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INTERPOLATING INEQUALITIES FOR FUNCTIONS OF POSITIVE SEMIDEFINITE MATRICES

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Abstract. Let A, B be positive semidefinite $n\times n$ matrices, and let $\alpha \in (0,1)$. We show that if f is an increasing submultiplicative function on $[0,\infty)$ with $f(0)=0$ such that $f(t)$ and $f^2(t^{1/2})$ are convex, then

$$\left|\left|\left| f(|AB|) \right|\right|\right|^2 \le f^4\Big(\frac{1}{(4\alpha(1-\alpha))^{1/4}}\Big)\Big(\left|\left|\left| \big(\alpha f(A)+(1-\alpha)f(B)\big)^2 \right|\right|\right| \times \left|\left|\left| \big((1-\alpha)f(A)+\alpha f(B)\big)^2 \right|\right|\right|\Big)$$

for every unitarily invariant norm. Moreover, if $\alpha \in [0,1]$ and X is an $n\times n$ matrix with $X \neq 0$, then

$$\left|\left|\left| f(|AXB|) \right|\right|\right|^2 \le \frac{f(\|X\|)}{\|X\|}\left|\left|\left| \alpha f^2(A)X+(1-\alpha)Xf^2(B) \right|\right|\right| \left|\left|\left| (1-\alpha)f^2(A)X+\alpha Xf^2(B) \right|\right|\right|$$

for every unitarily invariant norm. These inequalities present generalizations of recent results of Zou and Jiang and of Audenaert.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, let $\lambda_1(A),\dots,\lambda_n(A)$ be the eigenvalues of A repeated according to multiplicity. The

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singular values of A, denoted by $s_1(A), \ldots, s_n(A)$ are the eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity.

A matrix norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$ is said to be *unitarily invariant* if $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. A unitarily invariant norm is a symmetric gauge function of the singular values of the matrix; that is, $|||A||| = \Phi(s(A))$, where Φ is a symmetric gauge function defined on $\mathbb{R}^n$.

A useful property of unitarily invariant norms says that $|||ABC||| \leq \|A\| |||B||| \times \|C\|$ for all matrices $A, B, C \in \mathbb{M}_n(\mathbb{C})$. Here, $\|\cdot\|$ denotes the spectral (or the usual operator) norm. (For basic properties of unitarily invariant norms, we refer to [6], [12], or [15].)

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and arrange the components of x in decreasing order such that $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$, where $x_1^\downarrow \geq \cdots \geq x_n^\downarrow$. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we say that x is *weakly majorized* by y, written as $x \prec_w y$, if $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$, $k = 1, \ldots, n$.

It was shown in [10] that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$|||AB||| \leq \frac{1}{4}\left|\left|\left|(A+B)^2\right|\right|\right| \tag{1.1}$$

for every unitarily invariant norm. Recently, Zou and Jiang [20] obtained a natural generalization of the inequality (1.1). They showed that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite and $\alpha \in (0,1)$, then

$$|||AB|||^2 \leq \frac{1}{4\alpha(1-\alpha)}\left|\left|\left|\big(\alpha A + (1-\alpha)B\big)^2\right|\right|\right|\left|\left|\left|\big((1-\alpha)A + \alpha B\big)^2\right|\right|\right| \tag{1.2}$$

for every unitarily invariant norm.

The arithmetic-geometric mean inequality for unitarily invariant norms (see [9]) asserts that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$|||AB||| \leq \frac{1}{2}|||A^2 + B^2||| \tag{1.3}$$

for every unitarily invariant norm. A generalization of (1.3) (see, e.g., [2], [7], [18]) asserts that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$|||AXB||| \leq \frac{1}{2}|||A^2X + XB^2||| \tag{1.4}$$

for every unitarily invariant norm.

The Cauchy–Schwarz inequality for unitarily invariant norms (see [16]) asserts that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$|||AB|||^2 \leq |||A^2||| \, |||B^2||| \tag{1.5}$$

for every unitarily invariant norm. A generalization of (1.5) (see, e.g., [8]) asserts that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$|||AXB|||^2 \leq |||A^2X||| \, |||XB^2||| \tag{1.6}$$

for every unitarily invariant norm. (For more details about the Cauchy–Schwarz inequality, we refer to [14] and [17].)

Recently, Audenaert [4] showed that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite and $\alpha \in [0,1]$, then

$$|||AB|||^2 \le \left|\left|\left|\alpha A^2 + (1-\alpha)B^2\right|\right|\right|\left|\left|\left|(1-\alpha)A^2 + \alpha B^2\right|\right|\right| \tag{1.7}$$

for every unitarily invariant norm. The inequality (1.7) interpolates between the inequalities (1.3) and (1.5). In fact, letting $\alpha = 1/2$, we obtain (1.3), but letting $\alpha = 0$ or $\alpha = 1$, we obtain (1.5).

A generalization of (1.7) was recently obtained in [20]. The authors showed that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite and $\alpha \in [0,1]$, then

$$|||AXB|||^2 \le \left|\left|\left|\alpha A^2X + (1-\alpha)XB^2\right|\right|\right|\left|\left|\left|(1-\alpha)A^2X + \alpha XB^2\right|\right|\right| \tag{1.8}$$

for every unitarily invariant norm, which interpolates between the inequalities (1.4) and (1.6). A refinement of (1.8) for the Hilbert–Schmidt norm was recently given in [3]. (For related interpolating inequalities, the reader is referred to [1], [5], [13], and [19].)

In this article, we give interpolating norm inequalities for functions of matrices. In Sections 2 and 3, we generalize (1.2), (1.7), and (1.8) for certain functions of matrices. Henceforth, we assume that every function is continuous.

2. A generalization of the inequality (1.2)

In this section, we generalize (1.2) for submultiplicative convex functions of matrices. In order to do that, we need the following lemmas. The first lemma is given in [4, p. 6]. The second lemma is a well-known fact about singular values and increasing functions, while for the third lemma we refer our readers to [6, p. 42]. First, for two positive semidefinite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, let $C_{A,B}(t) = tA^2 + (1-t)B^2$, $t \in [0,1]$.

Lemma 2.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $\alpha \in [0,1]$, $r > 0$. Then*

$$\left(s_j\left(|AB|^{2r}\right)\right) \prec_w \left(s_j^r\left(C_{A,B}(\alpha)\right)s_j^r\left(C_{A,B}(1-\alpha)\right)\right).$$

Lemma 2.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$, and let f be a nonnegative increasing function on $[0,\infty)$. Then $f(s_j(A)) = s_j(f(|A|))$ for $j = 1, \dots, n$.*

Lemma 2.3. *Let $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$ such that $(a_1, \dots, a_n) \prec_w (b_1, \dots, b_n)$, and let f be an increasing convex function on $\mathbb{R}$. Then*

$$\left(f(a_1), \dots, f(a_n)\right) \prec_w \left(f(b_1), \dots, f(b_n)\right).$$

Lemma 2.4. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0,1]$. If f is an increasing submultiplicative convex function on $[0,\infty)$ with $f(0) = 0$, then*

$$\begin{aligned} &\left|\left|\left|f\left(|AXB|^{2r}\right)\right|\right|\right| \\ &\quad \le \left|\left|\left|f^p\left(\left(X^{1/2}C_{A,B}(\alpha)X^{1/2}\right)^r\right)\right|\right|\right|^{1/p}\left|\left|\left|f^q\left(\left(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\right)^r\right)\right|\right|\right|^{1/q} \end{aligned}$$

for every unitarily invariant norm.

Proof. For $j = 1, \ldots, n$, we have

$$\begin{aligned}
&\big(s_j\big(|AXB|^{2r}\big)\big)\\
&\quad = \big(s_j\big(\big|(AX^{1/2})(BX^{1/2})^*\big|^{2r}\big)\big)\\
&\quad \prec_w \big(s_j^r\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)s_j^r\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)\big)\\
&\qquad \text{(by Lemma 2.1).}
\end{aligned}\tag{2.1}$$

The monotonicity and convexity of f implies that

$$\begin{aligned}
&\big(s_j\big(f\big(|AXB|^{2r}\big)\big)\big)\\
&\quad = \big(f\big(s_j\big(|AXB|^{2r}\big)\big)\big) \quad \text{(by Lemma 2.2)}\\
&\quad \prec_w \big(f\big(s_j^r\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)s_j^r\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)\big)\big)\\
&\qquad \text{(by (2.1) and Lemma 2.3).}
\end{aligned}\tag{2.2}$$

Since f is submultiplicative, we have

$$\begin{aligned}
&f\big(s_j^r\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)s_j^r\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)\big)\\
&\quad \le f\big(s_j^r\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)\big)f\big(s_j^r\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)\big)\\
&\quad = s_j\big(f\big(\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)^r\big)\big)s_j\big(f\big(\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)^r\big)\big).
\end{aligned}\tag{2.3}$$

The inequalities (2.2) and (2.3) imply that

$$\begin{aligned}
&\big(s_j\big(f\big(|AXB|^{2r}\big)\big)\big)\\
&\quad \prec_w \big(s_j\big(f\big(\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)^r\big)\big)s_j\big(f\big(\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)^r\big)\big)\big).
\end{aligned}\tag{2.4}$$

In (2.4), applying Hölder's inequality for the symmetric gauge function Φ associated with $|||\cdot|||$ (see, e.g., [6, p. 88]), we have

$$\begin{aligned}
&\Phi\big(s_j\big(f\big(|AXB|^{2r}\big)\big)\big)\\
&\quad \le \Phi\big(s_j\big(f\big(\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)^r\big)\big)s_j\big(f\big(\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)^r\big)\big)\big)\\
&\quad \le \big(\Phi^{1/p}\big(s_j^p\big(f\big(\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)^r\big)\big)\big)\\
&\qquad \times \Phi^{1/q}\big(s_j^q\big(f\big(\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)^r\big)\big)\big)\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\big|\big|\big|f\big(|AXB|^{2r}\big)\big|\big|\big|\\
&\quad \le \big|\big|\big|f^p\big(\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)^r\big)\big|\big|\big|^{1/p}\big|\big|\big|f^q\big(\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)^r\big)\big|\big|\big|^{1/q},
\end{aligned}$$

as required. □

We also need the following two lemmas. The first lemma is the inequality (2.3) in [20], while the second lemma follows by using the polar decomposition and some basic properties of unitarily invariant norms.

Lemma 2.5. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$\big|\big|\big|X^{1/2}C_{A,B}(\alpha)X^{1/2}\big|\big|\big| \le \big|\big|\big|\alpha A^2X + (1-\alpha)XB^2\big|\big|\big|$$

for every unitarily invariant norm.

Lemma 2.6. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$, and let $r > 0$. If f is a nonnegative increasing function on $[0, \infty)$, then*

$$\left|\left|\left| f\left(|AXB^*|^{2r}\right) \right|\right|\right| = \left|\left|\left| f\left(\left||A|X|B|\right|^{2r}\right) \right|\right|\right|$$

for every unitarily invariant norm.

Now, we state our first main result.

Theorem 2.7. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0, 1]$. If f is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex, then*

$$\begin{aligned} &\left|\left|\left| f\left(|AXB|^{2r}\right) \right|\right|\right| \\ &\quad \le \left|\left|\left| f^p\left(\left|\alpha A^2 X + (1-\alpha) X B^2\right|^r\right) \right|\right|\right|^{1/p} \left|\left|\left| f^q\left(\left|(1-\alpha) A^2 X + \alpha X B^2\right|^r\right) \right|\right|\right|^{1/q} \end{aligned}$$

for every unitarily invariant norm.

Proof. Let $X = UDV^*$ be a singular value decomposition of X. In Lemma 2.4, replacing A, B, X by the matrices AU, BV, and D, respectively, we have

$$\begin{aligned} \left|\left|\left| f\left(|AXB|^{2r}\right) \right|\right|\right| &= \left|\left|\left| f\left(\left|(AU)D(BV)^*\right|^{2r}\right) \right|\right|\right| \\ &= \left|\left|\left| f\left(\left||AU|D|BV|\right|^{2r}\right) \right|\right|\right| \quad \text{(by Lemma 2.6)} \\ &\le \Big(\left|\left|\left| f^p\left(\left(D^{1/2} C_{|AU|,|BV|}(\alpha) D^{1/2}\right)^r\right) \right|\right|\right|^{1/p} \\ &\qquad \times \left|\left|\left| f^q\left(\left(D^{1/2} C_{|AU|,|BV|}(1-\alpha) D^{1/2}\right)^r\right) \right|\right|\right|^{1/q} \Big). \end{aligned} \tag{2.5}$$

Since $f^p(t^r)$ is convex, we have

$$\begin{aligned} &\left|\left|\left| f^p\left(\left(D^{1/2} C_{|AU|,|BV|}(\alpha) D^{1/2}\right)^r\right) \right|\right|\right| \\ &\quad \le \left|\left|\left| f^p\left(\left(\left|\alpha |AU|^2 D + (1-\alpha) D |BV|^2\right|^r\right)\right) \right|\right|\right| \quad \text{(by Lemmas 2.3 and 2.5)} \\ &\quad = \left|\left|\left| f^p\left(\left|\alpha U^* A^2 U D + (1-\alpha) D V^* B^2 V\right|^r\right) \right|\right|\right| \\ &\quad = \left|\left|\left| f^p\left(\left|U^*\left(\alpha A^2 UDV^* + (1-\alpha) UDV^* B^2\right) V\right|^r\right) \right|\right|\right| \\ &\quad = \left|\left|\left| f^p\left(\left|U^*\left(\alpha A^2 X + (1-\alpha) X B^2\right) V\right|^r\right) \right|\right|\right| \\ &\quad = \left|\left|\left| f^p\left(\left(V^* \left|\alpha A^2 X + (1-\alpha) X B^2\right| V\right)^r\right) \right|\right|\right| \\ &\quad = \left|\left|\left| V^* f^p\left(\left|\alpha A^2 X + (1-\alpha) X B^2\right|^r\right) V \right|\right|\right| \\ &\quad = \left|\left|\left| f^p\left(\left|\alpha A^2 X + (1-\alpha) X B^2\right|^r\right) \right|\right|\right|. \end{aligned} \tag{2.6}$$

Similarly, we have

$$\left|\left|\left| f^q\left(\left(D^{1/2} C_{|AU|,|BV|}(1-\alpha) D^{1/2}\right)^r\right) \right|\right|\right| \le \left|\left|\left| f^q\left(\left|(1-\alpha) A^2 X + \alpha X B^2\right|^r\right) \right|\right|\right|. \tag{2.7}$$

Now, the result follows from (2.5), (2.6), and (2.7). □

An application of Theorem 2.7 can be seen in the following corollary.

Corollary 2.8. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r \geq 1$, and $\alpha \in [0,1]$. If f is an increasing submultiplicative convex function on $[0, \infty)$ with $f(0) = 0$, then*

$$\begin{aligned} &\left|\left|\left| f\left(|AXB|^{2r}\right)\right|\right|\right| \\ &\quad \leq \left|\left|\left| f^p\left(\left|\alpha A^2X + (1-\alpha)XB^2\right|^r\right)\right|\right|\right|^{1/p} \left|\left|\left| f^q\left(\left|(1-\alpha)A^2X + \alpha XB^2\right|^r\right)\right|\right|\right|^{1/q} \end{aligned}$$

for every unitarily invariant norm. In particular, letting $p = q = 2$ and $r = 1$, we have

$$\begin{aligned} &\left|\left|\left| f\left(|AXB|^{2}\right)\right|\right|\right| \\ &\quad \leq \left|\left|\left| f^2\left(\left|\alpha A^2X + (1-\alpha)XB^2\right|\right)\right|\right|\right|^{1/2} \left|\left|\left| f^2\left(\left|(1-\alpha)A^2X + \alpha XB^2\right|\right)\right|\right|\right|^{1/2}. \end{aligned}$$

Proof. The convexity of $f(t)$ implies that $f^{\min(p,q)}(t^r)$ is also convex, and so the result follows directly from Theorem 2.7. $\square$

We also need the following lemma concerning singular values (see [11]).

Lemma 2.9. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $r \geq 0$. Then*

$$2s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \leq s_j\left((A+B)^{r+1}\right)$$

for $j = 1, \ldots, n$.

Remark 2.10. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $\alpha \in (0,1)$, $r > 0$. In Lemma 2.9, replacing A and B by αA and $(1-\alpha)B$, respectively, we have

$$s_j\left(A^{1/2}C^r_{A^{1/2},B^{1/2}}(\alpha)B^{1/2}\right) \leq \frac{1}{2\sqrt{\alpha(1-\alpha)}} s_j\left(C^{r+1}_{A^{1/2},B^{1/2}}(\alpha)\right)$$

for $j = 1, \ldots, n$, and so

$$\left|\left|\left| A^{1/2}C^r_{A^{1/2},B^{1/2}}(\alpha)B^{1/2}\right|\right|\right| \leq \frac{1}{2\sqrt{\alpha(1-\alpha)}} \left|\left|\left| C^{r+1}_{A^{1/2},B^{1/2}}(\alpha)\right|\right|\right|$$

for every unitarily invariant norm. In particular, letting $r = 1$, we have

$$\left|\left|\left| A^{1/2}C_{A^{1/2},B^{1/2}}(\alpha)B^{1/2}\right|\right|\right| \leq \frac{1}{2\sqrt{\alpha(1-\alpha)}} \left|\left|\left| C^{2}_{A^{1/2},B^{1/2}}(\alpha)\right|\right|\right| \tag{2.8}$$

for every unitarily invariant norm.

Our second main result in this section can be stated as follows. For a function $f : \mathbb{R} \to \mathbb{R}$, let $K(f; r, \alpha) = f^2\left(\frac{1}{(4\alpha(1-\alpha))^{r/2}}\right)$ for $r > 0$ and $\alpha \in (0,1)$.

Theorem 2.11. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in (0,1)$. If $f(t)$ is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex, then*

$$\begin{aligned} &\left|\left|\left| f\left(|AB|^{2r}\right)\right|\right|\right| \\ &\quad \leq K(f; r, \alpha) \left|\left|\left| f^p\left(C^{2r}_{A^{1/2},B^{1/2}}(\alpha)\right)\right|\right|\right|^{1/p} \left|\left|\left| f^q\left(C^{2r}_{A^{1/2},B^{1/2}}(1-\alpha)\right)\right|\right|\right|^{1/q} \end{aligned}$$

for every unitarily invariant norm.

Proof. In Theorem 2.7, letting $X = A^{1/2}B^{1/2}$, we have

$$\begin{aligned}\big|\big|\big|f\big(|AB|^{2r}\big)\big|\big|\big| &= \big|\big|\big|f\big(\big|A^{1/2}(A^{1/2}B^{1/2})B^{1/2}\big|^{2r}\big)\big|\big|\big| \\ &\leq \Big(\big|\big|\big|f^p\big(\big|\alpha A^{3/2}B^{1/2} + (1-\alpha)A^{1/2}B^{3/2}\big|^r\big)\big|\big|\big|^{1/p} \\ &\quad \times \big|\big|\big|f^q\big(\big|(1-\alpha)A^{3/2}B^{1/2} + \alpha A^{1/2}B^{3/2}\big|^r\big)\big|\big|\big|^{1/q}\Big) \\ &\quad \text{(by Theorem 2.7)} \\ &= \Big(\big|\big|\big|f^p\big(\big|A^{1/2}C_{A^{1/2},B^{1/2}}(\alpha)B^{1/2}\big|^r\big)\big|\big|\big|^{1/p} \\ &\quad \times \big|\big|\big|f^q\big(\big|A^{1/2}C_{A^{1/2},B^{1/2}}(1-\alpha)B^{1/2}\big|^r\big)\big|\big|\big|^{1/q}\Big). \end{aligned} \tag{2.9}$$

On the other hand,

$$\begin{aligned}&\big|\big|\big|f^p\big(\big|A^{1/2}C_{A^{1/2},B^{1/2}}(\alpha)B^{1/2}\big|^r\big)\big|\big|\big| \\ &\leq \bigg|\bigg|\bigg|f^p\bigg(\Big(\frac{C_{A^{1/2},B^{1/2}}(\alpha)}{\sqrt[4]{4\alpha(1-\alpha)}}\Big)^{2r}\bigg)\bigg|\bigg|\bigg| \quad \text{(by (2.8) and Lemma 2.3)} \\ &\leq f^p\Big(\frac{1}{(4\alpha(1-\alpha))^{r/2}}\Big)\big|\big|\big|f^p\big(C^{2r}_{A^{1/2},B^{1/2}}(\alpha)\big)\big|\big|\big| \\ &\quad \text{(since } f \text{ is submultiplicative)} \\ &= K^{p/2}(f;r,\alpha)\big|\big|\big|f^p\big(C^{2r}_{A^{1/2},B^{1/2}}(\alpha)\big)\big|\big|\big|. \end{aligned} \tag{2.10}$$

Similarly, we have

$$\begin{aligned}&\big|\big|\big|f^q\big(\big|A^{1/2}C_{A^{1/2},B^{1/2}}(1-\alpha)B^{1/2}\big|\big)\big|\big|\big| \\ &\quad \leq K^{q/2}(f;r,\alpha)\big|\big|\big|f^q\big(C^{2r}_{A^{1/2},B^{1/2}}(1-\alpha)\big)\big|\big|\big|. \end{aligned} \tag{2.11}$$

Now, the result follows from (2.9), (2.10), and (2.11). □

We need the following lemma, which is a Jensen-type inequality. A more general form of this lemma is given in Lemma 3.1.

Lemma 2.12. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $\alpha \in [0,1]$. If $f : \mathbb{R} \to \mathbb{R}$ is an increasing convex function with $f(0) = 0$, then*

$$\big|\big|\big|f\big(C_{A^{1/2},B^{1/2}}(\alpha)\big)\big|\big|\big| \leq \big|\big|\big|C_{f^{1/2}(A),f^{1/2}(B)}(\alpha)\big|\big|\big|$$

for every unitarily invariant norm.

Using Lemma 2.12, an application of Theorem 2.11 can be seen in the following corollary.

Corollary 2.13. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in (0,1)$. If f is an increasing submultiplicative function on $[0,\infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex, then*

$$\begin{aligned}&\big|\big|\big|f\big(|AB|^{2r}\big)\big|\big|\big| \\ &\quad \leq K(f;r,\alpha)\big|\big|\big|C_{f^{p/2}(A^{2r}),f^{p/2}(B^{2r})}(\alpha)\big|\big|\big|^{1/p}\big|\big|\big|C_{f^{q/2}(A^{2r}),f^{q/2}(B^{2r})}(1-\alpha)\big|\big|\big|^{1/q}\end{aligned}$$

for every unitarily invariant norm.

Proof. The convexity of the function $f^{\min(p,q)}(t^r)$ implies that $f^p(t^{2r})$ and $f^q(t^{2r})$ are convex, and so Lemma 2.12 implies that

$$|||f^p\big(C^{2r}_{A^{1/2},B^{1/2}}(\alpha)\big)||| \le |||C_{f^{p/2}(A^{2r}),f^{p/2}(B^{2r})}(\alpha)||| \tag{2.12}$$

and that

$$|||f^q\big(C^{2r}_{A^{1/2},B^{1/2}}(1-\alpha)\big)||| \le |||C_{f^{q/2}(A^{2r}),f^{q/2}(B^{2r})}(1-\alpha)|||. \tag{2.13}$$

Now, the result follows from Theorem 2.11, (2.12), and (2.13). □

The following result is a considerable generalization of the Zou–Jiang inequality (1.2) for functions of matrices. In fact, (1.2) can be retained by taking $f(t) = t$.

Corollary 2.14. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $\alpha \in (0,1)$. If f is an increasing submultiplicative function on $[0,\infty)$ with $f(0) = 0$ such that $f(t)$ and $f^2(t^{1/2})$ are convex, then*

$$|||f\big(|AB|\big)|||^2 \le K^2\Big(f;\frac{1}{2},\alpha\Big)|||C^2_{f^{1/2}(A),f^{1/2}(B)}(\alpha)|||\,|||C^2_{f^{1/2}(A),f^{1/2}(B)}(1-\alpha)|||$$

for every unitarily invariant norm.

Proof. The result follows from Theorem 2.11 by taking $p = q = 2$ and $r = \frac{1}{2}$. □

Corollary 2.15. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $p, q, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $\alpha \in (0,1)$. If f is an increasing submultiplicative convex function on $[0,\infty)$ with $f(0) = 0$, then*

$$\begin{aligned}&|||f\big(|AB|^2\big)|||\\ &\quad\le K(f;1,\alpha)|||C^p_{f^{1/2}(A^2),f^{1/2}(B^2)}(\alpha)|||^{1/p}|||C^q_{f^{1/2}(A^2),f^{1/2}(B^2)}(1-\alpha)|||^{1/q}\end{aligned}$$

for every unitarily invariant norm.

Proof. The convexity of the function $f(t)$ implies that $f^{\min(p,q)}(t)$ is convex. So, the result follows from Theorem 2.11 by taking $r = 1$ and applying Lemmas 2.3, 2.12. □

Specializing Theorem 2.11 to some particular functions, we have the following result.

Corollary 2.16. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $c, p, q, r, \alpha \in \mathbb{R}$ such that $c \ge 1, p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r \ge 1$, and $\alpha \in (0,1)$. Then*

$$\begin{aligned}&|||\big(|AB|^{2r} + cI\big)^m - c^m I|||\\ &\quad\le \mathcal{K}(r,\alpha)\Big(|||\big(\big(C_{A^{1/2},B^{1/2}}(\alpha) + cI\big)^{mr} - c^m I\big)^p|||^{1/p}\\ &\qquad\times |||\big(\big(C_{A^{1/2},B^{1/2}}(1-\alpha) + cI\big)^{mr} - c^m I\big)^q|||^{1/q}\Big)\end{aligned}$$

for every unitarily invariant norm and for $m = 2, 3, \ldots$, where $\mathcal{K}(r,\alpha) = ((\frac{1}{(4\alpha(1-\alpha))^{r/2}} + c)^m - c^m)^2$ and I is the identity matrix in $\mathbb{M}_n(\mathbb{C})$.

Proof. Let $f(t) = (t+c)^m - c^m$. Then the functions $f(t)$ and $f^{\min(p,q)}(t^r)$ are increasing submultiplicative convex functions on $[0,\infty)$ with $f(0) = 0$. So, the result follows by applying Theorem 2.11 to the function $f(t)$. □

3. Generalizations of the inequalities (1.7) and (1.8)

In this section, our goal is to generalize (1.7) and (1.8) for submultiplicative convex functions. First, we start with generalizing (1.7). In order to do that, we need the following lemma (see, e.g., [6, p. 119]). This lemma is a general form of Lemma 2.12.

Lemma 3.1. *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite and $|X|^2 + |Y|^2 \leq I$, and let f be an increasing convex function on $[0, \infty)$ with $f(0) = 0$. Then*

$$\left|\left|\left|f(X^*AX + Y^*BY)\right|\right|\right| \leq \left|\left|\left|X^*f(A)X + Y^*f(B)Y\right|\right|\right|$$

for every unitarily invariant norm.

The following result presents a generalization of (1.7).

Theorem 3.2. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, let X be a contraction, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0, 1]$. If f is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex, then*

$$\begin{aligned}\left|\left|\left|f\big(|AXB|^{2r}\big)\right|\right|\right| \leq \Big(&\left|\left|\left|X^{1/2}C_{f^{p/2}(A^{2r}),f^{p/2}(B^{2r})}(\alpha)X^{1/2}\right|\right|\right|^{1/p} \\ &\times \left|\left|\left|X^{1/2}C_{f^{q/2}(A^{2r}),f^{q/2}(B^{2r})}(1-\alpha)X^{1/2}\right|\right|\right|^{1/q}\Big)\end{aligned}$$

for every unitarily invariant norm.

Proof. The convexity of the function $f^{\min(p,q)}(t^r)$ implies that $f^p(t^r)$ is convex on $[0, \infty)$. Since $X^{1/2}$ is a contraction, Lemma 3.1 implies that

$$\begin{aligned}&\left|\left|\left|f^p\Big(\big(X^{1/2}C_{A,B}(\alpha)X^{1/2}\big)^r\Big)\right|\right|\right| \\ &= \left|\left|\left|f^p\Big(\big(\alpha X^{1/2}A^2X^{1/2} + (1-\alpha)X^{1/2}B^2X^{1/2}\big)^r\Big)\right|\right|\right| \\ &\leq \left|\left|\left|X^{1/2}\big(\alpha f^p(A^{2r}) + (1-\alpha)f^p(B^{2r})\big)X^{1/2}\right|\right|\right| \\ &= \left|\left|\left|X^{1/2}\big(C_{f^{p/2}(A^{2r}),f^{p/2}(B^{2r})}(\alpha)\big)X^{1/2}\right|\right|\right|.\end{aligned} \tag{3.1}$$

Similarly, we have

$$\begin{aligned}&\left|\left|\left|f^q\Big(\big(X^{1/2}C_{A,B}(1-\alpha)X^{1/2}\big)^r\Big)\right|\right|\right| \\ &\leq \left|\left|\left|X^{1/2}\big(C_{f^{q/2}(A^{2r}),f^{q/2}(B^{2r})}(1-\alpha)\big)X^{1/2}\right|\right|\right|.\end{aligned} \tag{3.2}$$

Now, the result follows from Lemma 2.4, (3.1), and (3.2). □

The following application of Theorem 3.2 is a considerable generalization of the Audenaert inequality (1.7) for functions of matrices. In fact, (1.7) can be retained by taking $f(t) = t$.

Corollary 3.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $\alpha \in [0, 1]$. If f is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^2(t^{1/2})$ are convex, then*

$$\left|\left|\left|f\big(|AB|\big)\right|\right|\right|^2 \leq \left|\left|\left|C_{f(A),f(B)}(\alpha)\right|\right|\right|\left|\left|\left|C_{f(A),f(B)}(1-\alpha)\right|\right|\right|$$

for every unitarily invariant norm.

Proof. The result follows from Theorem 3.2 by taking $X = I$, $p = q = 2$, and $r = \frac{1}{2}$. $\square$

Specializing Corollary 3.3 to some particular functions, we have the following result.

Corollary 3.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $c, \alpha \in \mathbb{R}$ with $c \geq 1$ and $\alpha \in [0,1]$. Then*

$$\begin{aligned}
&\left|\left|\left|\big(|AB| + cI\big)^m - c^m I\right|\right|\right|^2 \\
&\quad \leq \Big(\left|\left|\left|\alpha\big(\big((A+cI)^m - c^m I\big)\big)^2 + (1-\alpha)\big(\big((B+cI)^m - c^m I\big)\big)^2\right|\right|\right| \\
&\qquad \times \left|\left|\left|(1-\alpha)\big(\big((A+cI)^m - c^m I\big)\big)^2 + \alpha\big(\big((B+cI)^m - c^m I\big)\big)^2\right|\right|\right|\Big)
\end{aligned}$$

for every unitarily invariant norm and for $m = 2, 3, \ldots$.

Proof. The result follows by applying Corollary 3.3 to the function $f(t) = (t + c)^m - c^m$. $\square$

In the rest of this section, we generalize (1.8). Our first generalization can be stated as follows.

Theorem 3.5. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite and $X \neq 0$, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0,1]$. If f is an increasing submultiplicative function on $[0,\infty)$ with $f(0) = 0$ such that f and $f^{\min(p,q)}(t^r)$ are convex, then*

$$\begin{aligned}
\left|\left|\left|f\Big(\frac{|AXB|^{2r}}{\|X\|^{2r}}\Big)\right|\right|\right| \leq \frac{1}{\|X\|}\Big(&\left|\left|\left|\alpha f^p(A^{2r})X + (1-\alpha)Xf^p(B^{2r})\right|\right|\right|^{1/p} \\
&\times \left|\left|\left|(1-\alpha)f^q(A^{2r})X + \alpha Xf^q(B^{2r})\right|\right|\right|^{1/q}\Big)
\end{aligned}$$

for every unitarily invariant norm.

Proof. Let $Y = UDV^*$ be a singular value decomposition of $Y = \frac{X}{\|X\|}$. Then U, V are unitary matrices and D is a positive semidefinite contraction. Consequently,

$$\begin{aligned}
&\left|\left|\left|f\Big(\frac{|AXB|}{\|X\|}\Big)^{2r}\right|\right|\right| \\
&\quad = \left|\left|\left|f\big(|AYB|^{2r}\big)\right|\right|\right| \\
&\quad = \left|\left|\left|f\big(\big|(AU)D(BV)^*\big|^{2r}\big)\right|\right|\right| \\
&\quad = \left|\left|\left|f\big(\big||AU|D|BV|\big|^{2r}\big)\right|\right|\right| \quad \text{(by Lemma 2.6)} \\
&\quad \leq \Big(\left|\left|\left|D^{1/2}\big(C_{f^{p/2}(|AU|^{2r}),f^{p/2}(|BV|^{2r})}(\alpha)\big)D^{1/2}\right|\right|\right|^{1/p} \\
&\qquad \times \left|\left|\left|D^{1/2}\big(C_{f^{q/2}(|AU|^{2r}),f^{q/2}(|BV|^{2r})}(1-\alpha)\big)D^{1/2}\right|\right|\right|^{1/q}\Big) \\
&\qquad \text{(by Theorem 3.2)}. \qquad (3.3)
\end{aligned}$$

Now,

$$\begin{aligned}
&\left|\left|\left|D^{1/2}C_{f^{p/2}(|AU|^{2r}),f^{p/2}(|BV|^{2r})}(\alpha)D^{1/2}\right|\right|\right| \\
&\quad\le \frac{1}{2}\left|\left|\left|DC_{(U^*f^p(A^{2r})U)^{1/2},(V^*f^p(B^{2r})V)^{1/2}}(\alpha)\right.\right.\right. \\
&\qquad \left.\left.\left.+\, C_{(U^*f^p(A^{2r})U)^{1/2},(V^*f^p(B^{2r})V)^{1/2}}(\alpha)D\right|\right|\right| \\
&\qquad (\text{by } (1.4)) \\
&\quad = \left|\left|\left|\operatorname{Re}\big(\alpha U^*f^p(A^{2r})UD + (1-\alpha)DV^*f^p(B^{2r})V\big)\right|\right|\right| \\
&\quad \le \left|\left|\left|\alpha U^*f^p(A^{2r})UD + (1-\alpha)DV^*f^p(B^{2r})V\right|\right|\right| \\
&\quad = \frac{\left|\left|\left|\alpha f^p(A^{2r})X + (1-\alpha)Xf^p(B^{2r})\right|\right|\right|}{\|X\|}.
\end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}
&\left|\left|\left|D^{1/2}C_{f^{q/2}(A^{2r}),f^{q/2}(B^{2r})}(1-\alpha)D^{1/2}\right|\right|\right| \\
&\quad \le \frac{|||(1-\alpha)f^q(A^{2r})X + \alpha Xf^q(B^{2r})|||}{\|X\|}.
\end{aligned} \tag{3.5}$$

Now, the result follows from (3.3), (3.4), and (3.5). □

An application of Theorem 3.5 can be seen in the following corollary.

Corollary 3.6. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite and $X \neq 0$, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0, 1]$. If f is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex functions, then*

$$\begin{aligned}
&\left|\left|\left|f\big(|AXB|^{2r}\big)\right|\right|\right| \\
&\quad \le \frac{f(\|X\|^{2r})}{\|X\|}\Big(\left|\left|\left|\alpha f^p(A^{2r})X + (1-\alpha)Xf^p(B^{2r})\right|\right|\right|^{1/p} \\
&\qquad \times \left|\left|\left|(1-\alpha)f^q(A^{2r})X + \alpha Xf^q(B^{2r})\right|\right|\right|^{1/q}\Big)
\end{aligned}$$

for every unitarily invariant norm.

Proof. Let $Y = \frac{X}{\|X\|}$ and $a = \|X\|$. Then Y is a contraction with $\|Y\| = 1$, and so

$$\begin{aligned}
&\left|\left|\left|f\big(|AXB|^{2r}\big)\right|\right|\right| \\
&\quad = \left|\left|\left|f\big(a^{2r}|AYB|^{2r}\big)\right|\right|\right| \\
&\quad \le f(a^{2r})\left|\left|\left|f\big(|AYB|^{2r}\big)\right|\right|\right| \ (\text{since } f \text{ is submultiplicative}) \\
&\quad \le \frac{f(a^{2r})}{\|Y\|}\Big(\left|\left|\left|\alpha f^p(A^{2r})Y + (1-\alpha)Yf^p(B^{2r})\right|\right|\right|^{1/p} \\
&\qquad \times \left|\left|\left|(1-\alpha)f^q(A^{2r})Y + \alpha Yf^q(B^{2r})\right|\right|\right|^{1/q}\Big) \\
&\qquad (\text{by Theorem } 3.5)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(\|X\|^{2r})}{\|X\|}\Big(\big|\big|\big|\alpha f^p(A^{2r})X + (1-\alpha)Xf^p(B^{2r})\big|\big|\big|^{1/p} \\
&\quad \times \big|\big|\big|(1-\alpha)f^q(A^{2r})X + \alpha X f^q(B^{2r})\big|\big|\big|^{1/q}\Big),
\end{aligned}
$$

as required. □

The following application of Corollary 3.6 is a considerable generalization of the Zou–Jiang inequality (1.8) for functions of matrices. In fact, (1.8) can be retained by taking $f(t) = t$.

Corollary 3.7. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite and $X \neq 0$, and let $\alpha \in [0,1]$. If f is an increasing submultiplicative function on $[0,\infty)$ such that $f(t)$ and $f^2(t^{1/2})$ are convex with $f(0) = 0$, then*

$$
\begin{aligned}
\big|\big|\big|f\big(|AXB|\big)\big|\big|\big|^2 \le \frac{f(\|X\|)}{\|X\|}\Big(&\big|\big|\big|\alpha f^2(A)X + (1-\alpha)Xf^2(B)\big|\big|\big| \\
&\times \big|\big|\big|(1-\alpha)f^2(A)X + \alpha X f^2(B)\big|\big|\big|\Big)
\end{aligned}
$$

for every unitarily invariant norm.

Proof. The result follows from Corollary 3.6 by taking $r = \frac{1}{2}$ and $p = q = 2$. □

To give another application of Theorem 3.5, we need the following lemma.

Lemma 3.8. *Let $Y \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix, and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function with $f(0) \le 0$. Then*
(a) *$f(\alpha Y) \le \alpha f(Y)$ for $0 \le \alpha \le 1$,*
(b) *$f(\alpha Y) \ge \alpha f(Y)$ for $1 \le \alpha < \infty$.*

Proof. Let $Y = UDU^*$ be a spectral decomposition of Y with $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, and let $0 \le \alpha \le 1$. Then

$$
\begin{aligned}
f(\alpha Y) &= Uf(\alpha D)U^* \\
&= U \operatorname{diag}\big(f(\alpha\lambda_1), \dots, f(\alpha\lambda_n)\big)U^*.
\end{aligned} \tag{3.6}
$$

Also, we have

$$
\begin{aligned}
f(\alpha\lambda_j) &= f\big(\alpha\lambda_j + (1-\alpha)0\big) \\
&\le \alpha f(\lambda_j) + (1-\alpha)f(0) \quad \text{(since } f \text{ is convex)} \\
&\le \alpha f(\lambda_j) \quad \text{(since } f(0) \le 0\text{)}
\end{aligned} \tag{3.7}
$$

for $j = 1, \dots, n$. The relation (3.6) and the inequality (3.7) imply that

$$
\begin{aligned}
f(\alpha Y) &\le U \operatorname{diag}\big(\alpha f(\lambda_1), \dots, \alpha f(\lambda_n)\big)U^* \\
&= \alpha U f(D) U^* \\
&= \alpha f(Y).
\end{aligned}
$$

This proves part (a). The proof of part (b) follows from part (a) by replacing Y and α by αY and $\frac{1}{\alpha}$, respectively. □

An application of Theorem 3.5 can be seen in the following corollary.

Corollary 3.9. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are positive semidefinite and X is a contraction with $X \neq 0$, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0, 1]$. If f is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex, then*

$$\big|\big|\big| f\big(|AXB|^{2r}\big) \big|\big|\big| \le \big(\|X\|^{2r-1} \big|\big|\big| \alpha f^p(A^{2r})X + (1-\alpha)Xf^p(B^{2r}) \big|\big|\big|^{1/p} \\ \times \big|\big|\big| (1-\alpha)f^q(A^{2r})X + \alpha X f^q(B^{2r}) \big|\big|\big|^{1/q}\big)$$

for every unitarily invariant norm.

Proof. Since X is a contraction, we have $\|X\| \le 1$ and so $\frac{1}{\|X\|^{2r}} \ge 1$. In Lemma 3.8(b), letting $Y = A$ and $\alpha = \frac{1}{\|X\|^{2r}}$, we have

$$f\Big(\frac{|AXB|^{2r}}{\|X\|^{2r}}\Big) \ge \frac{f(|AXB|^{2r})}{\|X\|^{2r}},$$

and so

$$\Big|\Big|\Big| f\Big(\frac{|AXB|^{2r}}{\|X\|^{2r}}\Big) \Big|\Big|\Big| \ge \frac{|||f(|AXB|^{2r})|||}{\|X\|^{2r}}. \tag{3.8}$$

Now, the result follows from Theorem 3.5 and (3.8). $\square$

We conclude this article with the following remark.

Remark 3.10. Inequalities for arbitrary matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, which are equivalent to the inequalities given in Sections 2 and 3 for positive semidefinite matrices can be obtained by replacing the matrices A and B by $|A|$ and $|B|$, respectively, and then using Lemma 2.6. Indeed, results equivalent to Theorem 2.7 and Corollary 3.6 can be stated as follows. Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ with $X \neq 0$, and let $p, q, r, \alpha \in \mathbb{R}$ such that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, and $\alpha \in [0, 1]$. If f is an increasing submultiplicative function on $[0, \infty)$ with $f(0) = 0$ such that $f(t)$ and $f^{\min(p,q)}(t^r)$ are convex, then

(1)
$$\big|\big|\big| f\big(|AXB^*|^{2r}\big) \big|\big|\big| \le \big(\big|\big|\big| f^p\big(\big|\alpha|A|^2X + (1-\alpha)X|B|^2\big|^r\big) \big|\big|\big|^{1/p} \\ \times \big|\big|\big| f^q\big(\big|(1-\alpha)|A|^2X + \alpha X|B|^2\big|^r\big) \big|\big|\big|^{1/q}\big)$$

for every unitarily invariant norm,

(2)
$$\big|\big|\big| f\big(|AXB^*|^{2r}\big) \big|\big|\big| \\ \le \frac{f(\|X\|^{2r})}{\|X\|} \big(\big|\big|\big| \alpha f^p\big(|A|^{2r}\big)X + (1-\alpha)Xf^p\big(|B|^{2r}\big) \big|\big|\big|^{1/p} \\ \times \big|\big|\big| (1-\alpha)f^q\big(|A|^{2r}\big)X + \alpha Xf^q\big(|B|^{2r}\big) \big|\big|\big|^{1/q}\big)$$

for every unitarily invariant norm.

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