatorname{Re} z \le 1$

$$f(z) = e^{\delta(z^2-\eta^2)}D^{\frac{z}{q}}xD^{\frac{1-z}{q}}.$$

It is clear that

$$f(it) \in H^p(\mathcal{A})D^{\frac{1}{q}}, \quad \big\|f(it)\big\|_p^0 \le \|x\|_p,$$
$$f(1+it) \in D^{\frac{1}{q}}H^p(\mathcal{A}), \quad \big\|f(1+it)\big\|_p^1 \le e^{\delta(1-\eta^2)}\|x\|_p,$$
$$f(z) \in F\big(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A})\big), \quad |||f||| \le e^{\delta(1-\eta^2)}\|x\|_p, \qquad f(\eta) = D^{\frac{\eta}{q}}xD^{\frac{1-\eta}{q}}.$$

Thus, $i_p^\eta(x) \in C_\eta(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A}))$. It follows that

$$D^{\frac{\eta}{q}}H^p(\mathcal{A})D^{\frac{1-\eta}{q}} \subset C_\eta\big(H^p(\mathcal{A})D^{\frac{1}{q}}, D^{\frac{1}{q}}H^p(\mathcal{A})\big). \tag{3.7}$$

Therefore, by (3.6) and (3.7), the desired result holds. The second result follows immediately by virtue of Theorem 3.3. □

If $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$, then the duality between $i_p^\eta(H^p(\mathcal{A}))$ and $i_q^\eta(J(H^q(\mathcal{A})))$ is defined by $(i_p^\eta(a), i_q^\eta(b)) \mapsto \operatorname{tr}(ab)$, for which

$$\big(i_p^\eta\big(H^p(\mathcal{A})\big)\big)' = i_q^\eta\big(J\big(H^q(\mathcal{A})\big)\big)$$

isometrically (see Corollary 3.4 in [11]). Let $\mathrm{BMO}(\mathcal{A})$ be the noncommutative BMO space defined in [11]. We have $H^1(\mathcal{A})^* = \mathrm{BMO}(\mathcal{A})$ (see Theorem 3.10 in [11]).

Theorem 3.5. *Let* $0 \leq \eta \leq 1$. *Then*

$$C_{\frac{1}{p}}\big(\mathrm{BMO}(\mathcal{A}), H^1(\mathcal{A})\big) = i_p^\eta\big(H^p(\mathcal{A})\big), \quad \forall p \in (1,\infty).$$

Proof. By Theorem 3.3 and the reiteration theorem ([5, Theorem 4.6.1]), for $1 \leq p_0, p_1, p < \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$C_\theta\big(i_{p_0}^\eta\big(H^{p_0}(\mathcal{A})\big), i_{p_1}^\eta\big(H^{p_1}(\mathcal{A})\big)\big) = i_p^\eta\big(H^p(\mathcal{A})\big) \tag{3.8}$$

with equal norms.

Let $1 < r < p < \infty$, and let $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$. Then $1 < q < s < \infty$. By (3.8), the following holds with equivalent norms:

$$C_{\frac{r}{p}}\big(i_1^\eta\big(H^1(\mathcal{A})\big), i_s^\eta\big(H^s(\mathcal{A})\big)\big) = i_q^\eta\big(H^q(\mathcal{A})\big).$$

Applying Corollary 4.5.2 in [5], we obtain that

$$\begin{aligned} i_p^\eta\big(H^p(\mathcal{A})\big) &= \big(i_q^\eta\big(J\big(H^q(\mathcal{A})\big)\big)\big)' \\ &= \big(J\big(C_{\frac{r}{p}}\big(i_1^\eta\big(H^1(\mathcal{A})\big), i_s^\eta\big(H^s(\mathcal{A})\big)\big)\big)\big)' \\ &= C_{\frac{r}{p}}\big(\mathrm{BMO}(\mathcal{A}), i_r^\eta\big(H^r(\mathcal{A})\big)\big). \end{aligned}$$

Moreover, we deduce from (3.8) that

$$i_q^\eta\big(H^r(\mathcal{A})\big) = C_{\frac{q}{s}}\big(i_1^\eta\big(H^1(\mathcal{A})\big), i_p^\eta\big(H^p(\mathcal{A})\big)\big) = C_{1-\frac{q}{s}}\big(i_p^\eta\big(H^p(\mathcal{A})\big), i_1^\eta\big(H^1(\mathcal{A})\big)\big).$$

Using Wolff's interpolation theorem [23, Theorem 2], we obtain

$$C_{\frac{1}{p}}\big(\mathrm{BMO}(\mathcal{A}), H^1(\mathcal{A})\big) = i_p^\eta\big(H^p(\mathcal{A})\big).$$

□

Acknowledgments. Bekjan's work was partially supported by National Natural Science Foundation of China grant 11771372. Raikhan's work was partially supported by Science Committee project AP05131557 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

1. W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89** (1967), 578–642. Zbl 0183.42501. MR0223899. DOI 10.2307/2373237. 798, 802
2. T. N. Bekjan, *Noncommutative Hardy space associated with semi-finite subdiagonal algebras*, J. Math. Anal. Appl. **429** (2015), no. 2, 1347–1369. Zbl 1330.46063. MR3342521. DOI 10.1016/j.jmaa.2015.04.032. 799
3. T. N. Bekjan, *Szegö type factorization of Haagerup noncommutative Hardy spaces*, Acta Math. Sci. Ser. B (Engl. Ed.) **37** (2017), no. 5, 1221–1229. Zbl 1399.46094. MR3683891. DOI 10.1016/S0252-9602(17)30069-3. 799
4. T. N. Bekjan and K. N. Ospanov, *Complex interpolation of noncommutative Hardy spaces associated semifinite von Neumann algebra*, preprint, arXiv:1711.01532v2 [math.OA]. 799, 810
5. J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Grundlehren Math. Wiss. **223**, Springer, Berlin, 1976. Zbl 0344.46071. MR0482275. 809, 813
6. D. P. Blecher and L. E. Labuschagne, "Von Neumann algebraic H^p theory" in *Function Spaces*, Contemp. Math. **435**, Amer. Math. Soc., Providence, 2007, 89–114. Zbl 1145.46040. MR2359421. DOI 10.1090/conm/435/08369. 798
7. A. Connes, *Une classification des facteurs de type III*, Ann. Sci. Éc. Norm. Supér (4) **6** (1973), 133–252. Zbl 0274.46050. MR0341115. 803, 804

8. U. Haagerup, "L^p-spaces associated with an arbitrary von Neumann algebra" in *Algèbres d'opérateurs et leurs applications en physique mathématique (Marseille, 1977)*, Colloq. Internat. CNRS **274**, CNRS, Paris, 1979, 175–184. Zbl 0426.46045. MR0560633. 799
9. U. Haagerup, M. Junge, and Q. Xu, *A reduction method for noncommutative L_p-spaces and applications*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 2125–2165. Zbl 1200.46055. MR2574890. DOI 10.1090/S0002-9947-09-04935-6. 805, 806, 807
10. G. Ji, *A noncommutative version of H^p and characterizations of subdiagonal algebras*, Integral Equations Operator Theory **72** (2012), no. 1, 131–149. Zbl 1252.46066. MR2872610. DOI 10.1007/s00020-011-1920-1. 799, 804, 806
11. G. Ji, *Analytic Toeplitz algebras and the Hilbert transform associated with a subdiagonal algebra*, Sci. China Math. **57** (2014), no. 3, 579–588. Zbl 1314.46075. MR3166240. DOI 10.1007/s11425-013-4684-z. 799, 803, 808, 812
12. G. Ji, T. Ohwada, and K.-S. Saito, *Certain structure of subdiagonal algebras*, J. Operator Theory **39** (1998), no. 2, 309–317. Zbl 0999.46026. MR1620570. 802
13. G. Ji and K.-S. Saito, *Factorization in subdiagonal algebras*, J. Funct. Anal. **159** (1998), no. 1, 191–202. Zbl 0927.46033. MR1654186. DOI 10.1006/jfan.1998.3309. 799, 808
14. M. Junge, *Doob's inequality for noncommutative martingales*, J. Reine. Angew. Math. **549** (2002), 149–190. Zbl 1004.46043. MR1916654. DOI 10.1515/crll.2002.061. 807
15. M. Junge and Q. Xu, *Noncommutative Burkholder/Rosenthal inequalities*, Ann. Probab. **31** (2003), no. 2, 948–995. Zbl 1041.46050. MR1964955. DOI 10.1214/aop/1048516542. 803, 807
16. H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: Noncommutative L^p-spaces*, J. Funct. Anal. **56** (1984), no. 1, 29–78. Zbl 0604.46063. MR0735704. DOI 10.1016/0022-1236(84)90025-9. 799, 809, 810, 812
17. L. E. Labuschagne, *A noncommutative Szegö theorem for subdiagonal subalgebras of von Neumann algebras*, Proc. Amer. Math. Soc. **133** (2005), no. 12, 3643–3646. Zbl 1086.46044. MR2163602. DOI 10.1090/S0002-9939-05-08064-0. 798
18. L. E. Labuschagne, *Invariant subspaces for H^2 spaces of σ-finite algebras*, Bull. Lond. Math. Soc. **49** (2017), no. 1, 33–44. Zbl 1404.46055. MR3653099. DOI 10.1112/blms.12009. 799
19. M. Marsalli and G. West, *Noncommutative H^p-spaces*, J. Operator Theory **40** (1998), no. 2, 339–355. Zbl 1001.46041. MR1660390. 798, 799
20. G. K. Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras*, Acta Math. **130** (1973), 53–87. Zbl 0262.46063. MR0412827. DOI 10.1007/BF02392262. 799, 800
21. G. Pisier and Q. Xu, "Noncommutative L^p-spaces" in *Handbook of the Geometry of Banach Spaces, Vol. 2*, North-Holland, Amsterdam, 2003, 1459–1517. Zbl 1046.46048. MR1999201. DOI 10.1016/S1874-5849(03)80041-4. 799, 810
22. M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Math. **131** (1973), 249–310. Zbl 0268.46058. MR0438149. DOI 10.1007/BF02392041. 799, 803
23. T. H. Wolff, "A Note on interpolation spaces" in *Harmonic Analysis (Minneapolis, 1981)*, Lecture Notes in Math. **908**, Springer, Berlin, 1982. Zbl 0517.46054. MR0654187. 813
24. Q. Xu, *On the maximality of subdiagonal algebras*, J. Operator Theory **54** (2005), no. 1, 137–146. Zbl 1111.46039. MR2168864. 802, 804, 805, 806

[1]College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, People's Republic of China.
E-mail address: bekjant@yahoo.com; bek@xju.edu.cn

[2]Faculty of Mechanics and Mathematics, L. N. Gumilyov Eurasian National University, Astana 010008, Kazakhstan.
E-mail address: mady1029@yandex.kz