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QUANTITATIVE WEIGHTED BOUNDS FOR THE COMPOSITION OF CALDERÓN–ZYGMUND OPERATORS

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Abstract. Let T_1, T_2 be two Calderón–Zygmund operators, and let $T_{1,b}$ be the commutator of T_1 with symbol $b \in \mathrm{BMO}(\mathbb{R}^n)$. In this article, we establish the quantitative weighted bounds on $L^p(\mathbb{R}^n, w)$ with $w \in A_p(\mathbb{R}^n)$ for the composite operator $T_{1,b}T_2$.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let $A_p(\mathbb{R}^n)$ ($p \in [1, \infty)$) be the Muckenhoupt class of weight functions, that is, $w \in A_p(\mathbb{R}^n)$ if w is nonnegative and locally integrable, and

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)\,dx\Big)^{p-1} < \infty, \quad \text{if } p \in (1, \infty),$$

and

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)},$$

where the supremum is taken over all cubes in $\mathbb{R}^n$, and $[w]_{A_p}$ is called the A_p *constant of* w (see [7] for properties of $A_p(\mathbb{R}^n)$). In the last several years, there has been significant progress in the study of sharp weighted bounds with A_p weights for classical operators in harmonic analysis. The study was begun by Buckley

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[3], who proved that if $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, then the Hardy–Littlewood maximal operator M satisfies

$$\|Mf\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{1.1}$$

Moreover, the estimate (1.1) is sharp since the exponent $1/(p-1)$ cannot be replaced by a smaller one. Hytönen and Pérez [12] improved (1.1), and showed that

$$\|Mf\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} \big([w]_{A_p}[w^{-\frac{1}{p-1}}]_{A_\infty}\big)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n,w)}, \tag{1.2}$$

where here and in what follows, for a weight $u \in A_\infty(\mathbb{R}^n) = \bigcup_{p\ge 1} A_p(\mathbb{R}^n)$, $[u]_{A_\infty}$ is the A_∞ constant of u defined by (see [25])

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)\,dx.$$

It is obvious that (1.2) is more subtle than (1.1).

Let T be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K in the sense that, for all $f \in L^2(\mathbb{R}^n)$ with compact support and almost everywhere $x \in \mathbb{R}^n \setminus \operatorname{supp} f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \tag{1.3}$$

where K is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$. We say that T is a *Calderón–Zygmund operator* if K is a Calderón–Zygmund kernel, that is, K satisfies the size condition that

$$|K(x,y)| \lesssim |x-y|^{-n} \quad \text{if } x \neq y,$$

and the regularity condition that for any $x, y, y' \in \mathbb{R}^n$ with $|x-y| \ge 2|y-y'|$,

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \lesssim \frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}},$$

where $\varepsilon \in (0,1]$ is a constant. The sharp dependence of the weighted estimates of Calderón–Zygmund operators in terms of the $A_p(\mathbb{R}^n)$ constants was first considered by Petermichl [22], [23], who solved this question for Hilbert and Riesz transforms. Hytönen [10] proved that for a Calderón–Zygmund operator T and $w \in A_2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2(\mathbb{R}^n,w)} \lesssim_n [w]_{A_2} \|f\|_{L^2(\mathbb{R}^n,w)}. \tag{1.4}$$

This solved the so-called A_2 *conjecture*. Hytönen and Lacey [11] improved the estimate (1.4), and proved that for a Calderón–Zygmund operator T, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|Tf\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p}} \big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big) \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{1.5}$$

Here and in what follows, for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, $p' = p/(p-1)$, $\sigma = w^{-\frac{1}{p-1}}$. Lerner [16] gave a very simple proof of (1.5) by dominating the Calderón–Zygmund operator using sparse operators. (For other recent works about quantitative weighted bounds for singular integral operators, see [12], [13], [17], [18], [20] and the related references therein.)

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, let $m \in \mathbb{N}$, and let $T_{j,1}$ and $T_{j,2}$ with $j = 1, \dots, m$ be Calderón–Zygmund operators. Krantz and Li [15] considered the boundedness of a Toeplitz-type operator defined by

$$\mathcal{F}_b f(x) = \sum_{j=1}^{m} T_{j,1} S_b T_{j,2} f(x), \tag{1.6}$$

where, both here and in what follows, S_b is the multiplication operator with symbol b defined by

$$S_b g(x) = b(x) g(x).$$

Krantz and Li proved that if $f \in L^p(\mathbb{R}^n)$ such that $\sum_{j=1}^m T_{j,1} T_{j,2} f = 0$, then for $p \in (1, \infty)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|\mathcal{F}_b f\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\sum_{j=1}^{m} \|T_{j,1}\|_{L^p \to L^p} \Big) \Big(\sum_{j=1}^{m} \|T_{j,2}\|_{L^p \to L^p} \Big) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Observe that we can write

$$T_{j,1} S_b T_{j,2} f(x) = b(x) T_{j,1}(T_{j,2} f)(x) - T_{j,1;b}(T_{j,2} f)(x).$$

Therefore, $\sum_{j=1}^m T_{j,1} T_{j,2} f(x) = 0$ is equivalent to the fact that

$$\mathcal{F}_b f(x) = -\sum_{j=1}^{m} T_{j,1;b}(T_{j,2} f)(x),$$

where, both here and in what follows, for $b \in \mathrm{BMO}(\mathbb{R}^n)$ and a linear operator U, U_b is the commutator defined by

$$U_b f(x) = b(x) U f(x) - U(bf)(x).$$

Thus, the study of properties of $T_{j,1} S_b T_{j,2} f$ can be reduced to considering the operator $T_{j,1;b} T_{j,2}$. For $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, it follows from (1.5) and the quantitative weighted bounds for the commutators of Calderón–Zygmund operators (see [5], [6]) that, if T_1 and T_2 are two Calderón–Zygmund operators and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then

$$\begin{aligned} \|T_{1,b} T_2 f\|_{L^p(\mathbb{R}^n, w)} \lesssim{}& \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} [w]_{A_p}^{\frac{2}{p}} \big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}} \big)^2 \\ &\times \big([w]_{A_\infty} + [\sigma]_{A_\infty} \big) \|f\|_{L^p(\mathbb{R}^n, w)}. \end{aligned} \tag{1.7}$$

The main purpose of this article is to establish a weighted bound for $T_{1,b} T_2$ which is more refined than (1.7). Our main result can be stated as follows.

Theorem 1.1. *Let T_1 and T_2 be Calderón–Zygmund operators, and let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,*

$$\begin{aligned} \|T_{1,b} T_2 f\|_{L^p(\mathbb{R}^n, w)} \lesssim{}& \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} [w]_{A_p}^{\frac{1}{p}} \big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}} \big) \\ &\times \big([w]_{A_\infty} + [\sigma]_{A_\infty} \big)^2 \|f\|_{L^p(\mathbb{R}^n, w)}. \end{aligned} \tag{1.8}$$

Remark 1.2. As it is well known,

$$[w]_{A_\infty} \lesssim [w]_{A_p}, \qquad [\sigma]_{A_\infty} \lesssim [\sigma]_{A_{p'}} = [w]_{A_\infty}^{p'/p}.$$

Thus,

$$[w]_{A_\infty} + [\sigma]_{A_\infty} \lesssim [w]_{A_p}^{\frac{1}{p}} \big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big).$$

And the inequality (1.8) is more refined than the inequality (1.7).

Remark 1.3. Benea and Bernicot [2] considered the weighted bounds on $L^p(\mathbb{R}^n, w)$ for the composition of two Calderón–Zygmund operators. They proved that if T_1, T_2 are two Calderón–Zygmund operators and $T_1(1) = 0$, then for $r \in (1, \infty)$ and bounded functions f and g with compact supports, there exists a sparse family of cubes $\mathcal{S}$ such that

$$\Big|\int_{\mathbb{R}^n} g(x) T_1 T_2 f(x)\, dx\Big| \lesssim \sum_{Q\in\mathcal{S}} |Q| \Big(\frac{1}{|Q|}\int_Q \big|f(x)\big|^r dx\Big)^{\frac{1}{r}} \Big(\frac{1}{|Q|}\int_Q \big|g(y)\big|\, dy\Big),$$

which implies that for any $p \in (1, \infty)$, $r \in (1, p)$, and $w \in A_{p/r}(\mathbb{R}^n)$,

$$\|T_1 T_2 f\|_{L^p(\mathbb{R}^n, w)} \lesssim [w]_{A_{p/r}}^{\max\{\frac{1}{p-r}, 1\}} \|f\|_{L^p(\mathbb{R}^n, w)}. \tag{1.9}$$

Our argument in the proof of Theorem 1.1 does not require the assumption $T_1(1) = 0$, and is different from that used in [2]. In fact, repeating the proof of Theorem 1.1, we can verify that if T_1, T_2 are two Calderón–Zygmund operators, then for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|T_1 T_2 f\|_{L^p(\mathbb{R}^n, w)} \lesssim [w]_{A_p}^{\frac{1}{p}} \big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)\big([w]_{A_\infty} + [\sigma]_{A_\infty}\big) \|f\|_{L^p(\mathbb{R}^n, w)}.$$

Remark 1.4. To prove Theorem 1.1, we will employ the idea of Lerner [17], together with some variants. Precisely, by suitable estimate for the grand maximal operator $\mathscr{M}_{T_{1,b}T_2}$, we show that the bi-sublinear form $\int_{\mathbb{R}^n} |T_{1,b}T_2 f(x)||g(x)|\, dx$ can be dominated by the combination of three bi-sublinear sparse operators, which via the estimates for bi-sublinear sparse operators leads to (1.8).

In what follows, C always denotes a positive constant that is independent of the main parameters involved, but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. In particular, we use $A \lesssim_{n,p} B$ to denote that there exists a positive constant C depending only on n, p such that $A \le CB$. Constants with subscripts such as c_1 do not change from one occurrence to another. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ and $\lambda \in (0, \infty)$, we use $\ell(Q)$ ($\operatorname{diam} Q$) to denote the side length (diameter) of Q, and λQ to denote the cube with the same center as Q and whose side length is λ times that of Q. For a fixed cube Q, denote by $\mathcal{D}(Q)$ the set of dyadic cubes with respect to Q, that is, the cubes from $\mathcal{D}(Q)$ that are formed by repeated subdivision of Q and each of its descendants into 2^n congruent subcubes. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$

denotes the ball centered at x and having radius r. For $\beta \in [0,\infty)$, cube $Q \subset \mathbb{R}^n$, and a suitable function g, $\|g\|_{L(\log L)^\beta,Q}$ is the norm defined by

$$\|g\|_{L(\log L)^\beta,Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \frac{|g(y)|}{\lambda}\log^\beta\Big(\mathrm{e}+\frac{|g(y)|}{\lambda}\Big)\,dy \le 1\Big\}.$$

We denote $\|g\|_{L(\log L)^0,Q}$ by $\langle |g|\rangle_Q$. For $r \in (0,\infty)$, we set $\langle |g|\rangle_{r,Q} = (\langle |g|^r\rangle_Q)^{\frac{1}{r}}$.

2. Estimates for the grand maximal operator

As in [17], for a sublinear operator U, we define the corresponding bi-sublinear grand maximal operator $\mathscr{M}_U$ by

$$\mathscr{M}_U(f,g)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \big|U(f\chi_{\mathbb{R}^n\setminus 9Q})(\xi)\big|\big|g(\xi)\big|\,d\xi,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. Also, we define the grand maximal operator $\mathcal{M}_U$ by

$$\mathcal{M}_U f(x) = \sup_{Q\ni x}\operatorname*{ess\,sup}_{\xi\in Q}\big|U(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\big|.$$

Lerner [16, Section 3] proved that if T is a Calderón–Zygmund operator, then

$$\mathcal{M}_T f(x) \lesssim T^* f(x) + Mf(x). \tag{2.1}$$

This section is devoted to the estimates for the operators $\mathscr{M}_{T_1,bT_2}$. We begin with some preliminary lemmas.

Lemma 2.1. *Let $p_0 \in (1,\infty)$, let $\varrho \in [0,\infty)$, and let U be a sublinear operator. Suppose that*

$$\|Uf\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^n)},$$

and for all $\lambda > 0$,

$$\big|\{x \in \mathbb{R}^n : |Uf(x)| > \lambda\}\big| \lesssim \int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log^\varrho\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\,dx.$$

Then for cubes $Q_2 \subset Q_1 \subset \mathbb{R}^n$,

$$\big\|U(f\chi_{Q_2})\big\|_{L(\log L)^\beta,Q_1} \lesssim \|f\|_{L(\log L)^{\beta+\varrho+1},Q_2}.$$

For $\beta = 0$, Lemma 2.1 was proved in [9, Section 3]. For the case of $\beta > 0$, the proof is similar to the case of $\beta = 0$.

Lemma 2.2. *Let $s \in [0,\infty)$, and let T be a sublinear operator satisfying that for any $\lambda > 0$,*

$$\big|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\big| \lesssim \int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log^s\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\,dx.$$

Then for any $\varrho \in (0,1)$ and cube $Q \subset \mathbb{R}^n$,

$$\Big(\frac{1}{|Q|}\int_Q \big|T(f\chi_Q)(x)\big|^\varrho\,dx\Big)^{\frac{1}{\varrho}} \lesssim \|f\|_{L(\log L)^s,Q}.$$

For the proof of Lemma 2.2, see [8, p. 643].

Let $\beta \in [0,\infty)$. For a locally integrable function f, define the maximal function $M^{\sharp}_{L(\log L)^{\beta}} f$ by

$$M^{\sharp}_{L(\log L)^{\beta}} f(x) = \sup_{Q \ni x} \|f - \langle f\rangle_Q\|_{L(\log L)^{\beta},Q},$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. Obviously, $M^{\sharp}_{L(\log L)^{0}}$ is just $M^{\sharp}$, the Fefferman–Stein sharp maximal operator (see [7]). For $r \in (0,1)$, let $M^{\sharp}_r$ be the operator defined by

$$M^{\sharp}_r f(x) = \big[M^{\sharp}(|f|^r)(x)\big]^{1/r},$$

and let M_r be the maximal operator defined by $M_r f(x) = [M(|f|^r)(x)]^{1/r}$. It is well known that if $r \in (0,1)$, then (see [8]) for any $\lambda > 0$,

$$\big|\big\{x \in \mathbb{R}^n : M_r f(x) > \lambda\big\}\big| \lesssim \lambda^{-1} \sup_{t \ge 2^{-1/r}\lambda} t\big|\big\{x \in \mathbb{R}^n : \big|f(x)\big| > t\big\}\big|. \tag{2.2}$$

For $\beta \in [0,\infty)$, let $M_{L(\log L)^{\beta}}$ be the maximal operator defined by

$$M_{L(\log L)^{\beta}} g(x) = \sup_{Q \ni x} \|g\|_{L(\log L)^{\beta},Q}.$$

For simplicity, we denote $M_{L(\log L)^{1}}$ by $M_{L\log L}$. Carozza and Passarelli di Napoli [4, Theorem 2] proved that for $\alpha, \beta \in [0,\infty)$,

$$M_{L(\log L)^{\alpha}} M_{L(\log L)^{\beta}} f(x) \approx M_{L(\log L)^{\alpha+\beta+1}} f(x). \tag{2.3}$$

Also, we have that for any $\lambda > 0$,

$$\big|\big\{x \in \mathbb{R}^n : M_{L(\log L)^{\beta}} g(x) > \lambda\big\}\big| \lesssim \int_{\mathbb{R}^n} \frac{|g(x)|}{\lambda} \log^{\beta}\Big(\mathrm{e} + \frac{|g(x)|}{\lambda}\Big)\, dx. \tag{2.4}$$

Lemma 2.3. *Let Φ be an increasing function on $[0,\infty)$ satisfying that*

$$\Phi(2t) \le C\Phi(t), \quad t \in [0,\infty).$$

(i) *Let $\beta > 0$. Then*

$$\sup_{\lambda>0} \Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : M_{L(\log L)^{\beta}} f(x) > \lambda\big\}\big| \lesssim \sup_{\lambda>0} \Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : M^{\sharp}_{L(\log L)^{\beta}} f(x) > \lambda\big\}\big|,$$

provided that for any $R > 0$,

$$\sup_{0<\lambda<R} \Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : M_{L(\log L)^{\beta}} f(x) > \lambda\big\}\big| < \infty.$$

(ii) *Let $r \in (0,1)$. Then*

$$\sup_{\lambda>0} \Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : M_r f(x) > \lambda\big\}\big| \lesssim \sup_{\lambda>0} \Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : M^{\sharp}_r f(x) > \lambda\big\}\big|,$$

provided that for any $R > 0$,

$$\sup_{0<\lambda<R} \Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : M_r f(x) > \lambda\big\}\big| < \infty.$$

The proof of Lemma 2.3 is fairly similar to the proof of Corollary 7.4.6 in [7] (see also the proof of Theorem 2.2 in [9]). We omit the details for brevity.

For a Calderón–Zygmund operator T, let T^* be the maximal singular integral operator defined by

$$T^*f(x)=\sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon}K(x,y)f(y)\,dy\Big|.$$

For $b\in \mathrm{BMO}(\mathbb{R}^n)$, let T_b^* be the maximal commutator defined by

$$T_b^*f(x)=\sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon}\big(b(x)-b(y)\big)K(x,y)f(y)\,dy\Big|,$$

and let M_b be the commutator defined by

$$M_bf(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q\big|b(x)-b(y)\big|\big|f(y)\big|\,dy.$$

Lemma 2.4. *Let T_1 and T_2 be two Calderón–Zygmund operators. Then for each $\lambda>0$,*

$$\big|\big\{x\in\mathbb{R}^n: MT_2f(x)+T_1^*T_2f(x)>\lambda\big\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\,dx,$$

provided that f is bounded, compact supported, and has integral zero.

Proof. We first consider the operator MT_2. For $\beta\in[0,\infty)$, let $\Phi_\beta(t)=t\log^{-\beta}(\mathrm{e}+t^{-1})$. We claim that, for each bounded function f with compact support,

$$M^{\sharp}_{L(\log L)^\beta}(T_2f)(x)\lesssim M_{L(\log L)^{\beta+1}}f(x),\quad f\in\bigcup_{p\geq1}L^p(\mathbb{R}^n),\tag{2.5}$$

and

$$\sup_{\lambda>0}\Phi_{\beta+1}(\lambda)\big|\big\{x\in\mathbb{R}^n:\big|M_{L(\log L)^\beta}T_2f(x)\big|>\lambda\big\}\big|<\infty.\tag{2.6}$$

If we can prove these two estimates, then it would follow from Lemma 2.3 and the inequality (2.4) that

$$\big|\big\{x\in\mathbb{R}^n:\big|MT_2f(x)\big|>\lambda\big\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\,dx.\tag{2.7}$$

The proof of (2.5) is fairly standard. For each $x\in\mathbb{R}^n$, cube Q containing x, and function $f\in\bigcup_{p\geq1}L^p(\mathbb{R}^n)$, we decompose f as

$$f(y)=f(y)\chi_{6\sqrt{n}Q}(y)+f(y)\chi_{\mathbb{R}^n\setminus6\sqrt{n}Q}(y)=f_1(y)+f_2(y).$$

Let $x_Q\in Q$ such that $|T_2f_2(x_Q)|<\infty$. Then for all $y\in Q$,

$$\big|T_2f_2(y)-T_2f_2(x_Q)\big|\lesssim Mf(x).$$

On the other hand, by Lemma 2.1 and (2.3), we see that

$$\|T_2f_1\|_{L(\log L)^\beta,Q}\lesssim\|f\|_{L(\log L)^{\beta+1},6\sqrt{n}Q}\lesssim M_{L(\log L)^{\beta+1}}f(x),$$

and so (2.5) holds true.

To prove (2.6), we assume that $\operatorname{supp} f \subset B(0,R)$. Thus by Lemma 2.1, we know that $\|T_2f\|_{L(\log L)^\beta, B(0,3R)} < \infty$ and that

$$\begin{aligned}
&\int_{B(0,3R)} \big|T_2f(y)\big| \log^\beta\big(\mathrm{e} + \big|T_2f(y)\big|\big)\,dy \\
&\quad\lesssim \|T_2f\|_{L(\log L)^\beta, B(0,3R)} \log^\beta\big(\mathrm{e} + \|T_2f\|_{L(\log L)^\beta, B(0,3R)}\big) \\
&\qquad\times \int_{B(0,3R)} \frac{|T_2f(y)|}{\|T_2f\|_{L(\log L)^\beta, B(0,3R)}} \log^\beta\Big(\mathrm{e} + \frac{|T_2f(y)|}{\|T_2f\|_{L(\log L)^\beta, B(0,3R)}}\Big)\,dy \\
&\quad\lesssim \|T_2f\|_{L(\log L)^\beta, B(0,3R)} \log^\beta\big(\mathrm{e} + \|T_2f\|_{L(\log L)^\beta, B(0,3R)}\big).
\end{aligned}$$

Thus by (2.4),

$$\begin{aligned}
&\big|\big\{x \in \mathbb{R}^n : M_{L(\log L)^\beta}(\chi_{B(0,3R)}T_2f)(x) > \lambda/2\big\}\big| \\
&\quad\lesssim \int_{B(0,3R)} \frac{|T_2f(y)|}{\lambda} \log^\beta\Big(\mathrm{e} + \frac{|T_2f(y)|}{\lambda}\Big)\,dy \\
&\quad\lesssim \lambda^{-1}\log^\beta(\mathrm{e}+\lambda^{-1}) \int_{B(0,3R)} \big|T_2f(y)\big| \log^\beta\big(\mathrm{e} + \big|T_2f(y)\big|\big)\,dy \\
&\quad\lesssim \lambda^{-1}\log^\beta(\mathrm{e}+\lambda^{-1}) \|T_2f\|_{L(\log L)^\beta, B(0,3R)} \log^\beta\big(\mathrm{e} + \|T_2f\|_{L(\log L)^\beta, B(0,3R)}\big).
\end{aligned}$$

It is obvious that for any $x \in \mathbb{R}^n$,

$$M_{L(\log L)^\beta}(\chi_{\mathbb{R}^n\setminus B(0,3R)}T_2f)(x) \lesssim M_{L(\log L)^\beta}Mf(x) \lesssim M_{L(\log L)^{\beta+1}}f(x),$$

since $\operatorname{supp} f \subset B(0,R)$. This, via (2.4), implies that

$$\begin{aligned}
&\big|\big\{x \in \mathbb{R}^n : M_{L(\log L)^\beta}(\chi_{\mathbb{R}^n\setminus B(0,3R)}T_2f)(x) > \lambda/2\big\}\big| \\
&\quad\lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\beta+1}\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\,dx.
\end{aligned}$$

The estimate (2.6) now follows directly.

We turn our attention to the operator $T_1^*T_2$. The well-known Cotlar inequality states that

$$T_1^*g(x) \lesssim M_{\frac{1}{2}}T_1g(x) + Mg(x).$$

Applying (2.2), we know that

$$\begin{aligned}
&\sup_{\lambda>0} \Phi_1(\lambda)\big|\big\{x \in \mathbb{R}^n : M_{\frac{1}{2}}T_1T_2f(x) > \lambda\big\}\big| \\
&\quad\lesssim \sup_{\lambda>0} \Phi_1(\lambda)\lambda^{-1} \sup_{t \geq 2^{-2}\lambda} t\big|\big\{x \in \mathbb{R}^n : \big|T_1T_2f(x)\big| > t\big\}\big| \\
&\quad\lesssim \|T_2f\|_{L^1(\mathbb{R}^n)},
\end{aligned}$$

since $T_2f \in L^1(\mathbb{R}^n)$ when f is bounded, compact supported, and has integral zero. Recall (see [21]) that for $g \in \bigcup_{p\geq 1} L^p(\mathbb{R}^n)$,

$$M^\sharp_{\frac{1}{2}}(T_1g)(x) \lesssim Mg(x). \tag{2.8}$$

It then follows from Lemma 2.3 and (2.7) that

$$\begin{aligned}\big|\{x\in\mathbb{R}^n: M_{\frac12}T_1T_2f(x)>1\}\big| &\lesssim \sup_{\lambda>0}\Phi_1(\lambda)\big|\{x\in\mathbb{R}^n: M_{\frac12}(T_1T_2f)(x)>\lambda\}\big|\\ &\lesssim \sup_{\lambda>0}\Phi_1(\lambda)\big|\{x\in\mathbb{R}^n: MT_2f(x)>\lambda\}\big|\\ &\lesssim \int_{\mathbb{R}^n}|f(x)|\log\big(\mathrm{e}+|f(x)|\big)\,dx.\end{aligned}$$

This, along with (2.7), leads to the fact that

$$\big|\{x\in\mathbb{R}^n: T_1^*T_2f(x)>\lambda\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\,dx,$$

and this completes the proof of Lemma 2.4. □

Lemma 2.5. *Let T_1, T_2 be two Calderón–Zygmund operators, and let $b\in\mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. If $r\in(0,1)$, $U\in\{M_rT_{1,b}^*T_2, M_rM_bT_2\}$, then for each $\lambda>0$,*

$$\big|\{x\in\mathbb{R}^n: |Uf(x)|>\lambda\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log^2\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\,dx,$$

provided that f is bounded, compact supported, and has integral zero.

Proof. We only consider the operator $M_rT_{1,b}^*T_2$. At first, we claim that if f is bounded, compact supported, and has integral zero, then

$$\sup_{\lambda>0}\Phi_2(\lambda)\big|\{x\in\mathbb{R}^n: M_rT_{1,b}^*T_2f(x)>\lambda\}\big|<\infty \tag{2.9}$$

and

$$\sup_{\lambda>0}\Phi_2(\lambda)\big|\{x\in\mathbb{R}^n: M_rM_bT_2f(x)>\lambda\}\big|<\infty. \tag{2.10}$$

Since the proofs of these two inequalities are similar, we only prove (2.9). Recall (see [1]) that

$$\big|\{x\in\mathbb{R}^n: |T_{1,b}^*g(x)|>\lambda\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|g(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|g(x)|}{\lambda}\Big)\,dx. \tag{2.11}$$

For a bounded function f with compact support and integral zero, it is well known that $T_2f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Thus by (2.2),

$$\begin{aligned}&\sup_{\lambda>0}\Phi_2(\lambda)\big|\{x\in\mathbb{R}^n: M_rT_{1,b}^*T_2f(x)>\lambda\}\big|\\ &\quad\lesssim \sup_{\lambda>0}\Phi_2(\lambda)\lambda^{-1}\sup_{t\ge 2^{-1/r}\lambda}t\big|\{x\in\mathbb{R}^n: T_{1,b}^*T_2f(x)>t\}\big|\\ &\quad\lesssim \sup_{\lambda>0}\Phi_2(\lambda)\lambda^{-1}\sup_{t\ge 2^{-1/r}\lambda}t\int_{\mathbb{R}^n}\frac{|T_2f(x)|}{t}\log\Big(\mathrm{e}+\frac{|T_2f(x)|}{t}\Big)\,dx\\ &\quad\lesssim \int_{\mathbb{R}^n}|T_2f(x)|\log\big(\mathrm{e}+|T_2f(x)|\big)\,dx<\infty.\end{aligned}$$

We can now conclude the proof of Lemma 2.5. For $0 < r < \sigma < 1$, it was proved in [1] that there exist two operators W_1 and W_2 such that

$$T^*_{1,b}g(x) \le W_1g(x) + W_2g(x),$$
$$W_1g(x) \lesssim T^*_{1,b}g(x) + M_bg(x), \qquad W_2g(x) \lesssim M_bg(x),$$

and for $\sigma \in (r, 1)$,

$$\begin{aligned} M_r^\sharp(W_1g)(x) &\lesssim M_\sigma T^* g(x) + M_{L\log L}g(x), \\ M_r^\sharp(W_2g)(x) &\lesssim M_{L\log L}g(x). \end{aligned} \tag{2.12}$$

Observe that by Lemma 2.4 and (2.2),

$$\left|\left\{x \in \mathbb{R}^n : M_\sigma T_1^* T_2 f(x) > \lambda\right\}\right| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\, dx.$$

It then follows from Lemma 2.3 and inequalities (2.9), (2.12), (2.5), and (2.6) that

$$\begin{aligned} \left|\left\{x \in \mathbb{R}^n : M_rW_1T_2f(x) > 1\right\}\right| &\lesssim \sup_{\lambda>0} \Phi_2(\lambda)\left|\left\{x \in \mathbb{R}^n : M_r^\sharp(W_1T_2f)(x) > \lambda\right\}\right| \\ &\lesssim \sup_{\lambda>0} \Phi_2(\lambda)\left|\left\{x \in \mathbb{R}^n : M_\sigma T_1^* T_2 f(x) > \lambda\right\}\right| \\ &\quad + \sup_{\lambda>0} \Phi_2(\lambda)\left|\left\{x \in \mathbb{R}^n : M_{L\log L} T_2 f(x) > \lambda\right\}\right| \\ &\lesssim \int_{\mathbb{R}^n} |f(x)| \log^2\big(\mathrm{e} + |f(x)|\big)\, dx. \end{aligned}$$

Similarly, from Lemma 2.3 and inequalities (2.10), (2.12), (2.5), and (2.6), we obtain that

$$\left|\left\{x \in \mathbb{R}^n : M_rW_2T_2f(x) > 1\right\}\right| \lesssim \int_{\mathbb{R}^n} |f(x)| \log^2\big(\mathrm{e} + |f(x)|\big)\, dx.$$

Therefore,

$$\left|\left\{x \in \mathbb{R}^n : M_rT^*_{1,b}T_2f(x) > 1\right\}\right| \lesssim \int_{\mathbb{R}^n} |f(x)| \log^2\big(\mathrm{e} + |f(x)|\big)\, dx.$$

This completes the proof of Lemma 2.5. □

We are now ready to establish our main conclusion in this section.

Proposition 2.6. *Let T_1, T_2 be Calderón–Zygmund operators, and let $b \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. There exist three operators U_1, U_2, and U_3 such that*

(i) *U_1 is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;*
(ii) *for any bounded function f with compact support and any $\lambda > 0$,*

$$\left|\left\{x \in \mathbb{R}^n : |U_2f(x)| > \lambda\right\}\right| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\, dx,$$

while for any bounded function f with compact support and integral zero, and $\lambda > 0$,

$$\big|\{x \in \mathbb{R}^n : |U_3 f(x)| > \lambda\}\big| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^2\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\, dx;$$

(iii) *for any $q \in (1, \infty)$, bounded function f with compact support and integral zero, and bounded function g with compact support,*

$$\mathscr{M}_{T_{1,b}T_2}(f, g)(x) \lesssim \big(\max\{q, q'\}\big)^2 U_1 f(x) M_q g(x) + U_2 f(x) M_{L\log L} g(x) + U_3 f(x) M g(x).$$

Proof. Let $x \in \mathbb{R}^n$, and let $Q \subset \mathbb{R}^n$ be a cube containing x. Write

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \big|T_{1,b}T_2(f\chi_{\mathbb{R}^n\setminus 9Q})(\xi)\big|\big|g(\xi)\big|\,d\xi \\
&= \frac{1}{|Q|}\int_Q \big|T\big((b - \langle b\rangle_Q)\chi_{\mathbb{R}^n\setminus 3Q} T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(\xi)\big|\big|g(\xi)\big|\,d\xi \\
&\quad + \frac{1}{|Q|}\int_Q \big|b(\xi) - \langle b\rangle_Q\big|\big|T_1\big(\chi_{\mathbb{R}^n\setminus 3Q} T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(\xi)\big|\big|g(\xi)\big|\,d\xi \\
&\quad + \frac{1}{|Q|}\int_Q \big|T_{1,b}\big(\chi_{3Q} T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(\xi)\big|\big|g(\xi)\big|\,d\xi \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For the term I, we deduce from (2.1) that

$$\begin{aligned}
\mathrm{I} &\lesssim Mg(x) \inf_{y\in Q} \mathcal{M}_{T_1}\big((b - \langle b\rangle_Q)T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(y) \\
&\lesssim \Big[\frac{1}{|Q|}\int_Q \big(T_1^*\big((b - \langle b\rangle_Q)T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(y)\big)^{\frac13}\,dy\Big]^3 Mg(x) \\
&\quad + \Big[\frac{1}{|Q|}\int_Q \big(M\big((b - \langle b\rangle_Q)T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(y)\big)^{\frac13}\,dy\Big]^3 Mg(x).
\end{aligned}$$

On the other hand, a straightforward computation leads to the fact that

$$\begin{aligned}
&\Big[\frac{1}{|Q|}\int_Q \big(T_1^*\big((b - \langle b\rangle_Q)T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(y)\big)^{\frac13}\,dy\Big]^3 \\
&\lesssim \Big(\frac{1}{|Q|}\int_Q \big|T_{1,b}^*T_2 f(z)\big|^{\frac13}\,dz\Big)^3 + \Big(\frac{1}{|Q|}\int_Q \big|b(z) - \langle b\rangle_Q\big|^{\frac13}\big|T_1^*T_2 f(z)\big|^{\frac13}\,dz\Big)^3 \\
&\quad + \Big(\frac{1}{|Q|}\int_Q \big|T_{1,b}^*T_2(f\chi_{9Q})(z)\big|^{\frac13}\,dz\Big)^3 \\
&\quad + \Big(\frac{1}{|Q|}\int_Q \big|b(z) - \langle b\rangle_Q\big|^{\frac13}\big|T_1^*T_2(f\chi_{9Q})(z)\big|^{\frac13}\,dz\Big)^3.
\end{aligned}$$

It now follows from Lemmas 2.5 and 2.2 that

$$\Big(\frac{1}{|Q|}\int_Q \big|T_{1,b}^*T_2(f\chi_{9Q})(z)\big|^{\frac13}\,dz\Big)^3 \lesssim \|f\|_{L(\log L)^2, 9Q} \lesssim M_{L(\log L)^2} f(x)$$

and

$$\Big(\frac{1}{|Q|}\int_Q\big|b(z)-\langle b\rangle_Q\big|^{\frac13}\big|T_1^*T_2(f\chi_{9Q})(z)\big|^{\frac13}\,dz\Big)^3\lesssim\Big(\frac{1}{|Q|}\int_Q\big|T_1^*T_2(f\chi_{9Q})(z)\big|^{\frac12}\,dz\Big)^2$$
$$\lesssim M_{L\log L}f(x).$$

Therefore,

$$\Big[\frac{1}{|Q|}\int_Q\big(T_1^*\big((b-\langle b\rangle_Q)T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(y)\big)^{\frac13}\,dy\Big]^3$$
$$\lesssim M_{\frac12}T_{1,b}^*T_2f(x)+M_{\frac12}T_1^*T_2f(x)+M_{L(\log L)^2}f(x).$$

Similarly, we can obtain that

$$\Big[\frac{1}{|Q|}\int_Q\big(M\big((b-\langle b\rangle_Q)T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(y)\big)^{\frac13}\,dy\Big]^3$$
$$\lesssim M_{\frac12}M_bT_2f(x)+M_{\frac12}MT_2f(x)+M_{L(\log L)^2}f(x).$$

Combining the last two estimates yields

$$\begin{aligned}\mathrm{I}\lesssim\big(&M_{\frac12}T_{1,b}^*T_2f(x)+M_{\frac12}T_1^*T_2f(x)+M_{\frac12}M_bT_2f(x)\\&+MT_2f(x)+M_{L(\log L)^2}f(x)\big)Mg(x),\end{aligned}$$

since $M_{\frac12}Mh(x)\approx Mh(x)$. The generalization of Hölder's inequality (see [24, p. 64]), along with (2.1) and Lemma 2.2, gives us that

$$\begin{aligned}\mathrm{II}&\lesssim\big(\inf_{y\in Q}\mathcal{M}_{T_1}(T_2f)(y)+\inf_{y\in Q}\mathcal{M}_{T_1}\big(T_2(f\chi_{9Q})\big)(y)\big)M_{L\log L}g(x)\\&\lesssim\Big[\inf_{y\in Q}\mathcal{M}_{T_1}(T_2f)(y)+\Big(\frac{1}{|Q|}\int_Q\big|\mathcal{M}_{T_1}\big(T_2(f\chi_{9Q})\big)(y)\big|^{\frac12}\,dy\Big)^2\Big]M_{L\log L}g(x)\\&\lesssim\big(T_1^*T_2f(x)+MT_2f(x)+M_{L\log L}f(x)\big)M_{L\log L}g(x).\end{aligned}$$

Finally, we consider the term III. Let $\hat{q}=(1+q)/2$. By Hölder's inequality and the fact that T_1 is bounded on $L^{q'}(\mathbb{R}^n)$ with bound $C\max\{q,q'\}$, we deduce that

$$\begin{aligned}\mathrm{III}&\lesssim\frac{1}{|Q|}\int_Q\big|b(\xi)-\langle b\rangle_Q\big|\big|g(\xi)\big|\big|T_1\big(\chi_{3Q}T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(\xi)\big|\,d\xi\\&\quad+\frac{1}{|Q|}\int_Q\big|g(\xi)\big|\big|T_1\big((b-\langle b\rangle_Q)\chi_{3Q}T_2(f\chi_{\mathbb{R}^n\setminus 9Q})\big)(\xi)\big|\,d\xi\\&\lesssim\max\{q,q'\}\Big(\frac{1}{|3Q|}\int_{3Q}\big|b(\xi)-\langle b\rangle_Q\big|^{\hat{q}}\big|g(\xi)\big|^{\hat{q}}\,d\xi\Big)^{\frac{1}{\hat{q}}}\mathcal{M}_{T_2}f(x)\\&\quad+\max\{q,q'\}\Big(\frac{1}{|3Q|}\int_{3Q}\big|b(\xi)-\langle b\rangle_Q\big|^{q'}\,d\xi\Big)^{\frac{1}{q'}}\mathcal{M}_{T_2}f(x)M_qg(x)\\&\lesssim\big(\max\{q,q'\}\big)^2\big(T_2^*f(x)+Mf(x)\big)M_qg(x),\end{aligned}$$

where in the last inequality we have invoked the fact (see [7, p. 128]) that

$$\Big(\frac{1}{|3Q|}\int_{3Q}\big|b(\xi)-\langle b\rangle_Q\big|^{q'(q+1)}\,d\xi\Big)^{\frac{1}{q'(q+1)}}\lesssim q'(q+1)\lesssim\max\{q,q'\}.$$

Let

$$U_1f(x)=T_2^*f(x)+Mf(x),$$
$$U_2f(x)=T_1^*T_2f(x)+MT_2f(x)+M_{L\log L}f(x),$$

and

$$U_3f(x)=M_{\frac12}T^*_{1,b}T_2f(x)+M_{\frac12}T_1^*T_2f(x)+M_{\frac12}M_bT_2f(x)+MT_2f(x)+M_{L(\log L)^2}f(x).$$

Our desired conclusion then follows from Lemmas 2.4 and 2.5. □

3. Proof of Theorem 1.1

Let $\eta\in(0,1)$, and let $\mathcal{S}=\{Q_j\}$ be a family of cubes. We say that $\mathcal{S}$ is *η-sparse* if for each fixed $Q\in\mathcal{S}$, there exists a measurable subset $E_Q\subset Q$ such that $|E_Q|\ge\eta|Q|$ and the E_Q's are pairwise disjoint. Associated with the sparse family $\mathcal{S}$ and constants $\beta_1,\beta_2\in[0,\infty)$, we define the bi-sublinear sparse operator $\mathcal{A}_{\mathcal{S};L(\log L)^{\beta_1},L(\log L)^{\beta_2}}$ by

$$\mathcal{A}_{\mathcal{S};L(\log L)^{\beta_1},L(\log L)^{\beta_2}}(f,g)=\sum_{Q\in\mathcal{S}}|Q|\|f\|_{L(\log L)^{\beta_1},Q}\|g\|_{L(\log L)^{\beta_2},Q},$$

and the operator $\mathcal{A}_{\mathcal{S},L,L^r}$ by

$$\mathcal{A}_{\mathcal{S};L,L^r}(f,g)=\sum_{Q\in\mathcal{S}}|Q|\big\langle|f|\big\rangle_Q\langle g\rangle_{r,Q}.$$

We will now denote $\mathcal{A}_{\mathcal{S};L(\log L)^1,L(\log L)^{\beta_2}}$ by $\mathcal{A}_{\mathcal{S};L\log L,L(\log L)^{\beta_2}}$, and we will also denote $\mathcal{A}_{\mathcal{S};L(\log L)^{\beta_1},L(\log L)^0}$ by $\mathcal{A}_{\mathcal{S};L(\log L)^{\beta_1},L}$.

Theorem 3.1. *Let T_1 and T_2 be Calderón–Zygmund operators, and let $b\in\mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. Then for bounded function f with compact support and integral zero and bounded function g with compact support, there exists a $\frac12\frac1{9^n}$-sparse family of cubes $\mathcal{S}=\{Q\}$ such that for any $q\in(1,\infty)$,*

$$\int_{\mathbb{R}^n}\big|T_{1,b}T_2f(x)\big|\big|g(x)\big|\,dx\lesssim\mathcal{A}_{\mathcal{S};L(\log L)^2,L}(f,g)+\mathcal{A}_{\mathcal{S};L\log L,L\log L}(f,g)+\big(\max\{q,q'\}\big)^2\mathcal{A}_{\mathcal{S};L,L^q}(f,g).$$

Proof. We will employ the argument in [17]. For a fixed cube Q_0, define the local analogue of $\mathscr{M}_{T_{1,b}T_2}$ by

$$\mathscr{M}_{T_{1,b}T_2,Q_0}(f,g)(x)=\sup_{x\in Q\subset Q_0}\frac{1}{|Q|}\int_Q\big|T_{1,b}T_2(f\chi_{9Q_0\setminus 9Q})(y)\big|\big|g(y)\big|\,dy.$$

For $q\in(1,\infty)$, functions f and g, set

$$W^1_{f,g}(Q)=\big(\max\{q,q'\}\big)^2\big\langle|f|\big\rangle_{9Q}\big\langle|g|\big\rangle_{q,Q},$$

and

$$W^2_{f,g}(Q)=\|f\|_{L\log L,9Q}\|g\|_{L\log L,Q},\qquad W^3_{f,g}(Q)=\|f\|_{L(\log L)^2,9Q}\big\langle|g|\big\rangle_Q.$$

We claim that for each cube $Q_0 \subset \mathbb{R}^n$, there exist pairwise disjoint cubes $\{P_j\} \subset \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \le \frac{1}{2}|Q_0|$, and for almost everywhere $x \in Q_0$,

$$\int_{Q_0} \big|T_{1,b}T_2(f\chi_{9Q_0})(x)\big|\big|g(x)\big|\,dx \lesssim |Q_0| \sum_{k=1}^{3} W_{f,g}^k(Q_0) + \sum_j \int_{P_j} \big|T_{1,b}T_2(f\chi_{9P_j})(x)\big|\big|g(x)\big|\,dx. \quad (3.1)$$

If we can prove (3.1), then as in the proof of Theorem 3.1 in [17], we can obtain our desired conclusion by iterating the estimate (3.1) and applying a decomposition of $\mathbb{R}^n$.

We now prove (3.1). Let $E = E_1 \cup E_2$ with

$$E_1 = \big\{x \in Q_0 : \big|T_{1,b}T_2(f\chi_{9Q_0})(x)\big| > D\|f\|_{L(\log L)^2, 9Q_0}\big\}$$

and

$$E_2 = \Big\{x \in Q_0 : \mathscr{M}_{T_{1,b}T_2,Q_0}(f,g)(x) > D\sum_{k=1}^{3} W_{f,g}^k(Q_0)\Big\},$$

with D a positive constant. Note that by Proposition 2.6,

$$\begin{aligned} E_2 \subset &\big\{x \in \mathbb{R}^n : \big(\max\{q,q'\}\big)^2 U_1(f\chi_{9Q_0})(x) M_{q'}(g\chi_{Q_0})(x) > DW_{f,g}^1(Q_0)\big\} \\ &\cup \big\{x \in \mathbb{R}^n : U_2(f\chi_{9Q_0})(x) M_{L\log L}(g\chi_{Q_0})(x) > DW_{f,g}^2(Q_0)\big\} \\ &\cup \big\{x \in \mathbb{R}^n : U_3(f\chi_{9Q_0})(x) M(g\chi_{Q_0})(x) > DW_{f,g}^3(Q_0)\big\}. \end{aligned}$$

It then follows that

$$|E_1 \cup E_2| \le \frac{1}{2^{n+2}}|Q_0|,$$

if we choose D large enough.

Now on the cube Q_0, we apply the Calderón–Zygmund decomposition to χ_E at level $\frac{1}{2^{n+1}}$, and obtain pairwise disjoint cubes $\{P_j\} \subset \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \le |P_j \cap E| \le \frac{1}{2}|P_j|$$

and $|E \backslash \bigcup_j P_j| = 0$. Observe that $\sum_j |P_j| \le \frac{1}{2}|Q_0|$ and $P_j \cap E^c \neq \emptyset$. Therefore,

$$\frac{1}{|P_j|}\int_{P_j} \big|T_{1,b}T_2(f\chi_{9Q_0\backslash 9P_j})(\xi)\big|\big|g(\xi)\big|\,d\xi \le \inf_{y \in P_j} \mathscr{M}_{T_{1,b}T_2}(f,g)(y) \le D\sum_{k=1}^{3} W_{f,g}^k(Q_0).$$

The fact that $|E \backslash \bigcup_j P_j| = 0$ implies that

$$\int_{Q_0 \backslash \bigcup_j P_j} \big|T_{1,b}T_2(f\chi_{9Q_0})(\xi)\big|\big|g(\xi)\big|\,d\xi \le D\|f\|_{L(\log L)^2, 9Q_0} \big\langle |g| \big\rangle_{Q_0} |Q_0|.$$

Note that

$$\begin{aligned}\int_{Q_0}\big|T_{1,b}T_2(f\chi_{9Q_0})(\xi)\big|\big|g(\xi)\big|\,d\xi \le &\int_{Q_0\setminus\bigcup_j P_j}\big|T_{1,b}T_2(f\chi_{9Q_0})(\xi)\big|\big|g(\xi)\big|\,d\xi\\&+\sum_j\int_{P_j}\big|T_{1,b}T_2(f\chi_{9Q_0\setminus 9P_j})(\xi)\big|\big|g(\xi)\big|\,d\xi\\&+\sum_j\int_{P_j}\big|T_{1,b}T_2(f\chi_{9P_j})(\xi)\big|\big|g(\xi)\big|\,d\xi.\end{aligned}$$

The inequality (3.1) now follows. This completes the proof of Theorem 3.1. □

To prove Theorem 1.1, we will also need the following lemma, which is Theorem 2.3 in [12] (see also [14]).

Lemma 3.2.

(a) *Let $w\in A_\infty(\mathbb{R}^n)$ and $\tau_w=2^{11+n}[w]_{A_\infty}$. Then for any cube $Q\subset\mathbb{R}^n$ and $\delta\in(1,1+1/\tau_w)$,*

$$\Big(\frac{1}{|Q|}\int_Q w^\delta(x)\,dx\Big)^{\frac{1}{\delta}}\le\frac{2}{|Q|}\int_Q w(x)\,dx.$$

(b) *If a weight w satisfies the reverse Hölder inequality that*

$$\Big(\frac{1}{|Q|}\int_Q w^r(x)\,dx\Big)^{\frac{1}{r}}\le C_0\langle w\rangle_Q$$

for some constant $C_0>0$ and $r\in(1,\infty)$, then $w\in A_\infty(\mathbb{R}^n)$ with $[w]_{A_\infty}\lesssim_n C_0r'$.

Proof of Theorem 1.1. Again, we assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. We will employ the ideas in the proof of Theorem 1.4 in [20]. Let $p\in(1,\infty)$ and $w\in A_p(\mathbb{R}^n)$. We choose ε_1 such that $\frac{p\varepsilon_1}{p-1-\varepsilon_1}=\frac{1}{2\tau_\sigma}$, that is, $\varepsilon_1=\frac{p-1}{2p\tau_\sigma+1}$, and we choose $\varepsilon_2\in(0,1)$ such that $\frac{\varepsilon_2p'}{p'-1-\varepsilon_2}=\frac{1}{2\tau_w}$, namely, $\varepsilon_2=\frac{p'-1}{2p'\tau_w+1}$. For a sparse family $\mathcal{S}$, we consider the bi-sublinear sparse operator $\mathcal{A}_{\mathcal{S},L^{1+\varepsilon_1},L^{1+\varepsilon_2}}$ by

$$\mathcal{A}_{\mathcal{S},L^{1+\varepsilon_1},L^{1+\varepsilon_2}}(f,g)=\sum_{Q\in\mathcal{S}}\big\langle|f|\big\rangle_{1+\varepsilon_1,Q}\big\langle|g|\big\rangle_{1+\varepsilon_2,Q}|Q|.$$

We get by [19, Theorem 1.6] that

$$\mathcal{A}_{\mathcal{S},L^{1+\varepsilon_1},L^{1+\varepsilon_2}}(f,g)\lesssim[v]_{A_{r_1}}^{\frac{1}{1+\varepsilon_2}-\frac{1}{p'}}\big([u]_{A_\infty}^{\frac{1}{p}}+[v]_{A_\infty}^{\frac{1}{p'}}\big)\|f\|_{L^p(\mathbb{R}^n,w)}\|g\|_{L^{p'}(\mathbb{R}^n,\sigma)},$$

with $r_1=(\frac{1+\varepsilon_2}{\varepsilon_2p})'(\frac{p}{1+\varepsilon_1}-1)+1$, $u=w^{\frac{1+\varepsilon_1}{1+\varepsilon_1-p}}$, $v=\sigma^{\frac{1+\varepsilon_2}{1+\varepsilon_2-p'}}$. Let $\varrho=\frac{2\tau_\sigma+2}{2\tau_\sigma+1}$. We then have that for any cube I,

$$\Big(\frac{1}{|I|}\int_I u^\varrho(x)\,dx\Big)^{\frac{1}{\varrho}}\lesssim\frac{1}{|I|}\int_I u(x)\,dx,$$

which via Lemma 3.2(b) shows that $[u]_{A_\infty} \lesssim [\sigma]_{A_\infty}$. Similarly, we can verify that $[v]_{A_\infty} \lesssim [w]_{A_\infty}$. Note that

$$1+\varepsilon_2-p' = \frac{p\varepsilon_2-1-\varepsilon_2}{p-1}.$$

For each cube $I \subset \mathbb{R}^n$, we can verify that

$$\begin{aligned}
&\Big(\frac{1}{|I|}\int_I \sigma^{\frac{1+\varepsilon_2}{1+\varepsilon_2-p'}}(x)\,dx\Big)\Big(\frac{1}{|I|}\int_I \sigma^{-\frac{1+\varepsilon_2}{1+\varepsilon_2-p'}\frac{1}{r_1-1}}(x)\,dx\Big)^{r_1-1}\\
&\quad\lesssim \Big(\frac{1}{|I|}\int_I w(x)\,dx\Big)^{1+\frac{1}{2\tau_w}}\Big(\frac{1}{|I|}\int_I \sigma^{\frac{\frac{p-1}{p}}{1+\varepsilon_1}-1}(x)\,dx\Big)^{r_1-1}\\
&\quad\lesssim \Big(\frac{1}{|I|}\int_I w(x)\,dx\Big)^{1+\frac{1}{2\tau_w}}\Big(\frac{1}{|I|}\int_I \sigma(x)\,dx\Big)^{(r_1-1)(1+\frac{1}{2\tau_\sigma})}.
\end{aligned}$$

Thus, $[v]_{A_{r_1}}^{\frac{1}{1+\varepsilon_2}-\frac{1}{p'}} \lesssim [w]_{A_p}^{\frac{1}{p}}$, and

$$\mathcal{A}_{\mathcal{S};L^{1+\varepsilon_1},L^{1+\varepsilon_2}}(f,g) \lesssim [w]_{A_p}^{1/p}\big([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'}\big)\|f\|_{L^p(\mathbb{R}^n,w)}\|g\|_{L^{p'}(\mathbb{R}^n,\sigma)}. \tag{3.2}$$

Recall that for any $\epsilon \in (0,1]$ and cube $I \subset \mathbb{R}^n$,

$$\|f\|_{L(\log L)^\beta,I} \lesssim \frac{1}{\epsilon^\beta}\big\langle|f|\big\rangle_{I,1+\epsilon}.$$

It follows from (3.2) that

$$\begin{aligned}
&\mathcal{A}_{\mathcal{S};L(\log L)^2,L}(f,g)\\
&\quad\lesssim \frac{1}{\varepsilon_1^2}\mathcal{A}_{\mathcal{S};L^{1+\varepsilon_1},L^{1+\varepsilon_2}}(f,g)\\
&\quad\lesssim [\sigma]_{A_\infty}^2[w]_{A_p}^{1/p}\big([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'}\big)\|f\|_{L^p(\mathbb{R}^n,w)}\|g\|_{L^{p'}(\mathbb{R}^n,\sigma)}.
\end{aligned}\tag{3.3}$$

Also, we get from (3.2) that

$$\begin{aligned}
&\mathcal{A}_{\mathcal{S};L\log L,L\log L}(f,g)\\
&\quad\lesssim \frac{1}{\varepsilon_1\varepsilon_2}\mathcal{A}_{\mathcal{S};L^{1+\varepsilon_1},L^{1+\varepsilon_2}}(f,g)\\
&\quad\lesssim [\sigma]_{A_\infty}[w]_{A_\infty}[w]_{A_p}^{1/p}\big([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'}\big)\|f\|_{L^p(\mathbb{R}^n,w)}\|g\|_{L^{p'}(\mathbb{R}^n,\sigma)}
\end{aligned}\tag{3.4}$$

and

$$\frac{1}{\varepsilon_2^2}\mathcal{A}_{\mathcal{S};L,L^{1+\varepsilon_2}}(f,g) \lesssim [w]_{A_\infty}^2[w]_{A_p}^{1/p}\big([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'}\big)\|f\|_{L^p(\mathbb{R}^n,w)}\|g\|_{L^{p'}(\mathbb{R}^n,\sigma)}. \tag{3.5}$$

The estimates (3.3)–(3.5), via Theorem 3.1, imply that for bounded function f with compact support and integral zero, and bounded function g with compact support,

$$\begin{aligned}
\int_{\mathbb{R}^n}\big|T_{1,b}T_2f(x)\big|\big|g(x)\big|\,dx \lesssim{}& [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}}+[\sigma]_{A_\infty}^{\frac{1}{p}}\big)\\
&\times\big([w]_{A_\infty}+[\sigma]_{A_\infty}\big)^2\|f\|_{L^p(\mathbb{R}^n,w)}\|g\|_{L^{p'}(\mathbb{R}^n,\sigma)}.
\end{aligned}$$

This gives our desired conclusion. □

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References

1. A. M. Alphonse, *An end point estimate for maximal commutators*, J. Fourier Anal. Appl. **6** (2000), no. 4, 449–456. Zbl 0951.42006. MR1777383. DOI 10.1007/BF02510149. 141, 142
2. C. Benea and F. Bernicot, *Conservation de certaines propriétés à travers un contrôle épars d'un opérateur et applications au projecteur de Leray-Hopf*, preprint, arXiv:1703.00228v1 [math.CA]. 136, 149
3. S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340**, no. 1 (1993), 253–272. Zbl 0795.42011. MR1124164. DOI 10.2307/2154555. 134
4. N. Carozza and A. Passarelli di Napoli, *Composition of maximal operators*, Publ. Mat. **40** (1996), no. 2, 397–409. Zbl 0865.42016. MR1425627. DOI 10.5565/PUBLMAT_40296_11. 138
5. D. Chung, M. C. Pereyra, and C. Pérez, *Sharp bounds for general commutators on weighted Lebesgue spaces*, Trans. Amer. Math. Soc. **364**, no. 3 (2012), 1163–1177. Zbl 1244.42006. MR2869172. DOI 10.1090/S0002-9947-2011-05534-0. 135
6. W. Damian, M. Hormozi, and K. Li, *New bounds for bilinear Calderón-Zygmund operators and applications*, Rev. Mat. Iberoam. **34** (2018), no. 3, 1177–1210. MR3850284. 135
7. L. Grafakos, *Modern Fourier Analysis*, 2nd ed., Grad. Texts in Math. **250**, Springer, New York, 2009. Zbl 1158.42001. MR2463316. DOI 10.1007/978-0-387-09434-2. 133, 138, 139, 144
8. G. Hu and D. Li, *A Cotlar type inequality for the multilinear singular integral operators and its applications*, J. Math. Anal. Appl. **290** (2004), no. 2, 639–653. Zbl 1045.42008. MR2033048. DOI 10.1016/j.jmaa.2003.10.037. 138
9. G. Hu and D. Yang, *Weighted estimates for singular integral operators with nonsmooth kernels and applications*, J. Aust. Math. Soc. **85** (2008), no. 3, 377–417. Zbl 1171.42004. MR2476448. DOI 10.1017/S1446788708000657. 137, 139
10. T. Hytönen, *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2) **175** (2012), no. 3, 1473–1506. Zbl 1250.42036. MR2912709. DOI 10.4007/annals.2012.175.3.9. 134
11. T. Hytönen and M. T. Lacey, *The $A_p - A_\infty$ inequality for general Calderón-Zygmund operators*, Indiana Univ. Math. J. **61** (2012), no. 6, 2041–2092. Zbl 1290.42037. MR3129101. DOI 10.1512/iumj.2012.61.4777. 134
12. T. Hytönen and C. Pérez, *Sharp weighted bounds involving A_∞*, Anal. PDE. **6** (2013), no. 4, 777–818. Zbl 1283.42032. MR3092729. DOI 10.2140/apde.2013.6.777. 134, 147
13. T. Hytönen and C. Pérez, *The $L(\log L)^\epsilon$ endpoint estimate for maximal singular integral operators*, J. Math. Anal. Appl. **428** (2015), no. 1, 605–626. Zbl 1323.42021. MR3327006. DOI 10.1016/j.jmaa.2015.03.017. 134
14. T. Hytönen, C. Pérez, and E. Rela, *Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type*, J. Funct. Anal. **263** (2012), no. 12, 3883–3899. Zbl 1266.42045. MR2990061. DOI 10.1016/j.jfa.2012.09.013. 147
15. S. G. Krantz and S. Li, *Boundedness and compactness of integral operators on spaces of homogeneous type and applications, I*, J. Math. Anal. Appl. **258** (2001), no. 2, 629–641. Zbl 0990.47042. MR1835563. DOI 10.1006/jmaa.2000.7402. 135
16. A. K. Lerner, *On pointwise estimates involving sparse operators*, New York J. Math. **22** (2016), 341–349. Zbl 1347.42030. MR3484688. 134, 137

17. A. K. Lerner, *A weak type estimate for rough singular integrals*, preprint, arXiv:1705.07397v1 [math.CA]. 134, 136, 137, 145, 146
18. A. K. Lerner, S. Obmrosi, and I. P. Rivera-Ríos, *On pointwise and weighted estimates for commutators of Calderón-Zygmund operators*, Adv. Math. **319** (2017), 153–181. Zbl 1379.42007. MR3695871. DOI 10.1016/j.aim.2017.08.022. 134
19. K. Li, *Two weight inequalities for bilinear forms*, Collect. Math. **68** (2017), no. 1, 129–144. Zbl 1365.42012. MR3591468. DOI 10.1007/s13348-016-0182-2. 147
20. K. Li, C. Pérez, I. P. Rivera-Ríos, and L. Roncal, *Weighted norm inequalities for rough singular integral operators*, preprint, arXiv:1701.05170v3 [math.CA]. 134, 147
21. C. Pérez, *Weighted norm inequalities for singular integral operators*, J. Lond. Math. Soc. (2) **49** (1994), no. 2, 296–308. Zbl 0797.42010. MR1260114. DOI 10.1112/jlms/49.2.296. 140
22. S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math. **129** (2007), no. 5, 1355–1375. Zbl 1139.44002. MR2354322. DOI 10.1353/ajm.2007.0036. 134
23. S. Petermichl, *The sharp weighted bound for the Riesz transforms*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1237–1249. Zbl 1142.42005. MR2367098. DOI 10.1090/S0002-9939-07-08934-4. 134
24. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monogr. Textb. Pure Appl. Math. **146**, Marcel Dekker, New York, 1991. Zbl 0724.46032. MR1113700. 144
25. M. J. Wilson, *Weighted inequalities for the dyadic square function without dyadic A_∞*, Duke Math. J. **55** (1987), no. 1, 19–50. Zbl 0639.42016. MR0883661. DOI 10.1215/S0012-7094-87-05502-5. 134

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