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ON THE UNIT SPHERE OF POSITIVE OPERATORS

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Abstract. Given a C^*-algebra A, let $S(A^+)$ denote the set of positive elements in the unit sphere of A. Let H_1, H_2, H_3, and H_4 be complex Hilbert spaces, where H_3 and H_4 are infinite-dimensional and separable. In this article, we prove a variant of Tingley's problem by showing that every surjective isometry $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ (resp., $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$) admits a unique extension to a surjective complex linear isometry from $B(H_1)$ onto $B(H_2)$ (resp., from $K(H_3)$ onto $K(H_4)$). This provides a positive answer to a conjecture recently posed by Nagy.

1. Introduction

During the last thirty years, mathematicians have pursued an argument to prove or discard a positive solution to Tingley's problem (see the survey [23]). This problem, in which geometry and functional analysis interplay, is just as attractive as it is difficult. The concrete statement of the problem reads as follows. Let $S(X)$ and $S(Y)$ be the unit spheres of two normed spaces X and Y, respectively. Suppose that $\Delta : S(X) \to S(Y)$ is a surjective isometry. Does Δ admit an extension to a surjective real linear isometry from X onto Y?

A wide list of references, obtained during the last thirty years, encompasses positive solutions to Tingley's problem in the cases of sequence spaces (see [4]–[6]), spaces of measurable functions on a σ-finite measure space (see [26]–[28]), spaces of continuous functions (see [32]), finite-dimensional C^*-algebras (see [30], [31]),

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$K(H)$ spaces (see [24]), spaces of trace class operators (see [7]), and $B(H)$ spaces (see [12], [9], [8]). The most recent achievements in this line establish that a surjective isometry between the unit spheres of two arbitrary von Neumann algebras admits a unique extension to a surjective real linear isometry between the corresponding von Neumann algebras (see [10]), and an excellent contribution due to Mori [18] contains a complete positive solution to Tingley's problem for surjective isometries between the unit spheres of von Neumann algebra preduals. Readers interested in learning more details can consult our recent survey [23].

The particular setting of C^*-algebras, and especially the von Neumann algebra $B(H)$, of all bounded linear operators on a complex Hilbert space H, and its Hermitian subalgebras and subspaces, offer the optimal conditions in which to consider an interesting variant to Tingley's problem. Let us introduce some notation first. If B is a subset of a Banach space X, then we will write $S(B)$ for the intersection of B and $S(X)$. Given a C^*-algebra A, the symbol A^+ will denote the cone of positive elements in A, while $S(A^+)$ will stand for the sphere of positive norm 1 operators.

Problem 1.1. Let $\Delta : S(A^+) \to S(B^+)$ be a surjective isometry, where A and B are C^*-algebras. Does Δ admit an extension to a surjective complex linear isometry $T : A \to B$?

The hypotheses in Problem 1.1 are certainly weaker than the hypothesis in Tingley's problem. However, the required conclusion is also weaker, because the goal is to find a surjective linear isometry $T : A \to B$ satisfying $T|_{S(A^+)} \equiv \Delta$, and we do not care about the behavior of T on the rest of $S(A)$. For the moment, both problems seem to be independent.

Problem 1.1 can also be considered when A and B are replaced with the space $(C_p(H), \|\cdot\|_p)$ of all p-Schatten–von Neumann operators $(1 \leq p \leq \infty)$. For a finite-dimensional complex Hilbert space H and $\infty > p \geq 1$, Molnár and Nagy [16, Theorem 1] determined all surjective isometries on the space $(S(C_1(H)^+), \|\cdot\|_p)$. Molnár and Timmermann [17, Theorem 4] solved Problem 1.1 for the space $C_1(H)$ of trace class operators on an arbitrary complex Hilbert space H. Given p in the interval $(1, \infty)$ and $A = B = C_p(H)$, a complete solution to Problem 1.1 was obtained by Nagy in [19, Theorem 1].

Following the usual notation, for each complex Hilbert space H, we identify $C_\infty(H)$ with the space $B(H)$. In a very recent contribution, Nagy resumes the study of Problem 1.1 for $B(H)$. Applying deep geometric arguments in spectral theory and projective geometry, Nagy solves this problem in the case in which H is finite-dimensional. Concretely, if H is a finite-dimensional complex Hilbert space, and $\Delta : S(B(H)^+) \to S(B(H)^+)$ is an isometry, then Δ is surjective and there exists a surjective complex linear isometry $T : B(H) \to B(H)$ satisfying $T(x) = \Delta(x)$ for all $x \in S(B(H)^+)$ (see [20, Theorem]). In the third section of [20], Nagy conjectures that an infinite-dimensional version of his result holds true for surjective isometries on $S(B(H)^+)$.

In this article, we present an argument to prove Nagy's conjecture. Concretely, in Theorem 3.6 we prove that for any two complex Hilbert spaces H_1 and H_2, every surjective isometry $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ can be extended to a surjective complex linear isometry (actually, a *-isomorphism or a *-antiautomorphism) $T : B(H_1) \to B(H_2)$.

A closer look at the technical arguments in recent papers dealing with Tingley's problem (see, e.g., [8]–[10], [24], [30], [31]) reveals a common strategy based on a geometric tool asserting that a surjective isometry between the unit spheres of two Banach spaces X and Y preserves maximal convex sets of the corresponding spheres (see [3, Lemma 5.1(ii)], [29, Lemma 3.5]). This is a real obstacle in our setting, because this geometric tool is not applicable for a surjective isometry $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ where we can hardly identify a surjective isometry between the unit spheres of two normed spaces. We will develop independent arguments to prove Nagy's conjecture. In this article, we introduce new arguments built upon a recent abstract characterization of those elements in $S(B(H)^+)$ which are projections in terms of their distances to positive elements in $S(B(H)^+)$ (see [22]), and the Bunce–Wright Mackey–Gleason theorem (see [2, Theorem A]).

In Section 4, we also give a positive solution to Problem 1.1 in the case in which A and B are spaces of compact operators on separable complex Hilbert spaces (see Theorem 4.5). In this final section, the role played by the Bunce–Wright Mackey–Gleason theorem will be played by a theorem due to Aarnes [1, Corollary 2] which guarantees the linearity of quasistates on $K(H)$.

2. Basic background and precedents

In our recent note [22], we establish a geometric characterization of those elements in the unit sphere of an atomic von Neumann algebra M (or in the unit sphere of the space of compact operators on a separable complex Hilbert space) which are projections in terms of the unit sphere of positive operators around an element. Let us recall the basic definitions. Let E and P be subsets of a Banach space X. We define the *unit sphere around E in P* as the set

$$\mathrm{Sph}(E; P) := \big\{x \in P : \|x - b\| = 1 \text{ for all } b \in E\big\}.$$

If x is an element in X, we write $\mathrm{Sph}(x; P)$ for $\mathrm{Sph}(\{x\}; P)$. If E is a subset of a C^*-algebra A, we write $\mathrm{Sph}^+(E)$ or $\mathrm{Sph}^+_A(E)$ for the set $\mathrm{Sph}(E; S(A^+))$. For each element a in A, we write $\mathrm{Sph}^+(a)$ instead of $\mathrm{Sph}^+(\{a\})$.

We recall that a nonzero projection p in a C^*-algebra A is called *minimal* if $pAp = \mathbb{C}p$. A von Neumann algebra M is called *atomic* if it coincides with the weak*-closure of the linear span of its minimal projections. It is known that for every atomic von Neumann algebra M there exists a family $\{H_i\}_i$ of complex Hilbert spaces such that $M = \bigoplus_j^{\ell_\infty} B(H_j)$ (cf. [25, Section 2.2]). Every projection p in an atomic von Neumann algebra M is the least upper bound of the set of all minimal projections in M which are less than or equal to p.

Let a be a positive norm 1 element in an atomic von Neumann algebra M. In [22, Theorem 2.3] we prove that

$$a \text{ is a projection} \Leftrightarrow \mathrm{Sph}^+_M\big(\mathrm{Sph}^+_M(a)\big) = \{a\}.$$

This particularly holds true when $M = B(H)$. Theorem 2.5 in [22] assures that the same equivalence remains true for any positive element a in the unit sphere of $K(H_2)$, where H_2 is a separable complex Hilbert space. Since, for every $E \subseteq S(A^+)$, the set $\operatorname{Sph}^+_A(E)$ is completely determined by the metric structure of $S(A^+)$, the next results borrowed from [22] are direct consequences of the characterizations just commented. We recall first that, for a C^*-algebra A, the symbol $\mathcal{P}roj(A)$ will denote the set of all projections in A, and $\mathcal{P}roj(A)^*$ will stand for $\mathcal{P}roj(A)\backslash\{0\}$.

Corollary 2.1 ([22, Corollary 2.6]). *Let $\Delta : S(M^+) \to S(N^+)$ be a surjective isometry, where M and N are atomic von Neumann algebras. Then Δ maps $\mathcal{P}roj(M)^*$ to $\mathcal{P}roj(N)^*$, and the restriction $\Delta|_{\mathcal{P}roj(M)^*} : \mathcal{P}roj(M)^* \to \mathcal{P}roj(N)^*$ is a surjective isometry.*

Corollary 2.2 ([22, Corollary 2.7]). *Let H_2 and H_3 be separable complex Hilbert spaces, and let us assume that $\Delta : S(K(H_2)^+) \to S(K(H_3)^+)$ is a surjective isometry. Then Δ maps $\mathcal{P}roj(K(H_2))^*$ to $\mathcal{P}roj(K(H_3))^*$, and the restriction*

$$\Delta|_{\mathcal{P}roj(K(H_2))^*} : \mathcal{P}roj\big(K(H_2)\big)^* \to \mathcal{P}roj\big(K(H_3)\big)^*$$

is a surjective isometry.

Throughout this article, the closed unit ball and the dual space of a Banach space X will be denoted by $\mathcal{B}_X$ and X^*, respectively. The symbol X^{**} will stand for the second dual space of X. Given a subset $B \subset X$, we will write $\mathcal{B}_B$ for $\mathcal{B}_X \cap B$. We will write A_{sa} for the self-adjoint part of a C^*-algebra A, while the symbol $(A^*)^+$ will stand for the set of positive functionals on A. If A is unital, $\mathbf{1}$ will stand for its unit.

Suppose that a is a positive element in the unit sphere of a von Neumann algebra M. The *range projection* of a in M (denoted by $r(a)$) is the smallest projection p in M satisfying $ap = a$. It is known that the sequence $((1/n\mathbf{1} + a)^{-1}a)_n$ tends monotone-increasingly to $r(a)$, and hence it converges to $r(a)$ in the weak*-topology of M. Actually, $r(a)$ also coincides with the weak*-limit of the sequence $(a^{1/n})_n$ in M (see [21, 2.2.7, p. 23]). It is also known that the sequence $(a^n)_n$ converges to a projection $s(a) = s_M(a)$ in M, which is called the *support projection* of a in M. Let us observe that the support projection of a norm 1 element in M might be zero; however, for each positive element a in the unit sphere of the bidual space of a C^*-algebra A, we have $s_{A^{**}}(a) \neq 0$ (cf. [22, (2.3)]).

We recall next some known properties in C^*-algebra theory. Let p be a projection in a unital C^*-algebra A. Suppose that $x \in S(A)$ satisfies $pxp = p$. Then (see, e.g., [11, Lemma 3.1])

$$x = p + (\mathbf{1} - p)x(\mathbf{1} - p). \tag{2.1}$$

Suppose that $b \in A^+$ satisfies $pbp = 0$. Then (see [22, (2.2)])

$$pb = bp = 0. \tag{2.2}$$

If p is a nonzero projection in a C^*-algebra A, and a is an element in $S(A^+)$ satisfying $p \leq a$, then (see [22, (2.4)])

$$a = p + (\mathbf{1} - p)a(\mathbf{1} - p). \tag{2.3}$$

3. Surjective isometries between normalized positive elements of type I von Neumann factors

Throughout this section, H_1 and H_2 will be two complex Hilbert spaces. The main goal here is to determine when a surjective isometry $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ can be extended to a surjective complex linear isometry from $B(H_1)$ onto $B(H_2)$. The case in which $H_1 = H_2$ with $\dim(H_1) < \infty$ has been positively solved by Nagy [20]. In the just quoted reference, Nagy conjectures that the same statement holds true when H is infinite-dimensional. Corollary 2.1 above gives a generalization of [20, Claim 1] for arbitrary complex Hilbert spaces. Our next aim is to provide a proof of the whole conjecture posed by Nagy.

We recall next a tool that will be used throughout the rest of the article. Henceforth, let the symbol ℓ_2^n stand for an n-dimensional complex Hilbert space. If p is a rank 1 projection in $B(\ell_2^2)$, up to an appropriate representation, then we can assume that $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Given $t \in [0,1]$, the element $q_t = \left(\begin{smallmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{smallmatrix}\right)$ also is a projection in $B(\ell_2^2)$ and $\|p - q_t\| = \sqrt{1-t}$. Therefore, for each nontrivial projection p in $B(\ell_2^2)$ we can find another nontrivial projection q in $B(\ell_2^2)$ with $0 < \|p - q\| < 1$. Similar arguments show that if H is a complex Hilbert space with $\dim(H) \geq 2$, then, for each nontrivial projection p in $B(H)$, we can find another nontrivial projection q in $B(H)$ with $0 < \|p - q\| < 1$.

Let A and B be C^*-algebras. A linear map $\Phi : A \to B$ is called a *Jordan* **-homomorphism* if $\Phi(a^*) = \Phi(a)^*$ and $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$ for all $a, b \in A$, where $a \circ b$ denotes the natural Jordan product of a and b given by $a \circ b = \frac{1}{2}(ab + ba)$.

Elements a, b in a C^*-algebra A are called *orthogonal* (written $a \perp b$) if $ab^* = b^*a = 0$. It is known that $\|a + b\| = \max\{\|a\|, \|b\|\}$, for every $a, b \in A$ with $a \perp b$. Clearly, self-adjoint elements a, b in A are orthogonal if and only if $ab = 0$.

The following technical result will be needed for later purposes.

Lemma 3.1. *Suppose that $\Delta : \mathcal{P}roj(B(H_1)) \to \mathcal{P}roj(B(H_2))$ is a (unital) isometric order automorphism, where H_1 and H_2 are complex Hilbert spaces. Then Δ preserves orthogonality, that is, $\Delta(p)\Delta(q) = 0$ whenever $pq = 0$ in $\mathcal{P}roj(M)$. Furthermore, the same conclusion holds for an isometric order automorphism $\Delta : \mathcal{P}roj(K(H_1)) \to \mathcal{P}roj(K(H_2))$.*

Proof. Let e_1 and v_1 be orthogonal minimal projections in $B(H_1)$. By hypothesis, $\Delta(e_1)$ and $\Delta(v_1)$ are minimal projections, and $\Delta(e_1 + v_1)$ is a projection with $\Delta(e_1 + v_1) \geq \Delta(e_1), \Delta(v_1)$. Since $\|\Delta(e_1) - \Delta(v_1)\| = \|e_1 - v_1\| = 1$, [22, Lemma 2.1] assures the existence of a minimal projection $\widehat{e} \in B(H_2)^{**}$ such that one of the following statements holds:

(a) $\widehat{e} \leq \Delta(e_1)$ and $\widehat{e} \perp \Delta(v_1)$ in $B(H_2)^{**}$;
(b) $\widehat{e} \leq \Delta(v_1)$ and $\widehat{e} \perp \Delta(e_1)$ in $B(H_2)^{**}$.

Having in mind that $\Delta(e_1)$ and $\Delta(v_1)$ are minimal projections in $B(H_2)^{**}$, the above statements are equivalent to

(a) $\widehat{e} = \Delta(e_1)$ and $\widehat{e} \perp \Delta(v_1)$ in $B(H_2)^{**}$, and hence $\Delta(e_1) \perp \Delta(v_1)$;
(b) $\widehat{e} = \Delta(v_1)$ and $\widehat{e} \perp \Delta(e_1)$ in $B(H_2)^{**}$, and hence $\Delta(e_1) \perp \Delta(v_1)$.

Now let us take two arbitrary projections $p, q \in B(H_1)$ with $pq = 0$. We pick two arbitrary minimal projections $\widehat{e}_1 \leq \Delta(p)$ and $\widehat{v}_1 \leq \Delta(q)$. By hypothesis, there exist minimal projections e_1, v_1 in $B(H_1)$ satisfying $\Delta(e_1) = \widehat{e}_1$, $\Delta(v_1) = \widehat{v}_1$, $e_1 \leq p$, and $v_1 \leq q$. The condition $pq = 0$ implies that $e_1 v_1 = 0$. Applying the conclusion in the first paragraph, we deduce that $\Delta(e_1) = \widehat{e}_1 \perp \Delta(v_1) = \widehat{v}_1$. We have therefore proved that $\widehat{e}_1 \perp \widehat{v}_1$ whenever $\widehat{e}_1$ and $\widehat{v}_1$ are minimal projections with $\widehat{e}_1 \leq \Delta(p)$ and $\widehat{v}_1 \leq \Delta(q)$. Since in $B(H_2)$ the projection $\Delta(p)$ (resp., $\Delta(q)$) is the least upper bound of all minimal projections in $B(H_2)$ which are less than or equal to $\Delta(p)$ (resp., $\Delta(q)$), it follows that $\Delta(p) \perp \Delta(q)$.

If $\Delta : \mathcal{P}roj(K(H_1)) \to \mathcal{P}roj(K(H_2))$ is an isometric order automorphism, then the conclusion follows with similar arguments. □

In 1951, Kadison [14, Theorem 7] proved that a surjective linear isometry T from a unital C^*-algebra A onto another C^*-algebra B is of the form $T = u\Phi$, where u is a unitary element in B and Φ is a Jordan *-isomorphism from A onto B. In particular, every unital surjective linear isometry $T : A \to B$ is a Jordan *-isomorphism. Furthermore, if A is a factor von Neumann algebra, then T is a *-isomorphism or a *-anti-isomorphism. In our next result, we begin with weaker hypotheses.

Proposition 3.2. *Let $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ be a surjective isometry, where H_1 and H_2 are complex Hilbert spaces. Then Δ maps $\mathcal{P}roj(B(H_1))^*$ to $\mathcal{P}roj(B(H_2))^*$, and the restriction $\Delta|_{\mathcal{P}roj(B(H_1))^*} : \mathcal{P}roj(B(H_1))^* \to \mathcal{P}roj(B(H_2))^*$ is a surjective isometry and a unital order isomorphism. We further know that $\Delta|_{\mathcal{P}roj(B(H_1))^*}$ preserves orthogonality.*

Consequently, if $T : B(H_1) \to B(H_2)$ is a bounded complex linear mapping such that $T(S(B(H_1)^+)) = S(B(H_2)^+)$ and $T|_{S(B(H_1)^+)} : S(B(H_1)^+) \to S(B(H_2)^+)$ is an isometry, then T is a *-isomorphism or a *-antiautomorphism.

Proof. Most of the first statement is given by Corollary 2.1. Following an idea outlined by Nagy in [20, Proof of Claim 2], we will begin by proving that Δ is unital. By Corollary 2.1, $\Delta(\mathbf{1})$ is a nonzero projection. We recall that $\mathbf{1}$ is the unique nonzero projection in $B(H_2)$ whose distance to any other projection is 0 or 1. If $\Delta(\mathbf{1}) = q_0 \neq \mathbf{1}$, then there exists a nonzero projection $q_1 \in B(H_2)$ such that $0 < \|q_1 - q_0\| = \|\Delta(\mathbf{1}) - q_1\| < 1$. A new application of Corollary 2.1 to Δ^{-1} implies the existence of a nonzero projection $p_1 \in B(H_1)$ such that $\Delta(p_1) = q_1$. In this case, we have $p_1 \neq \mathbf{1}$ and $1 = \|\mathbf{1} - p_1\| = \|\Delta(\mathbf{1}) - \Delta(p_1)\| = \|q_0 - q_1\| < 1$, which is a contradiction.

Let us prove next that $\Delta|_{\mathcal{P}roj(B(H_1))^*}$ is an order automorphism. To this aim, let us pick $p, q \in \mathcal{P}roj(B(H_1))^*$ with $p \leq q$. Let v be a minimal projection in $B(H_2)$ such that $v \leq \mathbf{1} - \Delta(q) = \Delta(\mathbf{1}) - \Delta(q)$. The element $z = v + \frac{1}{2}(\mathbf{1} - v)$ lies in $S(B(H_2)^+)$. Pick $x \in S(B(H_1)^+)$ satisfying $\Delta(x) = z$. Since

$$\frac{1}{2} = \|z - \mathbf{1}\| = \big\|\Delta(x) - \Delta(\mathbf{1})\big\| = \|x - \mathbf{1}\|,$$

we deduce that x is invertible. Furthermore, since

$$1 \geq \|x - q\| = \big\|\Delta(x) - \Delta(q)\big\| = \big\|z - \Delta(q)\big\| \geq \big\|v\big(z - \Delta(q)\big)v\big\| = \|v\| = 1,$$

by Lemma 2.1 in [22] there exists a minimal projection e in $B(H_1)^{**}$ such that one of the following statements holds:

(a) $e \leq x$ and $e \perp q$ in $B(H_1)^{**}$;
(b) $e \leq q$ and $e \perp x$ in $B(H_1)^{**}$.

Case (b) is impossible because x is invertible in $B(H_1)$ (and hence in $B(H_1)^{**}$). Therefore $e \leq x$ and $e \perp q$, which implies that $e \perp p$, because $p \leq q$. Therefore, [22, Lemma 2.1] implies that $1 = \|x - p\| = \|\Delta(x) - \Delta(p)\| = \|z - \Delta(p)\|$. A new application of [22, Lemma 2.1] assures the existence of a minimal projection w in $B(H_2)^{**}$ such that one of the following statements holds:

(a) $w \leq z$ and $w \perp \Delta(p)$ in $B(H_2)^{**}$;
(b) $w \leq \Delta(p)$ and $w \perp z$ in $B(H_2)^{**}$.

As before, case (b) is impossible because z is invertible in $B(H_2)$. Therefore, $w \leq z = v + \frac{1}{2}(\mathbf{1} - v)$ and $w \perp \Delta(p)$. It can be easily deduced from the minimality of w in $B(H_2)^{**}$ and the minimality of v in $B(H_2)$ that $v = w \perp \Delta(p)$. We have therefore shown that $\Delta(p)$ is orthogonal to every minimal projection v in $B(H_2)$ with $v \leq \mathbf{1} - \Delta(q)$, and consequently $\mathbf{1} - \Delta(q) \leq \mathbf{1} - \Delta(p)$, or equivalently, $\Delta(p) \leq \Delta(q)$. The statement affirming that $\Delta|_{\mathcal{P}roj(B(H_1))^*}$ preserves orthogonality can be derived from Lemma 3.1.

To prove the final statement, let $T : B(H_1) \to B(H_2)$ be a linear mapping such that $T(S(B(H_1)^+)) = S(B(H_2)^+)$ and $T|_{S(B(H_1)^+)} : S(B(H_1)^+) \to S(B(H_2)^+)$ is an isometry. By applying the conclusion of the first statement, we deduce that $T|_{S(B(H_1)^+)}$ maps $\mathcal{P}roj(B(H_1))^*$ to $\mathcal{P}roj(B(H_2))^*$, and the restricted mapping $T|_{\mathcal{P}roj(B(H_1))^*} : \mathcal{P}roj(B(H_1))^* \to \mathcal{P}roj(B(H_2))^*$ is a surjective isometry and a unital order automorphism. Clearly, T preserves projections and orthogonality among them (just observe that the sum of two projections is a projection if and only if they are orthogonal). Since every Hermitian element in a von Neumann algebra can be approximated in norm by a finite real linear combination of mutually orthogonal projections (see [25, Proposition 1.3.1]), and by the above properties $T(a^2) = T(a)^2$ and $T(a) = T(a)^*$, whenever a is a finite real linear combination of mutually orthogonal projections, we deduce that $T(b^2) = T(b)^2$ and $T(b)^* = T(b)$ for every Hermitian element b in $B(H_1)$. It is well known that this is equivalent to saying that T is a Jordan *-isomorphism. The rest follows from [14, Corollary 11] because $B(H_1)$ is a factor. $\square$

We continue with an analogue of [20, Claim 3].

Lemma 3.3. *Let $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ be a surjective isometry, where H_1 and H_2 are complex Hilbert spaces. Let $p_0, p_1, \ldots, p_m$ be mutually orthogonal projections with $\sum_{k=0}^{m} p_k = \mathbf{1}$, and let $\lambda_1, \ldots, \lambda_m$ be real numbers in the interval $(0,1)$. Then $s_{B(H_2)}(\Delta(p_0 + \sum_{k=1}^{m} \lambda_k p_k)) = \Delta(p_0)$.*

Proof. Set $a = p_0 + \sum_{k=1}^{m} \lambda_k p_k$. Since $\Delta(\mathbf{1}) = \mathbf{1}$ and $\|\Delta(a) - \mathbf{1}\| = \|\Delta(a) - \Delta(\mathbf{1})\| = \|a - \mathbf{1}\| = \max\{1 - \lambda_k : k = 1, \ldots, m\} < 1$, we deduce that both a and $\Delta(a)$ are invertible elements.

Let $\widehat{v}$ be a minimal projection in $B(H_2)$. By Proposition 3.2, there exists a minimal projection v in $B(H_1)$ satisfying $\Delta(v) = \widehat{v}$. By the hypothesis on Δ and Proposition 3.2, we have $\|a - (\mathbf{1} - v)\| = 1$ if and only if $\|\Delta(a) - \Delta(\mathbf{1} - v)\| =$

$\|\Delta(a) - (\mathbf{1} - \Delta(v))\| = 1$. Combining the invertibility of a and $\Delta(a)$, and the minimality of v and $\Delta(v)$ with Lemma 2.1 in [22], we deduce that

$$v \leq p_0 \Leftrightarrow v \leq a \Leftrightarrow \big\|a-(\mathbf{1}-v)\big\| = 1 \Leftrightarrow \big\|\Delta(a)-\big(\mathbf{1}-\Delta(v)\big)\big\| = 1 \Leftrightarrow \Delta(v) \leq \Delta(a).$$

Therefore, a minimal projection v satisfies $v \leq p_0$ if and only if $v \leq a$ if and only if $\Delta(v) \leq \Delta(a)$ if and only if $\Delta(v) \leq \Delta(p_0)$.

Take a minimal projection $\widehat{v} \in B(H_2)$ such that $\widehat{v} = \Delta(v) \leq \Delta(p_0)$. We know from the above that $\widehat{v} \leq \Delta(a)$, and $v \leq a$. Since in $B(H_2)$ every projection q is the least upper bound of all minimal projections $\widehat{v}$ with $\widehat{v} \leq q$, we deduce that $\Delta(p_0) \leq \Delta(a)$, and hence $\Delta(p_0) \leq s_{B(H_2)}(\Delta(a))$. Another application of the above property shows that $\widehat{v} \leq \Delta(p_0)$ for every minimal projection $\widehat{v} \in B(H_2)$ with $\widehat{v} \leq s_{B(H_2)}(\Delta(a)) \leq \Delta(a)$. Therefore $s_{B(H_2)}(\Delta(a)) = \Delta(p_0)$. □

According to the usual notation, given a C^*-algebra A, the symbol $S(\mathrm{Inv}(A)^+)$ will denote the set of all positive invertible elements in $S(A)$. A projection p in a unital C^*-algebra A will be called *cominimal* if $1 - p$ is a minimal projection in A. The symbol comin-$\mathcal{P}roj(A)$ will stand for the set of all cominimal projections in A.

Theorem 3.4. *Let a be an invertible element in $S(B(H)^+)$, where H is an infinite-dimensional complex Hilbert space. Suppose that $s_{B(H)}(a) \neq 0$. Then the following statements hold:*

(a) $\mathrm{Sph}(a; \text{comin-}\mathcal{P}roj(B(H))) = \{p \in \text{comin-}\mathcal{P}roj(B(H)) : \mathbf{1}-p \leq s_{B(H)}(a)\}$;
(b) *the identity*

$$\mathrm{Sph}\big(\mathrm{Sph}\big(a; \text{comin-}\mathcal{P}roj\big(B(H)\big)\big); S\big(\mathrm{Inv}\big(B(H)\big)^+\big)\big) = \big\{x \in S\big(\mathrm{Inv}\big(B(H)\big)^+\big) : s_{B(H)}(a) \leq x\big\}$$

holds.

Proof. (a) Let v be a minimal projection in $B(H)$. Combining the invertibility of a and the minimality of v with [22, Lemma 2.1], it can be seen that

$$v \leq a \Leftrightarrow \big\|a - (\mathbf{1} - v)\big\| = 1.$$

Therefore, for each minimal projection v in $B(H)$ we have (cf. (2.3))

$$v \leq s_{B(H)}(a) \leq a \qquad \text{if and only if} \qquad \big\|a - (\mathbf{1} - v)\big\| = 1. \tag{3.1}$$

($\supseteq$) Take $p \in \text{comin-}\mathcal{P}roj(B(H))$ with $\mathbf{1} - p \leq s_{B(H)}(a)$. Applying (3.1) with $v = \mathbf{1} - p$, we get $\|a - p\| = 1$.

($\subseteq$) Now take $p \in \text{comin-}\mathcal{P}roj(B(H))$ with $\|a - (\mathbf{1} - (\mathbf{1} - p))\| = \|a - p\| = 1$. We deduce from (3.1) that $\mathbf{1} - p \leq s_{B(H)}(a) \leq a$.

(b) ($\supseteq$) Let us take $x \in S(\mathrm{Inv}(B(H))^+)$ satisfying $s_{B(H)}(a) \leq x$. For each $p \in \text{comin-}\mathcal{P}roj(B(H)))$ with $\|a - p\| = 1$, we know from (a) that $\mathbf{1} - p \leq s_{B(H)}(a) \leq x$. Applying the statement in (2.3), we have $\mathbf{1} - p \leq s_{B(H)}(x)$. A new application of (a) to the element x gives $\|x - p\| = 1$. This shows that x lies in $\mathrm{Sph}(\mathrm{Sph}(a; \text{comin-}\mathcal{P}roj(B(H))); S(\mathrm{Inv}(B(H))^+))$.

($\subseteq$) Take $x \in S(\mathrm{Inv}(B(H))^+)$ satisfying $\|x - p\| = 1$ for every projection p in $\mathrm{Sph}(a; \text{comin-}\mathcal{P}roj(B(H)))$. Applying (a), it can be seen that, for every minimal projection v in $B(H)$ with $v \leq s_{B(H)}(a)$ we have

$$\mathbf{1} - v \in \mathrm{Sph}\big(a; \text{comin-}\mathcal{P}roj\big(B(H)\big)\big),$$

and hence $\|x - (\mathbf{1} - v)\| = 1$. Since $x \in S(\mathrm{Inv}(B(H))^+)$ and v is minimal, it follows from (a) that $v \leq s_{B(H)}(x)$. We have proved that $v \leq s_{B(H)}(x) \leq x$ whenever v is a minimal projection with $v \leq s_{B(H)}(a)$. Therefore $s_{B(H)}(a) \leq x$. ☐

The next lemma is a simple observation.

Lemma 3.5. *Let $\Delta : S(A^+) \to S(B^+)$ be a surjective isometry, where A and B are unital C^*-algebras. Suppose that $\Delta(\mathbf{1}) = \mathbf{1}$. Then $\Delta(S(\mathrm{Inv}(A)^+)) = S(\mathrm{Inv}(B)^+)$.*

Proof. We observe that an element $b \in S(A^+)$ is invertible if and only if the inequality $\|a - \mathbf{1}\| < 1$ holds. Therefore, $b \in S(\mathrm{Inv}(A)^+)$ if and only if $\|b - \mathbf{1}\| < 1$ if and only if $\|\Delta(b) - \Delta(\mathbf{1})\| = \|\Delta(b) - \mathbf{1}\| < 1$ if and only if $\Delta(b) \in S(\mathrm{Inv}(B)^+)$. ☐

We are now in position to establish the main result of this section, which proves the conjecture posed by Nagy [20, Section 3].

Theorem 3.6. *Let $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ be a surjective isometry, where H_1 and H_2 are complex Hilbert spaces. Then there exists a surjective complex linear isometry (actually, a *-isomorphism or a *-antiautomorphism) $T : B(H_1) \to B(H_2)$ satisfying $\Delta(x) = T(x)$ for all $x \in S(B(H_1)^+)$.*

Proof. Proposition 3.2 implies that

$$\Delta|_{\mathcal{P}roj(B(H_1))^*} : \mathcal{P}roj\big(B(H_1)\big)^* \to \mathcal{P}roj\big(B(H_2)\big)^*$$

is a surjective isometry and a unital order isomorphism.

If $\dim(H_1)$ is finite, then it can be easily seen from the above that $\dim(H_1) = \dim(H_2)$ (just observe that $\dim(H)$ ($< \infty$) is precisely the cardinality of every maximal set of minimal projections in $B(H)$). In this case, the desired conclusion was established by Nagy [20, Theorem].

Let us assume that H_1 is infinite-dimensional. We define a vector measure $\mu : \mathcal{P}roj(B(H_1)) \to B(H_2)$ given by $\mu(0) = 0$ and $\mu(p) = \Delta(p)$ for all p in $\mathcal{P}roj(B(H_1))^*$. It is clear that $\mu(p) \in \mathcal{P}roj(B(H_2))$ for every p in $\mathcal{P}roj(B(H_1))$. In particular,

$$\big\{\big\|\mu(p)\big\| : p \in \mathcal{P}roj\big(B(H_1)\big)\big\} = \{0, 1\}. \tag{3.2}$$

We claim that μ is finitely additive, that is

$$\mu\Big(\sum_{j=1}^{m} p_j\Big) = \sum_{j=1}^{m} \mu(p_j), \tag{3.3}$$

for every family $\{p_1, \ldots, p_m\}$ of mutually orthogonal projections in $B(H_1)$. Namely, we can assume that $p_j \neq 0$ for every j. Lemma 3.1 and Proposition 3.2 assure that $\{\Delta(p_1), \ldots, \Delta(p_m)\}$ are mutually orthogonal projections in $B(H_2)$. We also know from Proposition 3.2 that $\mu(\sum_{j=1}^{m} p_j) = \Delta(\sum_{j=1}^{m} p_j)$ and $\mu(p_j) = \Delta(p_j)$

are projections in $B(H_2)$ with $\mu(\sum_{j=1}^m p_j) = \Delta(\sum_{j=1}^m p_j) \geq \mu(p_j) = \Delta(p_j)$ for all $j \in \{1, \dots, m\}$, and hence $\mu(\sum_{j=1}^m p_j) \geq \sum_{j=1}^m \mu(p_j)$. Since $\sum_{j=1}^m \mu(p_j)$ and $\sum_{j=1}^m p_j$ are the least upper bounds of $\{\Delta(p_1), \dots, \Delta(p_m)\}$ and $\{p_1, \dots, p_m\}$ in $B(H_2)$ and $B(H_1)$, respectively, and $\Delta|_{\mathcal{P}roj(B(H_1))^*}$ is an order isomorphism (see Proposition 3.2), we get $\mu(\sum_{j=1}^m p_j) = \sum_{j=1}^m \mu(p_j)$.

We have therefore shown that μ is a bounded finitely additive measure. We are in a position to apply the Bunce–Wright Mackey–Gleason theorem (see [2, Theorem A]), and thus there exists a unique bounded complex linear operator $T : B(H_1) \to B(H_2)$ satisfying

$$T(p) = \mu(p) = \Delta(p) \quad \text{for every } p \in \mathcal{P}roj\big(B(H_1)\big)^*. \tag{3.4}$$

Since $T|_{\mathcal{P}roj(B(H_1))^*} = \Delta|_{\mathcal{P}roj(B(H_1))^*} : \mathcal{P}roj(B(H_1))^* \to \mathcal{P}roj(B(H_2))^*$ is a surjective isometry and a unital order automorphism, the second part in Proposition 3.2 implies that T is a surjective isometry and a *-isomorphism or a *-anti-isomorphism.

It only remains to prove that $T(x) = \Delta(x)$ for every $x \in S(B(H_1))$. Let us begin with an element of the form $a = p_0 + \sum_{j=1}^m \lambda_j p_j$, where $\lambda_j \in (0,1)$, and $p_0, p_1, \dots, p_m$ are mutually orthogonal nonzero projections in $B(H_1)$ with $\sum_{j=0}^m p_j = \mathbf{1}$.

Under the condition that $\Delta(\mathbf{1}) = \mathbf{1}$, we can then apply Lemma 3.5 in order to deduce that $\Delta(S(\mathrm{Inv}(B(H_1))^+)) = S(\mathrm{Inv}(B(H_2))^+)$. Furthermore, since the sets $\mathrm{Sph}(a; \text{comin-}\mathcal{P}roj(B(H_1)))$ and

$$\mathrm{Sph}\big(\mathrm{Sph}\big(a; \text{comin-}\mathcal{P}roj\big(B(H_1)\big)\big); S\big(\mathrm{Inv}\big(B(H_1)^+\big)\big)\big)$$

are determined by the norm, the element a, the set $S(\mathrm{Inv}(B(H_1))^+)$, and the set $\mathrm{Sph}(a; \text{comin-}\mathcal{P}roj(B(H_1)))$, and all these structures are preserved by Δ, we deduce that

$$\Delta\big(\mathrm{Sph}\big(a; \text{comin-}\mathcal{P}roj\big(B(H_1)\big)\big)\big) = \mathrm{Sph}\big(\Delta(a); \text{comin-}\mathcal{P}roj\big(B(H_2)\big)\big)$$

and

$$\begin{aligned} &\Delta(\mathrm{Sph}\big(\mathrm{Sph}\big(a; \text{comin-}\mathcal{P}roj\big(B(H_1)\big)\big); S\big(\mathrm{Inv}\big(B(H_1)^+\big)\big)\big) \\ &\quad = \mathrm{Sph}\big(\mathrm{Sph}\big(\Delta(a); \text{comin-}\mathcal{P}roj\big(B(H_2)\big)\big); S\big(\mathrm{Inv}\big(B(H_2)^+\big)\big)\big). \end{aligned} \tag{3.5}$$

Lemma 3.3 implies that $s_{B(H_2)}(\Delta(a)) = \Delta(p_0)$. We have already commented that $\Delta(a)$ is invertible (cf. Lemma 3.5).

Now applying Theorem 3.4(b), we deduce that

$$\begin{aligned} &\mathrm{Sph}\big(\mathrm{Sph}\big(a; \text{comin-}\mathcal{P}roj\big(B(H_1)\big)\big); S\big(\mathrm{Inv}\big(B(H_1)^+\big)\big)\big)\big) \\ &\quad = \big\{x \in S\big(\mathrm{Inv}\big(B(H_1)\big)^+\big) : s_{B(H_1)}(a) = p_0 \leq x\big\} \\ &\quad = p_0 + \mathcal{B}_{\mathrm{Inv}((\mathbf{1}-p_0)B(H_1)^+(\mathbf{1}-p_0))} = p_0 + \mathcal{B}_{\mathrm{Inv}(B((\mathbf{1}-p_0)(H_1))^+)} \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Sph}\big(\mathrm{Sph}\big(\Delta(a); \text{comin-}\mathcal{P}roj\big(B(H_2)\big)\big); S\big(\mathrm{Inv}\big(B(H_2)^+\big)\big)\big) \\ &\quad = \Delta(p_0) + \mathcal{B}_{\mathrm{Inv}(B((\mathbf{1}-\Delta(p_0))(H_2))^+)}. \end{aligned}$$

To simplify the notation, let us denote $K_1 = (\mathbf{1} - p_0)(H_1)$ and $K_2 = (\mathbf{1} - \Delta(p_0))(H_2)$. By combining the above identities with (3.5), we can consider the following diagram of surjective isometries:

$$\begin{array}{ccc} p_0 + \mathcal{B}_{\mathrm{Inv}(B(K_1)^+)} & \xrightarrow{\ \Delta\ } & \Delta(p_0) + \mathcal{B}_{\mathrm{Inv}(B(K_2)^+)} \\ {\scriptstyle \tau_{-p_0}}\downarrow & & \uparrow{\scriptstyle \tau_{\Delta(p_0)}} \\ \mathcal{B}_{\mathrm{Inv}(B(K_1)^+)} & \overset{\Delta_a}{\dashrightarrow} & \mathcal{B}_{\mathrm{Inv}(B(K_2)^+)} \end{array} \tag{3.6}$$

where τ_z denotes the translation by z, and Δ_a is the surjective isometry making the above diagram commutative.

Let us observe the following property. For each unital C^*-algebra A, the set $\mathcal{B}_{\mathrm{Inv}(A^+)}$, of all positive invertible elements in the closed unit ball of A, is a convex subset with nonempty interior in A_{sa}. Actually, if $a, b \in \mathcal{B}_{\mathrm{Inv}(A^+)}$, then we know that $ta + (1-t)b \in \mathcal{B}_{A^+}$ for every $t \in [0,1]$ (see [25, Theorem 1.4.2]). By the invertibility of a, b, we can find positive constants m_1, m_2 such that $m_1\mathbf{1} \leq a$ and $m_2\mathbf{1} \leq b$. Therefore, $(tm_1 + (1-t)m_2)\mathbf{1} \leq ta + (1-t)b$, which guarantees that $ta + (1-t)b$ is invertible too. We note that the open unit ball in A_{sa} with center $\frac{1}{2}\mathbf{1}$ and radius $\frac{1}{2}$ is contained in $\mathcal{B}_{\mathrm{Inv}(A^+)}$. Since $\Delta_a : \mathcal{B}_{\mathrm{Inv}(B(K_1)^+)} \to \mathcal{B}_{\mathrm{Inv}(B(K_2)^+)}$ is a surjective isometry, we are in a position to apply Manckiewicz's theorem (see [15, Theorem 5, Remark 7]) to deduce the existence of a surjective real linear isometry $T_a : B(K_1)_{\mathrm{sa}} \to B(K_2)_{\mathrm{sa}}$ and a $z_0 \in B(K_2)_{\mathrm{sa}}$ such that

$$\Delta_a(x) = T_a(x) + z_0, \quad \text{for all } x \in \mathcal{B}_{\mathrm{Inv}(B(K_1)^+)}. \tag{3.7}$$

Since $\Delta(\mathbf{1}) = \mathbf{1}$, it follows from the construction above that $\Delta_a(\mathbf{1}_{B(K_1)}) = \mathbf{1}_{B(K_2)}$, and thus $T_a(\mathbf{1}_{B(K_1)}) + z_0 = \mathbf{1}_{B(K_2)}$.

Let us recall that an element s in $B(K_2)_{\mathrm{sa}}$ is called a *symmetry* if $s^2 = 1$. Actually, every symmetry in $B(K_2)_{\mathrm{sa}}$ is of the form $s = p_1 - (\mathbf{1}_{B(K_2)} - p_1)$, where p_1 is a projection. The real Jordan Banach (JB) algebras $B(K_1)_{\mathrm{sa}}$ and $B(K_2)_{\mathrm{sa}}$ (equipped with the natural Jordan product $x \circ y = \frac{1}{2}(xy + yx)$) are prototypes of JB-algebras in the sense employed in [33] and [13]. Since $T_a : B(K_1)_{\mathrm{sa}} \to B(K_2)_{\mathrm{sa}}$ is a surjective isometry, by applying [13, Theorem 1.4], we deduce the existence of a central symmetry $s \in B(K_2)_{\mathrm{sa}}$ and a unital Jordan *-isomorphism $\Phi_a : B(K_1)_{\mathrm{sa}} \to B(K_2)_{\mathrm{sa}}$ such that $T_a(x) = s\Phi_a(x)$, for all $x \in B(K_1)_{\mathrm{sa}}$. However, the unique central symmetries in $B(K_2)_{\mathrm{sa}}$ are $\mathbf{1}_{B(K_2)}$ and $-\mathbf{1}_{B(K_2)}$. Summing up, we have

$$\mathbf{1}_{B(K_2)} - z_0 = T_a(\mathbf{1}_{B(K_1)}) = s\mathbf{1}_{B(K_2)} = s = \pm\mathbf{1}_{B(K_2)}.$$

Then one (and only one) of the following statements holds:

(1) $z_0 = 0$, and thus $T_a(\mathbf{1}_{B(K_1)}) = \mathbf{1}_{B(K_2)}$, and T_a is a Jordan *-isomorphism;
(2) $z_0 = 2\mathbf{1}_{B(K_2)} \equiv 2(\mathbf{1} - \Delta(p_0))$, and thus $T_a(\mathbf{1}_{B(K_1)}) = -\mathbf{1}_{B(K_2)} \equiv -(\mathbf{1} - \Delta(p_0))$, and $\Phi_a = -T_a$ is a Jordan *-isomorphism.

We claim that case (2) is impossible; otherwise, by inserting the element $p_0 + \frac{1}{2}(\mathbf{1} - p_0)$ (where $\frac{1}{2}\mathbf{1}_{B(K_1)} \equiv \frac{1}{2}(\mathbf{1} - p_0) \in \mathcal{B}_{Inv(B(K_1)^+)} \cong \mathcal{B}_{Inv(B((\mathbf{1}-p_0)(H_1))^+)}$) in the diagram (3.6) (see also (3.7)) we get

$$\Delta\Big(p_0+\frac{1}{2}(\mathbf{1}-p_0)\Big)=\Delta(p_0)+\Delta_a\Big(\frac{1}{2}(\mathbf{1}-p_0)\Big)=\Delta(p_0)+T_a\Big(\frac{1}{2}(\mathbf{1}-p_0)\Big)+z_0$$
$$=\Delta(p_0)+2\big(\mathbf{1}-\Delta(p_0)\big)-\frac{1}{2}\Phi_a\big((\mathbf{1}-p_0)\big)$$
$$=\Delta(p_0)+2\big(\mathbf{1}-\Delta(p_0)\big)-\frac{1}{2}\big(\mathbf{1}-\Delta(p_0)\big)$$
$$=\Delta(p_0)+\frac{3}{2}\big(\mathbf{1}-\Delta(p_0)\big),$$

which proves that $\frac{3}{2}=\|\Delta(p_0)+\frac{3}{2}(\mathbf{1}-\Delta(p_0))\|=\|\Delta(p_0+\frac{1}{2}(\mathbf{1}-p_0))\|=1$, leading to a contradiction. Therefore, only case (1) holds, and hence T_a is a Jordan *-isomorphism.

We will prove next that

$$\Delta(q)=T_a(q),\quad \text{for every projection } q\leq \mathbf{1}-p_0. \tag{3.8}$$

Namely, take a projection $q\leq \mathbf{1}-p_0$. By inserting the element $b=p_0+q+\frac{1}{2}(\mathbf{1}-q-p_0)$ in the diagram (3.6) (see also (3.7)), we get

$$\Delta(b)=\Delta\Big(p_0+q+\frac{1}{2}(\mathbf{1}-q-p_0)\Big)=\Delta(p_0)+\Delta_a\Big(q+\frac{1}{2}(\mathbf{1}-q-p_0)\Big)$$
$$=\Delta(p_0)+T_a\Big(q+\frac{1}{2}(\mathbf{1}-q-p_0)\Big)=\Delta(p_0)+T_a(q)+\frac{1}{2}T_a(\mathbf{1}-q-p_0),$$

which assures that $s_{B(H_2)}(\Delta(b))=\Delta(p_0)+T_a(q)$. On the other hand, Lemma 3.3 implies that $s_{B(H_2)}(\Delta(b))=\Delta(s_{B(H_2)}(b))=\Delta(p_0+q)=$ (by (3.3)) $=\Delta(p_0)+\Delta(q)$. We have therefore shown that $\Delta(p_0)+T_a(q)=\Delta(p_0)+\Delta(q)$, which concludes the proof of (3.8).

Now, inserting our element $a=p_0+\sum_{j=1}^m\lambda_jp_j$ (where $\lambda_j\in\mathbb{R}^+$, and $p_0,p_1,\ldots,p_m$ are mutually orthogonal nonzero projections in $B(H_1)$ with $\sum_{j=0}^m p_j=\mathbf{1}$) in (3.6) (see also (3.7)), we deduce that

$$\Delta(a)=\Delta\Big(p_0+\sum_{j=1}^m\lambda_jp_j\Big)=\Delta(p_0)+\Delta_a\Big(\sum_{j=1}^m\lambda_jp_j\Big)=\Delta(p_0)+T_a\Big(\sum_{j=1}^m\lambda_jp_j\Big)$$
$$=\Delta(p_0)+\sum_{j=1}^m\lambda_jT_a(p_j)=(\text{by (3.8)})=\Delta(p_0)+\sum_{j=1}^m\lambda_j\Delta(p_j)$$
$$=(\text{by (3.4)})=T(p_0)+\sum_{j=1}^m\lambda_jT(p_j)=T(a).$$

Finally, it is well known that every positive element in the unit sphere of $B(H_1)$ can be approximated in norm by elements of the form $a=p_0+\sum_{j=1}^m\lambda_jp_j$, where $\lambda_j\in\mathbb{R}^+$, and $p_0,p_1,\ldots,p_m$ are mutually orthogonal nonzero projections in $B(H_1)$ with $\sum_{j=0}^m p_j=\mathbf{1}$. Therefore, since Δ and T are continuous and coincide on elements of the previous form, we deduce that $\Delta(x)=T(x)$, for every $x\in S(B(H_1)^+)$, which concludes the proof. □

4. Surjective isometries between spaces of normalized positive compact operators

Throughout this section, H_3 and H_4 will denote two separable infinite-dimensional complex Hilbert spaces. Our goal here will consist in studying surjective isometries $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$. We begin with a technical result.

Lemma 4.1. *Let $\Delta : \mathcal{B}_{B(H_1)^+} \to \mathcal{B}_{B(H_2)^+}$ be a surjective isometry, where H_1 and H_2 are complex Hilbert spaces. Suppose that $\Delta(\mathcal{P}roj(B(H_1))) = \mathcal{P}roj(B(H_2))$. Then there exists a surjective complex linear isometry (actually, a Jordan *-isomorphism) $T : B(H_1) \to B(H_2)$ such that one of the following statements holds:*

(a) $\Delta(x) = T(x)$, *for all* $x \in \mathcal{B}_{B(H_1)^+}$;
(b) $\Delta(x) = \mathbf{1} - T(x)$, *for all* $x \in \mathcal{B}_{B(H_1)^+}$.

Furthermore, since $B(H_1)$ and $B(H_2)$ are factors, we can also deduce that T is a *-isomorphism or a *-anti-isomorphism.

Proof. We consider the real Banach spaces $B(H_1)_{\mathrm{sa}}$ and $B(H_2)_{\mathrm{sa}}$ as JB-algebras in the sense employed in [33]. The proof is heavily based on a deep result due to Mankiewicz [15, Theorem 5, Remark 7] asserting that every bijective isometry between convex sets in normed linear spaces with nonempty interiors admits a unique extension to a bijective affine isometry between the corresponding spaces. Let us observe that $\mathcal{B}_{B(H_1)^+} \subset \mathcal{B}_{B(H_1)_{\mathrm{sa}}}$ and $\mathcal{B}_{B(H_2)^+} \subset \mathcal{B}_{B(H_2)_{\mathrm{sa}}}$ are convex sets with nonempty interiors (just observe that the open unit ball in $B(H)_{\mathrm{sa}}$ of radius $1/2$ and center $\frac{1}{2}\mathbf{1}$ is contained in $\mathcal{B}_{B(H)^+}$). Thus, by Mankiewicz's theorem, there exists a bijective real linear isometry $T : B(H_1)_{\mathrm{sa}} \to B(H_2)_{\mathrm{sa}}$ and a $z_0 \in \mathcal{B}_{B(H_2)^+}$ such that $\Delta(x) = T(x) + z_0$, for all $x \in \mathcal{B}_{B(H_1)^+}$. We denote by the same symbol T the bounded complex linear operator from $B(H_1)$ to $B(H_2)$ given by $T(x + iy) = T(x) + iT(y)$ for all $x, y \in B(H_1)_{\mathrm{sa}}$.

On the other hand, since, by hypothesis, Δ preserves projections, we infer that z_0 is a projection and $T(\mathcal{P}roj(B(H_1))) + z_0 = \Delta(\mathcal{P}roj(B(H_1))) = \mathcal{P}roj(B(H_2))$. The projections 0 and $\mathbf{1}$ are the unique projections in $B(H_1)$ (or in $B(H_2)$) whose distance to another projection is 0 or 1. If $z_0 = \Delta(0) \neq 0, \mathbf{1}$, then there exists a nontrivial projection q in $B(H_2)$ satisfying $0 < \|\Delta(0) - q\| < 1$. This implies that

$$\{0, 1\} \ni \left\|0 - \Delta^{-1}(q)\right\| = \left\|\Delta(0) - q\right\| \in (0, 1),$$

which is impossible. We have therefore proved that $z_0 = \Delta(0) \in \{0, \mathbf{1}\}$. Similar arguments show that $\Delta(\mathbf{1}) = T(\mathbf{1}) + z_0 \in \{0, \mathbf{1}\}$. Applying the fact that Δ is a bijection, we deduce that precisely one of the following statements holds:

(a) $\Delta(0) = z_0 = 0$ and $\Delta(\mathbf{1}) = \mathbf{1}$;
(b) $\Delta(0) = z_0 = \mathbf{1}$ and $\Delta(\mathbf{1}) = 0$.

If $z_0 = \Delta(0) = 0$ and $\Delta(\mathbf{1}) = T(\mathbf{1}) + z_0 = \mathbf{1}$, then the mapping $T : B(H_1)_{\mathrm{sa}} \to B(H_2)_{\mathrm{sa}}$ is a unital and surjective real linear isometry between JB-algebras. Applying [33, Theorem 4], we deduce that T is a Jordan isomorphism. In particular, the complex linear extension $T : B(H_1) \to B(H_2)$ is a complex linear Jordan *-isomorphism and $\Delta(x) = T(x)$, for all $x \in \mathcal{B}_{B(H_1)^+}$. We arrive at statement (a) in our conclusion.

If $\Delta(0) = z_0 = \mathbf{1}$ and $\Delta(\mathbf{1}) = T(\mathbf{1}) + z_0 = 0$, then we have $T(\mathbf{1}) = -\mathbf{1}$. Therefore, $-T : B(H_1)_{\rm sa} \to B(H_2)_{\rm sa}$ is a unital and surjective real linear isometry. The arguments in the previous case prove that the complex linear extension of $-T$, denoted by $-T : B(H_1) \to B(H_2)$, is a complex linear Jordan *-isomorphism and $\Delta(x) = \mathbf{1} - (-T(x))$, for all $x \in \mathcal{B}_{B(H_1)^+}$. We have therefore arrived at statement (b) in our conclusion.

The last statement follows from Corollary 11 in [14]. □

Corollary 2.2 admits a strengthened version which was established in [22].

Theorem 4.2 ([22, Theorem 2.8])**.** *Let H_3 be a separable infinite-dimensional complex Hilbert space. Then the identity*

$$\mathrm{Sph}^+_{K(H_3)}\big(\mathrm{Sph}^+_{K(H_3)}(a)\big) = \left\{ b \in S\big(K(H_3)^+\big) : \begin{array}{c} s_{K(H_3)}(a) \leq s_{K(H_3)}(b), \textit{ and} \\ \mathbf{1} - r_{B(H_3)}(a) \leq \mathbf{1} - r_{B(H_3)}(b) \end{array} \right\}$$

holds for every a in the unit sphere of $K(H_3)^+$.

We can now improve the conclusion of Corollary 2.2.

Proposition 4.3. *Let H_3 and H_4 be separable complex Hilbert spaces. Let us assume that H_3 is infinite-dimensional. Let $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$ be a surjective isometry. Then the following statements hold.*

(a) *The mapping Δ preserves projections, that is, $\Delta(\mathcal{P}roj(K(H_3))^*) = \mathcal{P}roj(K(H_4))^*$, and the restricted mapping $\Delta|_{\mathcal{P}roj(K(H_3))^*} : \mathcal{P}roj(K(H_3))^* \to \mathcal{P}roj(K(H_4))^*$ is a surjective isometry and an order automorphism. Furthermore, $\Delta(p)\Delta(q) = 0$ for every $p, q \in \mathcal{P}roj(K(H_3))^*$ with $pq = 0$.*

(b) *For every finite family $p_1, \ldots, p_n$ of mutually orthogonal minimal projections in $K(H_3)$, and $1 = \lambda_1 \geq \lambda_2, \ldots, \lambda_n \geq 0$, we have*

$$\Delta\Big(\sum_{j=1}^n \lambda_j p_j\Big) = \sum_{j=1}^n \lambda_j \Delta(p_j).$$

Proof. (a) The first part of the statement has been proved in Corollary 2.2. We will show next that Δ preserves the order between nonzero projections.

We claim that given $p, e_1 \in \mathcal{P}roj(K(H_3))^*$ with e_1 minimal and $e_1 \perp p$, we have

$$\Delta(p + e_1) \geq \Delta(p). \tag{4.1}$$

To prove the claim, let $m_0 \in \mathbb{N}$ denote the rank of the projection $\Delta(p) \in K(H_4)$. Since H_3 is infinite-dimensional, we can find a natural n with $n > m_0$ and mutually orthogonal minimal projections $e_2, \ldots, e_n$ such that $p + e_1 \perp e_j$ for all $j = 2, \ldots, n$.

We next apply Theorem 4.2 to the element $a = p + \sum_{j=1}^n \frac{1}{2} e_j$. Let us write $q_n = \sum_{j=1}^n e_j$. Clearly, q_n is a projection in $K(H_3)$ with $q_n \perp p$, and since $r_{B(H_3)}(a) = p + \sum_{j=1}^n e_j = p + q_n$, we have

$$\mathrm{Sph}^+_{K(H_3)}\big(\mathrm{Sph}^+_{K(H_3)}(a)\big) = \left\{ b \in S\big(K(H_3)^+\big) : \begin{array}{c} s_{K(H_3)}(a) = p \leq s_{K(H_3)}(b), \text{and} \\ \mathbf{1} - p - q_n \leq \mathbf{1} - r_{B(H_3)}(b) \end{array} \right\},$$

$$= \left\{ b \in S\big(K(H_3)^+\big) : \begin{array}{c} s_{K(H_3)}(a) = p \leq s_{K(H_3)}(b), \text{and} \\ b \leq p + q_n \end{array} \right\}$$
$$= p + \{x \in \mathcal{B}_{K(H_3)^+} : p \perp x \leq q_n\} = p + \mathcal{B}_{q_n K(H_3)^+ q_n},$$

and the set $\mathcal{B}_{q_n K(H_3)^+ q_n}$ can be C^*-isometrically identified with $\mathcal{B}_{B(\ell_2^n)^+}$.

Clearly, the restriction of Δ to $\text{Sph}^+_{K(H_3)}(\text{Sph}^+_{K(H_3)}(a))$ is a surjective isometry from this set onto $\text{Sph}^+_{K(H_4)}(\text{Sph}^+_{K(H_4)}(\Delta(a)))$. Similarly, by Theorem 4.2, we have

$$\text{Sph}^+_{K(H_4)}\big(\text{Sph}^+_{K(H_4)}\big(\Delta(a)\big)\big) = s_{K(H_4)}\big(\Delta(a)\big) + \mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}},$$

where $\widehat{q} = r_{B(H_4)}(\Delta(a)) - s_{K(H_4)}(\Delta(a)) \in B(H_4)$ and the set $\mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}}$ can be C^*-isometrically identified with $\mathcal{B}_{B(H)^+}$, where $H = \widehat{q}(H_4)$ is a complex Hilbert space whose dimension coincides with the rank of the projection $\widehat{q}$. Since every translation $x \mapsto \tau_z(x) = z + x$ is a surjective isometry, we can define a surjective isometry $\Delta_a : \mathcal{B}_{B(\ell_2^n)^+} \to \mathcal{B}_{B(H)^+}$ making the following diagram commutative:

$$\begin{array}{ccc}
\text{Sph}^+_{K(H_3)}\Big(\text{Sph}^+_{K(H_3)}(a)\Big) & \xrightarrow{\ \Delta\ } & \text{Sph}^+_{K(H_4)}\Big(\text{Sph}^+_{K(H_4)}(\Delta(a))\Big) \\
\Big\| & & \Big\| \\
p + \mathcal{B}_{q_n K(H_3)^+ q_n} & & s_{K(H_4)}(\Delta(a)) + \mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}} \\
\tau_{-p}\Big\downarrow & & \Big\uparrow \tau_{s_{K(H_4)}(\Delta(a))} \\
\mathcal{B}_{q_n K(H_3)^+ q_n} \cong \mathcal{B}_{B(\ell_2^n)^+} & \xrightarrow{\ \Delta_a\ } & \mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}} \cong \mathcal{B}_{B(H)^+}
\end{array}$$

Actually, $\mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}}$ can be identified with the set of the elements orthogonal to $s_{K(H_4)}(\Delta(a))$ inside the set $\mathcal{B}_{r_{B(H_4)}(\Delta(a))K(H_4)^+ r_{B(H_4)}(\Delta(a))}$.

Take a projection $p + r$ in $\text{Sph}^+_{K(H_3)}(\text{Sph}^+_{K(H_3)}(a))$ (clearly, r can be any projection in $K(H_3)$ with $r \leq q_n$). We know from Corollary 2.2 that $\Delta(p + r)$ is a projection in $\text{Sph}^+_{K(H_4)}(\text{Sph}^+_{K(H_4)}(\Delta(a)))$, and consequently

$$\Delta_a(r) = \Delta(p + r) - s_{K(H_4)}\big(\Delta(a)\big)$$

must be a projection. We have therefore shown that the map Δ_a above is a surjective isometry mapping projections to projections.

We deduce from Lemma 4.1 that $\dim(H) = n$, and by the same lemma there exists a complex linear (unital) Jordan *-isomorphism

$$T_a : q_n K(H_3) q_n \cong B(\ell_2^n) \to \widehat{q}K(H_4)^+\widehat{q} \cong B(\ell_2^n)$$

satisfying one of the following statements:

(1) $\Delta_a(x) = T_a(x)$, for all $x \in \mathcal{B}_{q_n K(H_3)^+ q_n}$;
(2) $\Delta_a(x) = \mathbf{1}_{\widehat{q}} - T_a(x)$, for all $x \in \mathcal{B}_{q_n K(H_3)^+ q_n}$, where $\mathbf{1}_{\widehat{q}} = r_{B(H_4)}(\Delta(a)) - s_{K(H_4)}(\Delta(a))$ is the unit of $\widehat{q}K(H_4)^+\widehat{q} \cong B(H)$.

We claim that case (2) is impossible. Actually, if case (2) holds, then

$$\begin{aligned}\Delta(p) &= s_{K(H_4)}\big(\Delta(a)\big) + \Delta_a(0)\\ &= s_{K(H_4)}\big(\Delta(a)\big) + \big(r_{B(H_4)}\big(\Delta(a)\big) - s_{K(H_4)}\big(\Delta(a)\big)\big) - T_a(0)\\ &= s_{K(H_4)}\big(\Delta(a)\big) + \big(r_{B(H_4)}\big(\Delta(a)\big) - s_{K(H_4)}\big(\Delta(a)\big)\big),\end{aligned}$$

where $(r_{B(H_4)}(\Delta(a)) - s_{K(H_4)}(\Delta(a)))$ and $s_{K(H_4)}(\Delta(a))$ are orthogonal, and the rank of $(r_{B(H_4)}(\Delta(a)) - s_{K(H_4)}(\Delta(a)))$ is precisely the dimension of H which is n. This shows that $\Delta(p)$ has rank greater than or equal to $n+1 > m_0$, which is impossible because m_0 is the rank of $\Delta(p)$.

Since case (1) holds, we have

$$\Delta(p+e_1) = s_{K(H_4)}\big(\Delta(a)\big) + T_a(e_1) \geq s_{K(H_4)}\big(\Delta(a)\big) = \Delta(p),$$

because $T_a(e_1)$ is a nonzero projection and $T_a(e_1) \perp s_{K(H_4)}(\Delta(a))$. This proves (4.1). We have also proved that

$$s_{K(H_4)}\big(\Delta(a)\big) = \Delta(p) \qquad \text{and} \qquad \Delta(p+q_n) = r_{B(H_4)}\big(\Delta(a)\big).$$

Now, let $p, q \in \mathcal{P}roj(K(H_3))^*$ with $p \leq q$. In our context, we can find mutually orthogonal minimal projections $e_1, \ldots, e_m$ in $K(H_3)$ satisfying $q = p + \sum_{j=1}^{m} e_j$. Applying (4.1) in a finite number of steps, we get

$$\Delta(p) \leq \Delta(p+e_1) \leq \cdots \leq \Delta\Big(p + \sum_{j=1}^{m} e_j\Big) = \Delta(q).$$

Now take $p, q \in \mathcal{P}roj(K(H_3))^*$ with $pq = 0$. Under these hypotheses, Lemma 3.1 assures that $\Delta(p)\Delta(q) = 0$.

(b) Let us apply the arguments in the proof of (a) to the element $a = p_1 + \sum_{j=2}^{n} \frac{1}{2} p_j$. Let $q_{n-1} = \sum_{j=2}^{n} p_j$ and $\widehat{q} = \Delta(q_{n-1}) = r_{B(H_4)}(\Delta(a)) - s_{K(H_4)}(\Delta(a))$. We deduce from the above arguments the existence of a surjective isometry

$$\Delta_a : \mathcal{B}_{q_{n-1}K(H_3)^+ q_{n-1}} \cong \mathcal{B}_{B(\ell_2^{n-1})^+} \to \mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}} \cong \mathcal{B}_{B(\ell_2^{n-1})^+}$$

making the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{Sph}^+_{K(H_3)}\Big(\mathrm{Sph}^+_{K(H_3)}(a)\Big) & \xrightarrow{\ \Delta\ } & \mathrm{Sph}^+_{K(H_4)}\Big(\mathrm{Sph}^+_{K(H_4)}(\Delta(a))\Big)\\
\Vert & & \Vert\\
p_1 + \mathcal{B}_{q_{n-1}K(H_3)^+ q_{n-1}} & & \Delta(p_1) + \mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}}\\
\tau_{-p_1}\big\downarrow & & \tau_{\Delta(p_1)}\big\uparrow\\
\mathcal{B}_{q_{n-1}K(H_3)^+ q_{n-1}} \cong \mathcal{B}_{B(\ell_2^{n-1})^+} & \xrightarrow{\ \Delta_a\ } & \mathcal{B}_{\widehat{q}K(H_4)^+\widehat{q}} \cong \mathcal{B}_{B(H)^+}
\end{array}$$

Since, by (a), $\Delta|_{\mathcal{P}roj(K(H_3))^*}$ is an order automorphism, the reasonings in (a) and Lemma 4.1 prove the existence of a complex linear (unital) Jordan *-isomorphism $T_a : B(\ell_2^{n-1}) \cong q_{n-1}K(H_3)q_{n-1} \to B(\ell_2^{n-1}) \cong \widehat{q}K(H_4)\widehat{q}$ satisfying

$$\Delta_a(x) = T_a(x), \quad \text{for all } x \in \mathcal{B}_{B(\ell_2^{n-1})^+} \cong \mathcal{B}_{q_{n-1}K(H_3)^+ q_{n-1}}.$$

Pick $j \in \{2, \ldots, n\}$. Since $\Delta|_{\mathcal{P}roj(K(H_3))^*}$ is an order automorphism and preserves orthogonality, the elements $\Delta(p_1)$, $\Delta(p_j)$, and $\Delta(p_1 + p_j)$ are nontrivial projections in $K(H_3)$, $\Delta(p_1)$ and $\Delta(p_j)$ are minimal, $\Delta(p_1) \perp \Delta(p_j)$, $\Delta(p_1 + p_j)$ is a rank 2 projection, and $\Delta(p_1 + p_j) \geq \Delta(p_j)$. We also know that p_j lies in $\mathcal{B}_{q_{n-1}K(H_3)^+ q_{n-1}}$, $T_a(p_j)$ is a minimal projection, $T_a(p_j) \perp \Delta(p_1)$, and $\Delta(p_1 + p_j) = \Delta(p_1) + T_a(p_j)$. By applying that $\Delta(p_1) \perp \Delta(p_j)$, we get

$$\Delta(p_j) = \Delta(p_1 + p_j)\Delta(p_j) = \big(\Delta(p_1) + T_a(p_j)\big)\Delta(p_j) = T_a(p_j)\Delta(p_j).$$

The minimality of $T_a(p_j)$ and $\Delta(p_j)$ assures that $T_a(p_j) = \Delta(p_j)$.

Finally, given $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the element $\sum_{j=1}^n \lambda_j p_j = p_1 + \sum_{j=2}^n \lambda_j p_j$ lies in the set $\mathrm{Sph}^+_{K(H_3)}(\mathrm{Sph}^+_{K(H_3)}(a))$ and hence

$$\begin{aligned}\Delta\Big(\sum_{j=1}^n \lambda_j p_j\Big) &= \Delta(p_1) + \Delta_a\Big(\sum_{j=2}^n \lambda_j p_j\Big) = \Delta(p_1) + T_a\Big(\sum_{j=2}^n \lambda_j p_j\Big)\\ &= \Delta(p_1) + \sum_{j=2}^n \lambda_j T_a(p_j) = \Delta(p_1) + \sum_{j=2}^n \lambda_j \Delta(p_j),\end{aligned}$$

which finishes the proof of (b). □

Our next corollary is a first consequence of Proposition 4.3.

Corollary 4.4. *Let H_3 and H_4 be separable complex Hilbert spaces. Let us assume that H_3 is infinite-dimensional. If $T : K(H_3) \to K(H_4)$ is a bounded (complex) linear mapping such that $T(S(K(H_3)^+)) = S(K(H_4)^+)$ and $T|_{S(K(H_3)^+)} : S(K(H_3)^+) \to S(K(H_4)^+)$ is a surjective isometry, then T is a *-isomorphism or a *-anti-isomorphism.*

Proof. Let $T : K(H_3) \to K(H_4)$ be a bounded linear map satisfying the hypothesis of the corollary. We observe that T must be bijective by hypothesis.

We observe that $T(\mathcal{P}roj(K(H_3))) = \mathcal{P}roj(K(H_4))$ (see Corollary 2.2), and by Proposition 4.3, T also preserves the order among projections. In particular, $T(p)T(q) = 0$ for every $p, q \in \mathcal{P}roj(K(H_3))^*$ with $pq = 0$ (just observe that the sum of two projections is a projection if and only if they are orthogonal), and thus $T(a^2) = T(a)^2$ and $T(a)^* = T(a)$, whenever a is a finite real linear combination of mutually orthogonal minimal projections in $K(H_3)$. The continuity of T and the norm density in $K(H_3)_{\mathrm{sa}}$ of elements which are finite real linear combinations of mutually orthogonal minimal projections in $K(H_3)$, imply that T is a Jordan *-isomorphism. The rest is clear from [14, Corollary 11] because $B(H_3)$ is a factor. □

In the main theorem of this section, we extend surjective isometries of the form $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$. In the proof, we will employ a technique based on the study of the linearity of "physical states" on $K(H)$ developed by Aarnes [1]. We recall that a *physical state* or a *quasistate* on a C^*-algebra A is a function $\rho : A_{\mathrm{sa}} \to \mathbb{R}$ whose restriction to each singly generated subalgebra of A_{sa} is a positive linear functional and

$$\sup\big\{\rho(a) : a \in \mathcal{B}_{A^+}\big\} = 1.$$

As remarked by Aarnes [1, p. 603], “It is far from evident that a physical state on A must be (real) linear on A_{sa}”; however, under a favorable hypothesis, linearity is automatic and not an extra assumption.

Theorem 4.5. *Let H_3 and H_4 be separable complex Hilbert spaces. Let us assume that H_3 is infinite-dimensional. Let $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$ be a surjective isometry. Then there exists a surjective complex linear isometry $T : K(H_3) \to K(H_4)$ satisfying $T(x) = \Delta(x)$ for all $x \in S(K(H_3)^+)$. We can further conclude that T is a *-isomorphism or a *-anti-isomorphism.*

Proof. Let a be an element in $S(K(H_3)^+)$, and let us consider the spectral resolution of a in the form $a = \sum_{n=1}^{\infty} \lambda_n p_n$, where $(\lambda_n)_n$ is a decreasing sequence in $\mathbb{R}_0^+$ converging to zero, $\lambda_1 = 1$, and $\{p_n : n \in \mathbb{N}\}$ is a family of mutually orthogonal minimal projections in $K(H_3)$. Applying Proposition 4.3(a), we deduce that $\{\Delta(p_n) : n \in \mathbb{N}\}$ is a family of mutually orthogonal minimal projections in $K(H_4)$. Keeping in mind that orthogonal elements are geometrically M-orthogonal, it can be easily deduced that the series $\sum_{n=1}^{\infty} \lambda_n \Delta(p_n)$ is norm-convergent. Furthermore, since by Proposition 4.3(b) and the hypothesis we have

$$\Big\|\Delta(a) - \sum_{n=1}^{m} \lambda_n \Delta(p_n)\Big\| = \Big\|\Delta(a) - \Delta\Big(\sum_{n=1}^{m} \lambda_n p_n\Big)\Big\| = \Big\|a - \sum_{n=1}^{m} \lambda_n p_n\Big\| = \lambda_{m+1},$$

it follows that

$$\Delta(a) = \Delta\Big(\sum_{n=1}^{\infty} \lambda_n p_n\Big) = \sum_{n=1}^{\infty} \lambda_n \Delta(p_n). \tag{4.2}$$

Combining (4.2) and Proposition 4.3(a), we can see that

$$a \perp b \quad \text{in } S\big(K(\ell_2)^+\big) \Rightarrow \Delta(a) \perp \Delta(b). \tag{4.3}$$

Every element b in $K(H_3)_{\mathrm{sa}}$ can be written uniquely in the form $b = b^+ - b^-$, where b^+, b^- are orthogonal positive elements in $K(H_3)$. Having this property in mind, we define a mapping $T : K(H_3)_{\mathrm{sa}} \to K(H_4)_{\mathrm{sa}}$ given by

$$T(b) := \|b^+\|\Delta\Big(\frac{b^+}{\|b^+\|}\Big) - \|b^-\|\Delta\Big(\frac{b^-}{\|b^-\|}\Big) \quad \text{if } \|b^+\|\|b^-\| \neq 0,$$
$$T(b) := \|b^+\|\Delta\Big(\frac{b^+}{\|b^+\|}\Big) \quad \text{if } \|b^+\| \neq 0, b^- = 0,$$
$$T(b) := \|b^-\|\Delta\Big(\frac{b^-}{\|b^-\|}\Big) \quad \text{if } \|b^-\| \neq 0, b^+ = 0, \text{and } T(0) = 0.$$

It follows from definition that

$$\big\|T(b)\big\| \leq \|b^+\| + \|b^-\| \leq 2\|b\|. \tag{4.4}$$

For each positive functional $\phi \in \mathcal{B}_{(K(H_4)^*)^+}$, we set $T_\phi := \phi \circ T : K(H_3)_{\mathrm{sa}} \to \mathbb{R}$, $T_\phi(x) = \phi(T(x))$. We claim that T_ϕ is a positive multiple of a physical state. Namely, it follows from (4.4) that $\sup\{|T_\phi(a)| : a \in \mathcal{B}_{A^+}\} \leq 2$. Therefore, we only have to show that the restriction of T_ϕ to each singly generated subalgebra of $K(H_3)_{\mathrm{sa}}$ is linear.

Let b be an element in $K(H_3)_{\rm sa}$. We will distinguish two cases.

Case (a): b has finite spectrum. In this case, b is a finite rank operator and $b=\sum_{n=1}^m \mu_n p_n$, where $\mu_1,\ldots,\mu_m\in\mathbb{R}\backslash\{0\}$, and $\{p_n : n=1,\ldots,m\}$ is a family of mutually orthogonal minimal projections in $K(H_3)$. Elements x, y in the subalgebra of $K(H_3)_{\rm sa}$ generated by b can be written in the form $x=\sum_{n=1}^m x(n)p_n$, and $y=\sum_{n=1}^m y(n)p_n$, where $x(n),y(n)\in\mathbb{R}$. Let us set $\Theta_x^+=\{n\in\{1,\ldots,m\}: x(n)\geq 0\}$ and $\Theta_x^-=\{n\in\{1,\ldots,m\}: x(n)<0\}$. Suppose that $x^+,x^-\neq 0$. By applying the definition of T, we obtain

$$\begin{aligned}
T(x) &= \|x^+\|\Delta\Big(\frac{x^+}{\|x^+\|}\Big)-\|x^-\|\Delta\Big(\frac{x^-}{\|x^-\|}\Big)\\
&= \|x^+\|\Delta\Big(\sum_{n\in\Theta_x^+}\frac{x(n)}{\|x^+\|}p_n\Big)-\|x^-\|\Delta\Big(\sum_{n\in\Theta_x^-}\frac{-x(n)}{\|x^-\|}p_n\Big)\\
&= \|x^+\|\sum_{n\in\Theta_x^+}\frac{x(n)}{\|x^+\|}\Delta(p_n)-\|x^-\|\sum_{n\in\Theta_x^-}\frac{-x(n)}{\|x^-\|}\Delta(p_n)=\sum_{n=1}^m x(n)\Delta(p_n),
\end{aligned}$$

where the penultimate equality follows from Proposition 4.3(b). In the remaining cases (i.e., $\|x^+\|\|x^-\|=0$), we also have $T(x)=\sum_{n=1}^m x(n)\Delta(p_n)$. Since similar conclusions hold for y, $x+y$ and αx with $\alpha\in\mathbb{R}$, we deduce that

$$\begin{aligned}
T(x+y) &= \sum_{n=1}^m\big(x(n)+y(n)\big)\Delta(p_n)=\sum_{n=1}^m x(n)\Delta(p_n)+\sum_{n=1}^m y(n)\Delta(p_n)\\
&= T(x)+T(y)
\end{aligned}$$

and

$$T(\alpha x)=\sum_{n=1}^m(\alpha x)(n)\Delta(p_n)=\alpha\sum_{n=1}^m x(n)\Delta(p_n)=\alpha T(x),$$

which shows that T is linear on the subalgebra generated by b.

Case (b): b has infinite spectrum. In this case, $b=\sum_{n=1}^\infty\lambda_n p_n$, where $(\lambda_n)_n$ is a decreasing sequence in $\mathbb{R}\backslash\{0\}$ converging to zero and $\{p_n : n\in\mathbb{N}\}$ is a family of mutually orthogonal minimal projections in $K(H_3)$. Elements x and y in the subalgebra of $K(H_3)_{\rm sa}$ generated by b can be written in the form $x=\sum_{n=1}^\infty x(n)p_n$ and $y=\sum_{n=1}^\infty y(n)p_n$, where $(x(n))$ and $(y(n))$ are null sequences in $\mathbb{R}$. Keeping in mind the notation employed in the previous paragraph, we deduce that if $x^+,x^-\neq 0$, then we have

$$\begin{aligned}
T(x) &= \|x^+\|\Delta\Big(\frac{x^+}{\|x^+\|}\Big)-\|x^-\|\Delta\Big(\frac{x^-}{\|x^-\|}\Big)\\
&= \|x^+\|\Delta\Big(\sum_{n\in\Theta_x^+}\frac{x(n)}{\|x^+\|}p_n\Big)-\|x^-\|\Delta\Big(\sum_{n\in\Theta_x^-}\frac{-x(n)}{\|x^-\|}p_n\Big)=\text{(by (4.2))}\\
&= \|x^+\|\sum_{n\in\Theta_x^+}\frac{x(n)}{\|x^+\|}\Delta(p_n)-\|x^-\|\sum_{n\in\Theta_x^-}\frac{-x(n)}{\|x^-\|}\Delta(p_n)=\sum_{n=1}^\infty x(n)\Delta(p_n).
\end{aligned}$$

In the remaining cases, the identity

$$T(x) = \sum_{n=1}^{\infty} x(n)\Delta(p_n) \tag{4.5}$$

also holds. It is therefore clear that T is linear on the subalgebra generated by b.

We have therefore proved that $T_\phi : K(H_3)_{sa} \to \mathbb{R}$ is a positive multiple of a physical state for every $\phi \in \mathcal{B}_{(K(H_4)^*)^+}$. Applying [1, Corollary 2] to the complex linear extension of T_ϕ from $K(H_3)$ to $\mathbb{C}$, it follows that

$$\phi\big(T(x+y)\big) = T_\phi(x+y) = T_\phi(x) + T_\phi(y) = \phi\big(T(x) + T(y)\big)$$

and

$$\phi\big(T(\alpha x)\big) = T_\phi(\alpha x) = \alpha T_\phi(x) = \phi\big(\alpha T(x)\big),$$

for all $x, y \in K(H_3)_{sa}$, $\alpha \in \mathbb{R}$, and $\phi \in \mathcal{B}_{(K(H_4)^*)^+}$. Since functionals in $\mathcal{B}_{(K(H_4)^*)^+}$ separate the points in $K(H_4)_{sa}$, we deduce that $T : K(H_3)_{sa} \to K(H_4)_{sa}$ is real linear. We denote by the same symbol T the complex linear extension of T from $K(H_3)$ to $K(H_4)$. We have thus obtained a complex linear map $T : K(H_3) \to K(H_4)$ satisfying $T(a) = \Delta(a)$ for all $a \in S(K(H_3)^+)$ (cf. (4.2) and (4.5)). Corollary 4.4 assures that $T : K(H_3) \to K(H_4)$ is an isometric *-isomorphism or *-anti-isomorphism. □

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