Banach J. Math. Anal. 13 (2019), no. 1, 151–173
https://doi.org/10.1215/17358787-2018-0020
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

SPECTRAL PICTURE FOR RATIONALLY MULTICYCLIC SUBNORMAL OPERATORS

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Communicated by J. A. Ball

Abstract. For a pure bounded rationally cyclic subnormal operator S on a separable complex Hilbert space $\mathcal{H}$, Conway and Elias showed that $\text{clos}(\sigma(S) \setminus \sigma_e(S)) = \text{clos}(\text{Int}(\sigma(S)))$. This article examines the property for rationally multicyclic (N-cyclic) subnormal operators. We show that there exists a 2-cyclic irreducible subnormal operator S with $\text{clos}(\sigma(S) \setminus \sigma_e(S)) \neq \text{clos}(\text{Int}(\sigma(S)))$. We also show the following. For a pure rationally N-cyclic subnormal operator S on $\mathcal{H}$ with the minimal normal extension M on $\mathcal{K} \supset \mathcal{H}$, let $\mathcal{K}_m = \text{clos}(\text{span}\{(M^*)^k x : x \in \mathcal{H}, 0 \leq k \leq m\}$. Suppose that $M|_{\mathcal{K}_{N-1}}$ is pure. Then $\text{clos}(\sigma(S) \setminus \sigma_e(S)) = \text{clos}(\text{Int}(\sigma(S)))$.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the space of bounded linear operators on $\mathcal{H}$. An operator $S \in \mathcal{L}(\mathcal{H})$ is *subnormal* if there exist a separable complex Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $M_z \in \mathcal{L}(\mathcal{K})$ such that $M_z\mathcal{H} \subset \mathcal{H}$ and $S = M_z|_{\mathcal{H}}$. By the spectral theorem of normal operators, we assume that

$$\mathcal{K} = \bigoplus_{i=1}^{m} L^2(\mu_i), \tag{1.1}$$

where $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_m$ (m may be ∞) are compactly supported finite positive measures on the complex plane $\mathbb{C}$, and M_z is multiplication by z on $\mathcal{K}$.

Received Mar. 14, 2018; Accepted Jun. 21, 2018.
First published online Sep. 28, 2018.
2010 *Mathematics Subject Classification.* Primary 47B20; Secondary 47A16, 30H99.
Keywords. spectral picture, multicyclic, subnormal operators.

For $H = (h_1, \dots, h_m) \in \mathcal{K}$ and $G = (g_1, \dots, g_m) \in \mathcal{K}$, we define

$$\big\langle H(z), G(z) \big\rangle = \sum_{i=1}^{m} h_i(z) \overline{g_i(z)} \frac{d\mu_i}{d\mu_1}, \qquad \big|H(z)\big|^2 = \big\langle H(z), H(z) \big\rangle. \tag{1.2}$$

The *inner product* of H and G in $\mathcal{K}$ is defined by

$$(H, G) = \int \big\langle H(z), G(z) \big\rangle \, d\mu_1(z). \tag{1.3}$$

The operator M_z is the minimal normal extension if

$$\mathcal{K} = \operatorname{clos}\big(\operatorname{span}(M_z^{*k} x : x \in \mathcal{H}, k \geq 0)\big). \tag{1.4}$$

We will always assume that M_z is the minimal normal extension of S and that $\mathcal{K}$ satisfies (1.1) to (1.4). (For details about the functional model above and basic knowledge of subnormal operators, we refer the reader to Chapter II of [8].)

For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, $\sigma_e(T)$ the essential spectrum of T, T^* its adjoint, $\ker(T)$ its kernel, and $\operatorname{Ran}(T)$ its range. For a subset $A \subset \mathbb{C}$, we set $\operatorname{Int}(A)$ for its interior, $\operatorname{clos}(A)$ for its closure, A^c for its complement, and $\bar{A} = \{\bar{z} : z \in A\}$. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0, 1)$. Let $\mathcal{P}$ denote the set of polynomials in the complex variable z. For a compact subset $K \subset \mathbb{C}$, let $\operatorname{Rat}(K)$ be the set of all rational functions with poles off K, and let $R(K)$ be the uniform closure of $\operatorname{Rat}(K)$.

A subnormal operator S on $\mathcal{H}$ is *pure* if, for every nonzero invariant subspace I of S ($SI \subset I$), the operator $S|_I$ is not normal. For $F_1, F_2, \dots, F_N \in \mathcal{H}$, let

$$R^2(S \mid F_1, F_2, \dots, F_N) = \operatorname{clos}\big\{r_1(S)F_1 + r_2(S)F_2 + \cdots + r_N(S)F_N\big\}$$

in $\mathcal{H}$, where $r_1, r_2, \dots, r_N \in \operatorname{Rat}(\sigma(S))$, and let

$$P^2(S \mid F_1, F_2, \dots, F_N) = \operatorname{clos}\big\{p_1(S)F_1 + p_2(S)F_2 + \cdots + p_N(S)F_N\big\}$$

in $\mathcal{H}$, where $p_1, p_2, \dots, p_N \in \mathcal{P}$. A subnormal operator S on $\mathcal{H}$ is *rationally multicyclic* (N-cyclic), where N denotes the number of cyclic vectors, if there are N vectors $F_1, F_2, \dots, F_N \in \mathcal{H}$ such that

$$\mathcal{H} = R^2(S \mid F_1, F_2, \dots, F_N),$$

and for any $G_1, \dots, G_{N-1} \in \mathcal{H}$,

$$\mathcal{H} \neq R^2(S \mid G_1, G_2, \dots, G_{N-1}).$$

Similarly, S is *multicyclic* (N-cyclic) if there are N vectors $F_1, F_2, \dots, F_N \in \mathcal{H}$ such that

$$\mathcal{H} = P^2(S \mid F_1, F_2, \dots, F_N),$$

and for any $G_1, \dots, G_{N-1} \in \mathcal{H}$,

$$\mathcal{H} \neq P^2(S \mid G_1, G_2, \dots, G_{N-1}).$$

In this case, $m \leq N$ where m is as in (1.1).

Let μ be a compactly supported finite positive measure on the complex plane $\mathbb{C}$, and let $\operatorname{spt}(\mu)$ denote the support of μ. For a compact subset K with $\operatorname{spt}(\mu) \subset K$, let $R^2(K,\mu)$ be the closure of $\mathrm{Rat}(K)$ in $L^2(\mu)$. Let $P^2(\mu)$ denote the closure of $\mathcal{P}$ in $L^2(\mu)$.

If S is rationally cyclic, then S is unitarily equivalent to multiplication by z on $R^2(\sigma(S),\mu_1)$, where $m=1$ and $F_1=1$. We may write $R^2(S \mid F_1) = R^2(\sigma(S),\mu_1)$. If S is cyclic, then S is unitarily equivalent to multiplication by z on $P^2(\mu_1)$. We may write $P^2(S \mid F_1) = P^2(\mu_1)$.

For a rationally N-cyclic subnormal operator S with cyclic vectors $F_1, F_2, \dots, F_N$ and $\lambda \in \sigma(S)$, we denote the map

$$E(\lambda) : \sum_{i=1}^N r_i(S)F_i \to \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \dots \\ r_N(\lambda) \end{bmatrix}, \tag{1.5}$$

where $r_1, r_2, \dots, r_N \in \mathrm{Rat}(\sigma(S))$. If $E(\lambda)$ is bounded from $\mathcal{K}$ to $(\mathbb{C}^N, \|\cdot\|_{1,N})$, where $\|x\|_{1,N} = \sum_{i=1}^N |x_i|$ for $x \in \mathbb{C}^N$, then every component on the right-hand side extends to a bounded linear functional on $\mathcal{H}$. We call λ a *bounded point evaluation* (*bpe*) for S, and we use $\mathrm{bpe}(S)$ to denote the set of bounded point evaluations for S. The set $\mathrm{bpe}(S)$ does not depend on the choices of cyclic vectors $F_1, F_2, \dots, F_N$ (see Corollary 1.1 in Mbekhta, Ourchane, and Zerouali [14]). A point $\lambda_0 \in \mathrm{int}(\mathrm{bpe}(S))$ is called an *analytic bounded point evaluation* (*abpe*) for S if there is a neighborhood $B(\lambda_0,\delta) \subset \mathrm{bpe}(S)$ of λ_0 such that $E(\lambda)$ is analytic as a function of λ on $B(\lambda_0,\delta)$ (equivalently, (1.5) is uniformly bounded for $\lambda \in B(\lambda_0,\delta)$). We use $\mathrm{abpe}(S)$ to denote the set of analytic bounded point evaluations for S. The set $\mathrm{abpe}(S)$ does not depend on the choices of cyclic vectors $F_1, F_2, \dots, F_N$ (see also Remark 3.1 in [14]). Similarly, for an N-cyclic subnormal operator S, we can define $\mathrm{bpe}(S)$ and $\mathrm{abpe}(S)$ if we replace $r_1, r_2, \dots, r_N \in \mathrm{Rat}(\sigma(S))$ in (1.5) by $p_1, p_2, \dots, p_N \in \mathcal{P}$.

For $N=1$, Thomson [18] proved a remarkable structural theorem for $P^2(\mu)$.

Thomson's theorem ([18, Theorem 5.8])**.** *There is a Borel partition* $\{\Delta_i\}_{i=0}^\infty$ *of* $\mathrm{spt}\,\mu$ *such that the space* $P^2(\mu|_{\Delta_i})$ *contains no nontrivial characteristic functions and*

$$P^2(\mu) = L^2(\mu|_{\Delta_0}) \oplus \left\{ \bigoplus_{i=1}^\infty P^2(\mu|_{\Delta_i}) \right\}.$$

Furthermore, if U_i *is the open set of analytic bounded point evaluations for* $P^2(\mu|_{\Delta_i})$ *for* $i \ge 1$*, then* U_i *is a simply connected region and the closure of* U_i *contains* Δ_i*.*

Conway and Elias [9] extend some results of Thomson's theorem to the space $R^2(K,\mu)$, while Brennan [5] expresses $R^2(K,\mu)$ as a direct sum that includes both Thomson's theorem and results of Conway and Elias [9]. For a compactly supported complex Borel measure ν of $\mathbb{C}$, by estimating the analytic capacity of the set $\{\lambda : |\mathcal{C}\nu(\lambda)| \ge c\}$, where $\mathcal{C}\nu$ is the Cauchy transform of ν (see Section 3 for a definition), Brennan [4] and Aleman, Richter, and Sundberg [1], [2] provide interesting alternative proofs of Thomson's theorem. Both their proofs rely on X and Tolsa's deep results on analytic capacity. There are other related research efforts for $N = 1$ in the literature, for example, Brennan [3], Hruscev [13], Brennan and Militzer [6], and Yang [21], among others.

Theorem 4.11 of Thomson [18] shows that $\text{abpe}(S) = \text{bpe}(S)$ for a cyclic subnormal operator S (see also Chapter VIII, Theorem 4.4 in [8]). Corollary 5.2 in Conway and Elias [9] proves that the result holds for rationally cyclic subnormal operators. For $N > 1$, Yang [22] extends the result to rationally N-cyclic subnormal operators. It is shown in Theorem 2.1 of Conway and Elias [9] that if S is a pure rationally cyclic subnormal operator, then

$$\text{clos}\big(\sigma(S) \setminus \sigma_e(S)\big) = \text{clos}\big(\text{Int}\big(\sigma(S)\big)\big). \tag{1.6}$$

This leads us to examine if (1.6) holds for a rationally N-cyclic subnormal operator.

A Gleason part of $R(K)$ is a maximal set Ω in $\mathbb{C}$ such that, for $x, y \in \Omega$, if e_x and e_y denote the evaluation functionals at x and y, respectively, then $\|e_x - e_y\|_{R(K)^*} < 2$. Olin and Thomson [17] show that a compact set K can be the spectrum of an irreducible subnormal operator if and only if $R(K)$ has only one nontrivial Gleason part Ω and $K = \text{clos}(\Omega)$. McGuire [16] and Feldman and McGuire [11] construct irreducible subnormal operators with a prescribed spectrum, approximate point spectrum, essential spectrum, and the (semi-)Fredholm index. Our first result is to construct a (rationally) 2-cyclic irreducible subnormal operator for a prescribed spectrum and essential spectrum. Consequently, we show that (1.6) may not hold for a (rationally) N-cyclic irreducible subnormal operator with $N > 1$.

Theorem 1.1. *Assume that K and K_e are two compact subsets of $\mathbb{C}$ such that $R(K)$ has only one nontrivial Gleason part Ω, $K = \text{clos}(\Omega)$, and such that $\partial K \subset K_e \subset K$. Then there exists a rationally 2-cyclic irreducible subnormal operator S such that $\sigma(S) = K$, $\sigma_e(S) = K_e$, and $\text{ind}(S - \lambda) = -1$ for $\lambda \in K \setminus K_e$. If, in particular, $\mathbb{C} \setminus K$ has only one component, then S can be constructed as a 2-cyclic irreducible subnormal operator.*

Note that if $K = \text{clos}(\mathbb{D})$ and $K_e = \partial\mathbb{D} \cup \text{clos}(\frac{1}{2}\mathbb{D})$, then K and K_e satisfy the conditions of Theorem 1.1 and

$$\begin{aligned}\text{clos}(K \setminus K_e) &= \left\{z : \frac{1}{2} \le |z| \le 1\right\} \\ &\ne \text{clos}\big(\text{Int}(K)\big) = \text{clos}(\mathbb{D}).\end{aligned}$$

The corollary below follows immediately.

Corollary 1.2. *There exists a 2-cyclic irreducible subnormal operator S such that (1.6) does not hold.*

In the second part of this article, we will investigate certain classes of rationally N-cyclic subnormal operators that have the property (1.6). Let S be a rationally N-cyclic subnormal operator on $\mathcal{H} = R^2(S \mid F_1, F_2, \dots, F_N)$. Let ψ be a smooth function with compact support. Define

$$\mathcal{K}_n^\psi = \text{clos}\{\psi^m x : x \in \mathcal{H}, 0 \le m \le n\}.$$

Then
$$\mathcal{H} \subset \mathcal{K}_1^\psi \subset \cdots \subset \mathcal{K}_n^\psi \subset \cdots \subset \mathcal{K}$$
and $M_z|_{\mathcal{K}_n^\psi}$ is a subnormal operator.

Definition 1.3. A *subnormal operator* satisfies the property (N,ψ) if the following conditions are met:

(1) S is a pure (rationally) N-cyclic subnormal operator on $\mathcal{H} = R^2(S \mid F_1, \ldots, F_N)$;
(2) ψ is a smooth function with compact support and $\text{Area}(\sigma(S) \cap \{\bar\partial\psi = 0\}) = 0$; if M_z on $\mathcal{K}$ is the minimal normal extension of S satisfying (1.1)–(1.4), then $M_z|_{\mathcal{K}_{N-1}^\psi}$ is also a pure subnormal operator.

Theorem 1.4. *Assume that $N > 1$ and that S is a pure subnormal operator on $\mathcal{H}$ satisfying the property (N,ψ). Then there exist bounded open subsets U_i for $1 \le i \le N$ such that*
$$\sigma_e(S) = \bigcup_{i=1}^N \partial U_i, \qquad \sigma(S) = \bigcup_{i=1}^N \text{clos}(U_i),$$
and for $i = 1, 2, \ldots, N$ and $\lambda \in U_i$,
$$\text{ind}(S - \lambda) = -i.$$
Consequently,
$$\sigma(S) = \text{clos}\big(\sigma(S) \setminus \sigma_e(S)\big) = \text{clos}\big(\text{Int}\big(\sigma(S)\big)\big).$$

An important special case is when $\psi = \bar z$. In Section 3, we will provide several examples of subnormal operators that satisfy the property (N,ψ). We prove Theorem 1.1 in Section 2 and Theorem 1.4 in Section 3.

2. Spectral pictures for irreducible rationally 2-cyclic subnormal operators

In this section, we prove our first main theorem.

Theorem 2.1. *Assume that K and K_e are two compact subsets of $\mathbb{C}$ such that $R(K)$ has only one nontrivial Gleason part Ω, $K = \text{clos}(\Omega)$, and such that $\partial K \subset K_e \subset K$. Then there exists a rationally 2-cyclic irreducible subnormal operator S such that $\sigma(S) = K$, $\sigma_e(S) = K_e$, and $\text{ind}(S - \lambda) = -1$ for $\lambda \in K \setminus K_e$. If, in particular, $\mathbb{C} \setminus K$ has only one component, then S can be constructed as a 2-cyclic irreducible subnormal operator.*

The proof of Theorem 2.1 depends on several technical lemmas. The operator $S(= T_1)$ is defined in (2.4). Lemma 2.3 shows that T_1 is an irreducible subnormal operator with $\sigma(T_1) = K$, $\sigma_e(T_1) = K_e$, and $\text{ind}(T_1 - \lambda) = -1$ for $\lambda \in K \setminus K_e$. We construct two rationally cyclic vectors for T_1 in Lemma 2.4 and Lemma 2.5.

In the remainder of the section, we assume that K is a compact subset of $\mathbb{C}$, $\text{Int}(K) \neq \emptyset$, and $R(K)$ has only one nontrivial Gleason part Ω with $K = \text{clos}(\Omega)$. Theorem 5 and Corollary 6 in McGuire [16] construct a representing measure ν of $R(K)$ at $z_0 \in \text{Int}(K)$ with support on ∂K such that S_ν on $R^2(K,\nu)$ is irreducible,

$\sigma(S_\nu) = K$, $\sigma_e(S_\nu) = \partial K$, and $\text{ind}(S_\nu - \lambda) = -1$ for $\lambda \in \text{Int}(K) = \sigma(S_\nu) \setminus \sigma_e(S_\nu)$. From Theorem 6.2 in Gamelin [12], we get

$$L^2(\nu) = R^2(K,\nu) \oplus N^2 \oplus \overline{R_0^2(K,\nu)}, \tag{2.1}$$

where $\overline{R_0^2(K,\nu)} = \{\bar{r} : r(z_0) = 0 \text{ and } r \in R^2(K,\nu)\}$. The operator M_z, multiplication by z on $L^2(\nu)$, can be written as the following matrix with respect to (2.1):

$$M_z = \begin{bmatrix} S_\nu & A & B \\ 0 & C & D \\ 0 & 0 & T_\nu^* \end{bmatrix},$$

where T_ν, multiplication by $\bar{z}$ on $\overline{R_0^2(K,\nu)}$, is an irreducible rationally cyclic subnormal operator with $\sigma(T_\nu) = \bar{K}$, $\sigma_e(T_\nu) = \partial \bar{K}$, and $\text{ind}(T_\nu - \lambda) = -1$ for $\lambda \in \text{Int}(\bar{K})$. If

$$S = \begin{bmatrix} S_\nu & A \\ 0 & C \end{bmatrix},$$

then S is the dual of T_ν. From the properties of dual subnormal operators (see, e.g., Conway [7] and Theorem 2.4 in Feldman and McGuire [11]), we see that S is an irreducible subnormal operator with $\sigma(S) = K$, $\sigma_e(S) = \partial K$, and $\text{ind}(S - \lambda) = -1$ for $\lambda \in \text{Int}(K)$. The following lemma, due to Cowen and Douglas [10, p. 194], allows us to choose eigenvectors for S^* in a coanalytic manner whenever the Fredholm index function for S is -1.

Lemma 2.2. *If $X \in L(\mathcal{H})$ and $\text{ind}(X - \lambda) = -1$ for all $\lambda \in G := \sigma(X) \setminus \sigma_e(X)$, then there exists a coanalytic function $h : G \to H$ that is not identically zero on any component of G such that $h(\lambda) \in \ker(X - \lambda)^*$. In particular, for every $x \in \mathcal{H}$, the function $\lambda \to (x, h(\lambda))$ is analytic on G.*

Using Lemma 2.2, we conclude that there exists a coanalytic function $k_\lambda \in \mathcal{H} := R^2(K,\nu) \oplus N^2$ such that $(S - \lambda)^* k_\lambda = 0$ on $\text{Int}(K)$. Let δ_λ be the point mass measure at λ. Let $K_e \subset K$ be a compact subset of $\mathbb{C}$ such that $\partial K \subset K_e$. Let $\{\lambda_n\} \subset K_e \cap \text{Int}(K)$ with $K_e \cap \text{Int}(K) \subset \text{clos}(\{\lambda_n\})$. Define

$$\mu = \nu + \sum_{n=1}^{\infty} c_n \delta_{\lambda_n}, \tag{2.2}$$

where $c_n > 0$ and $\sum_{n=1}^{\infty} c_n \|k_{\lambda_n}\|^2 = 1$. Let M_z^1 be the multiplication by z operator on $L^2(\mu)$.

Lemma 2.3. *Define an operator T from $\mathcal{H}$ to $L^2(\mu)$ by*

$$Tf(z) = \begin{cases} f(z), & z \in \partial K, \\ (f, k_{\lambda_n}), & z = \lambda_n. \end{cases} \tag{2.3}$$

Then T is a bounded linear one-to-one operator with closed range. Set $\mathcal{H}_1 = \text{Ran}(T)$. Then T is invertible from $\mathcal{H}$ to $\mathcal{H}_1$, $M_z^1 \mathcal{H}_1 \subset \mathcal{H}_1$, $S_1 = M_z^1|_{\mathcal{H}_1}$ is an irreducible subnormal operator such that $S_1 = TST^{-1}$, and M_z^1 is the minimal normal extension of S_1.

Proof. By definition, we get

$$\|f\|^2_{L^2(\nu)} \le \|Tf\|^2_{L^2(\mu)} = \|f\|^2_{L^2(\nu)} + \sum_{n=1}^{\infty} c_n \big|(f, k_{\lambda_n})\big|^2 \le 2\|f\|^2_{L^2(\nu)}.$$

Therefore, T is a bounded linear operator and invertible from $\mathcal{H}$ to $\mathcal{H}_1$. Since $(zf, k_{\lambda_n}) = \lambda_n (f, k_{\lambda_n})$, we see that $M_z^1 \mathcal{H}_1 \subset \mathcal{H}_1$ and $S_1 = TST^{-1}$. Since $(Tk_{\lambda_n})(\lambda_n) = \|k_{\lambda_n}\|^2 > 0$, clearly, we have

$$L^2(\mu) = \operatorname{clos}\big(\operatorname{span}\{\bar{z}^m x : x \in \mathcal{H}_1, m \ge 0\}\big).$$

Therefore, M_z^1 is the minimal normal extension of S_1.

It remains to prove that S_1 is irreducible. Let N_1 and N_2 be two reducing subspaces of S_1 such that $\mathcal{H}_1 = N_1 \oplus N_2$. Then for $f_1 \in N_1$ and $f_2 \in N_2$, we have

$$(z^n f_1, z^m f_2) = \int z^n \bar{z}^m f_1 \bar{f}_2 \, d\mu = 0$$

for $n, m = 0, 1, 2, \ldots$. This implies that $f_1(z)\bar{f}_2(z) = 0$, almost every μ. By the definition of T, we see that $(T^{-1}f_1)(z)\overline{(T^{-1}f_2)}(z) = 0$, almost every ν. Hence $\mathcal{H} = T^{-1}N_1 \oplus T^{-1}N_2$, where $T^{-1}N_1$ and $T^{-1}N_2$ are reducing subspaces for S. By the construction, T_ν is irreducible (see Corollary 6 in McGuire [16]), so S, as the dual T_ν, is irreducible (see, e.g., Theorem 2.4 in Feldman and McGuire [11]). This means that $N_1 = 0$ or $N_2 = 0$. The lemma is proved. □

We write the operator M_z^1 as

$$M_z^1 = \begin{bmatrix} S_1 & A_1 \\ 0 & T_1^* \end{bmatrix}. \tag{2.4}$$

Then T_1, as a dual of S_1, is irreducible.

Lemma 2.4. *Let μ be as in (2.2), and let $\mathcal{H}_1$ be as in Lemma 2.3. If*

$$F(z) = \begin{cases} \bar{z} - \bar{z}_0, & z \in \partial K, \\ 0, & z \in \operatorname{Int}(K) \end{cases} \tag{2.5}$$

and

$$G_n(z) = \begin{cases} k_{\lambda_n}(z), & z \in \partial K, \\ -1/c_n, & z = \lambda_n, \\ 0, & z = \lambda_m, m \ne n, \end{cases} \tag{2.6}$$

then

$$\mathcal{H}_1^\perp = \operatorname{clos}\big(\operatorname{span}\big\{r(\bar{z})F, G_j : 1 \le j < \infty, r \in \operatorname{Rat}(K)\big\}\big).$$

Proof. It is straightforward to check, from (2.1), (2.2), and (2.3), that $F, G_j \in \mathcal{H}_1^\perp$. Now let $H(z) \perp \operatorname{clos}(\operatorname{span}\{r(\bar{z})F, G_j, 1 \le j < \infty, r \in \operatorname{Rat}(K)\})$. Then

$$\int H(z) r(z) \bar{F}(z) \, d\mu = \int H(z) r(z)(z - z_0) \, d\nu = 0$$

for $r \in \operatorname{Rat}(K)$. From (2.1), we see that the function $H|_{\partial K} \in \mathcal{H}$. It follows from $\int H(z)\bar{G}_j(z) \, d\mu = 0$ that $H(\lambda_j) = (H|_{\partial K}, k_{\lambda_j})$. Thus, $H(z) \in \mathcal{H}_1$. The lemma is proved. □

Lemma 2.5. *If μ, T_1, F, and G_n are as in (2.2), (2.4), (2.5), and (2.6), respectively, then there exists a sequence of positive numbers $\{a_n\}$ satisfying*

$$\sum_{n=1}^{\infty} a_n \|G_n\| < \infty, \qquad G = \sum_{n=1}^{\infty} a_n G_n,$$

and

$$\mathcal{H}_1^{\perp} = \operatorname{clos}\big(\operatorname{span}\big\{r(\bar z)F(z) + p(\bar z)G(z) : r \in \operatorname{Rat}(K), p \in \mathcal{P}\big\}\big).$$

Therefore, T_1 is a rationally 2-cyclic irreducible subnormal operator with

$$\sigma(T_1) = \bar K, \qquad \sigma_e(T_1) = \bar K_e \qquad \text{and} \qquad \operatorname{ind}(T_1 - \lambda) = -1, \quad \lambda \in \bar K \setminus \bar K_e. \tag{2.7}$$

Proof. Note that

$$\int f(z)(z - \lambda_n)\bar k_{\lambda_n}(z)\, d\nu = 0$$

for $f \in \mathcal{H}$. We conclude, from (2.1), that $(\bar z - \bar\lambda_n)k_{\lambda_n}(z) \in \overline{R_0^2(K,\nu)}$. Hence, there are $\{r_n\} \subset R^2(K,\nu)$ such that

$$k_{\lambda_n}(z) = \frac{r_n(\bar z)}{\bar z - \bar\lambda_n}(\bar z - \bar z_0).$$

We will recursively choose $\{a_n\}$. First choose $a_1 = 1$. Then we assume that $a_1, a_2, \ldots, a_n$ have been chosen. Now we will choose a_{n+1}. Let

$$p_k(z) = \frac{\prod_{j\neq k, 1\le j\le n}(z - \bar\lambda_j)}{a_k \prod_{j\neq k, 1\le j\le n}(\bar\lambda_k - \bar\lambda_j)},$$

for $k = 1, 2, \ldots, n$. Denote

$$q_{1k}(z) = p_k(z) \sum_{j\neq k, 1\le j\le n} \frac{a_j}{z - \bar\lambda_j} r_j(z)$$

and

$$q_{2k}(z) = \frac{a_k(p_k(z) - p_k(\bar\lambda_k))}{z - \bar\lambda_k} r_k(z).$$

So $p_k \in \mathcal{P}$ and $q_{1k}, q_{2k} \in R^2(K,\nu)$ for $k = 1, 2, \ldots, n$. Clearly,

$$p_k(\bar z)\sum_{j=1}^{n} a_j G_j(z) - \big(q_{1k}(\bar z) + q_{2k}(\bar z)\big)(\bar z - \bar z_0) = \frac{r_k(\bar z)(\bar z - \bar z_0)}{\bar z - \bar\lambda_k}, \quad z \in \partial K.$$

Hence,

$$p_k(\bar z)\sum_{j=1}^{n} a_j G_j(z) - \big(q_{1k}(\bar z) + q_{2k}(\bar z)\big)F(z) = G_k(z), \quad \text{a.e. } \mu.$$

We have the following calculation:

$$\begin{aligned}
&\int \Big| p_k(\bar z) \sum_{j=1}^{n+1} a_j G_j(z) - \big(q_{1k}(\bar z) + q_{2k}(\bar z)\big) F(z) - G_k(z) \Big|^2 d\mu \\
&= \int \big| p_k(\bar z) a_{n+1} G_{n+1}(z) \big|^2 d\mu \\
&\le \Big(\frac{a_{n+1}}{a_k}\Big)^2 \frac{(4D^2)^{n-1}}{\prod_{j\neq k, 1\le j\le n} |\lambda_k - \lambda_j|^2} \|G_{n+1}\|^2,
\end{aligned}$$

where $D = \max\{|z| : z \in K\}$. Now set

$$\begin{aligned}
a_{n+1} = \ &\min\Big(\frac{1}{2^{n+1}}, \min_{1\le k\le n} \frac{a_k \prod_{j\neq k, 1\le j\le n} \min(1, |\lambda_k - \lambda_j|)}{4^n \max(1, D)^{n-1}}\Big) \\
&/ \max\big(1, \|G_{n+1}\|\big).
\end{aligned} \tag{2.8}$$

So we have chosen all $\{a_n\}$. From (2.8), we have the following calculation:

$$\begin{aligned}
&\Big\| p_k \sum_{i=n+2}^{\infty} a_j G_j \Big\| \\
&\le \frac{(2D)^{n-1}}{a_k \prod_{j\neq k, 1\le j\le n} |\lambda_k - \lambda_j|} \sum_{i=n+2}^{\infty} \frac{a_k \prod_{j\neq k, 1\le j\le i-1} \min(1, |\lambda_k - \lambda_j|)}{4^{i-1} \max(1, D)^{i-2}} \\
&\le \frac{1}{2^{n+2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\big\| p_k(\bar z) G - \big(q_{1k}(\bar z) + q_{2k}(\bar z)\big) F - G_k(z) \big\| \\
&\le \Big\| p_k(\bar z) \sum_{j=1}^{n+1} a_j G_j - \big(q_{1k}(\bar z) + q_{2k}(\bar z)\big) F - G_k(z) \Big\| + \Big\| p_k(\bar z) \sum_{j=n+2}^{\infty} a_j G_j \Big\| \\
&\le \frac{1}{2^n}.
\end{aligned}$$

Hence,

$$G_k \in \operatorname{clos}\big(\operatorname{span}\big\{ r(\bar z) F(z) + p(\bar z) G(z) : r \in \operatorname{Rat}(K), p \in \mathcal{P} \big\}\big), \quad k = 1, 2, \ldots.$$

Since T_1 is the dual of S_1, we see that $\sigma(M_z^1) \subset \sigma_e(S_1) \cup \overline{\sigma_e(T_1)}$ (see, e.g., Theorem 2.4 in Feldman and McGuire [11]), $\sigma_e(S_1) = \partial K$, and $\sigma_e(T_1) \supset \partial \bar K$. So (2.7) follows. This completes the proof. □

3. Spectral picture of a class of rationally multicyclic subnormal operators

In this section, we prove our second main theorem.

Theorem 3.1. *Assume that $N > 1$ and that S is a pure subnormal operator on $\mathcal{H}$ satisfying the property (N, ψ). Then there exist bounded open subsets U_i for $1 \le i \le N$ such that*

$$\sigma_e(S) = \bigcup_{i=1}^{N} \partial U_i, \qquad \sigma(S) = \bigcup_{i=1}^{N} \operatorname{clos}(U_i),$$

and for $i = 1, 2, \ldots, N$ and $\lambda \in U_i$,

$$\operatorname{ind}(S - \lambda) = -i.$$

Consequently,

$$\sigma(S) = \operatorname{clos}\big(\sigma(S) \setminus \sigma_e(S)\big) = \operatorname{clos}\big(\operatorname{Int}\big(\sigma(S)\big)\big).$$

Let U_k be the set of $\lambda \in \operatorname{Int}(\sigma(S))$ such that $\operatorname{Ran}(S - \lambda)$ is closed and $\dim(\ker(S - \lambda)^*) = k$, where $k = 1, 2, \ldots, N$. The proof of Theorem 3.1 depends on the construction of subsets $\{E_k^G\}$ for a given $G \perp K_{N-1}^{\psi}$ and $k = 1, 2, \ldots, N$ such that $\{E_k^G\}$ satisfies (3.16) and the conclusion of Theorem 3.9. We construct $E_N^G (= \Omega^G)$ in Lemma 3.5, $E_{N-1}^G (= \bigcup_{k=1}^{N} \Omega_k^G)$ in Lemma 3.6 and Corollary 3.7, and E_{N-2}^G in Corollary 3.8. Following the pattern, we can construct all subsets $\{E_k^G\}$.

First we provide some examples of subnormal operators that have the property (N, ψ) in Definition 1.3.

Example 3.2. Every pure subnormal operator S on $\mathcal{H}$ with finite-rank self-commutator has the property (N, ψ). Note that the structure of such subnormal operators has been established based on Xia's model (see Xia [19] and Yakubovich [20]).

Proof. Assume that M_z on $\mathcal{K}$ is the minimal normal extension satisfying (1.1)–(1.4). Define the self-commutator as

$$D = [S^*, S] = S^*S - SS^*.$$

The element $x \in \ker(D)$ if and only if $M_z^* x \in \mathcal{H}$. This implies that $S \ker(D) \subset \ker(D)$. Therefore,

$$S^* \operatorname{Ran}(D) \subset \operatorname{Ran}(D). \tag{3.1}$$

Let

$$\mathcal{H}_0 = \operatorname{clos}\big(\operatorname{span}\big(S^n f : f \in \operatorname{Ran}(D), n \ge 0\big)\big).$$

Then $S|_{\mathcal{H}_0}$ is N-cyclic subnormal, where $N \le \dim(\operatorname{Ran}(D))$.

On the other hand,

$$S^* S^n D = S S^* S^{n-1} D + D S^{n-1} D;$$

hence, we can recursively show that $S^* S^n \operatorname{Ran}(D) \subset \mathcal{H}_0$ since (3.1). So $S^* \mathcal{H}_0 \subset \mathcal{H}_0$. This implies that

$$S(\mathcal{H} \ominus \mathcal{H}_0) \subset \mathcal{H} \ominus \mathcal{H}_0$$

and $S|_{\mathcal{H}\ominus\mathcal{H}_0}$ is normal. Since S is pure, we conclude that $\mathcal{H} = \mathcal{H}_0$ and that S is N-cyclic. From (3.1), we see that there is a polynomial p such that

$$\bar{p}(S^*|_{\mathrm{Ran}(D)}) = 0.$$

Therefore,

$$p(S) : \mathcal{H} \to \ker(D).$$

Hence,

$$\left\|M_z^* p(S) f\right\| = \left\|M_z p(S) f\right\| = \left\|S p(S) f\right\| = \left\|S^* p(S) f\right\|$$

for $f \in \mathcal{H}$. This implies that $M_z^* p(M_z)\mathcal{H} \subset \mathcal{H}$. Let $\psi = \bar{z}p$. Then $\mathrm{Area}\{\bar{\partial}\psi = 0\} = \mathrm{Area}\{z : p(z) = 0\} = 0$, $\mathcal{K}_{N-1}^{\psi} = \mathcal{H}$, and S satisfies the property (N, ψ) in Definition 1.3. □

Example 3.3. In Lemma 2.5, if $K = \mathrm{clos}(\mathbb{D})$ and $K_e = (\partial\mathbb{D}) \cup (\frac{1}{2}\partial\mathbb{D})$, then the operator T_1 is a 2-cyclic irreducible subnormal operator satisfying the property $(2, \psi)$, where $\psi = |z|^4 - \frac{5}{4}|z|^2$.

Proof. For $f \in \mathcal{H}_1$, we get

$$\psi f = \left(|z|^2 - 1\right)\left(|z|^2 - \frac{1}{4}\right) f - \frac{1}{4} f = -\frac{1}{4} f$$

since $\mathrm{spt}(\mu) \subset K_e$. Hence, $\mathcal{K}_1^{\psi} = \mathcal{H}_1$. On the other hand,

$$\mathrm{Area}\{\bar{\partial}\psi = 0\} \le \mathrm{Area}\left(\{0\} \cup \left\{|z|^2 = \frac{5}{8}\right\}\right) = 0.$$

Therefore, the operator T_1 satisfies the property $(2, \psi)$. □

In the remainder of the section, we assume that $N > 1$ and that S is a pure rationally N-cyclic subnormal operator on $\mathcal{H} = R^2(S \mid F_1, F_2, \ldots, F_N)$, and that M_z on $\mathcal{K}$, which satisfies (1.1)–(1.4), is the minimal normal extension of S. Moreover, S satisfies the property (N, ψ) in Definition 1.3.

Lemma 3.4. *Assume that $1 \le k \le N$, that $\delta > 0$, that $B(\lambda_0, 2\delta) \subset \mathrm{Int}(\sigma(S))$, that I is an index subset of $\{1, 2, \ldots, N\}$ with size $N - k$, that $F = \sum_{i=1}^N r_i F_i$ where $r_i \in \mathrm{Rat}(\sigma(S))$, and that $\{a_{ls}(\lambda)\}_{1\le l\le N-k, 1\le s\le k}$ are analytic on $B(\lambda_0, 2\delta)$ such that*

$$\sup_{1\le s\le k, \lambda\in B(\lambda_0,\delta)} \left| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) \right| \le M\|F\| \tag{3.2}$$

and

$$F_{i_l}(z) = \sum_{s=1}^{k} a_{ls}(z) F_{j_s}(z), \quad a.e.\ \mu_1|_{B(\lambda_0,\delta)}, \tag{3.3}$$

where $i_l \in I$ and $j_s \notin I$. Then $\lambda_0 \in \bigcup_{i=k}^N U_k$.

Proof. From (3.3), we get

$$\int_{B(\lambda_0,\delta)} |F|^2\, d\mu_1 = \int_{B(\lambda_0,\delta)} \Big|\sum_{s=1}^{k}\Big(r_{j_s}(z) + \sum_{l=1}^{N-k} a_{ls}(z) r_{i_l}(z)\Big) F_{j_s}(z)\Big|^2\, d\mu_1.$$

Using (3.2) and the maximum modulus principle,

$$\sup_{1\le s\le k, \lambda\in B(\lambda_0,\delta)} \Big| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda)\Big| \le \frac{M}{\delta}\big\|(S-\lambda_0)F\big\|.$$

Hence,

$$\int |F|^2\, d\mu_1 \le \int_{B(\lambda_0,\delta)^c} |F|^2\, d\mu_1 + \Big(\sum_{j\notin I}\|F_j\|\Big)^2 \sup_{\substack{1\le s\le k\\ \lambda\in B(\lambda_0,\delta)}} \Big| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda)\Big|^2.$$

Therefore,

$$\|F\| \le M_1 \big\|(S-\lambda_0)F\big\|,$$

where

$$M_1^2 = \Big(1 + \Big(\sum_{j\notin I}\|F_j\|\Big)^2\Big)\Big(\frac{M}{\delta}\Big)^2.$$

So $\operatorname{Ran}(S-\lambda_0)$ is closed. On the other hand, there are k linearly independent $k_\lambda^j \in \mathcal{H}$ such that

$$r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) = \int \big\langle F(z), k_\lambda^j(z)\big\rangle\, d\mu_1(z),$$

where $j_s \notin I$ and $\lambda \in B(\lambda_0,\delta)$. This implies that

$$\dim\big(\ker(S-\lambda_0)^*\big) \ge k.$$

Therefore, $\lambda_0 \in \bigcup_{i=k}^{N} U_i$. □

Let ν be a compactly supported finite measure on $\mathbb{C}$. The transform

$$\mathcal{C}_\psi^i \nu(z) = \int \frac{(\psi(w)-\psi(z))^i}{w-z}\, d\nu(w)$$

is continuous at each point z with $|\nu|(\{z\}) = 0$ and $i > 0$. For $i = 0$, the transformation

$$\mathcal{C}_\psi^0(\nu) = \mathcal{C}(\nu) = \int \frac{1}{w-z}\, d\nu(w)$$

is the Cauchy transform of ν. Let $M^G(z)$ be the following $N \times N$ matrix:

$$M^G(z) = \big[\mathcal{C}_\psi^{i-1}\big(\langle F_j, G\rangle \mu_1\big)\big]_{N\times N},$$

where we assume that $G \perp \mathcal{K}_{N-1}^\psi$ (or, equivalently, that G) satisfies the conditions

$$\bar{\psi}^n G \perp \mathcal{H}, \quad n = 0, 1, 2, \dots, N-1. \tag{3.4}$$

The set $W^G \subset \mathbb{C}$ is defined by

$$W^G = \left\{ \lambda : \int \frac{1}{|z-\lambda|} \left| \langle F_i(z), G(z) \rangle \right| d\mu_1(z) < \infty, 1 \le i \le N \right\}.$$

Let

$$\Omega^G = \operatorname{Int}\big(\sigma(S)\big) \cap W^G \cap \big\{ \lambda : |\det(M^G(\lambda)| > 0 \big\}. \tag{3.5}$$

Then for $\lambda \in \Omega^G$, the matrix

$$\big[\mathcal{C}\big(\langle F_j \psi^{i-1}, G\rangle \mu_1\big)\big]_{N\times N} \tag{3.6}$$

is invertible. By construction, we see that

$$\det\big(M^G(z)\big) = 0, \quad \text{a.e. Area}\,|_{(\text{clos}(\Omega^G))^c}.$$

Lemma 3.5. *Using the above notation, we conclude that*

$$\Omega^G \subset \operatorname{abpe}(S).$$

Hence, by Lemma 3.4, we get $\Omega^G \subset U_N$.

Proof. Using (3.4), (3.5), and (3.6), we see that the lemma is a direct application of Theorem 2 in Yang [22]. □

Let $A = \{\lambda_n : \mu_1(\{\lambda_n\}) > 0\}$ be the set of atoms for μ_1. Now let us define the matrix $M_j^G(z)$ to be a submatrix of $M^G(z)$ by eliminating the first row and the jth column. Let $B_j^G(z)$ be the jth column of the matrix $M^G(z)$ by eliminating the first row. Define

$$\Omega_j^G = \big(\operatorname{Int}\big(\sigma(S)\big) \cap A^c \cap \big\{ z : \big|\det\big(M_j^G(z)\big)\big| > 0 \big\}\big) \setminus \operatorname{clos}(\Omega^G). \tag{3.7}$$

Note that $M_j^G(\lambda)$ is continuous at each $\lambda \in \Omega_j^G$. On Ω_j^G, we can define the vector-valued function

$$a_j(z) = \big[a_{ij}(z)\big]_{(N-1)\times 1} = \big(M_j^G(z)\big)^{-1} B_j^G(z). \tag{3.8}$$

Lemma 3.6. *If G, Ω^G, Ω_j^G, and $a_j(z)$ are as in (3.4), (3.5), (3.7), and (3.8), respectively, then for $\lambda_0 \in \Omega_j^G$, there exists $\delta > 0$ such that $a_j(z)$ equals an analytic vector-valued function on $B(\lambda_0, \delta) \subset \operatorname{Int}(\sigma(S))$ almost everywhere with respect to the area measure. Moreover,*

$$\begin{aligned}\mathcal{C}\big(\langle F_j, G\rangle \mu\big)(z) = &\sum_{k=1}^{j-1} a_{kj}(z) \mathcal{C}\big(\langle F_k, G\rangle \mu\big)(z) \\ &+ \sum_{k=j+1}^{N} a_{k-1,j}(z) \mathcal{C}\big(\langle F_k, G\rangle \mu\big)(z), \quad a.e.\ \text{Area}\,|_{B(\lambda_0,\delta)},\end{aligned} \tag{3.9}$$

and

$$\langle F_j, G\rangle = \sum_{k=1}^{j-1} a_{kj}(z) \langle F_k, G\rangle + \sum_{k=j+1}^{N} a_{k-1,j}(z) \langle F_k, G\rangle, \quad a.e.\ \mu|_{B(\lambda_0,\delta)}. \tag{3.10}$$

Proof. Without loss of generality, we assume that $j = N$. For $z \in \operatorname{Int}(\sigma(S)) \cap W^G \cap \Omega_N^G$, write

$$M^G(z) = \begin{bmatrix} A_N^G(z) & c_N^G(z) \\ M_N^G(z) & B_N^g(z) \end{bmatrix},$$

where

$$A_N^G(z) = \left[\mathcal{C}\big(\langle F_1, G\rangle\mu_1\big)(z), \mathcal{C}\big(\langle F_2, G\rangle\mu_1\big)(z), \ldots, \mathcal{C}\big(\langle F_{N-1}, G\rangle\mu_1\big)(z)\right]$$

and

$$c_N^G(z) = \mathcal{C}\big(\langle F_N, G\rangle\mu_1\big)(z).$$

By construction of Ω_N^G, we conclude that

$$\begin{aligned}\det\big(M^G(z)\big) &= \big(A_N^G(z)\big(M_N^G(z)\big)^{-1}B_N^G(z) - c_N^G(z)\big)\det\big(M_N^G(z)\big)\\ &= 0, \quad \text{a.e. Area}\,|_{\Omega_N^G}.\end{aligned}$$

Therefore,

$$c_N^G(z) = A_N^G(z)\big(M_N^G(z)\big)^{-1}B_N^G(z), \quad \text{a.e. Area}\,|_{\Omega_N^G}. \tag{3.11}$$

If $\nu_i = \langle F_i, G\rangle\mu_1$ and $H_{i,m}(z) = \frac{m^2}{\pi}\nu_i(B(z, \frac{1}{m}))$, then the functions $H_{i,m}(z)$ are bounded with compact supports. We have

$$\mathcal{C}(H_{i,m}\,dA)(w) = \int_{|\lambda - w| \ge \frac{1}{m}} \frac{1}{\lambda - w}\,d\nu_i(\lambda) + \int_{|\lambda - w| < \frac{1}{m}} \frac{m^2|\lambda - w|^2}{\lambda - w}\,d\nu_i(\lambda).$$

Hence,

$$\big|\mathcal{C}(H_{i,m}\,dA)(w) - \mathcal{C}\nu_i(w)\big| \le 2\int_{|w - z| < 1/m} \frac{1}{|w - z|}\,d|\nu_i|(z), \quad \text{a.e. Area}$$

and

$$\lim_{m\to\infty} \mathcal{C}(H_{i.m}\,dA)(w) = \mathcal{C}\nu_i(w), \quad \text{a.e. Area}\,.$$

Let $C_0 > 0$ be a constant such that $|\psi(z) - \psi(w)| \le C_0|z - w|$. We estimate $\mathcal{C}_\psi^1(\nu_i)$ as the following:

$$\begin{aligned}&\big|\mathcal{C}_\psi^1(H_{i,m}\,dA)(w) - \mathcal{C}_\psi^1\nu_i(w)\big|\\ &= \Big|\frac{m^2}{\pi}\int\int_{|z-\lambda|<\frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w}\,dA(z)\,d\nu_i(\lambda) - \mathcal{C}_\psi^1\nu_i(w)\Big|\\ &\le \Big|\int_{|\lambda - w| \ge \frac{1}{\sqrt{m}}} \Big(\frac{m^2}{\pi}\int_{|z-\lambda|<\frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w}\,dA(z) - \frac{\psi(\lambda) - \psi(w)}{\lambda - w}\Big)\,d\nu_i(\lambda)\Big|\\ &\quad + \Big|\frac{m^2}{\pi}\int_{|\lambda - w| < \frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w}\,dA(z)\,d\nu_i(\lambda)\Big|\\ &\quad + \Big|\int_{|\lambda - w| < \frac{1}{\sqrt{m}}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w}\,d\nu_i(\lambda)\Big|.\end{aligned}$$

Note that

$$\frac{m^2}{\pi}\int_{|\lambda-w|\ge\frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}}\frac{1}{z-w}\,dA(z)\,d\nu_i(\lambda)=\int_{|\lambda-w|\ge\frac{1}{\sqrt{m}}}\frac{1}{\lambda-w}\,d\nu_i(\lambda).$$

We get

$$\begin{aligned}
&\left|\mathcal{C}^1_\psi(H_{i,m}\,dA)(w)-\mathcal{C}^1_\psi\nu_i(w)\right|\\
&\quad\le\left|\frac{m^2}{\pi}\int_{|\lambda-w|\ge\frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}}\frac{\psi(z)-\psi(\lambda)}{z-w}\,dA(z)\,d\nu_i(\lambda)\right|\\
&\qquad+2C_0|\nu_i|\Big(B\Big(w,\frac{1}{\sqrt{m}}\Big)\Big)\\
&\quad\le\frac{m^2}{\pi}\int_{|\lambda-w|\ge\frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}}\frac{C_0|z-\lambda|}{|w-\lambda|-|z-\lambda|}\,dA(z)\,d\nu_i(\lambda)\\
&\qquad+2C_0|\nu_i|\Big(B\Big(w,\frac{1}{\sqrt{m}}\Big)\Big)\\
&\quad\le C_0\frac{\frac{1}{m}}{\frac{1}{\sqrt{m}}-\frac{1}{m}}|\nu_i|\Big(B\Big(w,\frac{1}{\sqrt{m}}\Big)^c\Big)+2C_0|\nu_i|\Big(B\Big(w,\frac{1}{\sqrt{m}}\Big)\Big)\\
&\quad\le\frac{C_0}{\sqrt{m}-1}\|\nu_i\|+2C_0|\nu_i|\Big(B\Big(w,\frac{1}{\sqrt{m}}\Big)\Big).
\end{aligned}$$

Therefore,

$$\lim_{m\to\infty}\mathcal{C}^1_\psi(H_{i,m}\,dA)(w)=\mathcal{C}^1_\psi\nu_i(w)$$

for $w\notin A$. For $\lambda_0\in\Omega^G_N$ and $\epsilon>0$, we can choose a $\delta>0$ and m_0 such that

$$\begin{aligned}
&\left|\mathcal{C}^1_\psi(H_{i,m}\,dA)(w)-\mathcal{C}^1_\psi\nu_i(w)\right|\\
&\quad\le 2C_0|\nu_i|\Big(B\Big(w,\frac{1}{\sqrt{m}}\Big)\Big)+\frac{C_0}{\sqrt{m}-1}\|\nu_i\|\\
&\quad\le 2C_0|\nu_i|\Big(B\Big(\lambda_0,\delta+\frac{1}{\sqrt{m}}\Big)\Big)+\frac{C_0}{\sqrt{m}-1}\|\nu_i\|\\
&\quad<\epsilon,
\end{aligned}$$

where $w\in B(\lambda_0,\delta)\setminus A$ and $m\ge m_0$. Since $\mathcal{C}^1_\psi\nu_i(w)$ is continuous at λ_0, δ can be chosen to ensure

$$\left|\mathcal{C}^1_\psi\nu_i(w)-\mathcal{C}^1_\psi\nu_i(\lambda_0)\right|<\epsilon,$$

where $w\in B(\lambda_0,\delta)\setminus A$. It is easy to verify that $\mathcal{C}^1_\psi(H_{i,m}\,dA)$ is a smooth function. For $k>1$, clearly $\mathcal{C}^k_\psi\nu_i(w)$ is a smooth function. Define

$$M^{Gm}_N(z)=\begin{bmatrix}\mathcal{C}^1_\psi(H_{1,m}\,dA) & \mathcal{C}^1_\psi(H_{2,m}\,dA) & \cdots & \mathcal{C}^1_\psi(H_{N-1,m}\,dA)\\ \mathcal{C}^2_\psi(\nu_1) & \mathcal{C}^2_\psi(\nu_2) & \cdots & \mathcal{C}^2_\psi(\nu_{N-1})\\ \cdots & \cdots & \cdots & \cdots\\ \mathcal{C}^{N-1}_\psi(\nu_1) & \mathcal{C}^{N-1}_\psi(\nu_2) & \cdots & \mathcal{C}^{N-1}_\psi(\nu_{N-1})\end{bmatrix}.$$

We can choose ϵ small enough so that

$$M_N^{Gm}(w) \qquad \text{and} \qquad M_N^G(w)$$

are invertible for $w \in B(\lambda_0, \delta) \setminus A$ and $m > m_0$. Define

$$B_N^{Gm}(z) = \begin{bmatrix} \mathcal{C}_\psi^1(H_{N,m}\, dA) \\ \mathcal{C}_\psi^2(\nu_N) \\ \cdots \\ \mathcal{C}_\psi^{N-1}(\nu_N) \end{bmatrix},$$

$$A_N^{Gm}(z) = \left[\mathcal{C}(H_{1,m}\, dA), \mathcal{C}(H_{2,m}\, dA), \ldots, \mathcal{C}(H_{N-1,m}\, dA)\right],$$

and

$$c_N^{Gm}(z) = \mathcal{C}(H_{N,m}\, dA)(z).$$

For a smooth function ϕ with compact support in $B(\lambda_0, \delta)$, using the definition (3.8) and Lebesgue's dominated convergence theorem, we get the following calculation:

$$\begin{aligned} &\int \bar\partial\phi(z) a_N(z)\, dA(z) \\ &= \lim_{m\to\infty} \int \bar\partial\phi(z)\big(\big(M_N^{Gm}(z)\big)^{-1} B_N^{Gm}(z)\big)\, dA(z) \\ &= -\lim_{m\to\infty} \int \phi(z)\bar\partial\big(\big(M_N^{Gm}(z)\big)^{-1} B_N^{Gm}(z)\big)\, dA(z) \\ &= \lim_{m\to\infty} \int \phi(z)\big(M_N^{Gm}(z)\big)^{-1}\big(\big(\bar\partial M_N^{Gm}(z)\big)\big(M_N^{Gm}(z)\big)^{-1} B_N^{Gm}(z) \\ &\quad - \bar\partial B_N^{Gm}(z)\big)\, dA(z). \end{aligned} \tag{3.12}$$

On the other hand,

$$\begin{aligned} &\bar\partial M_N^{Gm}(z) \\ &= \bar\partial\psi(z) \begin{bmatrix} -\mathcal{C}(H_{1,m}\, dA) & -\mathcal{C}(H_{2,m}\, dA) & \cdots & -\mathcal{C}(H_{N-1,m}\, dA) \\ -2\mathcal{C}_\psi^1(\nu_1) & -2\mathcal{C}_\psi^1(\nu_2) & \cdots & -2\mathcal{C}_\psi^1(\nu_{N-1}) \\ \cdots & \cdots & \cdots & \cdots \\ -(N-1)\mathcal{C}^{N-2}(\nu_1) & -(N-1)\mathcal{C}^{N-2}(\nu_2) & \cdots & -(N-1)\mathcal{C}^{N-2}(\nu_{N-1}) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\big(\bar\partial M_N^{Gm}(z)\big)\big(M_N^{Gm}(z)\big)^{-1} = -\bar\partial\psi(z) \begin{bmatrix} X \\ Y \end{bmatrix},$$

where the first block $X = A_N^{Gm}(z)(M_N^{Gm}(z))^{-1}$ is a $1 \times (N-1)$ matrix and the second block

$$Y = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & N-1 & 0 \end{bmatrix}$$

is an $(N-2)\times(N-1)$ matrix. Hence,

$$\begin{aligned}&\left(\bar\partial M_N^{Gm}(z)\right)\left(M_N^{Gm}(z)\right)^{-1}B_N^{Gm}(z)-\bar\partial B_N^{Gm}(z)\\ &=-\bar\partial\psi(z)\begin{bmatrix}A_N^{Gm}(z)(M_N^{Gm}(z))^{-1}B_N^{Gm}-c_N^{Gm}\\ 0\\ \dots\\ 0\end{bmatrix}.\end{aligned}$$

Using (3.11), we see that

$$\lim_{m\to\infty}\left(A_N^{Gm}(z)\left(M_N^{Gm}(z)\right)^{-1}B_N^{Gm}-c_N^{Gm}\right)=0,\quad \text{a.e. Area}\,|_{B(\lambda_0,\delta)}.$$

Since each component of the above vector function is less than

$$M\int\frac{1}{|w-z|}\,d|\nu_i|(z),\quad \text{a.e. Area}\,|_{B(\lambda_0,\delta)},$$

by applying Lebesgue's dominated convergence theorem to the last step of (3.12), we conclude that

$$\int\bar\partial\phi(z)a_N(z)\,dA(z)=0.$$

By Weyl's lemma, we see that $a_N(z)$ is analytic on $B(\lambda_0,\delta)$. From equation (3.8), we get

$$\mathcal{C}_\psi^1\langle F_N,G\rangle(\mu_1)(z)=\sum_{k=1}^{N-1}a_{kj}(z)\mathcal{C}_\psi^1\langle F_k,G\rangle(\mu_1)(z),\quad \text{a.e. Area}\,|_{B(\lambda_0,\delta)}.$$

The above equation implies (3.9) since

$$\bar\partial\mathcal{C}_\psi^1(\nu_i)(z)=-\bar\partial\psi(z)\mathcal{C}(\nu_i)(z),\quad \text{a.e. Area}\,.$$

For equation (3.10), let ϕ be a smooth function with compact support in $B(\lambda_0,\delta)$, and let ν be a compactly supported finite measure. Then we get

$$\int\bar\partial\phi(z)\mathcal{C}\nu(z)\,dA(z)=\pi\int\phi(z)\,d\nu(z).$$

On applying the above equation to the both sides of (3.9) for $j=N$ and using

$$\bar\partial\phi(z)a_{kj}(z)=\bar\partial\big(\phi(z)a_{kj}(z)\big),\quad z\in B(\lambda_0,\delta),$$

we conclude that

$$\int\phi\langle F_N,G\rangle\,d\mu_1=\int\phi\sum_{k=1}^{N-1}a_{kj}\langle F_k,G\rangle\,d\mu_1.$$

Hence (3.10) follows. This completes the proof of the lemma. □

Corollary 3.7. *Let G, Ω^G, and Ω_i^G be as in Lemma 3.6. Suppose that $G\perp\mathcal{K}_{N-1}^\psi$ satisfies (3.4). Then $\Omega_i^G\subset U_{N-1}\cup U_N$.*

Proof. Without loss of generality, we assume that $j = N$. From Lemma 3.6, for $\lambda_0 \in \Omega_N^G$, there exists $\delta > 0$ such that $B(\lambda_0, \delta) \subset \mathrm{Int}(\sigma(S))$ and (3.9) and (3.10) hold, which imply (3.3). For $r_1, r_2, \dots, r_N \in \mathrm{Rat}(\sigma(S))$, let

$$F = \sum_{i=1}^{N} r_i F_i.$$

Note that

$$r_i(\lambda)\mathcal{C}_\psi^k \langle F_i, G\rangle(\mu_1) = \mathcal{C}_\psi^k \langle r_i F_i, G\rangle(\mu_1)$$

since $G \perp \mathcal{K}_{N-1}^\psi$. Then

$$\sum_{i=1}^{N} r_i(\lambda)\mathcal{C}_\psi^k\big(\langle F_i, G\rangle\mu_1\big)(\lambda) = \mathcal{C}_\psi^k\big(\langle F, G\rangle\mu_1\big)(\lambda),$$

for $k = 1, 2, \dots, N-1$. Now using (3.9) for $\lambda \in B(\lambda_0, \delta) \setminus A$, we get

$$\sum_{i=1}^{N-1}\big(r_i(\lambda) + a_{Ni}(\lambda)r_N(\lambda)\big)\mathcal{C}_\psi^k\big(\langle F_i, G\rangle\mu_1\big)(\lambda) = \mathcal{C}_\psi^k\big(\langle F, G\rangle\mu_1\big)(\lambda),$$

or equivalently,

$$M_N^G(\lambda)\begin{bmatrix} r_1(\lambda) + a_{N1}(\lambda)r_N(\lambda) \\ r_2(\lambda) + a_{N2}(\lambda)r_N(\lambda) \\ \cdots \\ r_{N-1}(\lambda) + a_{N,N-1}(\lambda)r_N(\lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_\psi^1(\langle F, G\rangle\mu_1)(\lambda) \\ \mathcal{C}_\psi^2(\langle F, G\rangle\mu_1)(\lambda) \\ \cdots \\ \mathcal{C}_\psi^{N-1}(\langle F, G\rangle\mu_1)(\lambda) \end{bmatrix},$$

where the inverse of $M_N^G(\lambda)$ is bounded on $B(\lambda_0, \delta) \setminus A$ and a_{Ni} are analytic on $B(\lambda_0, \delta)$. Therefore, there exists a positive constant M such that

$$\sup_{1\le k\le N-1, \lambda\in B(\lambda_0, \frac{\delta}{2})} \big|r_k(\lambda) + a_{Nk}(\lambda)r_N(\lambda)\big| \le M\|F\|,$$

which implies (3.2). Hence, Theorem 3.1 implies that $\Omega_N^G \subset U_{N-1} \cup U_N$. □

Now let us recursively construct other sets such as Ω_{ij}^G for a given $G \perp \mathcal{K}_{N-1}^\psi$. We only describe the algorithm for $k = N-2$; the other cases follow recursively. Let $E_N^G = \Omega^G$ and $E_{N-1}^G = \bigcup_{i=1}^N \Omega_i^G$. Let M_{ij}^G be the $N-2$ by $N-2$ submatrix of M^G obtained by eliminating the first two rows and the ith and jth columns. Define

$$\Omega_{ij}^G = \big(\mathrm{Int}\big(\sigma(S)\big) \cap A^c \cap \big\{z : \big|\det\big(M_{ij}^G(z)\big)\big| > 0\big\}\big) \setminus \mathrm{clos}(E_N^G \cup E_{N-1}^G).$$

Without loss of generality, let us assume that $i = N-1$ and that $j = N$. Similar to Lemma 3.6, one can prove that for $\lambda_0 \in \Omega_{N-1,N}^G$, there exist $\delta > 0$, analytic functions $a_i(z)$ and $b_i(z)$ on $B(\lambda_0, \delta) \subset \mathrm{Int}(\sigma(S))$ such that

$$F_{N-1} = \sum_{i=1}^{N-2} a_i(z)F_i(z), \qquad F_N = \sum_{i=1}^{N-2} b_i(z)F_i(z), \quad \text{a.e. } \mu_1|_{B(\lambda_0,\delta)}, \tag{3.13}$$

and there exists a constant $M > 0$ such that

$$\sup_{1\le k\le N-2, \lambda\in B(\lambda_0,\frac{\delta}{2})} \left| r_k(\lambda) + a_k(\lambda) r_{N-1}(\lambda) + b_k(\lambda) r_N(\lambda)\right| \le M\|F\|, \tag{3.14}$$

where $r_1, r_2, \dots, r_N \in \text{Rat}(\sigma(S))$ and $F = \sum_{i=1}^N r_i F_i$. Equations (3.13) and (3.14) are the same as (3.2) and (3.3) for the case $k = N-2$. Let

$$E^G_{N-2} = \bigcup_{i<j}^{N} \Omega^G_{ij}. \tag{3.15}$$

Corollary 3.8. *Let E^G_{N-2} be as in (3.15). Suppose that $G \perp \mathcal{K}^\psi_{N-1}$ satisfies (3.4). Then*

$$E^G_{N-2} \subset U_{N-2} \cup U_{N-1} \cup U_N.$$

The proof is the same as in Corollary 3.7. Therefore, we can recursively construct E^G_k for $k = 1, 2, \dots, N$ such that

$$E^G_k \subset \bigcup_{i=k}^{N} U_i, \tag{3.16}$$

where the proof for $k = N$ is from Lemma 3.5, $k = N-1$ is from Corollary 3.7, and $k = N-2$ is from Corollary 3.8.

The following theorem proves, under the condition that S satisfies the property (N,ψ), that the set $\bigcup_{k=1}^N E^G_k$ is big.

Theorem 3.9. *Let E^G_i be constructed for $i = 1, 2, \dots, N$ as above. Suppose that $\{G_j\} \subset (\mathcal{K}^\psi_{N-1})^\perp$ is a dense subset. Then*

$$\text{spt}\, \mu_1 \subset \text{clos}\Big(\bigcup_{i=1}^{N}\bigcup_{j=1}^{\infty} E^{G_j}_i\Big).$$

Proof. First we prove that

$$\mu_1\Big(\text{Int}\big(\sigma(S)\big) \setminus \text{clos}\Big(\bigcup_{i=1}^{N}\bigcup_{j=1}^{\infty} E^{G_j}_i\Big)\Big) = 0.$$

Suppose that $B(\lambda_0,\delta) \subset \text{Int}(\sigma(S))$ and $B(\lambda_0,\delta) \cap \text{clos}(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E^{G_j}_i) = \emptyset$. Then by construction of $E^{G_j}_i$, we conclude that

$$\mathcal{C}^{N-1}_\psi\big(\langle F_i, G_j\rangle \mu_1\big)(z) = 0$$

on $B(\lambda_0,\delta)$, where $i = 1, 2, \dots, N$. By taking $\bar\partial$ in the sense of distribution, we see that

$$\mathcal{C}\big(\langle F_i, G_j\rangle \mu_1\big)(z) = 0$$

almost everywhere with respect to the area measure on $B(\lambda_0,\delta)$ since Area$(\{\bar\partial\psi = 0\}\cap\sigma(S)) = 0$, where $i = 1,2,\dots,N$. For a smooth function ϕ with compact support in $B(\lambda_0,\delta)$,

$$\int \phi(z)\langle F_i, G_j\rangle\, d\mu_1 = \frac{1}{\pi}\int \bar\partial\phi(z)\mathcal{C}\big(\langle F_i, G_j\rangle\mu_1\big)(z)\, dA(z) = 0.$$

Therefore,

$$\big\langle F_i(z), G_j(z)\big\rangle = 0, \quad \text{a.e. } \mu_1|_{B(\lambda_0,\delta)}, \tag{3.17}$$

where $i = 1,2,\dots,N$. From (1.4), we see that for $P\in\bigoplus_{k=1}^m L^2(\mu_k|_{B(\lambda_0,\delta)})$, (3.17) implies that $(P, G_j) = 0$, Therefore,

$$\bigoplus_{k=1}^m L^2(\mu_k|_{B(\lambda_0,\delta)}) \subset \mathcal{K}^\psi_{N-1}.$$

Hence, $\mu_1|_{B(\lambda_0,\delta)} = 0$ since $M_z|_{\mathcal{K}^\psi_{N-1}}$ is pure.

Now assume that $B(\lambda_0,\delta)\cap \operatorname{clos}(\operatorname{Int}(\sigma(S))) = \emptyset$. For $N > 1$, the function

$$\mathcal{C}^{N-1}_\psi\big(\langle F_i, G_j\rangle\mu_1\big)(z)$$

is continuous on $\mathbb{C}\setminus A$ and is zero on $\mathbb{C}\setminus\sigma(S)$. Hence,

$$\mathcal{C}^{N-1}_\psi\big(\langle F_i, G_j\rangle\mu_1\big)(z) = 0$$

on $B(\lambda_0,\delta)\setminus A$, where $i = 1,2,\dots,N$. Using the same proof as above, we see that $\mu_1|_{B(\lambda_0,\delta)} = 0$. This implies that $\operatorname{spt}\mu_1\subset\operatorname{clos}(\operatorname{Int}(\sigma(S)))$. The theorem is proved. □

Proof. Proof of Theorem 3.1 From (3.16) and Theorem 3.9, we get

$$\bigcup_{i=1}^N \partial U_i \subset \sigma_e(S)\subset \operatorname{spt}(\mu_1)\subset \operatorname{clos}\Big(\bigcup_{i=1}^N U_i\Big).$$

This implies that

$$\sigma_e(S) = \bigcup_{i=1}^N \partial U_i$$

since $\sigma_e(S)\cap U_i = \emptyset$. This completes the proof. □

For a positive finite measure μ with compact support on $\mathbb{C}$, define

$$P^2(\mu|1,\bar z,\dots,\bar z^{N-1}) = \operatorname{clos}\big\{p_1(z) + p_2(z)\bar z + \cdots + p_N(z)\bar z^{N-1} : p_1, p_2,\dots,p_N\in\mathcal{P}\big\}$$

and $S_{N,\mu}$ as the multiplication by z on $P^2(\mu|1,\bar z,\dots,\bar z^{N-1})$. Then $S_{N,\mu}$ is a multicyclic subnormal operator with the minimal normal extension M_μ, the multiplication by z, on $L^2(\mu)$.

Corollary 3.10. *Suppose that $S_{2,\mu}$ on $P^2(\mu|1,\bar z,\bar z^2)$ is pure. Then the operator $S_{1,\mu}$ on $P^2(\mu|1,\bar z)$ satisfies*

$$\sigma(S_{1,\mu}) = \operatorname{clos}\big(\sigma(S_{1,\mu})\setminus\sigma_e(S_{1,\mu})\big).$$

Proof. The result follows from Theorem 3.1 since

$$\mathcal{K}_1^{\bar z} = \operatorname{clos}\big(\operatorname{span}\big(\bar z^k P^2(\mu|1,\bar z) : 0 \le k \le 1\big)\big) = P^2(\mu|1,\bar z,\bar z^2)$$

and $S_{2,\mu}$ on $P^2(\mu|1,\bar z,\bar z^2)$ is pure. □

It seems strong to assume that $S_{2,\mu}$ on $P^2(\mu|1,\bar z,\bar z^2)$ is pure in the corollary. We believe that the condition can be reduced to assume that $S_{1,\mu}$ on $P^2(\mu|1,\bar z)$ is pure. However, we are not able to prove the result under the weaker conditions, so we will leave it as an open problem for further research.

Problem 3.11. *Does Corollary 3.10 hold under the weaker assumption that $S_{1,\mu}$ on $P^2(\mu|1,\bar z)$ is pure?*

Corollary 3.12. *Let S on $\mathcal{H}$ be a pure rationally N-cyclic subnormal operator with $\mathcal{H} = R^2(S|F_1, F_2, \dots, F_N)$, and let M_z be its minimal normal extension on $\mathcal{K}$ satisfying (1.1)–(1.4). Suppose that there exists a smooth function ψ on $\mathbb{C}$ such that $\operatorname{Area}(\{\bar\partial\psi = 0\} \cap \sigma(S)) = 0$ and $\psi(M_z)\mathcal{H} \subset \mathcal{H}$. Then there exist bounded open subsets U_i for $1 \le i \le N$ such that*

$$\sigma_e(S) = \bigcup_{i=1}^N \partial U_i, \qquad \sigma(S) \setminus \sigma_e(S) = \bigcup_{i=1}^N U_i,$$

and

$$\dim\ker(S-\lambda)^* = i,$$

for $\lambda \in U_i$.

Note that Examples 3.2 and 3.3 are special cases of Corollary 3.12. It seems that further results could be obtained for the special cases where S satisfies the conditions of Corollary 3.12. Moreover, we might be able to combine the methodology in McCarthy and Yang [15] to obtain the structural models for the class of subnormal operators, which might extend Xia's model for subnormal operators with finite-rank self-commutators.

Problem 3.13. *Can the structure of subnormal operators in Corollary 3.12 be characterized?*

Acknowledgment. The author would like to thank the referees for carefully reading the manuscript and providing helpful comments.

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